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Dear people,

Here is a limited edition preliminary version of a short paper I just wrote. I would like any comments or corrections you might have, because I'm planning to send it out to lots of people in ~3 weeks.

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Universal Links

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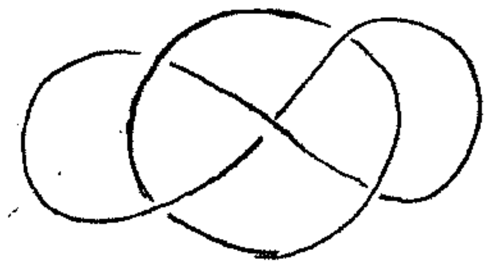
We define a link L in a closed, oriented three-manifold M to be <u>universal</u> if every closed, oriented three-manifold N contains a link $L' \subset N$ such that $N - L'$ is homeomorphic to a finite sheeted covering space of $M - L$. The link L is a <u>universally branching link</u> if every closed, oriented three-manifold N is a branched cover of M over L. Clearly, if L is a universally branching link it is a universal link.

<u>Theorem A</u>. <u>Universal links and universally branching links exist in S^3</u>.

Theorem A is an immediate consequence of Theorems B and C:

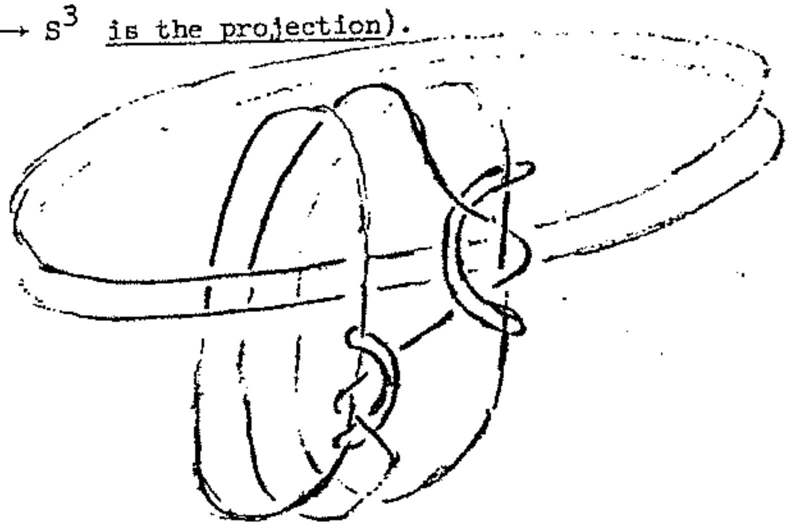
<u>Theorem B</u>. <u>The Whitehead link is universal. For every link</u> $L \subset S^3$, <u>there is a link</u> $L' \supset L$ <u>whose complement is a finite-sheeted covering of the Whitehead link complement</u>.



The Whitehead link is a universal link.

Figure 1

Theorem C. There is a link $L_1 \subset S^3$ (shown below) such that every closed oriented three-manifold M is a branched covering of S^3 over L_1 with branching index 1 or 2 for each component of $p^{-1}(L_1)$ in M. (where $p: M \longrightarrow S^3$ is the projection).



A universally branching link.

Figure 2

Questions. Is there a universal knot? Is the figure-eight knot universal? Is the Whitehead link a universally branching link?

Corollary 1. There is a discrete group Γ acting on hyperbolic space $\mathbb{H}^3$ with compact quotient such that every compact manifold N is homeomorphic to $\mathbb{H}^3/\Gamma'$ where Γ' is a subgroup of finite index in Γ.

Proof of Corollary: By applying Theorem B to the link L of Theorem C, we obtain a link L' whose complement has a hyperbolic structure which is a universal branching link, with branch indices 1 and 2.

By the theory of hyperbolic Dehn surgery (see [Thurston]), the orbifold O with underlying space S^3, singular locus L', and local group $\mathbb{Z}/n$ is hyperbolic for n sufficiently high. Take n to be any even integer such that this orbifold is hyperbolic. The fundamental group Γ of O is the discrete group of isometries of $\mathbb{H}^3$ needed for the corollary. □

Proof of Theorem B. Every closed oriented three-manifold contains a link whose complement is homeomorphic to the complement of a link in S^3, by [Lickorish] or [Wallace], so it suffices to prove the second assertion.

Let $L \subset S^3$ be an arbitrary link with k strands. Arrange L in the form of a closed braid, and add extra trivial loops if necessary so that each component of L goes around the braid the same number of times, say n. Make a further rearrangement in some cross-section of the braid to obtain a braid B such that each strand hits the cross-section at positions i, then $i+k$, then $i+2k$, etc., from left to right counted mod(kn). Our picture is this:

Every link can be arranged in this form, for sufficiently high n .

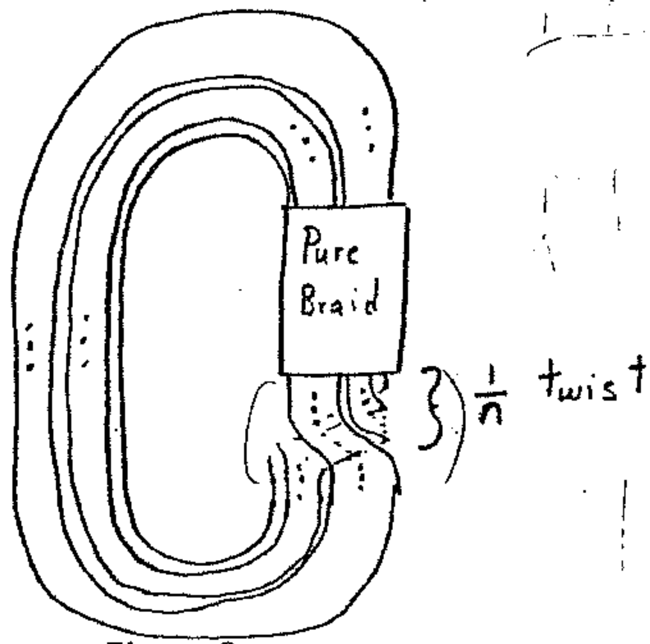


Figure 3

Every pure braid is a product of generators $\sigma_{k,\ell}$ $[k < \ell]$ and their inverses.

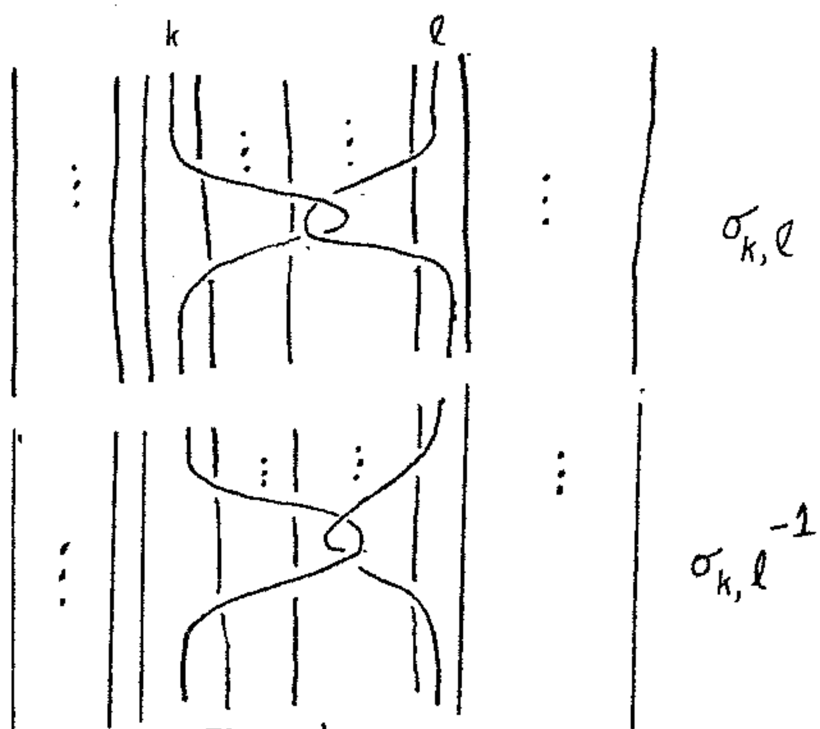


Figure 4

To achieve a single twist as in the picture above, use a circle which bounds a disk intersecting only these two strands:

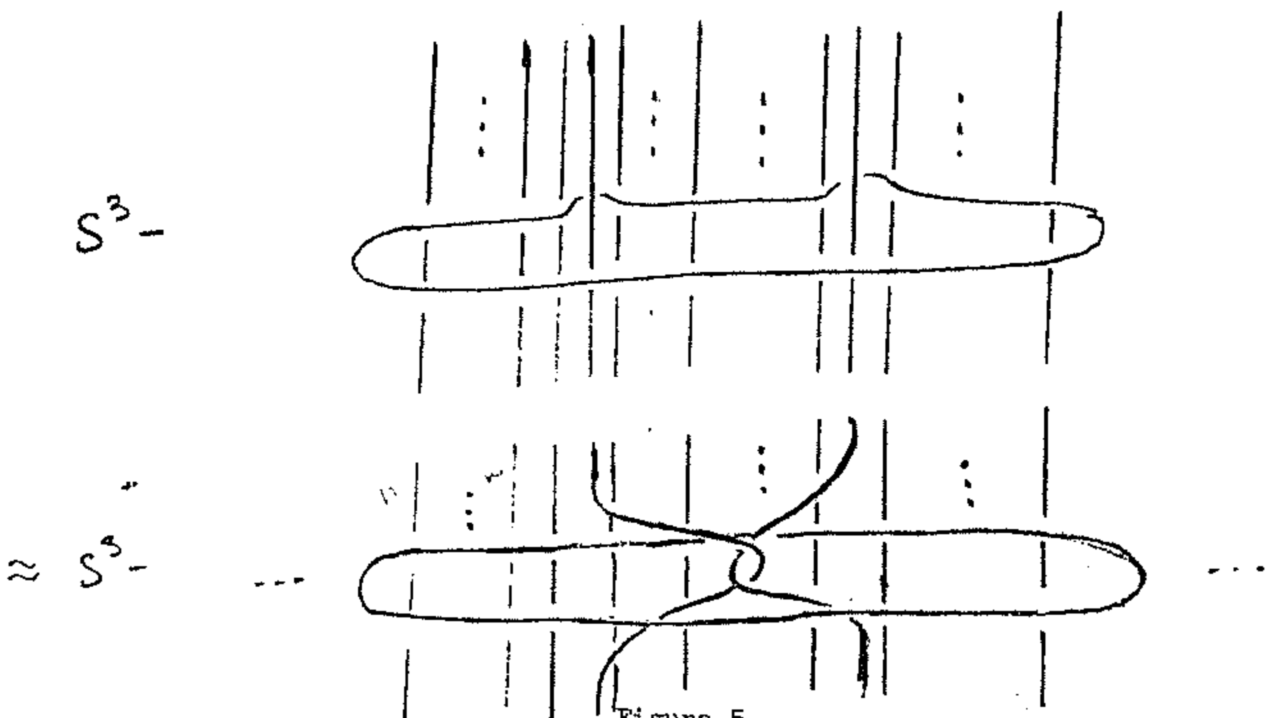


Figure 5

(To make a homeomorphism, cut along a spanning disk for C , make a full twist, and glue back.)

By adding a number of small circles linking the horizontal circles with the vertical circle, we can arbitrarily make a horizontal circle link any two strands we wish, without changing the complement.

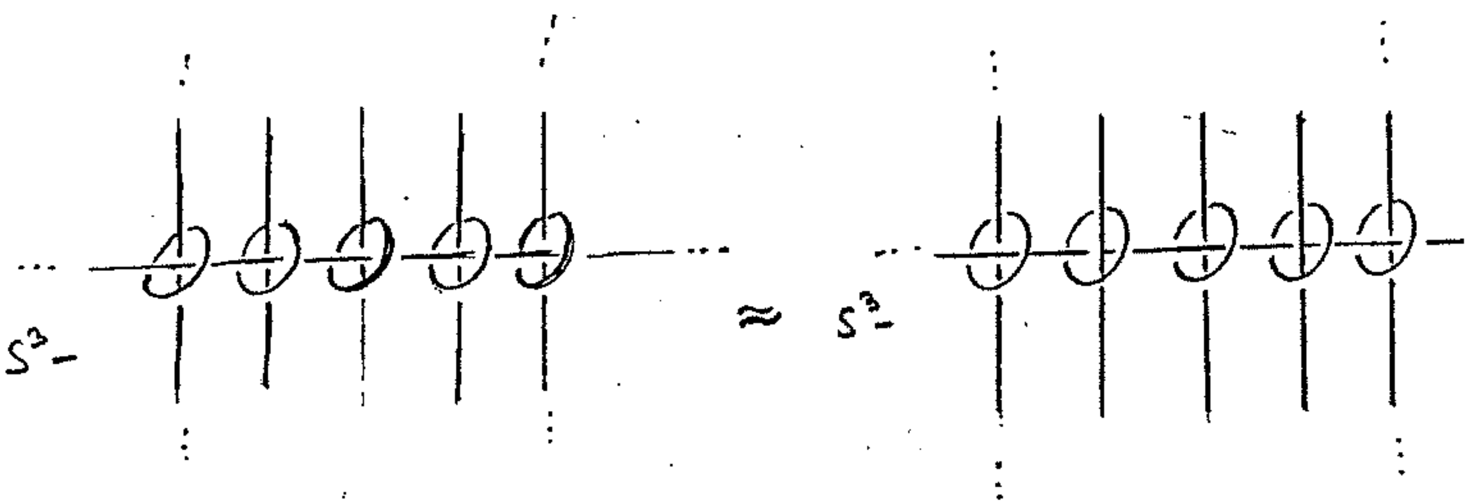


Figure 6

Therefore, the braid B can be obtained from a link of the following form:

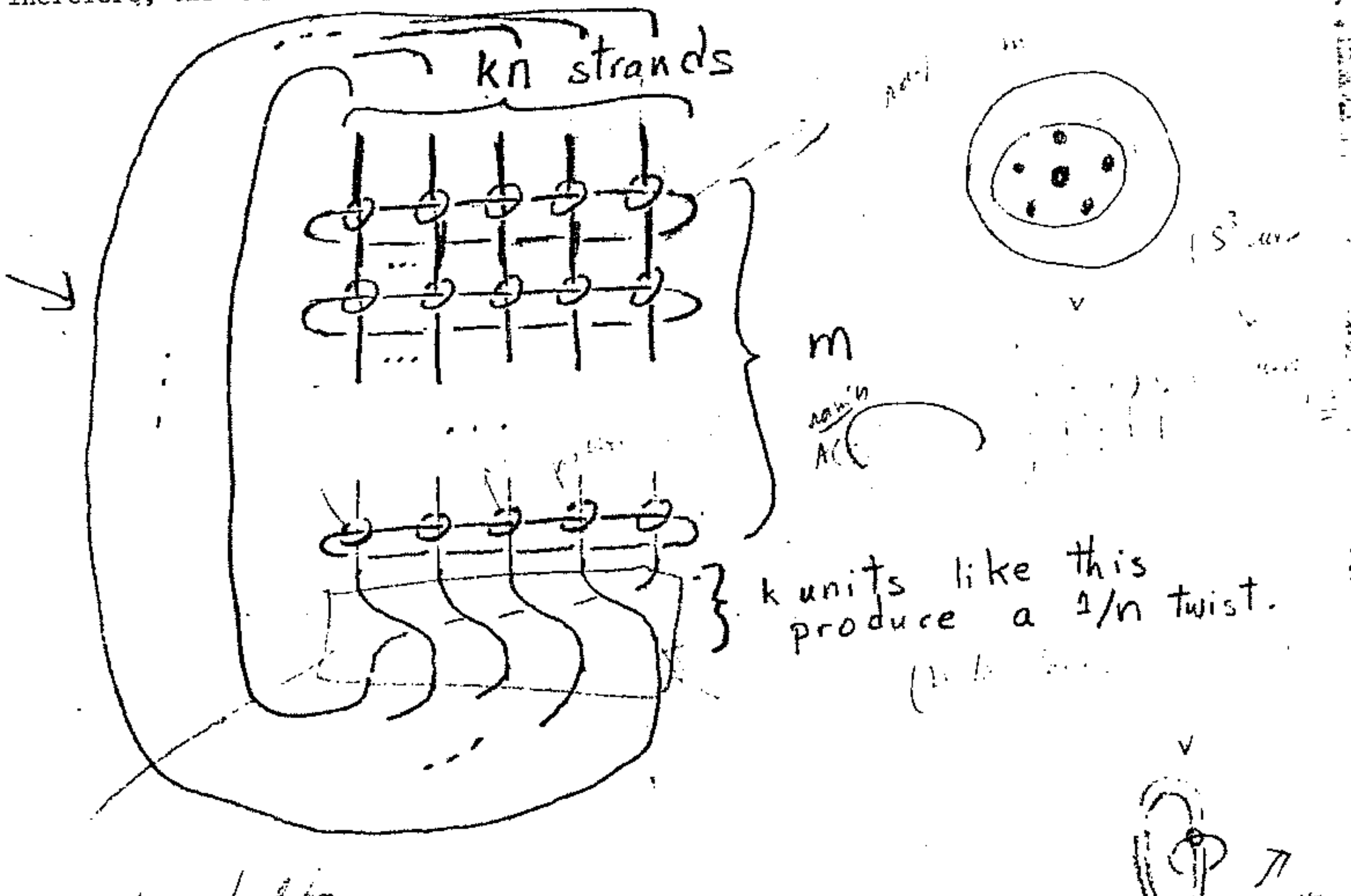


Figure 7 $L_{k,n,m}$

The link $L_{k,n,m}$ is contained in a thin neighborhood of an unknotted torus in S^3. Put in two circles representing the two axes of this torus, to obtain a link $L'_{k,n,m}$. The complements of the links $L'_{k,n,m}$ are often related via covering spaces of toruses over toruses. In fact, they are covering spaces of the complement of $L'_{1,1,0}$:

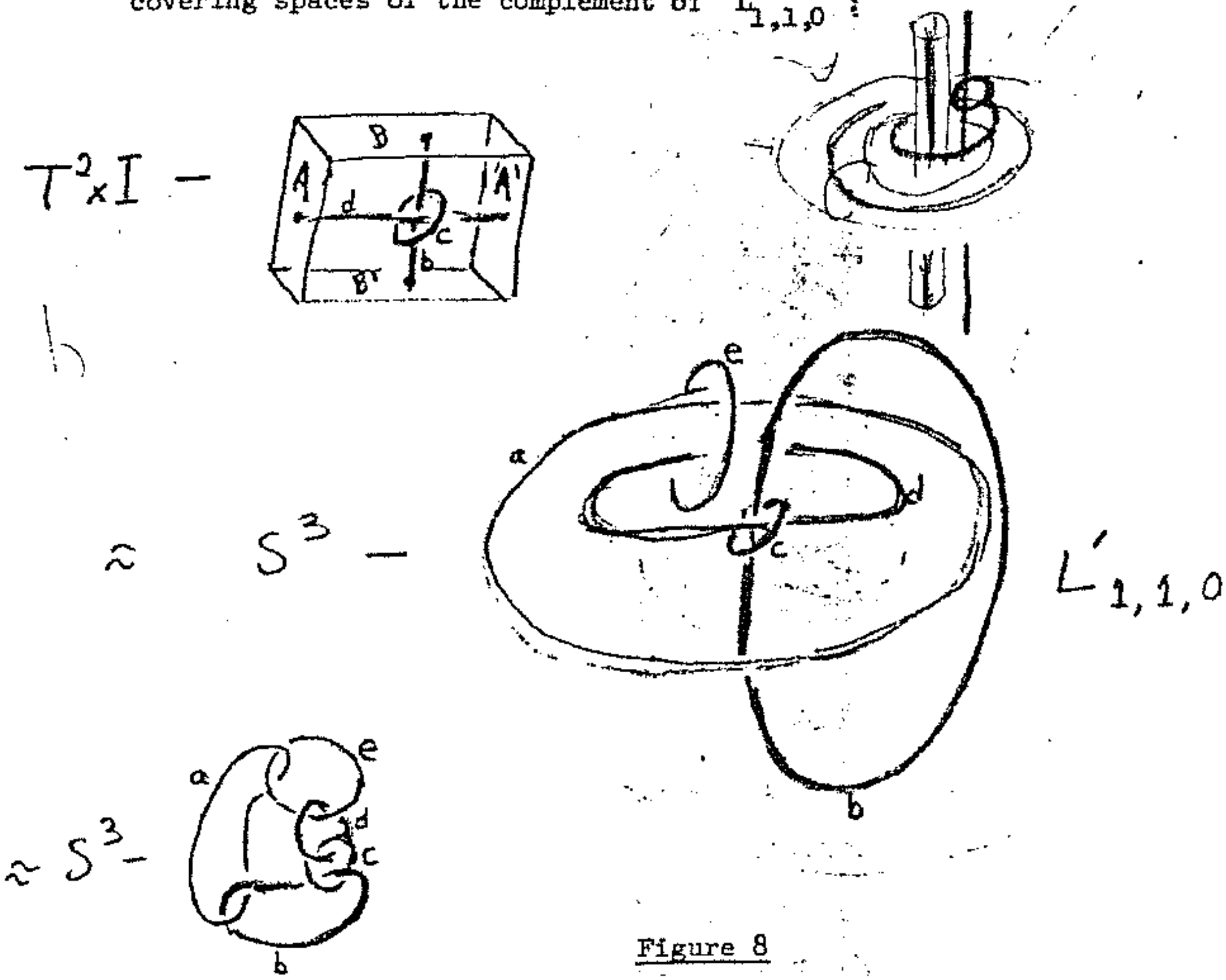


Figure 8

The last picture is obtained by performing a twist on a disk spanning component c , to unlink b and d .

This is our first universal link. Now add an additional component (Figure 9) and perform a twist about its spanning disk.

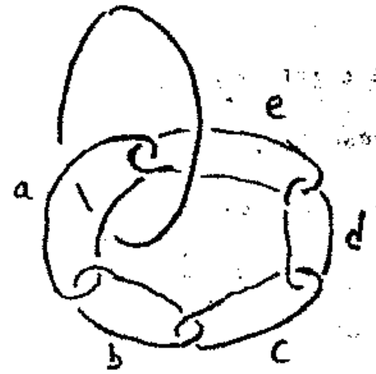


Figure 9

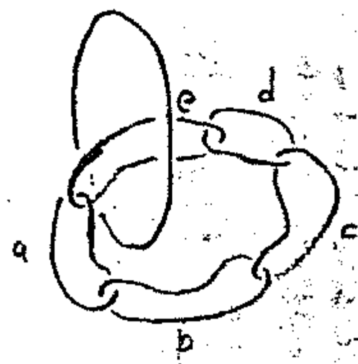


Figure 10

This link has the same complement as Fig. 9

The link above is a five-fold covering space of the Whitehead link.

OK

Figure 11

B

Proof of Theorem C. We begin with the theorem of [Alexander], sharpened by [Hilden] and [Montesinos] that every three-manifold is a branched cover of S^3 over some link L.

Proposition. L can be rearranged as a braid with the following form of projection, with some set of crossings.

A pattern for a closed braid with M·N crossings to be chosen arbitrarily.

Note how the pattern fits together at the top and bottom.

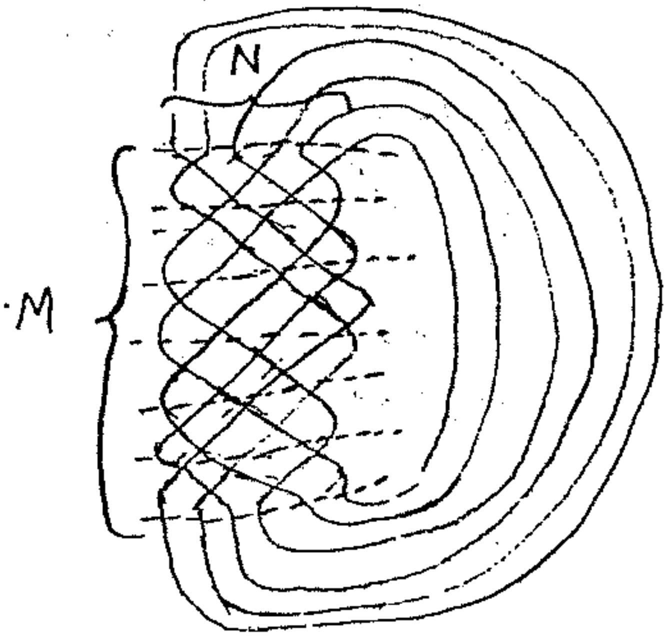


Figure 12

Proof of proposition. First arrange L in the braid form used for the proof of Theorem B, as a braid B which is a pure braid followed by a fractional twist. In the case the pure braid is trivial, it can be arranged in this pattern:

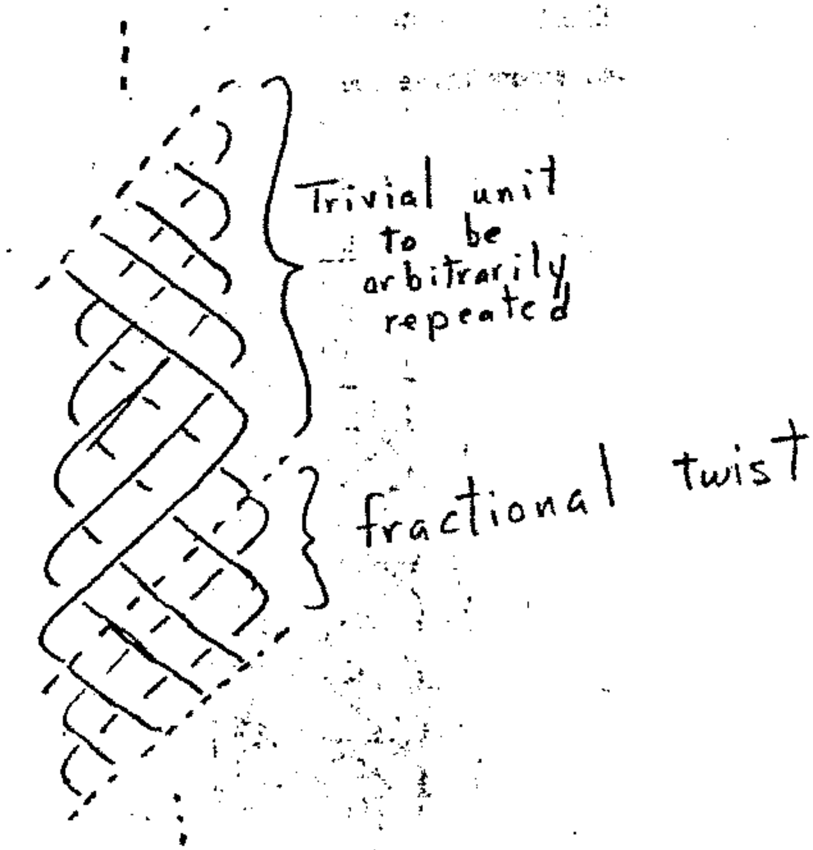
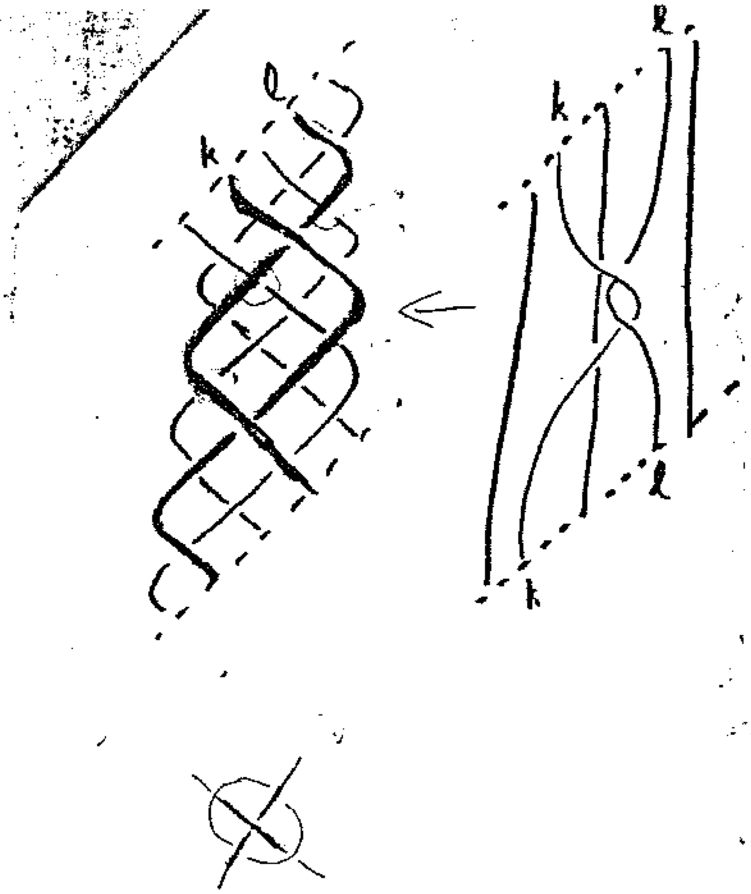


Figure 13

The trivial unit is formed by laying down strands one at a time. The order is right to left in the instance above, but any order will work.

The pure braid generators and their inverses can be realized by modifying crossings in the trivial unit above:



The braid generator $\sigma_{k,\ell}$ (or $\sigma^{-1}_{k,\ell}$) is formed by first laying down all strands except the kth and ℓth, then twisting these two together.

Figure 14

Proposition □

Here is a basic trick for modifying crossings with the use of branched coverings. Consider the following portion of a link $M = L \cup C$ in S^3

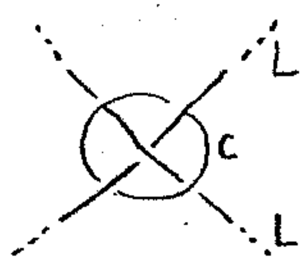


Figure 15

Form the two-fold branched cover of S^3 over C. In this branched cover, each component of L has two components lying above it. To see this branched cover, cut S^3 along a disk D spanning C, intersecting L in two points, to obtain D^3. Round out the two halves of the cut, so D^3 is embedded in S^3 as the complement of a small ball, and fill in this ball with another copy of D^3, identifying cuts properly; this is the two-fold branched cover.

In this two-fold cover, the union of the "big" components of L are in the form of L, with the single crossing changed.

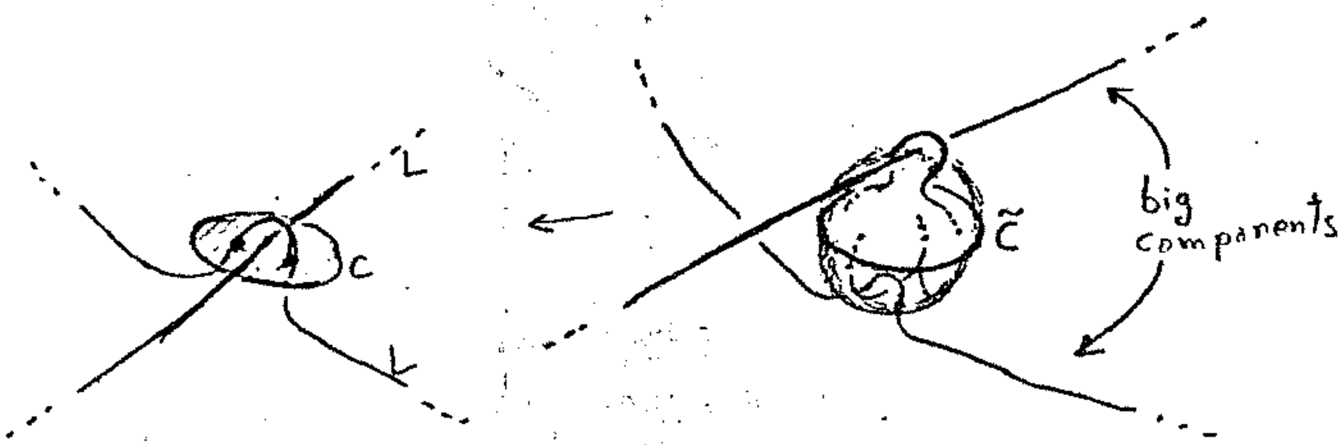
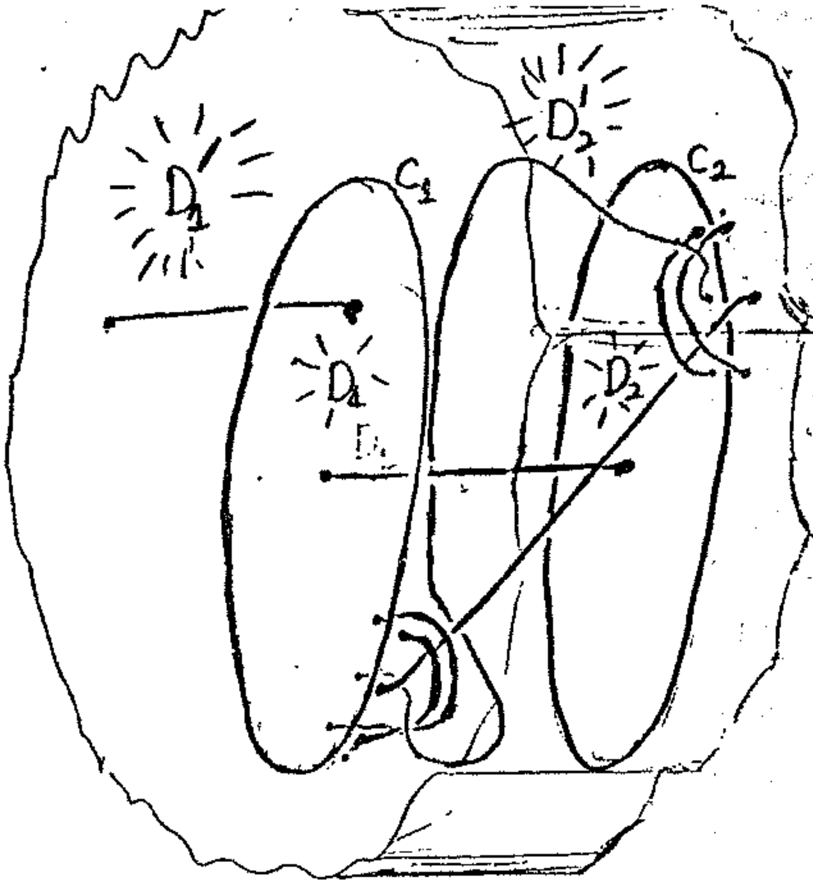


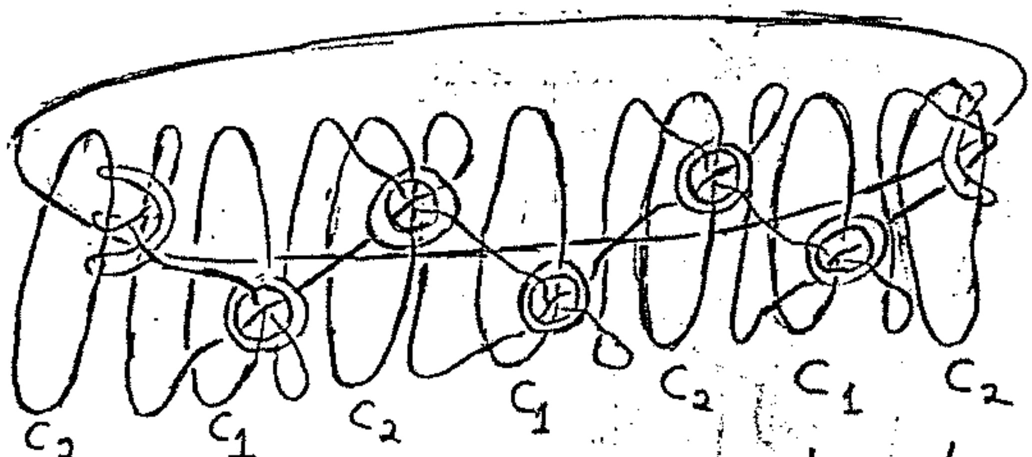
Figure 16

Applying this trick in our situation, we see that any branched covering of our link is a branched covering of a link in this form (of same length and width), a cylinder formed of a certain fabric.

Figure 19

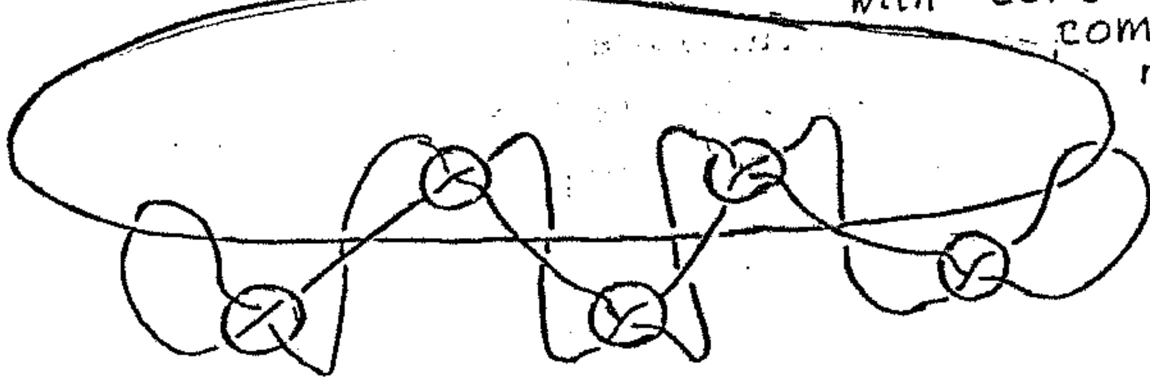


The basic unit



6 basic units glued together, forming a branched cover

with certain components removed



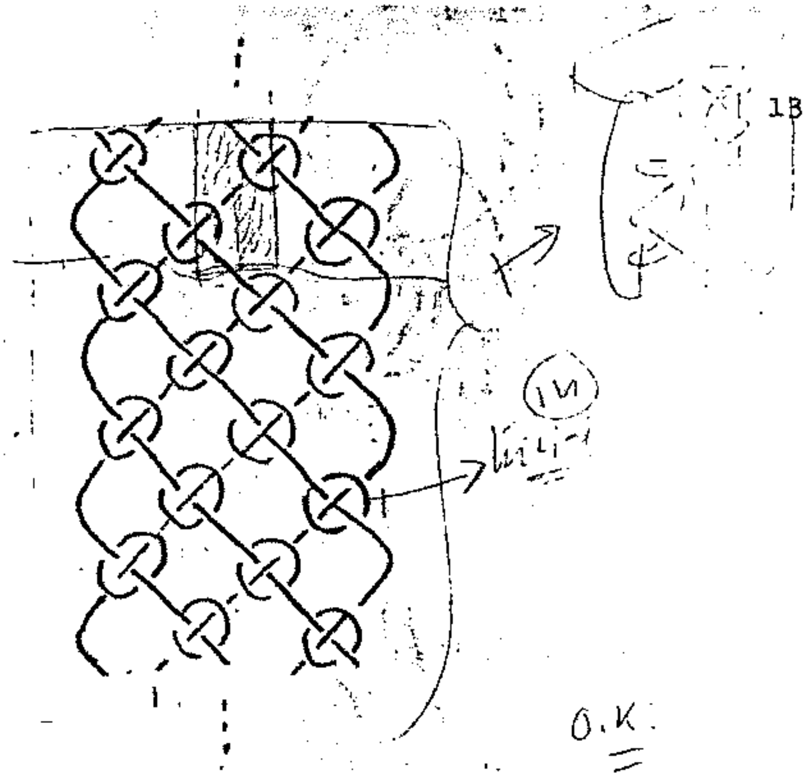


Figure 17

These links can be obtained from branched covering of the following link, L_0, as a union of some of the components of $p^{-1}(L_0)$:

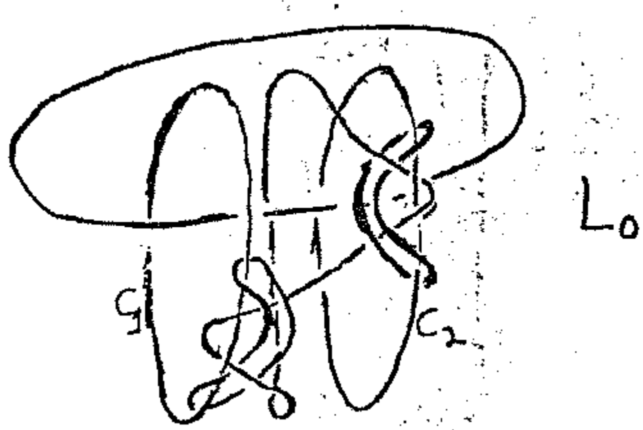


Figure 18

To obtain a fabric cylinder of arbitrary width w, form a branched covering by cutting along disks D_1 and D_2 spanning C_1 and C_2, round out the resulting manifold $(S^2 \times I)$, glue w copies together side by side in the three-sphere, and then close the left-hand and right-hand cuts.

Now we form the n-fold branched cover about the central axis, to obtain our arbitrary fabric cylinder.

In this construction, the branching index is 2 everywhere except at the central axis. To get a link L_1 so that every closed oriented three-manifold is a branched cover over L_1 with branching index always 1 or 2 , replace the central axis of L_0 by two parallel circles.

A cover with branching index n about a single circle can always be replaced by a cover with branching indices 1 and 2 about two parallel circles:

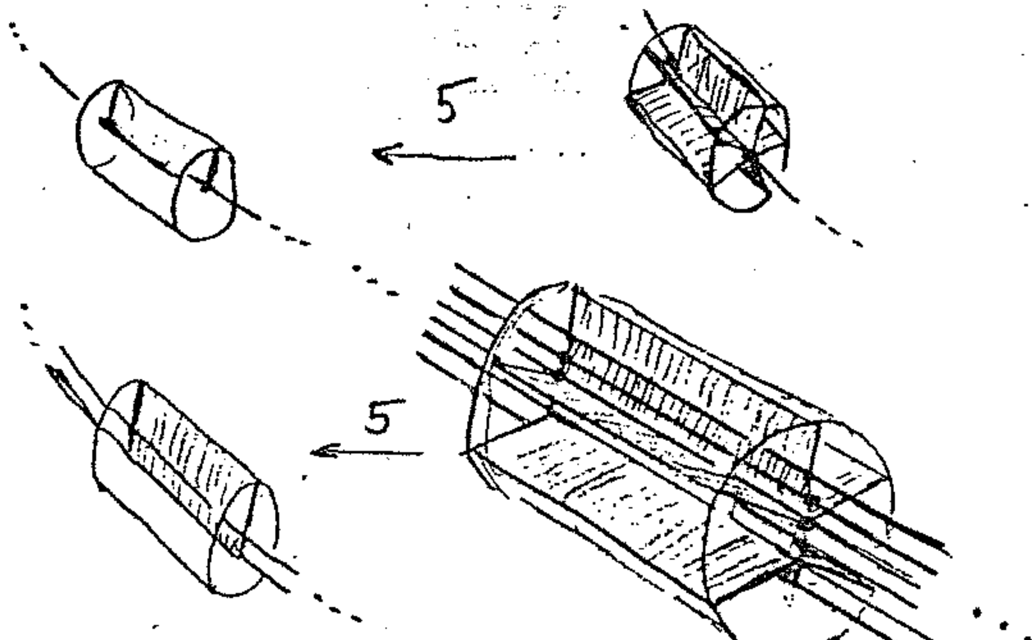


Figure 20

C □

It would be interesting to know more about which links are universal and universally branching links, and in particular, it would be nice to find a simpler example of a universally branching links. A variation of our construction for a universally branching link, making use of a rectangle of our fabric instead of a cylinder, yields the following example.

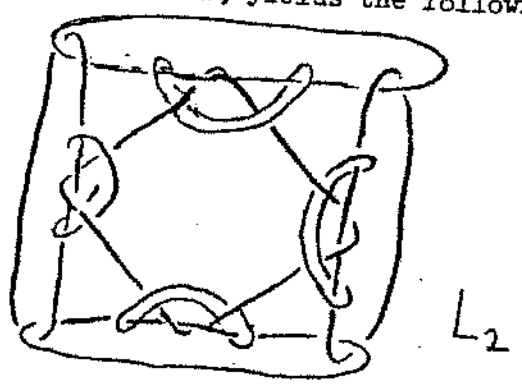


Figure 21

which is a two-fold branched cover of

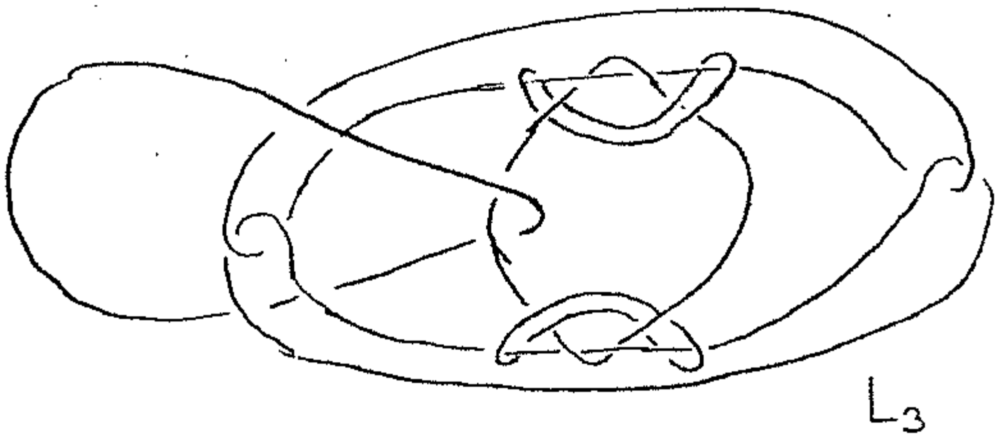


Figure 22

(The branching indices for L_2 or L_3 can be taken to be always 1 or 2.) Many other examples can be constructed, of course. There are probably some universally branching links which are reasonably simple, but I have not discovered any which are substantially simpler than these.

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