Some Roles of the Arcsine Distribution in Classical and non Classical Infinite Divisibility

Víctor Pérez-Abreu
CIMAT, Guanajuato, Mexico

Workshop on Infinite Divisibility and Branching Random Structures

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Notation

- $\mathrm{a}(x,s)$ density of **arcsine distribution** $\mathrm{a}(x,s)\mathrm{d}x$

$$\mathrm{a}(x,s) = \begin{cases} \frac{1}{\pi}(s - x^2)^{-1/2}, & |x| < \sqrt{s} \\ 0 & |x| \geq \sqrt{s}. \end{cases} \tag{1}$$

A_s random variable with density $\mathrm{a}(x,s)$ on $(-\sqrt{s}, \sqrt{s})$. $(A = A_1)$.

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- $\varphi(x;\tau)$ density of the **Gaussian distribution** $\varphi(x;\tau)\mathrm{d}x$ **zero mean and variance** $\tau > 0$

$$\varphi(x;\tau) = (2\pi\tau)^{-1/2}\mathrm{e}^{-x^2/(2\tau)},\ x \in \mathbb{R}. \tag{2}$$

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- Gaussian and exponential distributions are ID, but arcsine is not.

Representation of the Gaussian distribution

Fact

$$\varphi(x;\tau) = \frac{1}{2\tau}\int_0^\infty \mathrm{e}^{-s/(2\tau)}\mathrm{a}(x;s)\mathrm{d}s, \ \tau > 0, \ x \in \mathbb{R}. \tag{4}$$

Equivalently: If E_τ and A are independent random variables, then

$$Z_\tau \stackrel{L}{=} \sqrt{E_\tau}A.$$

Gaussian distribution is an exponential mixture of the arcsine distribution.

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- **<u>Goal:</u> show some implications of this representation in the construction of infinitely divisible distributions.**
- **<u>Motivation</u> comes from free infinite divisibility: construction of free ID distributions.**

Content of the talk

I. **Gaussian representation and infinite divisibility**

1. Simple consequences.
2. Power semicircle distributions (next talk by Octavio Arizmendi)

II. **Type G distributions again: a new look**

1. Lévy measure characterization (known).
2. New Lévy measure characterization using the Gaussian representation.

III. **Distributions of class A**

1. Lévy measure characterization.
2. Integral representation of type G distributions w.r.t. LP
3. Integral representation of distributions of class A w.r.t to LP.

IV **Non classical infinite divisibility**

1. Non classical convolutions
2. Free infinite divisibility
3. Bijection between classical and free ID distributions

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- **Examples:** X^2 is infinitely divisible if X is stable symmetric, normal inverse Gaussian, normal variance gamma, $t-$student.

I. A characterization of Exponential Distribution

- $G(\alpha,\beta)$, $\alpha > 0, \beta > 0$, gamma distribution with density

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- Y_{α}, $\alpha > 0$, random variable with gamma distribution $G(\alpha, \beta)$ independent of A. Let

$$X = \sqrt{Y_{\alpha}} A.$$

  Then X has an ID distribution if and only if $\alpha = 1$, in which case Y_1 has exponential distribution and X has Gaussian distribution.

- **PSD** (Kingman (63)) $PS(\theta, \sigma)$: $\theta \geq -3/2$, $\sigma > 0$

$$f_{\theta}(x\,;\sigma) = c_{\theta,\sigma}\left(\sigma^2 - x^2\right)^{\theta+1/2} \quad -\sigma < x < \sigma \tag{6}$$

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- Octavio's talk: symmetric Bernoulli, arcsine, semicircle and classical Gaussian are the only possible "Gaussian" distributions.

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Theorem (Kingman (63), Arizmendi- PA (10))

Let Y_α, $\alpha > 0$, r.v. with gamma distribution $G(\alpha, \beta)$ independent of r.v. S_θ with distribution $PS(\theta, 1)$. Let ,

$$X \stackrel{L}{=} \sqrt{Y_\alpha} S_\theta \tag{8}$$

When $\alpha = \theta + 2$, X has a Gaussian distribution.
Moreover, *the distribution of X is infinitely divisible iff $\alpha = \theta + 2$ in which case X has a classicial Gaussian distribution.*

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- Proof uses a simple kurtosis criteria (Octavio's talk)

I. Recursive representations

- S_θ is r.v. with distribution $PS(\theta, 1)$. For $\theta > -1/2$ it holds that

$$S_\theta \stackrel{L}{=} U^{1/(2(\theta+1))} S_{\theta-1} \qquad (9)$$

where U is r.v. with uniform distribution $U(0, 1)$ independent of r.v. $S_{\theta-1}$ with distribution $PS(\theta - 1, 1)$.

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- This fact and the Gaussian representation suggest that the arcsine distribution is a "nice small" distribution to mixture with.

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Recall: Definition and relevance

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  - $X_t = B_{V_t}$ has type G distribution

- $[X_t^2 = (B_{V_t})^2$ is always infinitely divisible].

II. Type G distributions: Lévy measure characterization

- If $V > 0$ is ID with Lévy measure ρ, then $\mu \overset{L}{=} \sqrt{V}Z$ is ID with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$

$$l(x) = \int_{\mathbb{R}_+} \varphi(x;s)\rho(\mathrm{d}s), \quad x \in \mathbb{R}. \tag{11}$$

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Theorem (Rosinski (91))

A symmetric distribution μ on $\mathbb{R}$ is type G iff is infinitely divisible and its Lévy measure is zero or $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where $l(x)$ is representable as

$$l(r) = g(r^2), \tag{12}$$

g is completely monotone on $(0, \infty)$ and $\int_0^\infty \min(1, r^2) g(r^2)\mathrm{d}r < \infty$.

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- In general $G(\mathbb{R})$ is the class of generalized type G distributions with Lévy measure (12).

II. Type G distributions: new characterization

- Using Gaussian representation in $I(x) = \int_{\mathbb{R}_+} \varphi(x;s)\rho(\mathrm{d}s)$:

$$I(x) = \int_0^\infty \mathrm{a}(x;s)\eta(s)\mathrm{d}s. \tag{13}$$

where $\eta(s) := \eta(s;\rho)$ is the completely monotone function

$$\eta(s;\rho) = \int_{\mathbb{R}_+} (2r)^{-1} e^{-s(2r)^{-1}} \rho(\mathrm{d}r). \tag{14}$$

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A symmetric distribution μ on $\mathbb{R}$ is type G iff it is infinitely divisible with Lévy measure ν zero or $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where
1) $l(x)$ is representable as (13),
2) η is a completely monotone function with $\int_0^\infty \min(1,s)\eta(s)\mathrm{d}s < \infty$.

Lemma

Let g be a real function. The following statements are equivalent:
(a) g is completely monotone on $(0, \infty)$ *with*

$$\int_0^\infty (1 \wedge r^2) g(r^2) \mathrm{d}r < \infty. \tag{15}$$

(b) There is a function $h(s)$ *completely monotone on* $(0, \infty)$, *with* $\int_0^\infty (1 \wedge s) h(s) \mathrm{d}s < \infty$ *and* $g(r^2)$ *has the arcsine transform*

$$g(r^2) = \int_0^\infty \mathrm{a}^+(r; s) h(s) \mathrm{d}s, \quad r > 0, \tag{16}$$

where

$$\mathrm{a}^+(r; s) = \begin{cases} 2\pi^{-1}(s - r^2)^{-1/2}, & 0 < r < s^{1/2}, \\ 0, & \textit{otherwise}. \end{cases} \tag{17}$$

II. Type G distributions: Summary new representation

- Lévy measure is a (special) mixture of arcsine measure: There is a completely monotone function $\eta(s)$ on $(0, \infty)$ such that

$$l(x) = \int_0^\infty \mathrm{a}(x; s)\eta(s)\mathrm{d}s. \qquad (18)$$

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- In free probability the (free) ID distribution with arcsine Lévy measure plays a key role (motivation).
- **Next problem**: Characterization of ID distributions which Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$ **is the arcsine transform**

$$l(x) = \int_0^\infty \mathrm{a}(x;s)\lambda(\mathrm{d}s). \tag{19}$$

III. Distributions of Class A

Definition

$A(\mathbb{R})$ is the class of A of distributions on $\mathbb{R}$: ID distributions with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where

$$l(x) = \int_{\mathbb{R}_+} \mathrm{a}(x; s)\lambda(\mathrm{d}s) \tag{20}$$

and λ is a Lévy measure on $\mathbb{R}_+ = (0, \infty)$.

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- Arizmendi, Barndorff-Nielsen, PA (10): Univariate and symmetric case (also in context of free ID).
- Further studied: Maejima, PA, Sato (11): $A(\mathbb{R}^d)$ including non-symmetric case, stochastic integral representation, relation to Upsilon transformations, comparison to other known ID classes.

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$A(\mathbb{R})$ is the class of A of distributions on $\mathbb{R}$: ID distributions with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where

$$l(x) = \int_{\mathbb{R}_+} \mathrm{a}(x; s)\lambda(\mathrm{d}s) \qquad (20)$$

and λ is a Lévy measure on $\mathbb{R}_+ = (0, \infty)$.

- Arizmendi, Barndorff-Nielsen, PA (10): Univariate and symmetric case (also in context of free ID).
- Further studied: Maejima, PA, Sato (11): $A(\mathbb{R}^d)$ including non-symmetric case, stochastic integral representation, relation to Upsilon transformations, comparison to other known ID classes.
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- $G(\mathbb{R}) \subset A(\mathbb{R})$.
- **How large is the class $A(\mathbb{R})$?**

- $ID(\mathbb{R}^d)$ **class of infinitely divisible distributions on** $\mathbb{R}^d$.

III. Recall some known classes of ID distributions

Characterization via Lévy measure

- $ID(\mathbb{R}^d)$ **class of infinitely divisible distributions on** $\mathbb{R}^d$.
- Polar decomposition of Lévy measure ν (univariate case $\xi = -1, 1$)

$$\nu(B) = \int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi) h_\xi(r)\mathrm{d}r, \quad B \in \mathcal{B}(\mathbb{B}^d). \tag{21}$$

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- $A(\mathbb{R}^d)$, **Class** $A(\mathbb{R})$**,** $h_\xi(r)$ is an arcsine transform.

III. Relations between classes

- $$T(\mathbb{R}^d) \cup B(\mathbb{R}^d) \cup L(\mathbb{R}^d) \cup G(\mathbb{R}^d) \subset U(\mathbb{R}^d)$$

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- Observation: Arcsine density $a(x; s)$ is increasing in $r \in (0, \sqrt{s})$

III. Relation between type G and type A distributions

- $\mu \in ID(\mathbb{R}^d)$, $X_t^{(\mu)}$ Lévy processes such that μ: $\mathcal{L}\left(X_1^{(\mu)}\right) = \mu$.

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Theorem (A, BN, PA (10); Maejima, PA, Sato (11).)

Let $\Psi : ID(\mathbb{R}^d) \rightarrow ID(\mathbb{R}^d)$ be the mapping given by

$$\Psi(\mu) = \mathcal{L}\left(\int_0^{1/2} \left(\log \frac{1}{s}\right)^{1/2} dX_s^{(\mu)}\right). \tag{22}$$

An ID distribution $\widetilde{\mu}$ belongs to $G(\mathbb{R}^d)$ iff there exists a type A distribution μ such that $\widetilde{\mu} = \Psi(\mu)$. That is

$$G(\mathbb{R}^d) = \Psi(A(\mathbb{R}^d)). \tag{23}$$

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- **Next problem: integral representation for type A distributions?**

III. Stochastic integral representations for some ID classes

- Jurek (85): $U(\mathbb{R}^d) = \mathcal{U}(ID(\mathbb{R}^d))$,

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- Jurek, Vervaat (83), Sato, Yamazato (83): $L(\mathbb{R}^d) = \Phi(ID_{\log}(\mathbb{R}^d))$

$$\Phi(\mu) = \mathcal{L}\left(\int_0^\infty \mathrm{e}^{-s}\mathrm{d}X_s^{(\mu)}\right),$$

$$ID_{\log}(\mathbb{R}^d) = \left\{\mu \in I(\mathbb{R}^d) : \int_{|x|>2} \log|x|\,\mu(\mathrm{d}x) < \infty\right\}.$$

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- Barndorff-Nielsen, Maejima, Sato (06): $B(\mathbb{R}^d) = \Upsilon(I(\mathbb{R}^d))$ and $T(\mathbb{R}^d) = \Upsilon(L(\mathbb{R}^d))$

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 \log\frac{1}{s}\mathrm{d}X_s^{(\mu)}\right).$$

Theorem (Maejima, PA, Sato (11))

Let $\Phi_{\cos} : ID(\mathbb{R}^d) \rightarrow ID(\mathbb{R}^d)$ *be the mapping*

$$\Phi_{\cos}(\mu) = \mathcal{L}\left(\int_0^1 \cos(\frac{\pi}{2}s)\mathrm{d}X_s^{(\mu)}\right), \quad \mu \in ID(\mathbb{R}^d). \tag{24}$$

Then

$$A(\mathbb{R}^d) = \Phi_{\cos}(ID(\mathbb{R}^d)). \tag{25}$$

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- **Upsilon transformations of Lévy measures:**

$$\Upsilon_\sigma(\rho)(B) = \int_0^\infty \rho(u^{-1}B)\sigma(\mathrm{d}u), \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{26}$$

[Barndorff-Nielsen, Rosinski, Thorbjørnsen (08)].

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- **Fractional transformations of Lévy measures:**

$$(\mathcal{A}_{q,p}^{\alpha,\beta}\nu)(C) = \frac{1}{\Gamma(p)}\int_0^\infty r^{-q-1}dr \int_{\mathbb{R}^d} 1_C(r\frac{x}{|x|})(|x|^\beta - r^\alpha)_+^{p-1}\nu(\mathrm{d}x),$$

$p, \alpha, \beta \in \mathbb{R}_+$, $q \in \mathbb{R}$ [Maejima, PA, Sato (in progress), Sato (10)].

- **Notation** μ probability measure on $\mathbb{R}$,

$$\mathbb{C}^{+}=\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}\,,\ \mathbb{C}^{-}=\{z\in\mathbb{C}:\mathrm{Im}(z)<0\}$$

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- **Cauchy (-Stieltjes) transform** $G_{\mu}(z) : \mathbb{C}^{+} \to \mathbb{C}^{-}$

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$$\mu(\mathrm{d}x) = -\frac{1}{\pi} \lim_{y \to 0^+} \mathrm{Im}\, G_\mu(x+iy)\mathrm{d}x$$

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- **Reciprocal Cauchy transform** $F_\mu(z) : \mathbb{C}^+ \to \mathbb{C}^+$,

$$F_\mu(z) = 1/G_\mu(z)$$

IV. Free analogous of classical cumulant transform

- Bercovici & Voiculescu (1993): There exists a domain $\Gamma = \cup_{\alpha>0}\Gamma_{\alpha,\beta_\alpha}$ where the right inverse F_μ^{-1} of F_μ exists $(F_\mu(F_\mu^{-1}(z)) = z)$

$$\Gamma_{\alpha,\beta} = \{z = x + iy : y > \beta,\ |x| < \alpha y\}, \alpha > 0, \beta > 0$$

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- **Voiculescu transform**

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- **Free cumulant transform**

$$C_\mu^{\boxplus}(z) = zF_\mu^{-1}(\frac{1}{z}) - 1$$

IV. Free convolution: Analytic approach

Bercovici & Voiculescu (1993)

- μ_1, μ_2 pm on $\mathbb{R}$: The **free additive convolution** $\mu_1 \boxplus \mu_2$ is the unique pm such that

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z)$$

or equivalently

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- **Classical cumulant transform**:

$$C^{*}_{\mu}(t) = \log \widehat{\mu}(t), \quad \forall t \in \mathbb{R}$$

$$\widehat{\mu}(t) = \int_{\mathbb{R}} \exp(itx)\mu_X(\mathrm{d}x), \quad \forall t \in \mathbb{R}.$$

IV. A difference with classical convolution

Example

Free convolution of atomic measures can be absolutely continuous

Symmetric Bernoulli measure

$$\mathrm{j}(\mathrm{d}x) = \frac{1}{2}\left(\delta_{\{-1\}}(\mathrm{d}x) + \delta_{\{1\}}(\mathrm{d}x)\right)$$

$\mathrm{a} = \mathrm{j} \boxplus \mathrm{j}$ is the **Arcsine measure on** $(-1, 1)$

$$\mathrm{a}(\mathrm{d}x) = \frac{1}{\pi\sqrt{1-x^2}} 1_{(-1,1)}(x)\mathrm{d}x$$

IV. Free infinite divisibility

Definition

A pm μ is **infinitely divisible** with respect to free convolution $\boxplus$ iff $\forall n \geq 1$, $\exists$ pm $\mu_{1/n}$ and

$$\mu = \mu_{1/n} \boxplus \mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}$$

$ID(\boxplus)$ is the class of all free infinitely divisible distributions.

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$ID(\boxplus)$ is the class of all free infinitely divisible distributions.

- If μ is $\boxplus$-infinitely divisible, μ has at most one atom.
- No nontrivial discrete distribution is $\boxplus$-infinitely divisible

IV. Free infinite divisibility

Theorem

Bercovici & Voiculescu (1993). The following are equivalent

a) μ *is free infinitely divisible*

b) ϕ_μ *has an analytic extension defined on* $\mathbb{C}^+$ *with values in* $\mathbb{C}^- \cup \mathbb{R}$

c) Barndorff-Nielsen & Thorjensen (2006): Lévy-Khintchine representation:

$$C_\mu^{\boxplus}(z) = \eta z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz 1_{[-1,1]}(x) \right) \rho(\mathrm{d}x), \; z \in \mathbb{C}^-$$

where (η, a, ρ) *is a Lévy triplet.*

IV. Relation between classical and free ID

- Classical Lévy-Khintchine representation $\mu \in ID(*)$

$$C_\mu^*(t) = \eta t - \frac{1}{2}at^2 + \int_{\mathbb{R}} \left(e^{itx} - 1 - tx1_{[-1,1]}(x) \right) \rho(\mathrm{d}x), \; t \in \mathbb{R}$$

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- **Bercovici-Pata bijection** (Ann. Math. 1999) $\Lambda : ID(*) \to ID(\boxplus)$

$$ID(*) \ \ni \mu \sim (\eta, a, \rho) \leftrightarrow \Lambda(\mu) \sim (\eta, a, \rho)$$

IV. Relation between classical and free ID

- Classical Lévy-Khintchine representation $\mu \in ID(*)$

$$C_\mu^*(t) = \eta t - \frac{1}{2}at^2 + \int_{\mathbb{R}} \left(e^{itx} - 1 - tx1_{[-1,1]}(x) \right) \rho(\mathrm{d}x), \;\; t \in \mathbb{R}$$

- Free Lévy-Khintchine representation $\nu \in ID(\boxplus)$

$$C_\nu^{\boxplus}(z) = \eta z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz1_{[-1,1]}(x) \right) \rho(\mathrm{d}x), \;\; z \in \mathbb{C}^-$$

- **Bercovici-Pata bijection** (Ann. Math. 1999) $\Lambda : ID(*) \to ID(\boxplus)$

$$ID(*) \;\ni \mu \sim (\eta, a, \rho) \leftrightarrow \Lambda(\mu) \sim (\eta, a, \rho)$$

- Λ preserves convolutions (and weak convergence)

$$\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$$

- For classical Gaussian measure $\gamma_{\eta,\sigma}$, $\mathrm{w}_{\eta,\sigma} = \Lambda(\gamma_{\eta,\sigma})$ is Wigner distribution on $(\eta - 2\sigma, \eta + 2\sigma)$ (**free Gaussian**) with free cumulant

$$C^{\boxplus}_{\mathrm{w}_{\eta,\sigma}}(z) = \eta z + \sigma z^2$$

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- Free stable distributions $S^{\boxplus} = \Lambda(S^*)$, S^* classical stable distributions.

IV. Examples of free infinitely divisible distributions

Images of classical ID distributions under Bercovici-Pata bijection

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- Free class A distributions $= \Lambda(A(\mathbb{R}))$

IV. New example of free ID distribution

Arizmendi, Barndorff-Nielsen and PA (2010)

- **Special symmetric Beta distribution**

$$\mathrm{b}(\mathrm{d}x) = \frac{1}{2\pi} |x|^{-1/2} (2 - |x|)^{1/2} \mathrm{d}x, \quad |x| < 2$$

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- **This was the motivation to study class A distributions**

μ_1, μ_2 probability measures on $\mathbb{R}$

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- **Boolean convolution:** $\mu_1 \uplus \mu_2$

$$K_{\mu_1 \uplus \mu_2}(z) = K_{\mu_1}(z) + K_{\mu_2}(z), \quad z \in \mathbb{C}^+,$$

$K_\mu(z) = z - F_\mu(z).$

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