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Tracking an Omnidirectional Evader with a Differential Drive Robot

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Abstract In this paper we consider the surveillance problem of tracking a moving evader by a nonholonomic mobile pursuer. We deal specifically with the situation in which the only constraint on the evader's velocity is a bound on speed (i.e., the evader is able to move omnidirectionally), and the pursuer is a nonholonomic, differential drive system having bounded speed.

We formulate our problem as a game. Given the evader's maximum speed, we determine a lower bound for the required pursuer speed to track the evader. This bound allows us to determine at the beginning of the game whether or not the pursuer can follow the evader based on the initial system configuration. We then develop the system model, and obtain optimal motion strategies for both players, which allow us to establish the long term solution for the game. We present an implementation of the system model, and motion strategies, and also present simulation results of the pursuit-evasion game.

Keywords Pursuit-Evasion · Tracking · Nonholonomic Constraints

A preliminary version of portions of this work appeared in [24].

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1 Introduction

In this paper, we consider the surveillance problem in which a nonholonomic mobile robot (the pursuer) tracks an unconstrained mobile evader, maintaining surveillance of the evader at all times, at a constant surveillance distance. We assume that both the pursuer and evader have full knowledge of the other's state, and that the speeds of the pursuer and evader are bounded (though they do not necessarily have the same bound). We consider here a purely kinematic problem, and neglect any effects due to dynamic constraints (e.g., acceleration bounds).

In our past research, we have developed motion strategies for other versions of the tracking problem. In [26] we have considered the case where both the pursuer and the evader are modeled as holonomic systems. We analyzed two main scenarios: one in which the distance between the pursuer and the evader is variable but the speed of both players is unbounded, and a second in which the speed of both the evader and the pursuer is bounded but the distance between the pursuer and the evader is constant. This work led to a sufficient condition for escape by the evader, and to pursuit strategies. In [27] we specifically addressed the combinatorial problem inherent to any strategy that considers visiting several locations in an environment with obstacles. We considered simultaneously bounded speed for both players and a variable distance separating them, but we used a simplified definition of visibility. In [27] we also provided complexity results for the problem.

The distinguishing feature of our current work is the consideration of nonholonomic constraints on the motion of the pursuer. Such constraints qualitatively change the solutions to the surveillance problems that we have previously considered. By imposing nonholonomic constraints on the motion of the pursuer, we are able to model a more realistic set of robotics surveillance problems, since most mobile robots are subject to such constraints, while many evaders are not. For example, the results we present here apply to problems for which the pursuer is a wheeled mobile robot tracking a human evader.

As is well known in mobile robotics research, constraints that are defined in terms of time derivatives of configuration variables and that cannot be integrated to eliminate these derivatives are known as nonholonomic constraints [18, 20]. Motion planning for robots with nonholonomic constraints has been an active research area since the 1990's (see, e.g., [2, 19, 22, 23, 32]). From the point of view of path planning, an important consequence of nonholonomic constraints is that the existence of a collision-free path in the configuration space does not necessarily imply the existence of a feasible trajectory for the constrained system [20].

The problem we consider in this paper is related to pursuit-evasion games. A great deal of previous research exists in the area of pursuit and evasion, particularly in the area of dynamics and control in the free space (without obstacles) [9, 14, 1]. The pursuit-evasion problem is often framed as a problem in noncooperative dynamic game theory [1].

A pursuit-evasion game can be defined in several ways. For example, one or more pursuers could be given the task of *finding* an evader [15, 12]. To solve this problem, the pursuer(s) must sweep the environment so that the evader is not able to eventually sneak into an area that has already been explored. Deterministic [28, 33, 8, 31, 34] and probabilistic algorithms [35, 10, 6] have been proposed to solve this problem. This problem is different from ours, since we assume that the pursuer is initially aware of the evader's position, with the goal of maintaining sight of the evader.

Alternatively, the pursuer(s) might have as a goal to actually "catch" the evader(s), that is, move to a contact configuration, or closer than a given distance. In the classical differential game, called the homicidal chauffeur problem [14], a faster pursuer (w.r.t. the evader) has as its objective to get closer than a given constant distance (the capture condition) from a slower but more agile evader. The pursuer is a vehicle with a minimum turning radius. The game takes place in the Euclidean plane without obstacles, and the evader aims to avoid the capture condition. In our problem, we also consider a faster pursuer and a more agile evader moving in an environment without obstacles. However, there are some important differences between the problem described in this paper and the homicidal chauffeur problem. In this paper the game is defined differently. The objective of the pursuer is to maintain the capture condition not to obtain it; that is, the pursuer aims to maintain a given constant distance from the evader, while the evader aims to break that constant surveillance distance. Also we consider that the pursuer is a Differential Drive Robot, i.e. the pursuer can rotate in place.

Our tracking problem consists of determining a pursuer motion strategy to always maintain surveillance of the evader by the pursuer (assuming surveillance in the initial state). The evader is under pursuer surveillance whenever the evader is at a constant distance L from the pursuer (L can be considered as the upper limit of the physical sensor used by the pursuer). It is pertinent to analyze this specific case for the following reasons: First, commercially available sensors (laser and cameras) have upper range limits. In particular, even in the *absence of obstacles*, if the evader is farther from the pursuer than the sensor range then its location is unknown, and the surveillance is broken. Second, our results are applicable to a variety of non-surveillance problems. For example, shared manipulation by a human and a nonholonomic robot imposes similar constraints on the two agents (maintaining a fixed relative distance between the agents).

Recent years have seen a growing interest in related problems within the robot motion planning community. For instance, a related problem, which has been extensively studied, consists in maintaining visibility of a moving evader in an environment with obstacles [21, 11, 17, 13, 3]. Game theory is proposed in [21] as a framework to formulate the tracking problem, and an online algorithm is presented. In [5], an algorithm is presented that operates by maximizing the probability of future visibility of the evader. This algorithm is also studied with more formalism in [21]. The work in [7] presents an approach that takes into account the positioning uncertainty of the robot pursuer. The approach presented in [25] computes a motion strategy by maximizing the *shortest distance to escape* —the shortest distance the evader needs to move in order to escape the pursuer's visibility region. In [11], a technique is proposed to track an evader without the need of a global map. Instead, a range sensor is used to construct a local map of the environment, and a combinatorial algorithm is then used to compute a motion for the pursuer at each iteration. In [13], a greedy approach to the surveillance problem was proposed, in which a local minimum risk function, called the vantage time, was optimized.

Others have studied an extended version of the problem involving multiple participants of each kind (evaders and pursuers). For example, [29] developed a method that attempts to minimize the total time in which the evaders escape surveillance. In a similar vein, [17] combined the application of mobile and static sensors, using a measure of the degree of occlusion, based on the average mean free path of a random line segment.

Pursuit-evasion has been found to be of use in a variety of interesting applications. For example, in [16], the authors noticed the similarity between pursuit-evasion games and mobile-routing for networking. Applying this similarity, they proposed motion planning algorithms for robotic routers to maintain connectivity between a mobile user and a base station.

In spite of these efforts, to date only heuristic solutions have been reported, both for the general tracking problem, and for the constrained tracking problem (the problem we consider in the present paper). Neither exact solutions nor correct and complete algorithms[1] to find them have previously been reported.

In this paper, we begin in Section 2 by developing the system equations for the pursuer-evader system. Using these equations together with the bound on evader speed, in Section 2.1, we deduce a lower bound for the pursuer speed that is required to maintain a constant distance from the evader in an environment without obstacles. In Section 3, we present the main contribution of this work, namely, (a) the determination of the conditions that permit each of the players to win, and (b) the corresponding motion strategies. These strategies are demonstrated in various scenarios in Section 4.

In what follows, we will refer to the line segment that connects the pursuer and evader as the *rod* due to an analogy with the motion planning problem studied in [30]. The evader controls the position of the rod's origin (x, y) and the control of the rod's orientation ϕ is shared by both players. We consider an antagonistic evader that moves continuously, and that full state feedback is available for both players.

[1] Such an algorithm is guaranteed to return a correct solution when one exists, or to report failure in finite time when a solution does not exist.

2 System Model

Figure 1 shows the geometric description of the system. The variables x_e, y_e, x_p, y_p denote the evader and pursuer positions with respect to the global reference frame. The variable θ is the angle of the pursuer's wheels with respect to the global x axis, and ϕ represents the angle between the rod and the global x axis. One can also interpret ϕ as the angular coordinate of the pursuer position relative to the evader in polar coordinates and therefore it may be considered to correspond to the sensor angle. Likewise, we can express the orientation of the evader relative to the pursuer as $\phi_p = \phi + \pi$. Note that $\dot{\phi}_p = \dot{\phi}$. The angle of the evader velocity vector with respect to the global x axis is denoted by ψ. Although all these quantities are time dependent, in what follows the explicit time dependence will be omitted, in order to simplify the notation.

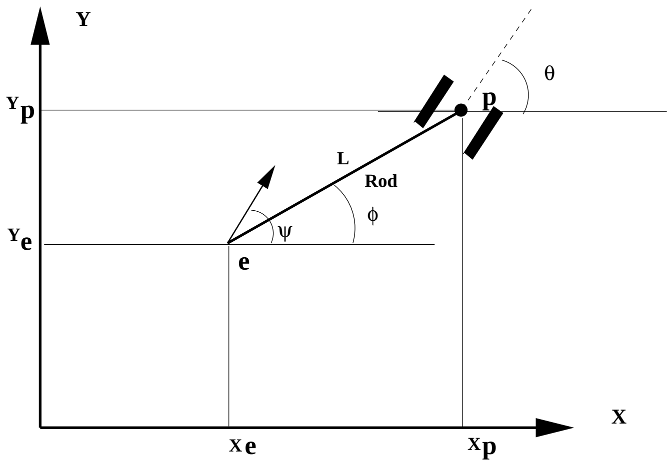

Fig. 1 The geometric model of the pursuer-evader system

For the evader state, the equations are simple, since the evader velocity is taken as an independent input to the system, under full control of the evader. Hence, we have

$$\dot{x}_e = V_e \cos\psi \quad (1)$$

$$\dot{y}_e = V_e \sin\psi \quad (2)$$

in which V_e is the linear velocity of the evader, and we use

$$u_1 = V_e \tag{3}$$
$$u_2 = \psi \tag{4}$$

under the constraint $|V_e| \leq V_e{}^{\max}$. The control u_2 appears as the argument to cos and sin functions in the state equations, and thus the state equations cannot be factored nicely into the form $A(q)u(q)$.

For the pursuer velocity, we have the usual parametrization using unicycle kinematics

$$\dot{x}_p = V_p \cos\theta \tag{5}$$

$$\dot{y}_p = V_p \sin\theta \tag{6}$$

with the constraint that $|V_p| \leq V_p^{\max}$. Since the pursuer is a differential drive robot, we use the usual assignment of control inputs [2]. For a robot with wheels of unit radius, the control inputs are therefore given by

$$u_3 = V_p = \frac{w_r(t) + w_l(t)}{2} \tag{7}$$

$$u_4 = \dot{\theta} = \frac{w_r(t) - w_l(t)}{2b} \tag{8}$$

in which b is the distance between the center of the robot and the wheel location. When $u_3 = 0$ and $u_4 \neq 0$, the robot rotates without translation, and when $u_3 \neq 0$ and $u_4 = 0$ the robot translates without rotation.

The bounds on the pursuer's speed derive from bounds on the rate at which the wheels can spin, and are thus naturally expressed as bounds on u_3 and u_4. In this paper, we will assume symmetric and equal bounds for the two wheels, $-w^{\max} \leq w_r, w_l \leq w^{\max}$. We denote these bounds by (considering the radius of the wheels equal to 1):

$$V_p^{\max} = u_3^{\max} = \max u_3 = \frac{1}{2}\max\{w_r(t) + w_l(t)\} = w^{\max}$$

$$u_4^{\max} = \max u_4 = \frac{1}{2b}\max\{w_r(t) - w_l(t)\} = \frac{1}{b}w^{\max}$$

so that $u_3^{\max}$ is the maximum forward linear speed of the pursuer and $u_4^{\max}$ is the maximum counterclockwise rate of rotation of the pursuer.

When the surveillance constraints are satisfied, the relationship between evader and pursuer positions is given by:

$$\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} x_e \\ y_e \end{pmatrix} + L \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \tag{9}$$

All pursuer velocities that maintain a constant distance L between the evader and the pursuer must therefore satisfy:

$$\begin{pmatrix} \dot{x}_p \\ \dot{y}_p \end{pmatrix} = \begin{pmatrix} \dot{x}_e \\ \dot{y}_e \end{pmatrix} + L\dot{\phi} \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \tag{10}$$

From equations 5, 6 and 10 we obtain the following expression for the evader velocity:

$$\begin{pmatrix} \dot{x}_e \\ \dot{y}_e \end{pmatrix} = V_p \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + L\dot{\phi} \begin{pmatrix} \sin\phi \\ -\cos\phi \end{pmatrix} \tag{11}$$

which can be re-written as

$$\begin{pmatrix} \dot{x}_e \\ \dot{y}_e \end{pmatrix} = \begin{pmatrix} \cos\theta & L\sin\phi \\ \sin\theta & -L\cos\phi \end{pmatrix} \begin{pmatrix} u_3 \\ \dot{\phi} \end{pmatrix} \tag{12}$$

If we define the matrix A as

$$A = \begin{pmatrix} \cos\theta & L\sin\phi \\ \sin\theta & -L\cos\phi \end{pmatrix} \tag{13}$$

we find

$$\det A = -L\cos(\theta - \phi) \tag{14}$$

which implies that the pursuer can follow the evader only when $(\theta - \phi) \neq \pm\frac{\pi}{2}$. In other words, the rod cannot have a relative angle to the pursuer wheels equal to $\pm\frac{\pi}{2}$ because this would require unbounded pursuer speed to maintain surveillance.

Using equations 12 and 13, the relationship between the speed of the evader and the linear velocity of the pursuer can be expressed as

$$\dot{x_e}^2 + \dot{y_e}^2 = (u_3 \quad \dot{\phi}) A^T A \begin{pmatrix} u_3 \\ \dot{\phi} \end{pmatrix} \tag{15}$$
$$= {u_3}^2 - 2u_3\dot{\phi}L\sin(\theta - \phi) + L^2\dot{\phi}^2 \tag{16}$$

The pursuer must be able to track the evader for any evader velocity that satisfies $\sqrt{\dot{x_e}^2 + \dot{y_e}^2} \leq V_e{}^{\max}$, and in particular, when the evader moves at maximum speed, the pursuer velocity must satisfy

$$f(u_3, \dot{\phi}) = {u_3}^2 - 2u_3\dot{\phi}L\sin(\theta - \phi) + L^2\dot{\phi}^2 \leq (V_e{}^{\max})^2$$

which defines the interior of an ellipse in the u_3-$\dot{\phi}$ plane (see Figure 2).

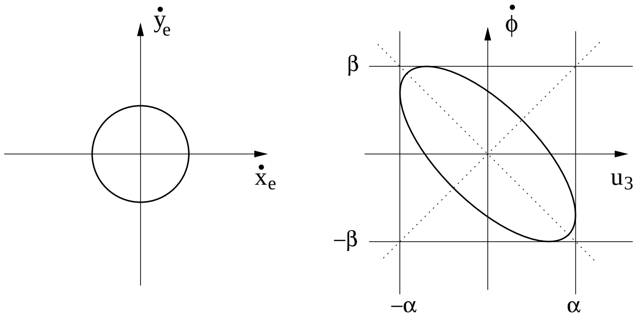

Fig. 2 Velocity bounds in $\dot{x}_e$-$\dot{y}_e$ plane and in the $u_3 - \dot{\phi}$ plane

2.1 Bounds for u_3 and $\dot{\phi}$ to maintain surveillance

To track the evader, u_3 must be able to attain all the values inside the projection of the ellipse onto the u_3 axis. Let α denote the maximal value for this projection of the ellipse (see figure 2). Then we have that $\alpha \leq \max |u_3| = V_p^{max}$.

To determine α we first solve for the value of $\dot{\phi}$ that corresponds to the value of f in equation 16 for the extremal of u_3

$$\frac{\partial f}{\partial \dot{\phi}} = 0 \rightarrow \dot{\phi} = \frac{u_3 \sin(\theta - \phi)}{L}$$

We now substitute this value into $f(u_3, \dot{\phi}) = (V_e{}^{\max})^2$, and solve for $u_3 = \alpha$ as follows

$$\begin{aligned}(V_e{}^{\max})^2 &= u_3{}^2 - 2u_3^2 \sin^2(\theta - \phi) + u_3{}^2 \sin^2(\theta - \phi) \\ &= u_3^2(1 - \sin^2(\theta - \phi))\end{aligned}$$

which implies that

$$\alpha = \frac{V_e{}^{\max}}{|\cos(\theta - \phi)|} = u_3^* \leq V_e^{max} \tag{17}$$

Using a similar analysis, we derive β as a bound on $\dot{\phi}$. In particular, we project the ellipse $f(u_3, \dot{\phi}) = (V_e{}^{\max})^2$ onto the $\dot{\phi}$ axis (see Figure 2), and after manipulations similar to those above we obtain

$$\beta = \frac{V_e{}^{\max}}{|L\cos(\theta - \phi)|} \leq \max |\dot{\phi}| \tag{18}$$

This implies that the pursuer must be able to choose its angular velocity to satisfy this constraint in order to track the evader.

From this analysis it follows that when the inequalities given in 17 and 18 hold, there is a control such that the pursuer can follow the evader moving at maximal velocity, whatever direction the evader chooses. In the next sections we present a detailed analysis that will allow us to determine, for each initial configuration of the system, which player may win the game, along with the corresponding winning strategies.

2.2 Determining u_3 and $\dot{\phi}$ to track the evader

If the evader's controls u_1, u_2 and the values of θ and ϕ are given, then the linear speed u_3^* of the pursuer required to maintain a constant distance L from the evader is in fact fixed. In Appendix A we derive an expression for this value of u_3^*, which is given by:

$$u_3^*(\phi, \theta, u_1, u_2) = \frac{u_1 \cos(u_2 - \phi)}{\cos(\theta - \phi)} \tag{19}$$

Notice that this expression takes its maximum value, which corresponds to the bound presented in 17, when $u_1 = V_e{}^{\max}$ and $u_2 = \psi = \phi$ or $u_2 = \psi = \phi + \pi$

In Appendix B we show that when the pursuer successfully tracks the evader (i.e., when $u_3 = u_3^*$), $\dot{\phi}$ is given by:

$$\dot{\phi}(\phi, \theta, u_1, u_2) = \frac{u_1 \sin(\theta - u_2)}{L\cos(\theta - \phi)} \tag{20}$$

Note that the bound presented in 18 is reached when $u_1 = V_e{}^{\max}$ and $u_2 = \psi = \theta \pm \frac{\pi}{2}$.

If we parametrize the configuration of the pursuer-evader system by the evader position, x_e, y_e, the angle of the rod with respect to the world coordinate frame, ϕ, and the orientation of the pursuer's wheels (heading) with respect to the world coordinate frame, θ, an alternative system model in state-space form is given by

$$\begin{pmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{\phi} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} u_1 \cos u_2 \\ u_1 \sin u_2 \\ \dfrac{u_1 \sin(\theta - u_2)}{L\cos(\theta - \phi)} \\ u_4 \end{pmatrix} \tag{21}$$

It is important to stress the fact that to maintain a constant distance between the evader and the pursuer, equations 19 and 20 in terms of the evader controls must be satisfied.

2.3 Bounds on u_4 when tracking the evader

For a given choice of u_3, there is a finite range of values that can be taken by u_4, since the differential drive robot wheel speeds are bounded (as described in Section 2). As the linear speed of the pursuer $|u_3|$ attains its maximum, the rate of rotation of the pursuer $|u_4|$ attains its minimum. For example, when $|u_3| = V_p^{\max}$, we necessarily have $\dot{\theta} = 0$. Using equations 7 and 8, the bounds on u_4 are most easily deduced by considering

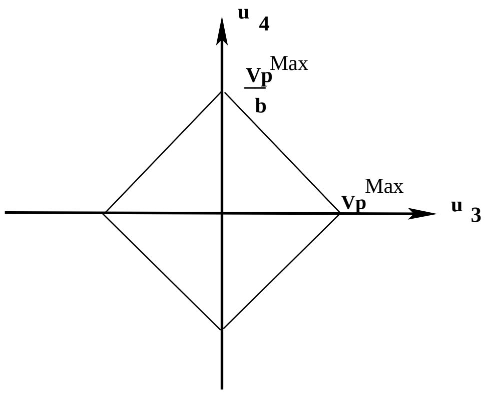

Fig. 3 Control space (u_3, u_4)

individually the case $u_3 < 0$ and $0 \leq u_3$. For $0 \leq u_3$ we have

$$-\frac{1}{b}(V_p^{\max} - u_3) \leq u_4 \leq \frac{1}{b}(V_p^{\max} - u_3)$$

and for $u_3 < 0$ we have

$$-\frac{1}{b}(V_p^{\max} + u_3) \leq u_4 \leq \frac{1}{b}(V_p^{\max} + u_3)$$

These four constraints are illustrated in Figure 3. They can be combined into the single expression

$$|\dot{\theta}| = |u_4(u_3)| \leq \frac{1}{b}\left(V_p^{max} - |u_3|\right) \tag{22}$$

This expression gives the maximum rate of rotation for the pursuer, given a specified linear speed u_3.

3 Evader and Pursuer Strategies

From the analysis presented in Section 2.1 we can establish that if at any time inequality 17 does not hold then the evader wins. But this analysis does not directly address the issue of determining which player wins the game for a given configuration of the system, nor does it determine winning strategies for the pursuer and evader. In this section, we consider these issues. We begin with an intuitive motivation for the pursuer and evader strategies, and then give a more formal treatment.

The result of the pursuit-evasion game depends critically on the rotation speeds for the rod ($\dot{\phi}$) and for the pursuers heading ($\dot{\theta}$). This may be seen in equation 19, which gives the linear velocity u_3^* for the pursuer, so that a constant value for L is maintained. As $|\cos(\theta - \phi)|$ increases, the required value for u_3 decreases, and reaches its minimum when $|\cos(\theta - \phi)| = 1$. On the other hand, when $|\cos(\theta - \phi)| = 0$ there is no pursuer with bounded maximum speed that can maintain surveillance, since $u_3^* \to \infty$. For this reason, a good strategy for the pursuer is to move θ (using $u_4 = \dot{\theta}$), so that $|(\theta - \phi)|$ decreases, while for the evader, a good strategy is to increase this value using $\dot{\phi}$, which depends only on its controls u_1, u_2 (equation 20). In particular, if $\max|\dot{\theta}| = \frac{1}{b}(V_p^{max} - |u_3|)$ is equal to $|\dot{\phi}|$, the pursuer will be able to compensate exactly the rotation that the evader tries to impose on the rod, keeping $|(\theta - \phi)|$ constant. In this case, the best strategy for the pursuer is uniquely determined, and consists in setting $u_3 = u_3^*$, and $u_4 = u_4^*$, with $|u_4^*| = \max|\dot{\theta}| = \frac{1}{b}(V_p^{max} - |u_3^*|)$. The sign for u_4^* must be chosen in such way that the angle between the rod and the pursuer's wheels $(\theta - \phi)$ moves away from $\pm\frac{\pi}{2}$. Specifically:

$$\begin{aligned} u_4^*(\theta, \phi) &= s(\theta, \phi) \max|\dot{\theta}| \\ &= s(\theta, \phi)\frac{1}{b}(V_p^{max} - |u_3^*(\theta, \phi)|) \end{aligned} \tag{23}$$

in which

$$s(\theta - \phi) = \begin{cases} -1 : & (\theta - \phi) \in (0, \frac{\pi}{2}) \bigcup (\pi, \frac{3\pi}{2}) \\ +1 : & (\theta - \phi) \in (\frac{\pi}{2}, \pi) \bigcup (\frac{3\pi}{2}, 2\pi) \end{cases} \tag{24}$$

and the value -1 corresponds to clockwise rotation while the value $+1$ corresponds to counterclockwise rotation.

In the case of the evader, the situation is more complex, since it can control $\dot{\phi}$ directly (equation 20), but also $\max|\dot{\theta}|$ indirectly, since it can maximize the required linear speed of the pursuer u_3^* (see equation (22)). In particular, one can establish the following lemma.

Lemma I:
Let $g(\phi, \theta, u_2) = (|\cos(\phi - u_2)| + \gamma|\sin(\theta - u_2)|)$ with $\gamma = b/L$. Then the following two converse conditions hold.

(i) $\max|\dot{\theta}| < |\dot{\phi}|$ if and only if $V_p^{\max}(|\cos(\theta - \phi)|) < |u_1| g(\phi, \theta, u_2)$

(ii) $\max|\dot{\theta}| \geq |\dot{\phi}|$ if and only if $V_p^{\max}(|\cos(\theta - \phi)|) \geq |u_1| g(\phi, \theta, u_2)$

Note that since b is the radius of the robot pursuer then $L \geq b$ otherwise the evader is in collision with the pursuer, hence $\gamma \in (0, 1]$.

Proof: The inequality $\max|\dot{\theta}| < |\dot{\phi}|$ can be expanded as follows

$$\max|\dot{\theta}| < |\dot{\phi}| \quad (25)$$

$$\frac{1}{b}\left(V_p^{\max} - |u_3^*|\right) < \left|\frac{u_1\sin(\theta-u_2)}{L\cos(\theta-\phi)}\right| \quad (26)$$

$$\frac{1}{b}\left(V_p^{\max} - \left|\frac{u_1\cos(u_2-\phi)}{\cos(\theta-\phi)}\right|\right) < \frac{|u_1\sin(\theta-u_2)|}{|L\cos(\theta-\phi)|} \quad (27)$$

$$V_p^{\max}\ (|\cos(\theta-\phi)|) < |u_1|\left(|\cos(\phi-u_2)| + \gamma|\sin(\theta-u_2)|\right) \quad (28)$$

in which inequality 26 follows from equation 22 (assuming the maximum value of $|\dot{\theta}|$) and from equation 20; inequality 27 follows from equation 19; and inequality 28 follows from straightforward manipulation.

The second part of the proof corresponding to $\max|\dot{\theta}| \geq |\dot{\phi}|$ is analogous to the one presented above, yielding

$$V_p^{\max}\ (|\cos(\theta-\phi)|) \geq |u_1|\left(|\cos(\phi-u_2)| + \gamma|\sin(\theta-u_2)|\right) \quad (29)$$

■

From this result, we can infer that a good strategy for the evader is to choose $(u_1, u_2) = (u_1^*, u_2^*)$ at every time instant, such that

$$(u_1^*, u_2^*) = \arg\max_{u_1,u_2} |u_1| g(\phi,\theta,u_2) \quad (30)$$

If $V_p^{\max}(|\cos(\theta-\phi)|) = u_1^* g(\phi,\theta,u_2^*) - \epsilon$ at any time, for an arbitrarily small $\epsilon > 0$, then this choice will be the only one that will allow the evader to move $\cos(\theta-\phi)$ towards 0. As we will now show, these choices, i.e., (u_1^*, u_2^*) and (u_3^*, u_4^*), actually represent winning strategies for the evader and pursuer respectively, and in some sense they may be considered as equilibrium strategies for the game [14].

We now proceed with a formal development of the conditions that determine the winner of the game and the players' strategies. The following lemma gives conditions on the value of u_2 that maximizes $g(\phi,\theta,u_2)$.

Lemma II:
Consider the following functions:

$$\psi_1 = \arctan\left(\frac{\sin\phi - \gamma\cos\theta}{\cos\phi + \gamma\sin\theta}\right)$$

$$\psi_2 = \arctan\left(\frac{\sin\phi + \gamma\cos\theta}{\cos\phi - \gamma\sin\theta}\right)$$

$$\psi_3 = \arctan\left(\frac{-\sin\phi - \gamma\cos\theta}{-\cos\phi + \gamma\sin\theta}\right)$$

$$\psi_4 = \arctan\left(\frac{-\sin\phi + \gamma\cos\theta}{-\cos\phi - \gamma\sin\theta}\right)$$

The evader control u_2 that maximizes $g(\phi,\theta,u_2)$ for given values of ϕ and θ is given by

$$u_2 = \begin{cases} \psi_1 \text{ or } \psi_4 = \psi_1 + \pi : & (\theta-\phi) \in [0,\pi] \\ \psi_2 \text{ or } \psi_3 = \psi_2 + \pi : & (\theta-\phi) \in [\pi, 2\pi] \end{cases}$$

The proof of this lemma is given in Appendix C.

Next, we give some monotonicity properties of some functions that will be used later to establish our main result:

Lemma III:
Define the following functions:

$$K(\theta,\phi) = \begin{cases} \psi_4(\theta,\phi), \text{If } (\theta-\phi) \in [0,\pi] \\ \psi_3(\theta,\phi), \text{If } (\theta-\phi) \in [\pi,2\pi] \end{cases}$$

$$u_3^{**}(\theta,\phi) = u_3^*(V_p^{max}, K(\theta,\phi), \theta, \phi),$$

$$u_4^{**}(\theta,\phi) = s(\theta,\phi)\max|\dot{\theta}| = s(\theta,\phi)\frac{1}{b}(V_p^{max} - |u_3^{**}(\theta,\phi)|)$$

with $s(\theta,\phi)$ given by equation 24.

If $(\theta-\phi) \in (0,\frac{\pi}{2})\bigcup(\pi,\frac{3\pi}{2})$ then $g(\phi,\theta,K)$ and $|\dot{\phi}(V_p^{max},K,\theta,\phi)|$ increase monotonically (w.r.t $(\theta-\phi)$), and $|u_4^{**}(\theta,\phi)| = \max|\dot{\theta}|$ decreases monotonically (w.r.t $(\theta-\phi)$).

Symmetrically, if $(\theta-\phi) \in (\frac{\pi}{2},\pi)\bigcup(\frac{3\pi}{2},2\pi)$then $g(\phi,\theta,K)$ and $|\dot{\phi}(V_p^{max},K,\theta,\phi)|$ decrease monotonically (w.r.t $(\theta-\phi)$), and $|u_4^{**}(\theta,\phi)| = \max|\dot{\theta}|$ increases monotonically (w.r.t $(\theta-\phi)$).

Proof: We note here the two main properties of this proof.

- The possible admissible values of the ratio $\gamma \in (0,1]$ do not affect the monotonicity of $g(\phi,\theta,K)$.
- All the possible values of $(\theta-\phi)$ are considered, i.e., all four quadrants are covered.

The proof of Lemma III appears in Appendix D.

As it was mentioned above, the selected evader control must produce a rod rotation that brings the rod perpendicular to the pursuer wheels (pursuer heading) without bringing the rod and the pursuer heading (pursuer wheels) closer to parallelism. Let's call this sense of rotation Desirable Evader Sense of Rotation (DESR). Conversely, the selected pursuer control must produce a pursuer heading rotation that brings the pursuer wheels (pursuer heading) closer to parallelism with the rod, without bringing the pursuer heading (pursuer wheels) and the rod closer to perpendicularity. Let's call this sense of rotation Desirable Pursuer Sense of Rotation (DPSR). Refer to figure 13, which shows (DPSR) w.r.t the rod's orientation.

Thus, it is necessary to analyze the sense of rotation (clockwise or counterclockwise) of both the pursuer heading (directly controlled by the pursuer with u_4) and the rod (controlled by the evader by virtue of u_2). The desirable sense of rotation of both u_4^{**} and $\dot{\phi}(V_p^{max}, K, \theta, \phi)$ is provided in the following corollary.

Corollary of Lemma III:
If $(\theta-\phi) \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$ then the DPSR is clockwise (cw), i.e. $sgn(u_4^{**}) = s(\theta, \phi) = -1$ (as given by Eq. 24), and the DESR is also clockwise (cw), i.e. $sgn(\dot{\phi}(\psi_i)) = -1$. Conversely, if $(\theta - \phi) \in (\frac{\pi}{2}, \pi) \cup (\frac{3\pi}{2}, 2\pi)$ then the DPSR is counterclockwise (ccw) $sgn(u_4^{**}) = s(\theta, \phi) = +1$ (as given by Eq. 24), and the DESR is also counterclockwise (ccw) $sgn(\dot{\phi}(\psi_i)) = +1$. Thus, the following controls must be applied by the players according to the value of $(\theta - \phi)$, to obtain the required sense of rotation of both u_4^{**} and $\dot{\phi}(V_p^{max}, K, \theta, \phi)$.

The evader control u_2^*, is given in table 1.

$(\theta-\phi)$	$sgn(\dot{\phi}(V_p^{max}, K, \theta, \phi))$	$u_2^* = \psi_i$
$(\theta-\phi) \in [0, \frac{\pi}{2}]$	$sgn(\dot{\phi}(V_p^{max}, K, \theta, \phi)) = -1$	$u_2^* = \psi_4$
$(\theta-\phi) \in [\frac{\pi}{2}, \pi]$	$sgn(\dot{\phi}(V_p^{max}, K, \theta, \phi)) = +1$	$u_2^* = \psi_4$
$(\theta-\phi) \in [\pi, \frac{3\pi}{2}]$	$sgn(\dot{\phi}(V_p^{max}, K, \theta, \phi)) = -1$	$u_2^* = \psi_3$
$(\theta-\phi) \in [\frac{3\pi}{2}, 2\pi]$	$sgn(\dot{\phi}(V_p^{max}, K, \theta, \phi)) = +1$	$u_2^* = \psi_3$

Table 1 Evader control u_2 and rod's sense of rotation

The sign (sense of rotation) of $u_4^{**} = \max|\dot{\theta}| = \frac{1}{b}(V_p^{max} - |u_3^{**}|)$ is defined by equation 24.

The proof of the Corollary of Lemma III appears in Appendix E.

Remark 1: Note that with this definition for $u_2^* = \psi_i$ one gets precisely $u_2^* = K(\theta, \phi)$.

Remark 2: The DPSR for u_4^{**} also holds for u_4^*.

Remark 3: The control u_1 that maximizes $|u_1| * g$ is $u_1^* = V_e^{max}$.

The next theorem represents our main result.

Theorem I:
Let

$$u_1^* = V_e^{max}$$
$$u_2^* = K(\theta, \phi)$$

and let u_3^* be as given by equation 19, and u_4^* as given by equation 23. Now define

$$M(V_e^{max}, V_p^{max}, \theta, \phi) = |\dot{\phi}(u_1^*, u_2^*)| - \frac{1}{b}(V_p^{max} - |u_3^{**}|)$$

The manifold $M(V_e^{max}, V_p^{max}, \theta, \phi) = 0$ partitions the space spanned by $V_e^{max}, V_p^{max}, \theta, \phi$ into 2 regions, one in which the pursuer can maintain surveillance indefinitely, and another in which the evader can eventually escape.

If $M(V_e^{max}, V_p^{max}, \theta, \phi) > 0$ at the beginning of the game, then the evader eventually wins at some time $t > t_0$ if the strategy $(u_1, u_2) = (u_1^*, u_2^*)$ is applied at all times, regardless of the strategy applied by the pursuer. Otherwise, if at the beginning of the game $M(V_e^{max}, V_p^{max}, \theta, \phi) \leq 0$, the pursuer wins, if the strategy $(u_3, u_4) = (u_3^*, u_4^*)$ is applied at all times, regardless of the strategy applied by the evader.

Proof:

The theorem can be proved based on the maximization of g and the monotonicity of $\max\{g\}$, $|\dot{\phi}|$, $\max|\dot{\theta}|$ and $|\cos(\theta-\phi)|$. Because $|\cos(\theta-\phi)|$ behaves differently depending on which quadrant contains $\theta - \phi$, we consider individually the four quadrants. Here, we consider the case for which $\theta - \phi \in [0, \frac{\pi}{2}]$. The proofs of the other cases are analogous and are included in Appendix F.

The proof proceeds as follows. We first consider the case when the pursuer applies the strategy $(u_3, u_4) = (u_3^{**}, u_4^{**})$ and the evader applies the strategy $(u_1, u_2) = (u_1^*, u_2^*)$. We show that under these strategies the pursuer wins if $\max|\dot{\theta}(t_0)| \geq |\dot{\phi}(t_0)|$ (i.e., if at the beginning of the game $M(V_e^{max}, V_p^{max}, \theta, \phi) \leq 0$), else the evader wins. Second, we show that if the pursuer applies the strategy $(u_3, u_4) = (u_3^*, u_4^*)$ and $|\dot{\phi}(t_0)| \leq \max|\dot{\theta}(t_0)|$, then the pursuer wins regardless of the strategy applied by the evader. Symmetrically, if the evader applies the strategy $(u_1, u_2) = (u_1^*, u_2^*)$ and $|\dot{\phi}(t_0)| > \max|\dot{\theta}(t_0)|$ then the evader wins regardless of the pursuer strategy. We now develop the details.

Assume that $M(V_e^{max}, V_p^{max}, \theta, \phi) \leq 0$ at the beginning of the game. We analyze this inequality first considering $M(V_e^{max}, V_p^{max}, \theta, \phi) < 0$. Second, we analyze $M(V_e^{max}, V_p^{max}, \theta, \phi) = 0$. Then $\max|\dot{\theta}(t_0)| > |\dot{\phi}(t_0)|$ and $V_p^{max}|\cos(\theta-\phi)| > |u_1^*| \cdot g(\phi, \theta, u_2^*)$. By Lemmas II and III, $g(\phi, \theta, u_2^*)$ is maximal and varies monotonically by applying the optimal $u_2^* = \psi_4$. By Lemma III, $g(\phi, \theta, u_2^*)$ monotonically decreases and $|\cos(\theta-\phi)|$ monotonically increases as $(\theta-\phi)$ varies from $\frac{\pi}{2}$ to 0.

If $\max|\dot{\theta}(t_0)| > |\dot{\phi}(t_0)|$ then for $\epsilon \to 0$ we have $(\theta(t_0) - \phi(t_0)) > (\theta(t_0+\epsilon) - \phi(t_0+\epsilon))$, and by Lemma III, $\max|\dot{\theta}|$ monotonically increases and $|\dot{\phi}|$ monotonically decreases as $(\theta-\phi)$ varies from $\frac{\pi}{2}$ to 0. Hence $\forall t > t_0$, $\max|\dot{\theta}(t)| > |\dot{\phi}(t)|$, and therefore $\forall t > t_0, (\theta(t_0) - \phi(t_0)) > (\theta(t) - \phi(t))$, and $(\theta(t) - \phi(t))$ decreases monotonically until it reaches 0 and the pursuer wins.

Now assume that $M(V_e^{max}, V_p^{max}, \theta, \phi) > 0$ at the beginning of the game. Then $\max|\dot\theta(t_0)| < |\dot\phi(t_0)|$ and $V_p^{max}|\cos(\theta-\phi)| < |u_1^*| \cdot g(\phi,\theta,u_2^*)$. Then by Lemma III, $g(\phi,\theta,u_2^*)$ monotonically increases and $|\cos(\theta-\phi)|$ monotonically decreases as $(\theta-\phi)$ varies from 0 to $\frac{\pi}{2}$.

If $\max|\dot\theta(t_0)| < |\dot\phi(t_0)|$ then for $\epsilon \to 0$ we have $(\theta(t_0)-\phi(t_0)) < (\theta(t_0+\epsilon)-\phi(t_0+\epsilon))$, and by Lemma III, $\max|\dot\theta|$ monotonically decreases and $|\dot\phi|$ monotonically increases as $(\theta-\phi)$ varies from 0 to $\frac{\pi}{2}$.

Hence $\forall t > t_0,\ \max|\dot\theta(t)| < |\dot\phi(t)|$, and therefore $(\theta(t_0)-\phi(t_0)) < (\theta(t)-\phi(t))$ and $(\theta(t)-\phi(t))$ increases monotonically until it reaches $\frac{\pi}{2}$ and the evader wins.

In the case of equality, i.e., $\max|\dot\theta(t_0))| = |\dot\phi(t_0)|$ (equivalently, $V_p^{max}|\cos(\theta-\phi)| = |u_1^*|\cdot g(\phi,\theta,u_2^*)$), for $\epsilon \to 0$ we will have $(\theta(t_0)-\phi(t_0)) = (\theta(t_0+\epsilon)-\phi(t_0+\epsilon))$. By Lemma III, $\max|\dot\theta|$ and $|\dot\phi|$ remain constant for a given value of $(\theta-\phi)$. Hence $\forall t > t_0,\ \max|\dot\theta(t)| = |\dot\phi(t)|$ and $(\theta(t)-\phi(t)) = (\theta(t_0)-\phi(t_0))$. Therefore, the values of both $g(\phi,\theta,u_2=K)$ and $|\cos(\theta-\phi)|$ remain constant and the pursuer wins. Note that this result obtains regardless of the quadrant in which $(\theta-\phi)$ lies.

We now show that, if $\max|\dot\theta(t_0)| < |\dot\phi(t_0)|$ then the evader wins regardless the pursuer strategy, whenever the evader applies $(u_1,u_2) = (u_1^*,u_2^*)$.

If $u_3 \neq u_3^*$, the pursuer immediately loses, it cannot maintain the constant distance.

If $u_3 = u_3^*$ but $u_4 \neq u_4^*$ then for all time t we will have $|u_4| = |\dot\theta(t)| < |u_4^*| = \max|\dot\theta(t)| < |\dot\phi(t)|$. Hence, $(\theta(t)-\phi(t))$ under (u_1^*,u_2^*,u_3^*,u_4) is closer to $\pm\frac{\pi}{2}$ than $(\theta(t)-\phi(t))$ under $(u_1^*,u_2^*,u_3^*,u_4^*)$. Therefore, if $\max|\dot\theta(t_0)| < |\dot\phi(t_0)|$, the evader wins regardless the pursuer strategy.

Symmetrically, if $|\dot\phi(t_0)| \le \max|\dot\theta(t_0)|$ then the pursuer wins regardless the evader strategy. By Lemma I, the condition $|\dot\phi(t_0)| \le \max|\dot\theta(t_0)|$ is equivalent to

$$g(\phi(t_0),\theta(t_0),u_2(t_0))|u_1(t_0))| \le V_p^{max}|\cos(\theta(t_0)-\phi(t_0))|$$

Let us assume that the evader uses (u_1^*,u_2^*). These evader controls maximize $g(\phi(t),\theta(t),u_2(t))|u_1(t)|$. The inequality

$$g(\phi(t),\theta(t),u_2(t))|u_1(t)| \le V_p^{max}|\cos(\theta(t)-\phi(t))|$$

is also equivalent to (again by Lemma I):

$$\underbrace{\frac{|u_1^*\sin(\theta-u_2^*)|}{|L\cos(\theta-\phi)|}}_{|\dot\phi(t)|} - \underbrace{\frac{1}{b}\left(V_p^{\max} - \overbrace{\left|\frac{u_1^*\cos(u_2^*-\phi)}{\cos(\theta-\phi)}\right|}^{u_3^{**}(u_1,u_2)}\right)}_{|u_4^{**}| = \max|\dot\theta(t)|} \le 0$$

Thus, the evader's controls (u_1^*,u_2^*) maximize the difference $|\dot\phi(u_1(t),u_2(t))| - \max|u_4(u_1(t),u_2(t))|$.

If the evader uses $(u_1,u_2) \neq (u_1^*,u_2^*)$ then

$$\forall t:\ g(\phi(t),\theta(t),u_2(t))|u_1(t)| < g(\phi(t),\theta(t),u_2^*(t))|u_1^*(t)|$$

and

$$|\dot\phi(u_1(t),u_2(t))| - \max|u_4^*(u_1(t),u_2(t))| < |\dot\phi(u_1^*(t),u_2^*(t))| - \max|u_4^{**}(u_1^*(t),u_2^*(t))| \le 0$$

If the evader uses at all times the controls that maximize the difference $|\dot\phi(t)| - \max|\dot\theta(t)|$, and this difference is still negative or equal to zero, then no evader control will make the difference greater than zero. Therefore

$$\forall t:\ g(\phi(t),\theta(t),u_2(t))|u_1(t)| < V_p^{max}|\cos(\theta(t)-\phi(t))|$$

and

$$|\dot\phi(u_1(t_0),u_2(t_0))| < |u_4^*(u_3^*(t_0))| = \max|\theta(t_0)|$$

The result follows.

■

Corollary of Theorem I:
For $|M(V_e^{max},V_p^{max},\theta,\phi)| < \epsilon$, i.e., in a neighborhood of the manifold described above, for $\epsilon \to 0$, the strategies: $(u_3,u_4) = (u_3^*,u_4^*)$ for the pursuer and $(u_1,u_2) = (u_1^*,u_2^*)$ for the evader are the only equilibrium strategies for the game. The application of these strategies are necessary and sufficient conditions for guaranteeing that the corresponding player wins.
Note that when both players follow these strategies we have $(u_3^*,u_4^*) = (u_3^{**},u_4^{**})$.

Proof:
If each of V_e^{max}, V_p^{max}, and $(\theta-\phi)$ are given, and $\max|\dot\theta(t_0)| = |\dot\phi(t_0)|$ then there is no valid control, other than $|u_4^*| = \max|\dot\theta|$, which guarantees that the pursuer will win. Any other value of u_4 makes $|\dot\theta|$ smaller, yielding $|\dot\theta(t_0)| < |\dot\phi(t_0)|$, which corresponds to the evader winning. Hence, that the pursuer use strategy $u_4^* = \max|\dot\theta(t_0)|$ is both a necessary and sufficient condition for the pursuer to win.

Suppose now that $\max|\dot{\theta}(t_0)| < |\dot{\phi}(t_0)|$. By Lemma I, this is equivalent to:

$$V_p^{max}|\cos(\theta(t_0) - \phi(t_0))| <$$
$$|u_1(t_0)| \cdot g(\phi(t_0), \theta(t_0), u_2(t_0))$$

If V_p^{max}, $(\theta - \phi)$ are given, and $u_1 = u_1^*$, $u_2 = u_2^*$ are such that $|u_1^*| \cdot g(\phi, \theta, u_2^*)$ has the minimum value for the inequality to hold, then there do not exist controls other than (u_1^*, u_2^*) that maintain the condition. Any other controls make $|u_1| \cdot g(\phi, \theta, u_2)$ smaller, and consequently the inequality will change to:

$$V_p^{max}|\cos(\theta(t_0) - \phi(t_0))| \geq |u_1| \cdot g(\phi, \theta, u_2)$$

and the pursuer will win. Hence, that the evader apply the strategy (u_1^*, u_2^*) that maximizes $|u_1| \cdot g(\phi, \theta, u_2)$ is both a necessary and sufficient condition for the evader to win.

■

Remark 4: By Corollary of Theorem I, the players' strategies may be considered Local Equilibrium Strategies (LES). As the system moves away from the manifold, the winning player's choice of controls is constrained (so that the system remains in the corresponding region), but is not unique. The constraints are: for the pursuer $u_3 = u_3^*$ and $|u_4| \geq |\dot{\phi}|$, for the evader $|u_1|g(\phi, \theta, u_2) > V_p^{max}|\cos(\theta - \phi)|$. If the LES are followed, however, there are some interesting properties that are obtained. If the pursuer wins, the LES eventually leads to an alignment of its wheels with the rod (i.e., $\theta = \phi$ or $\theta = \phi + \pi$), which allows it to maintain surveillance with the minimal effort (i.e., minimal $|V_p|$). Notice that at the moment that the pursuer heading reaches parallelism with the rod, it is possible for the pursuer to keep this parallelism by applying $u_4 = \dot{\theta} = \dot{\phi}$, thus avoiding oscillations. If the evader wins, the LES progressively increases the required value of u_3 up to ∞, so that it allows it to escape from any pursuer with bounded speed.

Remark 5: If V_p^{max} and V_e^{max} are given as inputs to the problem, the LES create a partition over the value of $(\theta - \phi)$ into two sets which define the winning of the game. On the other hand, for a given initial $(\theta - \phi)$ and if either V_p^{max} or V_e^{max} is given, these motion strategies allow the calculation of the remaining minimum value of V_e^{max} or V_p^{max} respectively, which correspond to the minimal capabilities for the players to win.

Remark 6: It is interesting to note that, when the evader wins g converges to 2 (for $\gamma = 1$) and $|\cos(\theta - \phi)|$ converges to 0, and when the pursuer wins g converges to $\sqrt{2}$ (for $\gamma = 1$) and $|\cos(\theta - \phi)|$ converges to 1.

4 Simulations

In this section, we present numerical simulations to illustrate the pursuer's and the evader's motion strategies. Since the pursuer is a nonholonomic system, we use numerical integration to compute the approximate paths for the pursuer.

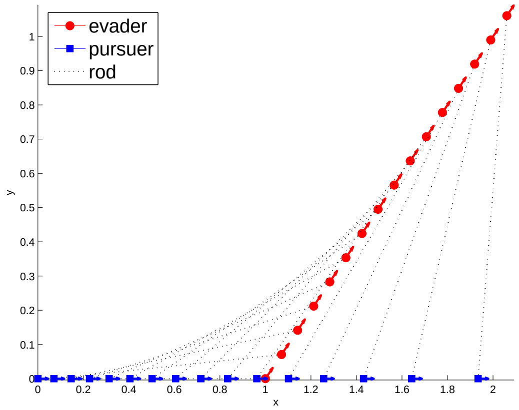

Fig. 4 Evader wins, pursuer follows a straight line trajectory $u_4 = 0$

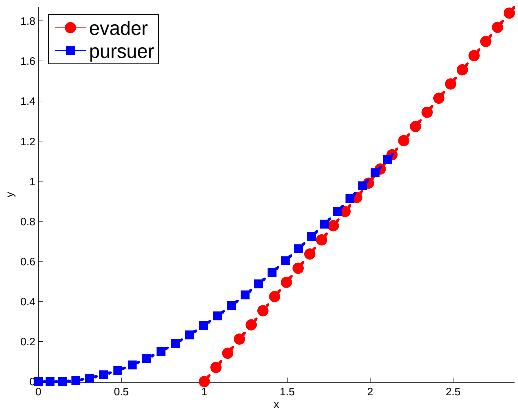

Fig. 5 Pursuer wins by making $\theta = \phi$

In all figures the evader is shown with a (red) circle the pursuer with a (blue) square, the rod is represented with a dotted line segment. The (blue) arrows emerging from the pursuer show the heading of the pursuer (heading of the wheels of the differential drive robot pursuer), and the (red) arrows emerging from the evader show the direction of the evader velocity vector.

In Figure 4, the initial system configuration is $\theta = 0$ and $\phi = \pi$. The evader chooses its velocity vector at

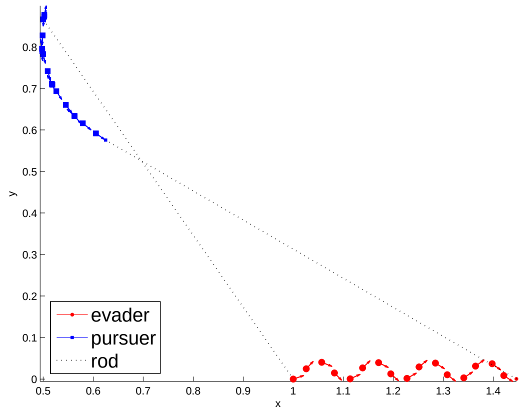

Fig. 6 Evader moves tracing a sinusoidal path, pursuer wins by pointing its heading parallel to the rod

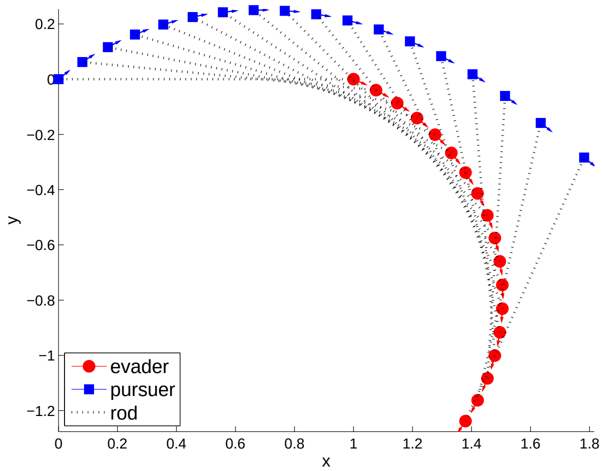

Fig. 8 Evader wins using optimal $u_2 = \phi$ yielding $(\theta - \phi) = \frac{\pi}{2}$

a constant orientation ($u_2 = \psi = \frac{\pi}{4}$). The pursuer does not change its heading ($u_4 = 0$), but it uses the required u_3^* and $\dot{\phi}$ to follow the evader. The pursuer is able to follow the evader for a short period of time until the rod orientation gets close to being perpendicular to the pursuer heading.

In Figure 5, again the initial system configuration is $\theta = 0$ and $\phi = \pi$ and the evader chooses at all time its velocity vector at a constant orientation ($u_2 = \psi = \frac{\pi}{4}$). But this time the pursuer changes u_4 to point its heading to be parallel to the rod, yielding a pursuer win. Note that in these first two simulations the evader has followed a sub-optimal strategy.

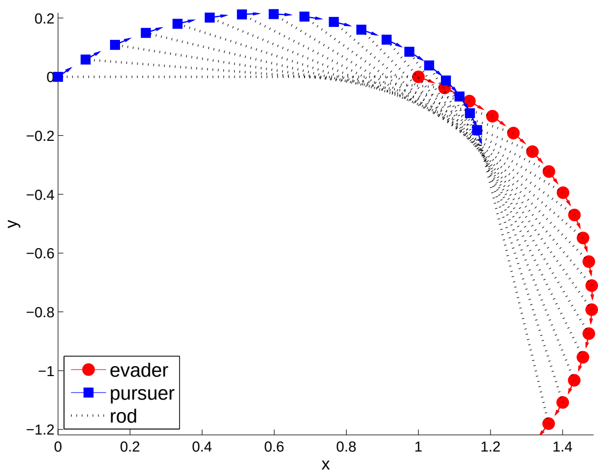

Fig. 7 Pursuer wins by making $\theta = \phi + \pi$, evader uses optimal $u_2 = \psi$ but is maximal velocity $V_e{}^{\max}$ is insufficient for winning

In Figure 6 the evader moves tracing a sinusoidal path, the pursuer wins by making $\theta = \phi$. Again the evader has followed a sub-optimal strategy.

Figures 7 and 8 show optimal pursuer and evader motion strategies. In the two simulations presented in these figures, at the beginning of the game $\gamma = 1$, $\theta = 40$ degrees and $\phi = 180$ degrees. In both of these simulations, the evader uses the optimal $u_2^* = \psi_i$. The pursuer uses $u_4^{**} = s(\theta, \phi) \max |\dot{\theta}|$ trying to make it parallel to the rod orientation.

In figure 7, the pursuer wins since $\max |\dot{\theta}(t_0)| > |\dot{\phi}(t_0)|$, hence at the end the pursuer is able to point its heading parallel to the rod orientation $(\theta - \phi) = \pi$.

In figure 8, the evader wins since $|\dot{\phi}(t_0)| > \max |\dot{\theta}(t_0)|$, hence at the end the evader is able to get the rod orientation perpendicular to the pursuer heading $(\theta - \phi) = \frac{\pi}{2}$.

From inequality 27, it is possible to compute a smallest critical value of the evader velocity V_e^{max} or even a ratio of the pursuer and evader velocities, which determines either the evader or pursuer winning, the critical ratio is defined by $\rho = \frac{V_e{}^{max}}{V_p{}^{max}} = \frac{|\cos(\theta - \phi)|}{g}$.

For the simulations shown in figures 7 and 8, $V_p{}^{\max} = 1$. Then the critical evader velocity determining the winner of the game is $V_e^{max} = 0.4226$. When the evader wins $V_e^{max} = 0.43$, and when the evader loses $V_e^{max} = 0.41$.

Figures 9 and 10 show additional two simulations in which both players use optimal motion strategies. In these simulations $b = 0.25$ and $L = 0.5$, thus $\gamma = 0.5$. At the beginning of the game $\theta = 80$ degrees, and $\phi = 125$ degrees. In figure 9 the pursuer wins, in figure 10 the evader wins. For the initial system configuration corresponding to this simulations, the critical value for

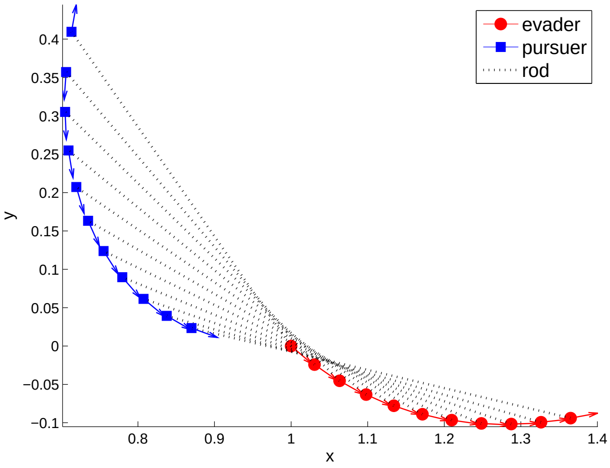

Fig. 9 The initial conditions are identical to those for the example of figure 10. Here, the pursuer wins since $\rho \leq 0.5054$ (see text).

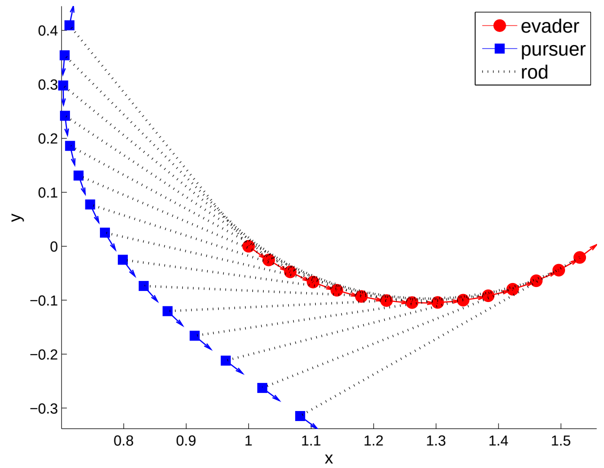

Fig. 10 The initial conditions are identical to those for the example of figure 9. Here, the evader wins since $\rho > 0.5054$ (see text).

the ratio ρ is given by $\rho = \frac{V_e{}^{max}}{V_p{}^{max}} = \frac{|\cos(\theta - \phi)|}{g} = 0.5054$. If $\rho > 0.5054$ then evader wins, else when $\rho \leq 0.5054$ the pursuer wins. Note that this ratio can be computed based only on θ, ϕ and the optimal $u_2^* = \psi_i$.

Finally, figures 11 and 12 show a simulation in which the evader moves randomly, the pursuer wins aligning its wheels parallel to the rod ($\overrightarrow{V_P} \parallel \overrightarrow{L}$), figure 12 shows the initial and final orientations of the pursuer wheels w.r.t the rod. Figures 11 and 12 exemplify that whenever Theorem I is satisfied for the pursuer, the pursuer can track the evader even if the evader follows a random motion. If the pursuer is able to track an evader following the optimal policy, it shall be able to track an evader following any other policy. Hence, it is not needed to know the policy chosen by the evader at every instant of time.

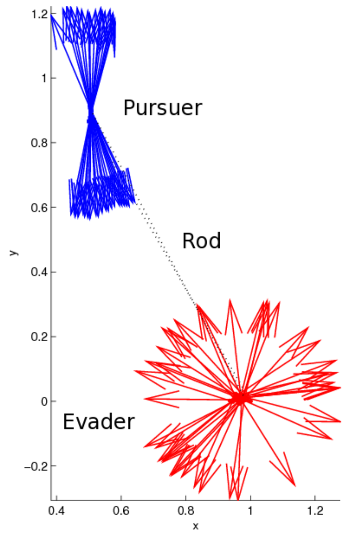

Fig. 11 Evader moves randomly, pursuer wins aligning its wheels parallel to the rod

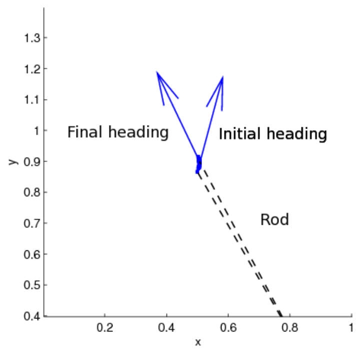

Fig. 12 Initial and final pursuer heading, at the end the pursuer wheels are aligned with the rod

5 Conclusions and Future Work

In this paper we have considered the surveillance problem of tracking of a moving evader by a differential drive pursuer (nonholonomic robot).

We have analyzed the case in which the pursuer and the evader move in an environment without obstacles. We have shown that in order to maintain a constant distance to the evader, the linear speed of the pursuer is totally determined for every configuration of the system, so that its only degree of freedom is its rotation angle. We have derived a lower bound for the pursuer speed to follow the evader. We have obtained optimal motion strategies for both players, in the sense that they require the minimal capabilities of the players for winning. We have also obtained the long term solution for the game, and we have presented simulation results of the game of pursuit. The methodology proposed in this paper basically has two main components: 1) Find the optimal policies for each player for a given criterion. 2) Show that these policies induce a monotonic evolution under the condition defining the winner. This methodology might be used to solve other related problems (e.g., tracking and omnidirectional evader with a car-like robot).

In our current research, we are using the results obtained in this paper as a base to solve the more general problem of tracking an evader to a variable but bounded distance. As future work, we want to consider acceleration bounds on the pursuer and the evader.

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A Determination of u_3

From equation 10 we obtain the following two expressions for $\dot{\phi}(t)$

$$\dot{\phi} = \frac{\dot{x}_e - \dot{x}_p}{L \sin\phi} \tag{31}$$

$$\dot{\phi} = \frac{\dot{y}_p - \dot{y}_e}{L \cos\phi} \tag{32}$$

Substituting equations 5 in 31 and 6 in 32 we obtain

$$\dot{\phi} = \frac{\dot{x}_e - V_p \cos\theta}{L \sin\phi} \tag{33}$$

$$\dot{\phi} = \frac{V_p \sin\theta - \dot{y}_e}{L \cos\phi} \tag{34}$$

Equating equations 33 and 34 and solving for $u_3 = V_p$ we obtain:

$$u_3 = V_p = \frac{\dot{x}_e \cos\phi + \dot{y}_e \sin\phi}{\cos(\theta - \phi)} \tag{35}$$

Finally, substituting equations 1, 2 in 35, we obtain

$$u_3 = \frac{V_e \cos(\psi - \phi)}{\cos(\theta - \phi)} \tag{36}$$

B Determination of $\dot{\phi}$

If the coordinates of the pursuer are x_p, y_p, the following coordinate transformations relate the pursuer's Cartesian coordinates to its polar coordinates relative to the evader:

$$x_p - x_e = L \cos\phi \tag{37}$$

$$y_p - y_e = L \sin\phi \tag{38}$$

Using equations 37 and 38 we obtain an expression for $\tan\phi$, and we differentiate this to obtain an expression for $\dot{\phi}$.

$$\begin{aligned} &\frac{d}{dt}\tan\phi = \frac{d}{dt}\frac{y_p - y_e}{x_p - x_e} \\ &\dot{\phi}\sec^2\phi = \\ &\frac{1}{x_p - x_e}\dot{y}_p - \frac{1}{x_p - x_e}\dot{y}_e - \frac{(y_p - y_e)}{(x_p - x_e)^2}\dot{x}_p + \frac{(y_p - y_e)}{(x_p - x_e)^2}\dot{x}_e \end{aligned} \tag{39}$$

Making the substitution given by equation 37 we obtain

$$\begin{aligned} &\dot{\phi}\sec^2\phi = \\ &\frac{1}{L\cos\phi}\left(\dot{y}_p - \dot{y}_e - \frac{y_p - y_e}{x_p - x_e}\dot{x}_p + \frac{y_p - y_e}{x_p - x_e}\dot{x}_e\right) \\ &\dot{\phi}\sec^2\phi = \frac{1}{L\cos\phi}\left(\dot{y}_p - \dot{y}_e - (\dot{x}_p - \dot{x}_e)\tan\phi\right) \end{aligned} \tag{40}$$

$$\dot{\phi} = \frac{1}{L}\left(\dot{y}_p - \dot{y}_e\right)\cos\phi - \frac{1}{L}\left(\dot{x}_p - \dot{x}_e\right)\sin\phi \tag{41}$$

If we now define the pursuer velocity as $\dot{x}_p = V_p \cos\theta$ and $\dot{y}_p = V_p \sin\theta$ we obtain the first form for $\dot{\phi}$

$$\dot{\phi} = \frac{1}{L}V_p \sin(\theta - \phi) - \frac{1}{L}(\dot{y}_e \cos\phi - \dot{x}_e \sin\phi) \tag{42}$$

If we parametrize the evader velocity by magnitude and angle, $\dot{x}_e = V_e \cos\psi$ and $\dot{y}_e = V_e \sin\psi$, we obtain

$$\dot{\phi} = \frac{1}{L}V_p \sin(\theta - \phi) - \frac{1}{L}V_e \sin(\psi - \phi) \tag{43}$$

Since $u_1 = V_e$, $u_2 = \psi$ and $u_3 = V_p$ equation 43 can also be expressed as follows:

$$\dot{\phi} = \frac{1}{L}[u_3 \sin(\theta - \phi) - u_1 \sin(u_2 - \phi)] \tag{44}$$

Equation 44 indicates for every instant of time the rate of rotation of the rod to maintain a constant distance from the evader, and shows that the state variable $\dot{\phi}$ depends on both the evader and pursuer controls.

Now, we can express the state transition model of our system in its first form:

$$\begin{pmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{\phi} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} u_1 \cos u_2 \\ u_1 \sin u_2 \\ \frac{1}{L}[u_3 \sin(\theta - \phi) - u_1 \sin(u_2 - \phi)] \\ u_4 \end{pmatrix} \tag{45}$$

Substituting equation 36 in equation 43 we can obtain a simplified form for $\dot{\phi}$:

$$\dot{\phi} = \frac{V_e \cos(\psi - \phi)}{L\cos(\theta - \phi)}\sin(\theta - \phi) - \frac{1}{L}V_e \sin(\psi - \phi) \tag{46}$$

Setting $v = \theta - \phi$ and $u = \psi - \phi$ we obtain

$$\dot{\phi} = \frac{V_e}{L}\left[\frac{\sin(v)\cos(u) - \cos(v)\sin(u)}{\cos(\theta-\phi)}\right] \tag{47}$$

and using the trigonometric identity: $\sin(u-v) = \sin(u)\cos(v) - \cos(u)\sin(v)$ and replacing the values of u and v, we obtain

$$\dot{\phi} = \frac{V_e \sin(\theta-\psi)}{L\cos(\theta-\phi)} \tag{48}$$

C Proof of Lemma II

Lemma II:
Consider the following functions:

$$\psi_1 = \arctan\left(\frac{\sin\phi - \gamma\cos\theta}{\cos\phi + \gamma\sin\theta}\right)$$
$$\psi_2 = \arctan\left(\frac{\sin\phi + \gamma\cos\theta}{\cos\phi - \gamma\sin\theta}\right)$$
$$\psi_3 = \arctan\left(\frac{-\sin\phi - \gamma\cos\theta}{-\cos\phi + \gamma\sin\theta}\right)$$
$$\psi_4 = \arctan\left(\frac{-\sin\phi + \gamma\cos\theta}{-\cos\phi - \gamma\sin\theta}\right)$$

The evader control u_2 that maximizes $g(\phi, \theta, u_2)$ for given values of ϕ and θ is given by

$$u_2 = \begin{cases} \psi_1 \text{ or } \psi_4 = \psi_1 + \pi : & (\theta - \phi) \in [0, \pi] \\ \psi_2 \text{ or } \psi_3 = \psi_2 + \pi : & (\theta - \phi) \in [\pi, 2\pi] \end{cases}$$

Proof:
Recall that

$$g(\phi, \theta, \psi) = |\cos(\phi - \psi)| + \gamma|\sin(\theta - \psi)|$$

in which $0 < \gamma \le 1$. Since $\max\{|a|+|b|\} = \max\{a+b, a-b, -a+b, -a-b\}$ we have

$$\max_{\psi} g(\phi,\theta,\psi) =$$
$$\max_{\psi}\{g_1(\phi,\theta,\psi), g_2(\phi,\theta,\psi), g_3(\phi,\theta,\psi), g_4(\phi,\theta,\psi)\}$$

in which

$$g_1(\phi,\theta,\psi) = \cos(\phi-\psi) + \gamma\sin(\theta-\psi)$$
$$g_2(\phi,\theta,\psi) = \cos(\phi-\psi) - \gamma\sin(\theta-\psi)$$
$$g_3(\phi,\theta,\psi) = -\cos(\phi-\psi) + \gamma\sin(\theta-\psi)$$
$$g_4(\phi,\theta,\psi) = -\cos(\phi-\psi) - \gamma\sin(\theta-\psi)$$

Thus, we proceed by finding the maximizers for each of the g_i.

For the case of g_1, using basic trigonometric identities we obtain the following

$$\begin{aligned} g_1 &= \cos(\phi-\psi) + \gamma\sin(\theta-\psi) \\ &= \cos\phi\cos\psi + \sin\phi\sin\psi + \gamma(\sin\theta\cos\psi - \cos\theta\sin\psi) \\ &= \cos\psi(\cos\phi + \gamma\sin\theta) + \sin\psi(\sin\phi - \gamma\cos\theta) \\ &= A_1\sin\psi + B_1\cos\psi \end{aligned}$$

in which A_1 and B_1 do not depend on ψ and are given by

$$A_1 = \sin\phi - \gamma\cos\theta, \qquad B_1 = \cos\phi + \gamma\sin\theta$$

	A_i	B_i
$i=1$	$\sin\phi - \gamma\cos\theta$	$\cos\phi + \gamma\sin\theta$
$i=2$	$\sin\phi + \gamma\cos\theta$	$\cos\phi - \gamma\sin\theta$
$i=3$	$-\sin\phi - \gamma\cos\theta$	$-\cos\phi + \gamma\sin\theta$
$i=4$	$-\sin\phi + \gamma\cos\theta$	$-\cos\phi - \gamma\sin\theta$

Table 2 Coefficients A_i and B_i for $g_i(\phi, \theta, \psi)$

Repeating this process for each of g_2, g_3 and g_4, we obtain the general form

$$g_i = A_i\sin\psi + B_i\cos\psi$$

with A_i and B_i given in Table 2.

This expression for g_i can be rewritten as a sinusoid (see, e.g., [4])

$$g_i(\phi,\theta,\psi) = A_i\sin\psi + B_i\cos\psi = \sqrt{{A_i}^2 + {B_i}^2}\sin\left(\psi + \tan^{-1}\left(\frac{B_i}{A_i}\right)\right) \tag{49}$$

Since neither A_i nor B_i depend on ψ, the maximizer satisfies

$$\psi + \tan^{-1}\left(\frac{B_i}{A_i}\right) = \frac{\pi}{2} \tag{50}$$

and using the identity $\tan^{-1}(B_i/A_i) = \pi/2 - \tan^{-1}(A_i/B_i)$ we obtain

$$\psi = \tan^{-1}\frac{A_i}{B_i} \tag{51}$$

This immediately yields the four values of ψ_i given in the Lemma. To ensure that these values yield a maximum, we examine the second partial derivative of g_i with respect to ψ, which is given by

$$\frac{\partial^2 g_i}{\partial {\psi_i}^2} = -A_i\sin\psi_i\dot{\psi}_i^2 - B_i\cos\psi_i\dot{\psi}_i^2$$
$$\frac{\partial^2 g_i}{\partial {\psi_i}^2} = (-A_i\sin\psi_i - B_i\cos\psi_i)\dot{\psi}_i^2$$
$$\frac{\partial^2 g_i}{\partial {\psi_i}^2} = -(A_i\sin\psi_i + B_i\cos\psi_i)\dot{\psi}_i^2$$
$$\frac{\partial^2 g_i}{\partial {\psi_i}^2} = -g_i\dot{\psi}_i^2 \tag{52}$$

Since a maximum of g_i satisfies $\frac{\partial^2 g_i}{\partial {\psi_i}^2} > 0$, equation 52 implies that $g_i > 0$ must hold. We will show this below.

We note here that a simple relationship holds between ψ_1 and ψ_4, and between ψ_2 and ψ_3. In particular, since

$$\psi_1 = \arctan\left(\frac{A_1}{B_1}\right)$$

and

$$\psi_4 = \arctan\left(\frac{-A_1}{-B_1}\right)$$

we have $\psi_4 = \psi_1 + \pi$. Likewise, $\psi_3 = \psi_2 + \pi$.

Now, we proceed to evaluate each ψ_i in the corresponding g_i. We will show that if $(\theta - \phi) \in [0, \pi]$ then g is maximized by ψ_1 and ψ_4, and if $(\theta - \phi) \in [\pi, 2\pi]$ then g is maximized by ψ_2 and ψ_3. We begin with the case of g_4.

$$g_4 = \sqrt{{A_4}^2 + {B_4}^2} \sin\left(\psi + \arctan\left(\frac{B_4}{A_4}\right)\right)$$

Plugging ψ_4 in g_4, that is $g_4(\psi_4)$, we obtain

$$g_4(\psi_4) = \sqrt{{A_4}^2 + {B_4}^2} \sin\left(\arctan\left(\frac{A_4}{B_4}\right) + \arctan\left(\frac{B_4}{A_4}\right)\right)$$

Note that ψ_4 maximizes g_4 at all times, so that the argument of the sine

$$\arctan\left(\frac{A_4}{B_4}\right) + \arctan\left(\frac{B_4}{A_4}\right) = \frac{\pi}{2}.$$

since this maximizes the sine, that is:

$$\sin\left(\arctan(\frac{A_4}{B_4}) + \arctan(\frac{B_4}{A_4})\right) = 1$$

Thus

$$g_4(\psi_4) = \sqrt{{A_4}^2 + {B_4}^2}$$

Evaluating A_4 and B_4 inside the square root:

$$\begin{aligned} g_4(\psi_4) &= \sqrt{(-\sin\phi + \gamma\cos\theta)^2 + (-\cos\phi - \gamma\sin\theta)^2} \\ &= \sqrt{1 + \gamma^2 + 2\gamma\sin(\theta - \phi)} \end{aligned} \quad (53)$$

For g_3 starting from

$$g_3 = \sqrt{{A_3}^2 + {B_3}^2} \sin\left(\psi + \arctan\left(\frac{B_3}{A_3}\right)\right)$$

and following a similar procedure that with g_4, we get:

$$g_3(\psi_3) = \sqrt{1 + \gamma^2 - 2\gamma\sin(\theta - \phi)} \quad (54)$$

Likewise for g_2 and g_1

$$g_2(\psi_2) = \sqrt{1 + \gamma^2 - 2\gamma\sin(\theta - \phi)}$$

$$g_1(\psi_1) = \sqrt{1 + \gamma^2 + 2\gamma\sin(\theta - \phi)}$$

To summarizes the forms of g_i evaluated over the respective ψ_i are given by:

$$\begin{aligned} g_1(\psi_1) &= \sqrt{1 + \gamma^2 + 2\gamma\sin(\theta - \phi)} \\ g_2(\psi_2) &= \sqrt{1 + \gamma^2 - 2\gamma\sin(\theta - \phi)} \\ g_3(\psi_3) &= \sqrt{1 + \gamma^2 - 2\gamma\sin(\theta - \phi)} \\ g_4(\psi_4) &= \sqrt{1 + \gamma^2 + 2\gamma\sin(\theta - \phi)} \end{aligned}$$

If $(\theta - \phi) \in [0, \pi]$ then ψ_1 and ψ_4 maximize g since the $\sin(\theta - \phi) \geq 0$. Likewise, if $(\theta - \phi) \in [\pi, 2\pi]$ then ψ_2 and ψ_3 maximize g since the $\sin(\theta - \phi) \leq 0$.

In the domain $(\theta - \phi) \in [0, \pi]$, g_4 and g_1 are positive (different of zero), since the argument on the square root of equation 53, that is $1 + \gamma^2 + 2\gamma\sin(\theta - \phi) > 0$.

In the domain $(\theta - \phi) \in [\pi, 2\pi]$, g_3 and g_2 are positive (different of zero), since the argument on the square root of equation 54, that is $1 + \gamma^2 - 2\gamma\sin(\theta - \phi) > 0$.

■

D Proof of Lemma III

Lemma III:
Define the following functions:

$$K(\theta, \phi) = \begin{cases} \psi_4(\theta, \phi), \text{If } (\theta - \phi) \in [0, \pi] \\ \psi_3(\theta, \phi), \text{If } (\theta - \phi) \in [\pi, 2\pi] \end{cases}$$

$$u_3^{**}(\theta, \phi) = u_3^*(V_p^{max}, K(\theta, \phi), \theta, \phi),$$

$$u_4^{**}(\theta, \phi) = s(\theta, \phi) \max|\dot{\theta}| = s(\theta, \phi)\frac{1}{b}(V_p^{max} - |u_3^{**}(\theta, \phi)|)$$

with $s(\theta, \phi)$ given by equation 24.

If $(\theta - \phi) \in (0, \frac{\pi}{2}) \bigcup (\pi, \frac{3\pi}{2})$ then $g(\phi, \theta, K)$ and $|\dot{\phi}(V_p^{max}, K, \theta, \phi)|$ increase monotonically (w.r.t $(\theta - \phi)$), and $|u_4^{**}(\theta, \phi)| = \max|\dot{\theta}|$ decreases monotonically (w.r.t $(\theta - \phi)$).

Symmetrically, if $(\theta - \phi) \in (\frac{\pi}{2}, \pi) \bigcup (\frac{3\pi}{2}, 2\pi)$then $g(\phi, \theta, K)$ and $|\dot{\phi}(V_p^{max}, K, \theta, \phi)|$ decrease monotonically (w.r.t $(\theta - \phi)$), and $|u_4^{**}(\theta, \phi)| = \max|\dot{\theta}|$ increases monotonically (w.r.t $(\theta - \phi)$).

Proof:
This lemma is proved by cases. Since the cases are analogous, we only present in detail the case of ψ_4, and we provide a sketch of the proofs for the other cases. Table 1 summarizes the sense of rotation (counterclockwise +1 or clockwise -1) for $\dot{\phi}(V_p^{max}, K, \theta, \phi)$

In all cases, as a first step, we show that $|u_4^{**}(u_3^{**})|$ monotonically increases and $|\dot{\phi}(V_e^{max}, K, \theta, \phi)|$ monotonically decreases, if $(\theta - \phi) \in (\frac{\pi}{2}, \pi) \bigcup (\frac{3\pi}{2}, 2\pi)$. Symmetrically, $|u_4^{**}(u_3^{**})|$ monotonically decreases and $|\dot{\phi}(V_p^{max}, K, \theta, \phi)|$ monotonically increases, if $(\theta - \phi) \in (0, \frac{\pi}{2}) \bigcup (\pi, \frac{3\pi}{2})$. In a second step, we show that if $(\theta - \phi) \in (0, \frac{\pi}{2}) \bigcup (\pi, \frac{3\pi}{2})$ then $g(\phi, \theta, K)$ monotonically increases.

Analysis for ψ_4:

For ψ_4 there are two cases, which we refer to here as **Case A** and **Case B** for convenience. We now proceed individually with each of these.

Case A: $(\theta - \phi) \in I$ **quadrant**

Analyzing $\dot{\phi}(V_e^{max}, \psi_4, \theta, \phi)$

First, we obtain the variations of $(\theta - \psi_4)$.

$$\psi_4 = \arctan\left(\frac{-\sin\phi + \gamma\cos\theta}{-\cos\phi - \gamma\sin\theta}\right)$$

Subtracting ψ_4 from θ in both sides of the equation.

$$\theta - \psi_4 = \theta - \arctan\left(\frac{-\sin\phi + \gamma\cos\theta}{-\cos\phi - \gamma\sin\theta}\right)$$

Expressing θ as $\arctan\left(\frac{\sin\theta}{\cos\theta}\right)$

$$\theta - \psi_4 = \arctan\left(\frac{\sin\theta}{\cos\theta}\right) - \arctan\left(\frac{-\sin\phi + \gamma\cos\theta}{-\cos\phi - \gamma\sin\theta}\right)$$

Computing the tangent in both sides of the equation.

$$\tan(\theta - \psi_4) = \tan\left(\arctan\left(\frac{\sin\theta}{\cos\theta}\right) - \arctan\left(\frac{-\sin\phi + \gamma\cos\theta}{-\cos\phi - \gamma\sin\theta}\right)\right)$$

Using an equivalence of $\tan(u - v)$

$$\tan(\theta - \psi_4) = \frac{\tan\left(\arctan\left(\frac{\sin\theta}{\cos\theta}\right)\right) - \tan\left(\arctan\left(\frac{-\sin\phi + \gamma\cos\theta}{-\cos\phi - \gamma\sin\theta}\right)\right)}{1 + \tan\left(\arctan\left(\frac{\sin\theta}{\cos\theta}\right)\right)\tan\left(\arctan\left(\frac{-\sin\phi + \gamma\cos\theta}{-\cos\phi - \gamma\sin\theta}\right)\right)}$$

This can be further simplified as

$$\tan(\theta-\psi_4) = \frac{\left(\frac{\sin\theta}{\cos\theta}\right) - \left(\frac{-\sin\phi+\gamma\cos\theta}{-\cos\phi-\gamma\sin\theta}\right)}{1+\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{-\sin\phi+\gamma\cos\theta}{-\cos\phi-\gamma\sin\theta}\right)}$$
$$= \frac{-(\sin\theta\cos\phi-\cos\theta\sin\phi+\gamma)}{-(\sin\theta\sin\phi+\cos\theta\cos\phi)}$$
$$= \frac{-\sin(\theta-\phi)-\gamma}{-\cos(\theta-\phi)}$$

Now we obtain the extremal values of $(\theta-\psi_4)$ for $(\theta-\phi) \in I$ quadrant. If $(\theta-\phi) \in I$ quadrant then the extremal values of $(\theta-\psi_4)$ can be computed as follows:

$$(\theta-\psi_4) = \arctan\left(\frac{-\sin(\theta-\phi)-\gamma}{-\cos(\theta-\phi)}\right)$$

Note that the admissible values of the ratio γ satisfy $\gamma \in (0,1]$. If $(\theta-\phi)=0$ then $-\sin(\theta-\phi)-\gamma \in [-1,0)$ and $-\cos(\theta-\phi)=-1$. If $(\theta-\phi)=\frac{\pi}{2}$ then $-\sin(\theta-\phi)-\gamma \in [-2,-1]$ and $-\cos(\theta-\phi)=0$.

If $(\theta-\phi)=0$ then according to the value of the ratio γ, the possibles values of $(\theta-\psi_4)$ are within the extremal values:

$$(\theta-\psi_4) = \arctan\left(\frac{0}{-1}\right) \to \pi$$

$$(\theta-\psi_4) = \arctan\left(\frac{-1}{-1}\right) = \frac{5\pi}{4}$$

If $(\theta-\phi)=\frac{\pi}{2}$ then regardless of the value of the ratio γ, the value of $(\theta-\psi_4)$ is:

$$(\theta-\psi_4) = \arctan\left(\frac{-1}{0}\right) = \frac{3\pi}{2}$$

$$(\theta-\psi_4) = \arctan\left(\frac{-2}{0}\right) = \frac{3\pi}{2}$$

Thus, if $(\theta-\phi) \in [0,\frac{\pi}{2}]$ then the extremal values of $(\theta-\psi_4) \in (\pi,\frac{3\pi}{2}]$. Note that the admissible values for the ratio $\gamma \in (0,1]$ can produce a bigger interval, but always within the interval $(\pi,\frac{3\pi}{2}]$, as the ratio γ becomes smaller, the lower limit of the interval approaches π.

It is important to stress that if $(\theta-\phi)$ increases $(\theta-\psi_4)$ also increases and vice-versa, they have the same sense of variation.

Now we show that $|\dot{\phi}(V_e^{max},\psi_4,\theta,\phi)|$ monotonically increases or decreases. Recall that

$$\dot{\phi}(V_e^{max},\psi_4,\theta,\phi) = \frac{V_e^{max}\sin(\theta-\psi_4)}{L\cos(\theta-\phi)}$$

If $(\theta-\phi) \in I$ quadrant varying from 0 to $\frac{\pi}{2}$ then $(\theta-\psi_4)$ increases within the interval $(\pi,\frac{3\pi}{2}]$, therefore

- $\cos(\theta-\phi) \geq 0$, and its value is decreasing monotonically
- $\sin(\theta-\psi_4) < 0$, and its value is decreasing monotonically
- $|\sin(\theta-\psi_4)| > 0$, and its value is increasing monotonically

Hence, $|\dot{\phi}(V_e^{max},\psi_4,\theta,\phi)|$ monotonically increases. However, notice that $\dot{\phi}(V_e^{max},\psi_4,\theta,\phi)$ *is negative*, meaning that it produces a rod clockwise rotation. Symmetrically, if $(\theta-\phi) \in I$ quadrant varying from $\frac{\pi}{2}$ to 0 then $(\theta-\psi_4)$ decreases. Hence, $|\dot{\phi}(V_e^{max},\psi_4,\theta,\phi)|$ monotonically decreases.

Analyzing $u_3^{**}(V_e^{max},\psi_4,\theta,\phi)$ and $|u_4^{**}(|u_3^{**}|)|$

Using a procedure similar to what has been performed above, we obtain

$$(\psi_4-\phi) = \arctan\left(\frac{\gamma\cos(\theta-\phi)}{-1-\gamma\sin(\theta-\phi)}\right)$$

Recalling again that $\gamma \in (0,1]$, if $(\theta-\phi)=0$ then $\gamma\cos(\theta-\phi) \in (0,1]$ and $(-1-\gamma\sin(\theta-\phi))=-1$. If $(\theta-\phi)=\frac{\pi}{2}$ then $\gamma\cos(\theta-\phi)=0$ and $(-1-\gamma\sin(\theta-\phi)) \in [-2,-1)$.

If $(\theta-\phi)=0$ then according to the value of the ratio γ, the possibles values of $(\psi_4-\phi)$ are within the extremal values:

$$(\psi_4-\phi) = \arctan\left(\frac{1}{-1}\right) = \frac{3\pi}{4}$$

$$(\psi_4-\phi) = \arctan\left(\frac{0}{-1}\right) \to \pi$$

If $(\theta-\phi)=\frac{\pi}{2}$ then regardless of the value of the ratio γ, the value of $(\psi_4-\phi)$ is:

$$(\psi_4-\phi) = \arctan\left(\frac{0}{-2}\right) = \pi$$

$$(\psi_4-\phi) = \arctan\left(\frac{0}{-1}\right) \to \pi$$

Thus, if $(\theta-\phi) \in [0,\frac{\pi}{2}]$ then the extremal values of $(\psi_4-\phi) \in [\frac{3\pi}{4},\pi]$. The admissible values for the ratio $\gamma \in (0,1]$ can produce a smaller interval, but always within the interval $[\frac{3\pi}{4},\pi]$, as the ratio γ becomes smaller, the lower limit of the interval approaches π. If $(\theta-\phi)$ increases $(\psi_4-\phi)$ also increases and vice-versa, they have the same sense of variation.

Now we show that $|u_4^{**}(|u_3^{**}(V_e^{max},\psi_4,\theta,\phi)|)|$ monotonically increases or decreases. The term $|u_4^{**}(|u_3^{**}|)|$ can be expressed as follows:

$$|u_4^{**}(|u_3^{**}|)| = \max|\dot{\theta}| = \frac{1}{b}\left(V_p^{max} - |u_3^{**}(V_e^{max},\psi_4,\theta,\phi)|\right)$$

We will obtain the variation of $|u_4^{**}|$ using $|u_3^{**}|$. Recall that:

$$u_3^{**}(V_e^{max},\psi_4,\theta,\phi) = \frac{V_e^{max}\cos(\psi_4-\phi)}{\cos(\theta-\phi)}$$

If $(\theta-\phi) \in I$ quadrant varying from 0 to $\frac{\pi}{2}$ then $(\psi_4-\phi)$ increases within the interval $[\frac{3\pi}{4},\pi]$. Therefore

- $\cos(\theta-\phi) \geq 0$, and its value is decreasing monotonically
- $\cos(\psi_4-\phi) < 0$, and its value is decreasing monotonically
- $|\cos(\psi_4-\phi)| > 0$, and its value is increasing monotonically

Hence, $|u_3^{**}(V_e^{max},\psi_4,\theta,\phi)|$ monotonically increases and consequentially $|u_4^{**}(|u_3^{**}|)|$ monotonically decreases. Notice that, $u_3^{**}(V_e^{max},\psi_4,\theta,\phi)$ *is negative*, meaning that it produces a pursuer backwards motion.

Symmetrically, if $(\theta-\phi) \in I$ quadrant varying from $\frac{\pi}{2}$ to 0 then $(\psi_4-\phi)$ decreases. Hence, $|u_3^{**}(V_e^{max},\psi_4,\theta,\phi)|$ monotonically decreases and $|u_4^{**}(|u_3^{**}|)|$ monotonically increases.

Finally, we show that, $g_4(\psi_4,\theta,\phi)$ monotonically increases or decreases. If $(\theta-\phi) \in I$ quadrant and $(\theta-\phi)$ varies from 0 to $\frac{\pi}{2}$ then $\sin(\theta-\phi) \geq 0$ in equation 53 (Lemma II), and its value is monotonically increasing. Hence, the value of $g_4(\phi,\theta,\psi_4)$ is monotonically *increasing*. Symmetrically, if $(\theta-\phi)$ varies from $\frac{\pi}{2}$ to 0 then $\sin(\theta-\phi) \geq 0$ in the equation 53 (Lemma II), and its value is monotonically decreasing. Therefore, the value of $g_4(\phi,\theta,\psi_4)$ is monotonically *decreasing*.

Case B: $(\theta-\phi) \in II$
Analyzing $\dot{\phi}(V_e^{max},\psi_4,\theta,\phi)$

First, we obtain the extremal values of $(\theta-\psi_4)$ for $(\theta-\phi) \in II$ quadrant.

$$(\theta-\psi_4) = \arctan\left(\frac{-\sin(\theta-\phi)-\gamma}{-\cos(\theta-\phi)}\right)$$

If $(\theta-\phi)=\frac{\pi}{2}$ then $-\sin(\theta-\phi)-\gamma \in (-2,-1]$ and $-\cos(\theta-\phi)=0$. If $(\theta-\phi)=\pi$ then $-\sin(\theta-\phi)-\gamma \in [-1,0)$ and $-\cos(\theta-\phi)=1$. If $(\theta-\phi)=\frac{\pi}{2}$ then regardless of the value of the ratio γ, the value of $(\theta-\psi_4)$ is:

$$(\theta-\psi_4)=\arctan\left(\frac{-1}{0}\right)=\frac{3\pi}{2}$$

$$(\theta-\psi_4)=\arctan\left(\frac{-2}{0}\right)\to\frac{3\pi}{2}$$

If $(\theta-\phi)=\pi$ then according to the value of the ratio γ, the possibles values of $(\theta-\psi_4)$ are within the extremal values:

$$(\theta-\psi_4)=\arctan\left(\frac{-1}{1}\right)=\frac{7\pi}{4}$$

and

$$(\theta-\psi_4)=\arctan\left(\frac{0}{1}\right)\to 2\pi$$

Thus, if $(\theta-\phi)\in[\frac{\pi}{2},\pi]$ then the extremal values of $(\theta-\psi_4)\in[\frac{3\pi}{2},2\pi)$. The admissible values for the ratio $\gamma\in(0,1]$ can produce a bigger interval, but always within the interval $[\frac{3\pi}{2},2\pi)$, as the ratio γ becomes smaller, the upper limit of the interval approaches 2π. If $(\theta-\phi)$ increases $(\theta-\psi_4)$ also increases and vice-versa, they have the same sense of variation.

Now we show that $|\dot{\phi}(V_e^{max},\psi_4,\theta,\phi)|$ monotonically increases or decreases. Recall that

$$\dot{\phi}(V_e^{max},\psi_4,\theta,\phi)=\frac{V_e^{max}\sin(\theta-\psi_4)}{L\cos(\theta-\phi)}$$

If $(\theta-\phi)\in II$ quadrant varying from $\frac{\pi}{2}$ to π then $(\theta-\psi_4)$ increases within the interval $[\frac{3\pi}{2},2\pi)$. Therefore

- $\cos(\theta-\phi)\le 0$, and its value is decreasing monotonically
- $|\cos(\theta-\phi)|\ge 0$, and its value is increasing monotonically
- $\sin(\theta-\psi_4)<0$, and its value is increasing monotonically
- $|\sin(\theta-\psi_4)|>0$, and its value is decreasing monotonically

Hence, $|\dot{\phi}(V_e^{max},\psi_4,\theta,\phi)|$ monotonically decreases, and the value of $\dot{\phi}(V_e^{max},\psi_4,\theta,\phi)$ *is positive*, meaning that it produces a rod counterclockwise rotation.

Symmetrically, if $(\theta-\phi)\in II$ quadrant varying from π to $\frac{\pi}{2}$ then $(\theta-\psi_4)$ decreases. Hence, $|\dot{\phi}(V_e^{max},\psi_4,\theta,\phi)|$ monotonically increases.

Analyzing $u_3^{**}(V_e^{max},\psi_4,\theta,\phi)$ and $|u_4^{**}(|u_3^{**}|)|$

If $(\theta-\phi)\in II$ quadrant then the extremal values of $(\psi_4-\phi)$ can be computed as follows:

$$(\psi_4-\phi)=\arctan\left(\frac{\gamma\cos(\theta-\phi)}{-1-\gamma\sin(\theta-\phi)}\right)$$

Since the ratio $\gamma\in(0,1]$, if $(\theta-\phi)=\pi$ then $\gamma\cos(\theta-\phi)\in[-1,0)$ and $(-1-\gamma\sin(\theta-\phi))=-1$. If $(\theta-\phi)=\frac{\pi}{2}$ then $\gamma\cos(\theta-\phi)=0$ and $(-1-\gamma\sin(\theta-\phi))\in[-2,-1)$.

If $(\theta-\phi)=\pi$ then according to the value of the ratio γ, the possibles values of $(\psi_4-\phi)$ are within the extremal values:

$$(\psi_4-\phi)=\arctan\left(\frac{0}{-1}\right)\to\pi$$

and

$$(\psi_4-\phi)=\arctan\left(\frac{-1}{-1}\right)=\frac{5\pi}{4}$$

If $(\theta-\phi)=\frac{\pi}{2}$ then regardless of the value of the ratio γ, the value of $(\psi_4-\phi)$ is:

$$(\psi_4-\phi)=\arctan\left(\frac{0}{-2}\right)=\pi$$

$$(\psi_4-\phi)=\arctan\left(\frac{0}{-1}\right)=\pi$$

Thus, if $(\theta-\phi)\in[\frac{\pi}{2},\pi]$ then the extremal values of $(\psi_4-\phi)\in[\pi,\frac{5\pi}{4}]$. The admissible values of the ratio $\gamma\in(0,1]$ can produce a smaller interval but always within the interval $[\pi,\frac{5\pi}{4}]$, as the ratio γ becomes smaller, the upper limit of the interval approaches π.

If $(\theta-\phi)$ increases $(\psi_4-\phi)$ also increases and vice-versa, they have the same sense of variation.

Now we show that $|u_4^{**}(|u_3^{**}|)|$ monotonically increases or decreases by examining its variation in terms of the value of $|u_3^{**}(V_e^{max},\psi_4,\theta,\phi)|$. The term $|u_4^{**}(|u_3^{**}|)|$ can be expressed as follows:

$$|u_4^{**}(u_3^{**})|=\max|\dot{\theta}|=\frac{1}{b}\left(V_p^{max}-|u_3^{**}(V_e^{max},\psi_4,\theta,\phi)|\right)$$

Recall that:

$$u_3^{**}(V_e^{max},\psi_4,\theta,\phi)=\frac{V_e^{max}\cos(\psi_4-\phi)}{\cos(\theta-\phi)}$$

If $(\theta-\phi)\in II$ quadrant varying from $\frac{\pi}{2}$ to π then $(\psi_4-\phi)$ increases within the interval $(\pi,\frac{5\pi}{4}]$. Therefore

- $\cos(\theta-\phi)\le 0$, and its value is decreasing monotonically
- $|\cos(\theta-\phi)|\ge 0$, and its value is increasing monotonically
- $\cos(\psi_4-\phi)<0$, and its value is increasing monotonically
- $|\cos(\psi_4-\phi)|>0$, and its value is decreasing monotonically

Hence, $|u_3^{**}(V_e^{max},\psi_4,\theta,\phi)|$ monotonically decreases and consequentially $|u_4^{**}(|u_3^{**}|)|$ monotonically increases, and we have $u_3^{**}(V_e^{max},\psi_4,\theta,\phi)$ *is positive*, meaning that it produces a forward motion.

Symmetrically, if $(\theta-\phi)\in II$ quadrant varying from π to $\frac{\pi}{2}$ then $(\psi_4-\phi)$ decreases. Hence, $|u_3^{**}(V_e^{max},\psi_4,\theta,\phi)|$ monotonically increases and $|u_4^{**}(u_3^{**})|$ monotonically decreases.

Finally, we show that $g_4(\phi,\theta,\psi_4)$ monotonically increases or decreases. If $(\theta-\phi)\in II$ quadrant and $(\theta-\phi)$ varies from $\frac{\pi}{2}$ to π then $\sin(\theta-\phi)\ge 0$, and its value is monotonically decreasing. Therefore, the value of $g_4(\phi,\theta,\psi_4)$ is monotonically *decreasing*. Symmetrically, if $(\theta-\phi)$ varies from π to $\frac{\pi}{2}$ then $\sin(\theta-\phi)\ge 0$ in the equation 53, and its value is monotonically increasing. Therefore, the value of $g_4(\phi,\theta,\psi_4)$ is monotonically *increasing*.

Analysis for ψ_3

$$(\theta-\psi_3)=\arctan\left(\frac{-\sin(\theta-\phi)+\gamma}{-\cos(\theta-\phi)}\right)$$

If $(\theta-\phi)\in[\pi,\frac{3\pi}{2}]$ then $(\theta-\psi_3)\in(0,\frac{\pi}{2}]$ and $\dot{\phi}(V_e^{max},\psi_3,\theta,\phi)$ *is negative*, producing a rod clockwise rotation. If $(\theta-\phi)\in[\frac{3\pi}{2},2\pi]$ then $(\theta-\psi_3)\in[\frac{\pi}{2},\pi)$ and $\dot{\phi}(V_e^{max},\psi_3,\theta,\phi)$ *is positive*, meaning that it produces a rod counterclockwise rotation.

$$(\psi_3-\phi)=\arctan\left(\frac{-\gamma\cos(\theta-\phi)}{-1+\gamma\sin(\theta-\phi)}\right)$$

If $(\theta-\phi) \in [\pi, \frac{3\pi}{2}]$ then $(\psi_3-\phi) \in [\frac{3\pi}{4}, \pi]$ and $u_3^{**}(V_e^{max}, \psi_3, \theta, \phi)$ is *positive*, meaning that it produces a pursuer forward motion. If $(\theta-\phi) \in [\frac{3\pi}{2}, 2\pi]$ then $(\psi_3-\phi) \in [\pi, \frac{5\pi}{4}]$ and $u_3^{**}(V_e^{max}, \psi_3, \theta, \phi)$ is *negative*, causing the pursuer to move backward.

We show now that $g_3(\psi_3, \theta, \phi)$ monotonically increases or decreases. If $(\theta-\phi) \in III$ quadrant, and $(\theta-\phi)$ varies from π to $\frac{3\pi}{2}$ then $\sin(\theta-\phi) \leq 0$ in equation 54 (Lemma II), and its value is monotonically decreasing. Consequently, $-\sin(\theta-\phi) \geq 0$ in equation 54 (Lemma II), and its value is monotonically increasing. Hence, the value of $g_3(\psi_3, \theta, \phi)$ is monotonically *increasing*. Symmetrically, if $(\theta-\phi)$ varies from $\frac{3\pi}{2}$ to π then the value of $g_3(\psi_3, \theta, \phi)$ is monotonically *decreasing*. If $(\theta-\phi) \in IV$ quadrant, and $(\theta-\phi)$ varies from $\frac{3\pi}{2}$ to 2π then $\sin(\theta-\phi) \leq 0$ (in equation 54, Lemma II), and its value is monotonically increasing. Consequently, $-\sin(\theta-\phi) \geq 0$ (in equation 54, Lemma II), and its value is monotonically decreasing. Therefore, the value of $g_3(\psi_3, \theta, \phi)$ is monotonically *decreasing*. Symmetrically, if $(\theta-\phi)$ varies from 2π to $\frac{3\pi}{2}$ then the value of $g_3(\psi_3, \theta, \phi)$ is monotonically *increasing*.

Analysis of ψ_2

$$(\theta-\psi_2) = \arctan\left(\frac{\sin(\theta-\phi)-\gamma}{\cos(\theta-\phi)}\right)$$

If $(\theta-\phi) \in [\pi, \frac{3\pi}{2}]$ then $(\theta-\psi_2) \in (\pi, \frac{3\pi}{2}]$, and $\dot{\phi}$ is positive, which generates a counterclockwise rotation. If $(\theta-\phi) \in [\frac{3\pi}{2}, 2\pi]$ then $(\theta-\psi_2) \in [\frac{3\pi}{2}, 2\pi)$, and $\dot{\phi}(V_e^{max}, \psi_2, \theta, \phi)$ is negative, which generates a clockwise rotation.

$$(\psi_2-\phi) = \arctan\left(\frac{\gamma\cos(\theta-\phi)}{1-\gamma\sin(\theta-\phi)}\right)$$

If $(\theta-\phi) \in [\pi, \frac{3\pi}{2}]$ then $(\psi_2-\phi) \in [\frac{7\pi}{4}, 2\pi]$, then $u_3^{**}(V_e^{max}, \psi_2, \theta, \phi)$ is negative, which generates a backward pursuer motion. If $(\theta-\phi) \in [\frac{3\pi}{2}, 2\pi]$ then $(\psi_2-\phi) \in [0, \frac{\pi}{4}]$, and $u_3^{**}(V_e max, \psi_2, \theta, \phi)$ is positive, which generates a forward pursuer motion.

Analysis of ψ_1

$$(\theta-\psi_1) = \arctan\left(\frac{\sin(\theta-\phi)+\gamma}{\cos(\theta-\phi)}\right)$$

If $(\theta-\phi) \in [0, \frac{\pi}{2}]$ then $(\theta-\psi_1) \in (0, \frac{\pi}{2}]$ and $\dot{\phi}(V_e^{max}, \psi_1, \theta, \phi)$ is positive, which generates a counterclockwise rotation. If $(\theta-\phi) \in [\frac{\pi}{2}, \pi]$ then $(\theta-\psi_1) \in [\frac{\pi}{2}, \pi)$ and $\dot{\phi}(V_e^{max}, \psi_1, \theta, \phi)$ is negative, which generates a clockwise rotation.

$$(\psi_1-\phi) = \arctan\left(\frac{-\gamma\cos(\theta-\phi)}{1+\gamma\sin(\theta-\phi)}\right)$$

If $(\theta-\phi) \in [0, \frac{\pi}{2}]$ then $(\psi_1-\phi) \in [\frac{7\pi}{4}, 2\pi]$ and $u_3^{**}(V_e^{max}, \psi_1, \theta, \phi)$ is positive, which generates a forward pursuer motion. If $(\theta-\phi) \in [\frac{\pi}{2}, \pi]$ then $(\psi_1-\phi) \in [0, \frac{\pi}{4}]$ and $u_3^{**}(V_e^{max}, \psi_1, \theta, \phi)$ is negative, which generates backward pursuer motion.

■

E Proof of Corollary of Lemma III

Corollary of Lemma III:
If $(\theta-\phi) \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$ then the DPSR is clockwise (cw), i.e. $sgn(u_4^{**}) = s(\theta, \phi) = -1$ (as given by Eq. 24), and the DESR is also clockwise (cw), i.e. $sgn(\dot{\phi}(\psi_i)) = -1$. Conversely, if $(\theta-\phi) \in (\frac{\pi}{2}, \pi) \cup (\frac{3\pi}{2}, 2\pi)$ then the DPSR is counterclockwise (ccw) $sgn(u_4^{**}) = s(\theta, \phi) = +1$ (as given by Eq. 24), and the DESR is also counterclockwise (ccw) $sgn(\dot{\phi}(\psi_i)) = +1$. Thus, the following controls must be applied by the players according to the value of $(\theta-\phi)$, to obtain the required sense of rotation of both u_4^{**} and $\dot{\phi}(V_p^{max}, K, \theta, \phi)$.

The evader control u_2^*, is given in table 1.

The sign (sense of rotation) of $u_4^{**} = \max|\dot{\theta}| = \frac{1}{b}(V_p^{max} - |u_3^{**}|)$ is defined by equation 24.

Proof:
Figure 13 illustrates the sense of rotation for the pursuer strategy. In that figure the orientation of the pursuer heading θ is measured with respect to the value of ϕ, the rod's orientation. The pursuer rotates its heading either clockwise θ^- or counterclockwise θ^+ based on the direction that requires a smaller rotation to reach a parallel alignment of the robot heading with respect to the rod orientation, that is $(\theta-\phi) = \{0, \pi, 2\pi\}$, without bringing the pursuer heading (pursuer wheels) and the rod closer to perpendicularity (i.e. $(\theta-\phi) = \{\frac{\pi}{2}, \frac{3\pi}{2}\}$.

If $(\theta-\phi)$ is within the first or third quadrant then the pursuer rotates clockwise θ^-, and if $(\theta-\phi)$ is within the second or fourth quadrant then the pursuer rotates counterclockwise θ^+.

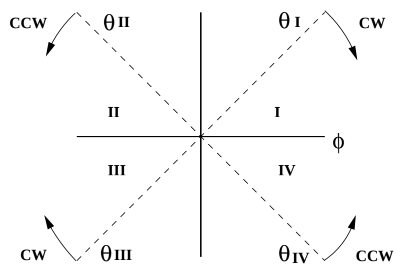

Fig. 13 Sense of rotation

Symmetrically, the evader must select a rod's counterclockwise or a clockwise rotation, based on the current state of rod and pursuer wheels orientation. The evader control ψ_i must produce a rod rotation that brings the rod perpendicular to the pursuer wheels (pursuer heading) without bringing the rod and the pursuer heading (pursuer wheels) closer to parallelism (i.e.$(\theta-\phi) = \{0, \pi, 2\pi\}$). Therefore, if $(\theta-\phi)$ is in the first or third quadrant the evader should rotate the rod clockwise (ϕ^-), and if $(\theta-\phi)$ is in the second or fourth quadrant the evader should rotate the rod counterclockwise (ϕ^+). Furthermore, the evader control must maximize g.

By Lemma II, if $(\theta-\phi) \in [0, \pi]$ then g is maximized by ψ_1 and ψ_4, and if $(\theta-\phi) \in [\pi, 2\pi]$ then g is maximized by ψ_2 and ψ_3. Hence, If $(\theta-\phi) \in [0, \pi]$ then the evader must select either ψ_1 or ψ_4, and if $(\theta-\phi) \in [\pi, 2\pi]$ then the evader must select either ψ_2 or ψ_3

Refer to table 1. If $(\theta-\phi) \in [0, \frac{\pi}{2}]$ the evader must select ψ_4, since it produces a rod's clockwise rotation, if $(\theta-\phi) \in [\frac{\pi}{2}, \pi]$, the evader must also select ψ_4, since it produces a counterclockwise rotation. Similarly, if $(\theta-\phi) \in [\pi, \frac{3\pi}{2}]$ the evader must select ψ_3, since it produces a rod's clockwise rotation, and if $(\theta-\phi) \in [\frac{3\pi}{2}, 2\pi]$ the evader must also select ψ_3. Hence, ψ_1 and ψ_2 are not used.

■

F Proof of remaining cases of Theorem I

Case II $(\theta - \phi) \in [\frac{\pi}{2}, \pi]$

First, assume that $M(V_e^{max}, V_p^{max}, \theta, \phi) < 0$ at the beginning of the game. This implies that $\max|\dot{\theta}(t_0)| > |\dot{\phi}(t_0)|$ and $V_p^{max}|\cos(\theta - \phi)| > |u_1^*| \cdot g(\phi, \theta, u_2^*)$. By Lemmas II and III $g(\phi, \theta, u_2^*)$ is maximal and varies monotonically by applying the optimal $u_2^* = \psi_4$. By Lemma III $g(\phi, \theta, u_2^*)$ monotonically decreases and $|\cos(\theta - \phi)|$ monotonically increases as $(\theta - \phi)$ varies from $\frac{\pi}{2}$ to π.

If $\max|\dot{\theta}(t_0)| > |\dot{\phi}(t_0)|$ then $(\theta(t_0) - \phi(t_0)) < (\theta(t_0 + \epsilon) - \phi(t_0 + \epsilon))$ for $\epsilon \to 0$. By Lemma III $\max|\dot{\theta}|$ monotonically increases and $|\dot{\phi}|$ monotonically decreases as $(\theta - \phi)$ varies from $\frac{\pi}{2}$ to π. Hence $\forall t > t_0,\ \max|\dot{\theta}(t)| > |\dot{\phi}(t)|$. Therefore $\forall t > t_0, (\theta(t_0) - \phi(t_0)) < (\theta(t) - \phi(t))$, and $(\theta(t) - \phi(t))$ increases monotonically until it reaches π yielding a pursuer winning.

Now assume that $M(V_e^{max}, V_p^{max}, \theta, \phi) > 0$ at the beginning of the game. This implies that $\max|\dot{\theta}(t_0)| < |\dot{\phi}(t_0)|$ and $V_p^{max}|\cos(\theta - \phi)| < |u_1^*| \cdot g(\phi, \theta, u_2^*)$. Then by Lemma III $g(\phi, \theta, u_2^*)$ monotonically increases by applying the optimal $u_2^* = \psi_4$, and $|\cos(\theta - \phi)|$ monotonically decreases as $(\theta - \phi)$ varies from π to $\frac{\pi}{2}$.

If $\max|\dot{\theta}(t_0)| < |\dot{\phi}(t_0)|$ then $(\theta(t_0) - \phi(t_0)) > (\theta(t_0 + \epsilon) - \phi(t_0 + \epsilon))$ for $\epsilon \to 0$. By Lemma III $\max|\dot{\theta}|$ monotonically decreases and $|\dot{\phi}|$ monotonically increases as $(\theta - \phi)$ varies from π to $\frac{\pi}{2}$. Hence $\forall t > t_0,\ \max|\dot{\theta}(t)| < |\dot{\phi}(t)|$. Therefore $(\theta(t_0) - \phi(t_0)) > (\theta(t) - \phi(t))$ and $(\theta(t) - \phi(t))$ decreases monotonically until it reaches $\frac{\pi}{2}$ yielding an evader winning.

Case III $(\theta - \phi) \in [\pi, \frac{3\pi}{2}]$

First, assume that $M(V_e^{max}, V_p^{max}, \theta, \phi) < 0$ at the beginning of the game. This implies that $\max|\dot{\theta}(t_0)| > |\dot{\phi}(t_0)|$ and $V_p^{max}|\cos(\theta - \phi)| > |u_1^*| \cdot g(\phi, \theta, u_2^*)$. By Lemmas II and III $g(\phi, \theta, u_2^*)$ is maximal and varies monotonically by applying the optimal $u_2^* = \psi_3$. By Lemma III $g(\phi, \theta, u_2^*)$ monotonically decreases and $|\cos(\theta - \phi)|$ monotonically increases as $(\theta - \phi)$ varies from $\frac{3\pi}{2}$ to π.

If $\max|\dot{\theta}(t_0)| > |\dot{\phi}(t_0)|$ then $(\theta(t_0) - \phi(t_0)) > (\theta(t_0 + \epsilon) - \phi(t_0 + \epsilon))$ for $\epsilon \to 0$. By Lemma III $\max|\dot{\theta}|$ monotonically increases and $|\dot{\phi}|$ monotonically decreases as $(\theta - \phi)$ varies from $\frac{3\pi}{2}$ to π. Hence $\forall t > t_0,\ \max|\dot{\theta}(t)| > |\dot{\phi}(t)|$. Therefore $\forall t > t_0, (\theta(t_0) - \phi(t_0)) > (\theta(t) - \phi(t))$, and $(\theta(t) - \phi(t))$ decreases monotonically until it reaches π yielding a pursuer winning.

Now assume that $M(V_e^{max}, V_p^{max}, \theta, \phi) > 0$ at the beginning of the game. This implies that $\max|\dot{\theta}(t_0)| < |\dot{\phi}(t_0)|$ and $V_p^{max}|\cos(\theta - \phi)| < |u_1^*| \cdot g(\phi, \theta, u_2^*)$. Then by Lemma III $g(\phi, \theta, u_2^*)$ monotonically increases by applying the optimal $u_2^* = \psi_3$, and $|\cos(\theta - \phi)|$ monotonically decreases as $(\theta - \phi)$ varies from π to $\frac{3\pi}{2}$.

If $\max|\dot{\theta}(t_0)| < |\dot{\phi}(t_0)|$ then $(\theta(t_0) - \phi(t_0)) < (\theta(t_0 + \epsilon) - \phi(t_0 + \epsilon))$ for $\epsilon \to 0$. By Lemma III $\max|\dot{\theta}|$ monotonically decreases and $|\dot{\phi}|$ monotonically increases as $(\theta - \phi)$ varies from π to $\frac{3\pi}{2}$. Hence $\forall t > t_0,\ \max|\dot{\theta}(t)| < |\dot{\phi}(t)|$. Therefore $(\theta(t_0) - \phi(t_0)) < (\theta(t) - \phi(t))$ and $(\theta(t) - \phi(t))$ increases monotonically until it reaches $\frac{3\pi}{2}$ yielding an evader winning.

Case IV $(\theta - \phi) \in [\frac{3\pi}{2}, 2\pi]$

First, assume that $M(V_e^{max}, V_p^{max}, \theta, \phi) < 0$ at the beginning of the game. This implies that $\max|\dot{\theta}(t_0)| > |\dot{\phi}(t_0)|$ and $V_p^{max}|\cos(\theta - \phi)| > |u_1^*| \cdot g(\phi, \theta, u_2^*)$. By Lemmas II and III $g(\phi, \theta, u_2^*)$ is maximal and varies monotonically by applying the optimal $u_2^* = \psi_3$. By Lemma III $g(\phi, \theta, u_2^*)$ monotonically decreases and $|\cos(\theta - \phi)|$ monotonically increases as $(\theta - \phi)$ varies from $\frac{3\pi}{2}$ to 2π.

If $\max|\dot{\theta}(t_0)| > |\dot{\phi}(t_0)|$ then $(\theta(t_0) - \phi(t_0)) < (\theta(t_0 + \epsilon) - \phi(t_0 + \epsilon))$ for $\epsilon \to 0$. By Lemma III $\max|\dot{\theta}|$ monotonically increases and $|\dot{\phi}|$ monotonically decreases as $(\theta - \phi)$ varies from $\frac{3\pi}{2}$ to 2π. Hence $\forall t > t_0,\ \max|\dot{\theta}(t)| > |\dot{\phi}(t)|$. Therefore $\forall t > t_0, (\theta(t_0) - \phi(t_0)) < (\theta(t) - \phi(t))$, and $(\theta(t) - \phi(t))$ increases monotonically until it reaches 2π yielding a pursuer winning.

Now assume that $M(V_e^{max}, V_p^{max}, \theta, \phi) > 0$ at the beginning of the game. This implies that $\max|\dot{\theta}(t_0)| < |\dot{\phi}(t_0)|$ and $V_p^{max}|\cos(\theta - \phi)| < |u_1^*| \cdot g(\phi, \theta, u_2^*)$. Then by Lemma III $g(\phi, \theta, u_2^*)$ monotonically increases by applying the optimal $u_2^* = \psi_3$, and $|\cos(\theta - \phi)|$ monotonically decreases as $(\theta - \phi)$ varies from 2π to $\frac{3\pi}{2}$.

If $\max|\dot{\theta}(t_0)| < |\dot{\phi}(t_0)|$ then $(\theta(t_0) - \phi(t_0)) > (\theta(t_0 + \epsilon) - \phi(t_0 + \epsilon))$ for $\epsilon \to 0$. By Lemma III $\max|\dot{\theta}|$ monotonically decreases and $|\dot{\phi}|$ monotonically increases as $(\theta - \phi)$ varies from 2π to $\frac{3\pi}{2}$. Hence $\forall t > t_0,\ \max|\dot{\theta}(t)| < |\dot{\phi}(t)|$. Therefore $(\theta(t_0) - \phi(t_0)) > (\theta(t) - \phi(t))$ and $(\theta(t) - \phi(t))$ decreases monotonically until it reaches $\frac{3\pi}{2}$ yielding an evader winning.

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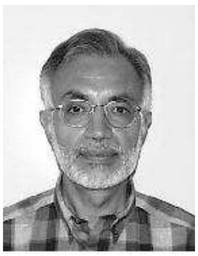

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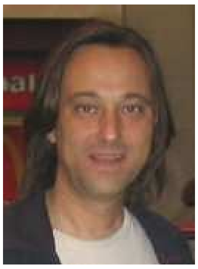

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