Arcsine Measure and Infinite Divisibility

Universität Ulm Mathematics Colloquium

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October 25, 2011

A simple fact

- $\varphi(x;\tau)$ *Gaussian kernel,* $\tau > 0$:

$$\varphi(x;\tau) = (2\pi\tau)^{-1/2} \mathrm{e}^{-x^2/(2\tau)}, \; x \in \mathbb{R}. \tag{1}$$

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- Then

$$\varphi(x;\tau) = \int_0^\infty f_\tau(s)\mathrm{a}(x;s)\mathrm{d}s,\ \tau > 0,\ x \in \mathbb{R}.$$

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4. **What is an infinitely divisible distribution?**

Plan

I. **Preliminaries on infinite divisibility of probability measures**
   1. Lévy-Khintchine representation for the Fourier transform
   2. Lévy processes
   3. Examples

II. **Gaussian representation and infinite divisibility**
   1. Simple consequences
   2. Ultraspherical distributions

III. **Type G distributions again: a new look**
   1. Lévy measure characterization (known).
   2. New Lévy measure characterization using the Gaussian representation

IV. **Distributions of class A**
   1. Lévy measure characterization
   2. Integral representation of type G distributions
   3. Integral representation of distributions of class A

V. **General framework**

- $\mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathbb{R}^d)$: Probability measures on $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

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- f_μ is the *density of* μ or *of a r.v.* X, $\mathcal{L}(X) = \mu$.

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- Similarly for $\mu_1 * \mu_2 * ... * \mu_n$

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- Let $ID(\mathbb{R})$ be the class of all infinitely divisible distributions on $\mathbb{R}$.

Theorem

A $\mu \in \mathcal{P}(\mathbb{R})$ is in $ID(\mathbb{R})$ iff its Fourier transform has the Lévy-Khintchine representation

$$\widehat{\mu}(s) = \exp\left\{\eta s - \frac{1}{2}as^2 + \int_{\mathbb{R}} \left(e^{isx} - 1 - sx1_{[-1,1]}(x)\right)\nu(\mathrm{d}x)\right\}, \ s \in \mathbb{R},$$

(Lévy) triplet (η, a, ν) is unique and such that:

i) $\eta \in \mathbb{R}$;

ii) $a \geq 0$ is the Gaussian part;

iii) ν is a measure (called Lévy measure) with: $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (1 \wedge |x|^2)\nu(\mathrm{d}x) < \infty.$$

(The Lévy measure ν is not necessary a finite measure).

I. Lévy Processes

Definition

A stochastic processes $X = \{X(t) : t \geq 0\}$ is a *Lévy process* if:

i) $\mathbb{P}(X(0) = 0) = 1$.

ii) X has *independent increments*.

iii) X has *stationary increments*.

iv) With probability one the function $t \to X(t)$ is right continuos with left limits (r.c.l.l.).

Theorem

Given a Lévy process $X = \{X(t) : t \geq 0\}$ *there is a unique* $\mu \in ID(\mathbb{R})$ *with*

$$\mathcal{L}(X(1)) = \mu.$$

If μ *has triplet* (η, a, ν), *then* $\forall\ t > 0$,

$$\mathcal{L}(X(t)) = \mu_t \in ID(\mathbb{R})$$

with triplet $(t\eta, ta, t\nu)$.

I. Role of the Lévy measure in the Lévy Process

- (η, a, ν) is also called the *triplet of the Lévy process* $X = \{X(t) : t \geq 0\}$, $\mathcal{L}(X(1)) = \mu \in ID(\mathbb{R})$ (with triplet (η, a, ν)).

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- $\mathrm{d}t\nu(\mathrm{d}x)$ is called *control measure of* $N(t, A)$.

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- Many interesting classes of ID distributions are characterized by integral representations (later today).
- **Open problem: what is the largest class of $ID(\mathbb{R})$ that can be represented as integral with respect to Lévy process?**

I. Infinitely divisibility in the positive real line

- $\mathcal{P}(\mathbb{R}_+)$ probability measures on $\mathbb{R}_+$, $ID(\mathbb{R}_+) = \mathcal{P}(\mathbb{R}_+) \cap ID(\mathbb{R})$.

Theorem

$\mu \in ID(\mathbb{R}_+)$ iff its Lévy triplet (η, a, ν) satisfies: $a = 0$

$$\eta_0 = \eta - \int_{|x|\le 1} x\nu(\mathrm{d}x) \ge 0$$

$\nu((-\infty, 0] = 0$ and

$$\int_{\mathbb{R}} (1 \wedge |x|)\nu(\mathrm{d}x) < \infty.$$

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$$\widehat{\mu}(s) = \exp\left\{\eta_0 s + \int_{\mathbb{R}} (e^{isx} - 1)\nu(\mathrm{d}x)\right\}, \quad s \in \mathbb{R}.$$

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- Associated Lévy process $\{V(t); t \geq 0\}$ is nondecresing (w.p. 1) and is called *subordinator* corresponding to $\mu = \mathcal{L}(V(1))$.

I. Infinitely divisible distributions: Examples

The Gaussian distribution is ID

- *Gaussian distribution* $\mathrm{N}(\eta, \tau)$ has density

$$\varphi(x; \eta, \tau) = (2\pi\tau)^{-1/2} \mathrm{e}^{-(x-\eta)^2/(2\tau)}, \ x \in \mathbb{R}.$$

- Lévy measure is zero ($\nu \equiv 0$).
- $\eta \in \mathbb{R}$ is the mean and $\tau > 0$ is the variance:

$$\eta = \int_{\mathbf{R}} x\varphi(x; \tau)\mathrm{d}x, \quad \tau = \int_{\mathbf{R}} (x - \eta)^2 \varphi(x; \tau)\mathrm{d}x.$$

- The distribution is symmetric around zero when $\eta = 0$, i.e. $\varphi(-x; 0, \tau) = \varphi(x; 0, \tau)$.
- The corresponding Lévy process is the *Brownian motion* $B(t)$, $t \geq 0$.
- *Brownian motion is the only Lévy process without jumps.*

- *Poisson distribution* $\mathrm{P}(\lambda)$, $\lambda > 0$, is a discrete distribution

$$p_k = \frac{\lambda^k}{k!} e^{-\lambda}, \; k = 0, 1, 2, \ldots$$

- Gaussian part is zero $(\tau = 0)$, $\eta = \lambda$ and the Lévy measure is

$$\nu(\mathrm{d}x) = \lambda \delta_1(\mathrm{d}x).$$

- The corresponding Lévy process is the *Poisson process* $N(t), t \geq 0$.
- It has jumps of size 1 and the expected number of jumps in an interval of length t is λt.
- Several ID distributions can be constructed from the Poisson process.

- Let $N = \{N(t); t \geq 0\}$ be a Poisson process of parameter $\lambda > 0$.

I. Infinitely divisible distributions: Examples

Compound Poisson distributions

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$$X(t) = \sum_{j=1}^{N(t)} Y_j$$

is a Lévy process with Lévy triplet: $\tau = 0$,

$$\eta = \int_{|x|\leq 1} x\mu(\mathrm{d}x);$$

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- *Every ID distribution is a limit of compound Poisson distributions.*

I. Infinitely divisible distributions: Examples

The Gamma distribution is ID

- *Gamma distribution* $\mathrm{G}(\alpha,\beta)$, $\alpha \geq 0$,$\beta \geq 0$, has density

$$g_{\alpha,\beta}(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} 1_{[0,\infty)}(x)$$

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- $\tau = 0$, $\eta = \int_{|x|\leq 1} x\nu(\mathrm{d}x)$ and Lévy measure is

$$\nu(\mathrm{d}x) = l(x)\mathrm{d}x, \quad l(x) = \alpha \frac{e^{-x/\beta}}{x} 1_{[0,\infty)}(x)$$

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- The *Lévy density* $l(x)$ is a completely monotone function in $x > 0$.
- $\alpha = \beta = 1$, associated Lévy process is the *Gamma process* $\gamma(t)$.

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- *Probabilistic interpretation*: GGC is the smallest subclass of $ID(\mathbb{R}_+)$ that is closed under convolution and weak convergence and containing the Gamma distributions.

II. Representation of the Gauss distribution

- $\varphi(x;\tau)$ density of the Gaussian distribution *zero mean and variance* $\tau > 0$

$$\varphi(x;\tau) = (2\pi\tau)^{-1/2} \mathrm{e}^{-x^2/(2\tau)}, \; x \in \mathbb{R}. \qquad (3)$$

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- $\mathrm{a}(x,s)$ density of *arcsine distribution* $\mathrm{a}(x,s)\mathrm{d}x$
$$\mathrm{a}(x,s) = \begin{cases} \frac{1}{\pi}(s - x^2)^{-1/2}, & |x| < \sqrt{s} \\ 0 & |x| \geq \sqrt{s}. \end{cases} \tag{5}$$
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A_s random variable with density $\mathrm{a}(x,s)$ on $(-\sqrt{s}, \sqrt{s})$. $(A = A_1)$.
- **Arcsine distribution is not ID**.

II. A representation of the Gaussian distribution

Fact

$$\varphi(x;\tau) = \frac{1}{2\tau}\int_0^\infty \mathrm{e}^{-s/(2\tau)}\mathrm{a}(x;s)\mathrm{d}s,\ \tau > 0,\ x \in \mathbb{R}. \tag{4}$$

Equivalently: If E_τ and A are independent random variables, then

$$Z_\tau \stackrel{L}{=} \sqrt{E_\tau}A.$$

Gaussian distribution is a exponential superposition of the arcsine distribution.

II. Simple consequences of the Gaussian representation

- **Variance mixture of Gaussians:** V positive r.v. $X \stackrel{L}{=} \sqrt{V}Z$, V and Z independent.

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Corollary

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- **Examples:** X^2 is infinitely divisible if X is stable symmetric, normal inverse Gaussian, normal variance gamma, $t-$student.

II. A characterization of Exponential Distribution

Theorem

Y_α, $\alpha > 0$, random variable with gamma distribution $G(\alpha, \beta)$ independent of A. Let

$$X = \sqrt{Y_\alpha} A.$$

Then X has an ID distribution if and only if $\alpha = 1$, in which case Y_1 has exponential distribution and X has Gaussian distribution.

- (Kingman (63)) $USP(\theta, \sigma)$: $\theta \geq -3/2$, $\sigma > 0$

$$f_\theta(x\,;\sigma) = c_{\theta,\sigma}\left(\sigma^2 - x^2\right)^{\theta+1/2} \qquad -\sigma < x < \sigma \tag{6}$$

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- $\theta = \infty$ is Gaussian distribution: *Poincaré´s theorem*: $(\theta \to \infty)$

$$f_\theta(x\,;\sqrt{(\theta+2)/2}\sigma) \to \frac{1}{\sqrt{2\pi}\sigma}\exp(-x^2/(2\sigma^2)).$$

II. Other Gaussian representations

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Theorem (Kingman (63))

Let Y_α, $\alpha > 0$, r.v. with gamma distribution $G(\alpha, \beta)$ independent of r.v. S_θ with distribution $USP(\theta, 1)$. Let

$$X \stackrel{L}{=} \sqrt{Y_\alpha} S_\theta \tag{8}$$

When $\alpha = \theta + 2$, X has a Gaussian distribution.

Moreover, *the distribution of X is infinitely divisible iff $\alpha = \theta + 2$ in which case X has a classical Gaussian distribution.*

- S_θ is r.v. with distribution $USP(\theta, 1)$. For $\theta > -1/2$ it holds that

$$S_\theta \stackrel{L}{=} U^{1/(2(\theta+1))} S_{\theta-1}$$

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- In particular, the *semicircle distribution is a mixture of the arcsine*

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- *This fact and the Gaussian representation suggest that the arcsine distribution is a "nice small" distribution to mixture with.*

III. Type G distributions

Definition and relevance

- *Definition*: A mixture of Gaussians $X = \sqrt{V}Z$ has a *type G* distribution if $V > 0$ has an ID divisible distribution.

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- Several well-known ID distributions are type G.
- $X_t^2 = (B_{V_t})^2$ is always infinitely divisible.
- **Open problem the ID of $(B_{V_t})^2$ as a process.**

III. Type G distributions: Lévy measure characterization

- If $V > 0$ is ID with Lévy measure ρ, then $\mu \stackrel{L}{=} \sqrt{V}Z$ is ID with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$

$$l(x) = \int_{\mathbb{R}_+} \varphi(x; s)\rho(\mathrm{d}s), \quad x \in \mathbb{R}. \tag{9}$$

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Theorem (Rosinski (91))

A symmetric distribution μ on $\mathbb{R}$ is type G iff is infinitely divisible and its Lévy measure is zero or $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where $l(x)$ is representable as

$$l(r) = g(r^2), \tag{10}$$

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- In general $G(\mathbb{R})$ is the class of *generalized type G* distributions with Lévy measure (10).

III. Type G distributions: new characterization

- Using Gaussian representation in $I(x) = \int_{\mathbb{R}_+} \varphi(x;s)\rho(\mathrm{d}s)$:

$$I(x) = \int_0^\infty \mathrm{a}(x;s)\eta(s)\mathrm{d}s. \tag{11}$$

where $\eta(s) := \eta(s;\rho)$ is the completely monotone function

$$\eta(s;\rho) = \int_{\mathbb{R}_+} (2r)^{-1}\, e^{-s(2r)^{-1}} \rho(\mathrm{d}r). \tag{12}$$

III. Type G distributions: new characterization

- Using Gaussian representation in $l(x) = \int_{\mathbb{R}_+} \varphi(x;s)\rho(\mathrm{d}s)$:

$$l(x) = \int_0^\infty \mathrm{a}(x;s)\eta(s)\mathrm{d}s. \tag{11}$$

where $\eta(s) := \eta(s;\rho)$ is the completely monotone function

$$\eta(s;\rho) = \int_{\mathbb{R}_+} (2r)^{-1}\, e^{-s(2r)^{-1}} \rho(\mathrm{d}r). \tag{12}$$

Theorem

A symmetric distribution μ on $\mathbb{R}$ is type G iff it is infinitely divisible with Lévy measure ν zero or $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where $l(x)$ is representable as (11) and η is a completely monotone function with

$$\int_0^\infty \min(1,s)\eta(s)\mathrm{d}s < \infty.$$

Lemma

Let g be a real function. The following statements are equivalent:
(a) g is completely monotone on $(0, \infty)$ with

$$\int_0^\infty (1 \wedge r^2) g(r^2) \mathrm{d}r < \infty. \tag{13}$$

(b) There is a function $h(s)$ completely monotone on $(0, \infty)$, with $\int_0^\infty (1 \wedge s) h(s) \mathrm{d}s < \infty$ and $g(r^2)$ has the arcsine transform

$$g(r^2) = \int_0^\infty \mathrm{a}^+(r; s) h(s) \mathrm{d}s, \quad r > 0, \tag{14}$$

where

$$\mathrm{a}^+(r; s) = \begin{cases} 2\pi^{-1}(s - r^2)^{-1/2}, & 0 < r < s^{1/2}, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

III. Type G distributions: Summary new representation

- Lévy measure is a (special) mixture of arcsine measure: There is a completely monotone function $\eta(s)$ on $(0, \infty)$ such that

$$l(x) = \int_0^\infty \mathrm{a}(x; s)\eta(s)\mathrm{d}s. \tag{16}$$

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- This is not the finite range mixture of the arcsine measure.
- Not type G : Compound Poisson distribution with Lévy measure the arcsine or semicircle measures.
- **Next problem**: Characterization of ID distributions when Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$ **is the arcsine transform**

$$l(x) = \int_0^\infty \mathrm{a}(x;s)\lambda(\mathrm{d}s). \tag{17}$$

IV. Distributions of Class A

Definition

$A(\mathbb{R})$: ID distributions with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where

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- $G(\mathbb{R}) \subset A(\mathbb{R})$
- **How large is the class** $A(\mathbb{R})$?

- ϖ a measure on $\{-1,1\}$, $h_\xi : \mathbb{R} \to \mathbb{R}_+$, $\xi = 1$ or -1,

$$\nu(B) = \int_{\mathbb{S}} \varpi(\mathrm{d}\xi) \int_0^\infty 1_E(r\xi) h_\xi(r) \mathrm{d}r, \quad E \in \mathcal{B}(\mathbb{R}). \qquad (19)$$

- $U(\mathbb{R})$, *Jurek class*: $h_\xi(r)$ is decreasing in $r > 0$.
- $L(\mathbb{R})$, *Selfdecomposable class*: $h_\xi(r) = r^{-1} g_\xi(r)$ and $g_\xi(r)$ decreasing in $r > 0$.
- $B(\mathbb{R})$, *Bondesson class:* $h_\xi(r)$ completely monotone in $r > 0$.
- $T(\mathbb{R})$, *Thorin class:* $h_\xi(r) = r^{-1} g_\xi(r)$ and $g_\xi(r)$ completely monotone in $r > 0$.
- $G(\mathbb{R})$, *Generalized type G class* $h_\xi(r) = g_\xi(r^2)$ and $g_\xi(r)$ completely monotone in $r > 0$.
- $A(\mathbb{R})$, *Class* $A(\mathbb{R})$, $h_\xi(r)$ is an **arcsine transform**.

IV. Relations between classes

- $$T(\mathbb{R}) \cup B(\mathbb{R}) \cup L(\mathbb{R}) \cup G(\mathbb{R}) \subset U(\mathbb{R})$$

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- Observation: Arcsine density $a(x; s)$ is increasing in $r \in (0, \sqrt{s})$

IV. Relation between type G and type A distributions

- $\mu \in ID(\mathbb{R})$, $X_t^{(\mu)}$ Lévy processes such that $\mathcal{L}\left(X_1^{(\mu)}\right) = \mu$.

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$$\Psi(\mu) = \mathcal{L}\left(\int_0^{1/2} \left(\log \frac{1}{s}\right)^{1/2} dX_s^{(\mu)}\right). \tag{20}$$

An ID distribution $\widetilde{\mu}$ belongs to $G(\mathbb{R})$ iff there exists a type A distribution μ such that $\widetilde{\mu} = \Psi(\mu)$. That is

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$$G(\mathbb{R}) = \Psi(A(\mathbb{R})). \tag{21}$$

- *This is a stochastic interpretation of the fact that for a generalized type G distribution its Lévy measure is mixture of arcsine measure*

$$l(x) = \int_0^\infty \mathrm{a}(x; s)\eta(s)\mathrm{d}s.$$

IV. Stochastic integral representations for some ID classes

- **Next problem: integral representation for type A distributions?**
- Jurek (85): $U(\mathbb{R}) = \mathcal{U}(ID(\mathbb{R}))$,

$$\mathcal{U}(\mu) = \mathcal{L}\left(\int_0^1 s\mathrm{d}X_s^{(\mu)}\right).$$

- Jurek, Vervaat (83), Sato, Yamazato (83): $L(\mathbb{R}) = \Phi(ID_{\log}(\mathbb{R}))$

$$\Phi(\mu) = \mathcal{L}\left(\int_0^\infty \mathrm{e}^{-s}\mathrm{d}X_s^{(\mu)}\right),$$

$$ID_{\log}(\mathbb{R}) = \left\{\mu \in ID(\mathbb{R}) : \int_{|x|>2} \log|x|\,\mu(\mathrm{d}x) < \infty\right\}.$$

- Barndorff-Nielsen, Maejima, Sato (06): $B(\mathbb{R}) = \Upsilon(ID(\mathbb{R}))$ and $T(\mathbb{R}) = \Upsilon(L(\mathbb{R}))$

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 \log\frac{1}{s}\mathrm{d}X_s^{(\mu)}\right).$$

Theorem (Maejima, PA, Sato (11))

Let $\Phi_{\cos} : ID(\mathbb{R}) \rightarrow ID(\mathbb{R})$ *be the mapping*

$$\Phi_{\cos}(\mu) = \mathcal{L}\left(\int_0^1 \cos(\frac{\pi}{2}s)\mathrm{d}X_s^{(\mu)}\right), \quad \mu \in ID(\mathbb{R}). \tag{22}$$

Then

$$A(\mathbb{R}) = \Phi_{\cos}(ID(\mathbb{R})). \tag{23}$$

.

- **Upsilon transformations of Lévy measures:**

$$\Upsilon_\sigma(\rho)(B) = \int_0^\infty \rho(u^{-1}B)\sigma(\mathrm{d}u), \quad B \in \mathcal{B}(\mathbb{R}). \tag{24}$$

[Barndorff-Nielsen, Rosinski, Thorbjørnsen (08)].

V. General framework

- **Upsilon transformations of Lévy measures:**

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[Barndorff-Nielsen, Rosinski, Thorbjørnsen (08)].

- **Fractional transformations of Lévy measures:**

$$(\mathcal{A}_{q,p}^{\alpha,\beta}\nu)(C) = \frac{1}{\Gamma(p)}\int_0^\infty r^{-q-1}dr \int_{\mathbb{R}} 1_C(r\frac{x}{|x|})(|x|^\beta - r^\alpha)_+^{p-1}\nu(\mathrm{d}x),$$

$p, \alpha, \beta \in \mathbb{R}_+$, $q \in \mathbb{R}$ [Maejima, PA, Sato (11), Sato (11)].

V. Fractional transformations of measures

- $p > 0, \alpha > 0, \beta > 0,\ q \in \mathbb{R}$

$$(\mathcal{A}_{q,p}^{\alpha,\beta}\nu)(C) = \frac{1}{\Gamma(p)} \int_0^\infty r^{-q-1} \mathrm{d}r \int_{\mathbb{R}} 1_C(r\frac{x}{|x|})(|x|^\beta - r^\alpha)_+^{p-1} \nu(\mathrm{d}x).$$

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- Associated classes of infinitely divisible distributions

$$A_{q,p}^{\alpha}(\mathbb{R}) = \mathcal{A}_{q,p}^{\alpha,\beta}(ID(\mathbb{R})).$$

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- **How large is the class $A_{q,p}^{\alpha}(\mathbb{R})$?**

V. Fractional transformations of measures

- Flexibility in choice of parameters

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Teorema

$$U(\mathbb{R}) \subset A^{\alpha}_{q,p}(\mathbb{R}) \qquad \textit{if } 0 < p \leq 1,\ q \leq -1, \tag{25}$$

$$A^{\alpha}_{q,p}(\mathbb{R}) \subset U(\mathbb{R}) \qquad \textit{if } p \geq 1,\ -1 \leq q < 2. \tag{26}$$

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  - Uniform $(q = -1, p = 1)$ then (25) and (26).
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- **We do not know if there are stochastic integrals representations for** $q \geq 1$.

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- **Special case:** $\alpha > 0, p > 0$, $q = -\alpha$. Let $\Phi_{\alpha,p} : ID(\mathbb{R}) \to ID(\mathbb{R})$

$$\Phi_{\alpha,p}(\mu) = \mathcal{L}\left(c_{p+1}^{-1/(\alpha p)} \int_0^{c_{p+1}} \left(c_{p+1}^{1/p} - s^{1/p} \right)^{1/\alpha} dX_s^{(\mu)} \right). \qquad (27)$$

with $c_p = 1/\Gamma(p)$. Then $A^{\alpha}_{-\alpha,p}(\mathbb{R}) = \Phi_{\alpha,p}(ID(\mathbb{R}))$.

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Example

If $p = 1/2$, $\alpha = 1$, $(q = -1)$

$$A^1_{-1,1/2}(\mathbb{R}) = \Phi_{1,1/2}(ID(\mathbb{R})),$$

$$\Phi_{1,1/2}(\mu) = \frac{\pi}{4} \int_0^{2/\sqrt{\pi}} \left(\frac{4}{\pi} - s^2 \right) dX_s^{(\mu)}, \quad \mu \in ID(\mathbb{R}).$$

Talk based on joint works

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