Free Convolutions in Free Probability

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Plan of the Seminar

1. Motivation: Asymptotic spectrum of random matrices.
2. Non-commutative probability spaces and free independence.
3. Free additive convolution: analytic approach.
   - 3.1 The Cauchy transform and its reciprocal.
   - 3.2 Voiculescu & free cumulants transforms.
   - 3.3 Free additive convolution of measures.
   - 3.4 Examples.
4. Free multiplicative convolution: analytic approach.
   - 4.1 S-transforms.
   - 4.2 Free multiplicative convolution of measures.
   - 4.3 Examples.
5. Overview Talks 2 and 3: Random Matrices and Infinite Divisibility (classical and free)

I. Pioneering work on Random Matrices by Eugene Wigner

Ann Math. 1955, 1957, 1958

- Ensemble of random matrices: Sequence $X = (X_n)_{n\geq 1}$, where X_n is $n \times n$ matrix with random entries.

- **Wigner random matrices**:

$$X_n(k,j) = X_n(j,k) = \frac{1}{\sqrt{n}} \begin{cases} Z_{j,k}, & \text{if } j < k \\ Y_j, & \text{if } j = k \end{cases}$$

  $\{Z_{j,k}\}_{j\leq k}, \{Y_j\}_{j\geq 1}$ independent sequences of i.i.d. random variables with assumptions on the first two moments: $\mathbb{E}Z_{1,2} = \mathbb{E}Y_1 = 0, \quad \mathbb{E}Z_{1,2}^2 = 1.$

- $\lambda_{n,1} \leq \ldots \leq \lambda_{n,n}$ eigenvalues of X_n, $n \geq 1$.

- **Empirical spectral distribution (ESD)** of X_n**:**

$$\widehat{F}_n(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{\lambda_{n,j}\leq x\}}.$$

- What is the limit of $\widehat{F}_n$ **(ASD)** when $n \to \infty$?

I. Pioneering work on RMT by E. Wigner

Ann Math. 1955, 1957, 1958

Asymptotic spectral distribution (ASD): $\widehat{F}_n$ converges, as $n \to \infty$, to **semicircle distribution on** $(-2, 2)$.

Theorem (Wigner)

$\forall f \in C_b(\mathbb{R})$ and $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left| \int f(x) \mathrm{d}\widehat{F}_n(x) - \int f(x) \mathrm{w}(x) \mathrm{d}x \right| > \varepsilon \right) = 0.$$

where $\mathrm{w}(x)\mathrm{d}x$ is the semicircle distribution on $(-2, 2)$

$$\mathrm{w}(x) = \frac{1}{2\pi} \sqrt{4 - x^2}, \quad |x| \le 2.$$

I. Marchenko-Pastur (1967): Wishart type RM

- $X_n = X_{p\times n}$ with i.i.d. entries under moment assumptions.
- Covariance matrix $(p \times p)$ $S_n = \frac{1}{n} X_n X_n^*$, with eigenvalues $0 \le \lambda_{p,1} \le ... \le \lambda_{p,p}$ and ESD $\widehat{F}_p(\lambda)$.
- If $n/p \to c > 0$, $\widehat{F}_p$ converges to Marchenko-Pastur (MP)

$$\mathrm{m}_c(\mathrm{d}x) = \begin{cases} f_c(x)\mathrm{d}x, & \textit{if} \quad c \ge 1 \\ (1-c)\delta_0(\mathrm{d}x) + f_c(x)\mathrm{d}x, & \textit{if} \quad 0 < c < 1, \end{cases}$$

$$f_c(x) = \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\mathbf{1}_{[a,b]}(x),$$

$$a = (1-\sqrt{c})^2, \;\; b = (1+\sqrt{c})^2.$$

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- *Wireless communication* (Talatar, 99): if $C(p,n)$ is capacity of MIMO system of n receiver & p transmitter antennas,

$$\frac{C(p,n)}{p} \to \int_a^b \log_2(1+cRx)\,\mathrm{m}_c(\mathrm{d}x) = K(c,R)$$
$$C(p,n) \sim pK(c,R).$$

I. Motivation to study RMT and Free Probability

From the Blog of Terence Tao (Free Probability, 2010):

1. The significance of free probability to random matrix theory lies in the fundamental observation that *random matrices which are independent in the classical sense, also tend to be independent in the free probability sense*, in the large limit.

2. This is only possible because of the highly *non-commutative nature* of these matrices; it is not possible for non-trivial commuting independent random variables to be *freely independent.*

3. Because of this, many tedious computations in random matrix theory, particularly those of an algebraic or enumerative combinatorial nature, can be done more quickly and systematically by using the *framework of free probability.*

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2. **Classical analogous:**
   2.1 If R_1 & R_2 are *real independent* r.v. with distributions μ_1 & μ_2, the distribution of $R_1 + R_2$ is the *classical convolution*

$$\mu_1 * \mu_2(E) = \int_{\mathbb{R}} \mu_1(E - x)\mu_2(\mathrm{d}x) = \int_{\mathbb{R}} \mu_2(E - x)\mu_1(\mathrm{d}x).$$

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   3.2 Free convolution: **analytic** tools similar to classical case.

I. Asymptotically free random matrices

Some facts about classical independence

- Two real random variables X_1 & X_2 are **independent** iff $\forall$ bounded Borel functions $(B_b(\mathbb{R}))$ f, g

$$\mathbb{E}(f(X_1)g(X_2)) = \mathbb{E}(f(X_1))\mathbb{E}(g(X_2))$$

$$\mathbb{E}\left([f(X_1) - \mathbb{E}(f(X_1)]\,[g(X_2) - \mathbb{E}(g(X_2)]\right) = 0.$$

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- iff (X_1 & X_2 are bounded) $\forall$ $n, m \geq 1$

$$\mathbb{E}\left[(X_1^n - \mathbb{E}X_1^n)(X_2^m - \mathbb{E}X_2^m)\right] = 0$$

$$\mathbb{E}X_1^n X_2^m = \mathbb{E}X_1^n \mathbb{E}X_2^m.$$

  Then independence allows to compute all joint moments, so the moments of $X_1 + X_2$.

I. Asymptotically free random matrices

Voiculescu (1991)

- For an ensemble of Hermitian random matrices $\mathbf{X} = (X_n)_{n\geq 1}$ define "expectation" τ as the linear functional τ, $(\tau(\mathbf{I}) = 1)$

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- Two Hermitian ensembles $\mathbf{X}_1$ & $\mathbf{X}_2$ are *asymptotically free* (AF) if $\forall\ r \in Z_+$ & polynomials $p_i(\cdot)$, $q_i(\cdot)$, $1 \leq i \leq r$ with

$$\tau(p_i(\mathbf{X}_1)) = \tau(q_i(\mathbf{X}_2)) = 0,$$

we have

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- If $\mathbf{X}_1$ & $\mathbf{X}_2$ are AF and *commute*, *one has to be deterministic.*
- *Non-commutative word!*

I. Asymptotically free RM: Examples

1. **X** and $\mathrm{I} = (\mathrm{I}_n)$ are AF.
2. If **X** and **Y** are *independent Wigner ensembles*, they are AF.
3. If **X** and **Y** are *independent standard Gaussian ensembles*, then $\mathbf{XX}^*$ and $\mathbf{YY}^*$ are AF.
4. If **X** and **Y** *independent Wishart ensembles*, they are AF.
5. If **U** and **V** are independent unitarily ensembles, they are AF.
6. If A, B are *deterministic* ensembles whose ASD have compact support & U is an unitary ensemble, then UAU^* & B are AF.

II. Free probability: Non-commutative probability spaces

Definitions

(i) A *non-commutative probability space* $(\mathcal{A}, \tau)$ is a unital algebra $\mathcal{A}$ over $\mathbb{C}$ with a linear functional $\tau : \mathcal{A} \to \mathbb{C}$ with $\tau(\mathbf{1}) = 1$.

Elements of $\mathcal{A}$ are called *non-commutative random variables*.

(ii) $(\mathcal{A}, \tau)$ is C^**-probability space* if $\mathcal{A}$ is a C^*-algebra and τ is a positive linear functional.

(iii) $(\mathcal{A}, \tau)$ is W^**-probability space* if $\mathcal{A}$ is a W^*-algebra and τ is a normal faithful trace.

II. Non-commutative r.v. with a given distribution

Fact

(i) **Given a p.m.** μ *on* $\mathbb{R}$ *with* **bounded support**, *there exist a* C^**-probability space* $(\mathcal{A}, \tau)$ *and a self-adjoint* $\mathbf{a} \in \mathcal{A}$ *with*

$$\tau(f(\mathbf{a})) = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x), \quad \forall f \in C_b(\mathbb{R}).$$

Fact

(ii) **Given a p.m.** μ *on* $\mathbb{R}$, *there exists a* W^**-probability space* $(\mathcal{A}, \tau)$ *and self-adjoint operator* $\mathbf{a}$ *on a Hilbert space* H *such that*

$$f(\mathbf{a}) \in \mathcal{A} \quad \forall f \in B_b(\mathbb{R}), \tag{1}$$

$$\tau(f(\mathbf{a})) = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x), \quad \forall f \in B_b(\mathbb{R}).$$

If (1) holds, it is said that $\mathbf{a}$ *is affiliated with* $\mathcal{A}$.

II. Free Random Variables

Definitions

(i) A family of W^*-subalgebras $\{\mathcal{A}_i\}_{i \in I} \subset \mathcal{A}$ in a W^*-probability space is *free* if

$$\tau(\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n) = 0$$

whenever $\tau(\mathbf{a}_j) = 0$, $\mathbf{a}_j \in \mathcal{A}_{i_j}$, and $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$.

(iii) If $\{\mathcal{A}_i\}_{i \in I}$ is a family of free W^*-subalgebras & $\mathbf{a}_i$ is affiliated with $\mathcal{A}_i$, $i \in I$, the r.v. $\{\mathbf{a}_i\}_{i \in I}$ are called *freely independent.*

Fact

Given *μ_1 & μ_2 p.m. on $\mathbb{R}$, there exist a W^*-probability space, W^*-subalgebras $\mathcal{A}_1, \mathcal{A}_2$ and self-adjoint operators* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *on a Hilbert space H affiliated with $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, such that*

(i) $\mathbf{a}_i$ *has distribution μ_i*

(i) $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *are freely independent.*

II. Free independence allows to compute joint moments

Example

Computation of $\tau(\mathbf{abab})$ when $\mathbf{a}$ & $\mathbf{b}$ are freely independent:
Suppose $\{\mathbf{a}_1, \mathbf{a}_3\}$ and $\{\mathbf{a}_2, \mathbf{a}_4\}$ are freely independent. Since

$$\tau(\mathbf{a}_i - \tau(\mathbf{a}_i)1_{\mathcal{A}}) = 0,$$

$$\tau(\mathbf{a}_1 - \tau(\mathbf{a}_1)1_{\mathcal{A}})\tau(\mathbf{a}_2 - \tau(\mathbf{a}_2)1_{\mathcal{A}})\tau(\mathbf{a}_3 - \tau(\mathbf{a}_3)1_{\mathcal{A}})\tau(\mathbf{a}_4 - \tau(\mathbf{a}_4)1_{\mathcal{A}}) = 0.$$

Computations yield

$$\begin{aligned}\tau(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4) &= \tau(\mathbf{a}_1\mathbf{a}_3)\tau(\mathbf{a}_2)\tau(\mathbf{a}_4) + \tau(\mathbf{a}_1)\tau(\mathbf{a}_3)\tau(\mathbf{a}_2\mathbf{a}_4) \\ &\quad - \tau(\mathbf{a}_1)\tau(\mathbf{a}_2)\tau(\mathbf{a}_3)\tau(\mathbf{a}_4).\end{aligned}$$

In particular if $\mathbf{a}_1 = \mathbf{a}_3 = \mathbf{a}$ and $\mathbf{a}_2 = \mathbf{a}_4 = \mathbf{b}$

$$\tau(\mathbf{abab}) = \tau(\mathbf{a})^2\tau(\mathbf{b}^2) + \tau(\mathbf{a}^2)\tau(\mathbf{b})^2 - \tau(\mathbf{a})^2\tau(\mathbf{b})^2 \neq \tau(\mathbf{a})^2\tau(\mathbf{b})^2.$$

II. Application: Free Central Limit Theorem

Theorem

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots$ *be a sequence of independent free random variables with the same distribution with all moments. Assume that* $\tau(\mathbf{a}_1) = 0$ *and* $\tau(\mathbf{a}_1^2) = 1$. *Then the distribution of*

$$\mathbf{Z}_m = \frac{1}{\sqrt{m}}(\mathbf{a}_1 + \ldots + \mathbf{a}_m)$$

converges to the semicircle distribution as $m \to \infty$.

- *Idea of proof:* Show that the moments $\tau(\mathbf{Z}_m^k)$, $k \geq 1$, converge to the moments of the semicircle distribution $m_{2k+1} = 0$ and
$$m_{2k} = \frac{1}{k+1}\binom{2k}{k}$$
using *combinatorics of noncrossing partitions*.

II. Free Additive and Multiplicative Convolution

Definition

Let $\mathbf{a}_1, \mathbf{a}_2$ be free random variables with distributions μ_1 & μ_2. The distribution of $\mathbf{a}_1 + \mathbf{a}_2$ is the *free additive convolution* of μ_1 and μ_2 and it is denoted by

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Definition

Let μ_1 have positive support. Then $\mathbf{a}_1$ is a positive self-adjoint operator and the distribution of $\mathbf{a}_1^{1/2}$ is uniquely determined by μ_1. The distribution of the self-adjoint operator $\mathbf{a}_1^{1/2}\mathbf{a}_2\mathbf{a}_1^{1/2}$ is determined by μ_1 and μ_2. This measure is the *free multiplicative convolution* of μ_1 and μ_2 and it is denoted by

$$\mu_1 \boxtimes \mu_2.$$

II. Free Additive and Multiplicative Convolutions

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Questions and purpose of the talk:

1) Can $\mu_1 \boxtimes \mu_2$ & $\mu_1 \boxtimes \mu_2$ be considered merely as two new types of "convolutions" in the set of probability measures on $\mathbb{R}$?

2) What are the analytic tools to study them?

III. Free additive convolution: Analytic approach

Recall the classical convolution case

- Fourier transform of probability measure μ on $\mathbb{R}$

$$\widehat{\mu}(s) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sx} \mu(\mathrm{d}x), \quad s \in \mathbb{R}.$$

III. Free additive convolution: Analytic approach

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- $(\mu_n)_{n\geq 1}$ converges weakly to μ if and only if $\exists\ \alpha, \beta$ such that $\phi_{\mu_n}(z) \to \phi_\mu(z)$ in compact sets of $\Gamma_{\alpha,\beta}$.

III. Useful transformations of the Cauchy transform

Free cumulant transforms

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- ϕ_μ, C_μ and R_μ linearize free additive convolution.

III. Free additive convolution

Analytic definition

- For μ_1 & μ_2 p.m. on $\mathbb{R}$, $\mu_1 \boxplus \mu_2$ is the unique p.m. on $\mathbb{R}$ such that

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z), \quad z \in \Gamma^{\mu_1}_{\alpha_1,\beta_1} \cap \Gamma^{\mu_2}_{\alpha_2,\beta_2}$$

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- If $(X_n^1)_{n\geq 1}, (X_n^2)_{n\geq 1}$ are asymptotically free random matrices with ASD μ_1 & μ_2, then $(X_n^1 + X_n^2)_{n\geq 1}$ has ASD $\mu_1 \boxplus \mu_2$.

IV. Example: free convolution of Wigners

Semicircle distribution w_{m,σ^2} *on* $(m-2\sigma, m+2\sigma)$ *centered at* m

$$w_{m,\sigma^2}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-(x-m)^2}1_{[m-2\sigma,m+2\sigma]}(x).$$

Cauchy transform:

$$G_{\mathrm{w}_{m,\sigma^2}}(z) = \frac{1}{2\sigma^2}\left(z-\sqrt{(z-m)^2-4\sigma^2}\right),$$

Free cumulant transform:

$$C_{\mathrm{w}_{m,\sigma^2}}(z) = mz + \sigma^2 z.$$

$\boxplus$-*convolution of Wigner distributions is a Wigner distribution*:

$$\mathrm{w}_{m_1,\sigma_1^2} \boxplus \mathrm{w}_{m_2,\sigma_2^2} = \mathrm{w}_{m_1+m_2,\sigma_1^2+\sigma_2^2}.$$

III. Free additive convolutions: Examples

Marchenko-Pastur distribution

$c > 0$

$$\mathrm{m}_c(\mathrm{d}x) = (1-c)_+\delta_0 + \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\ 1_{[a,b]}(x)\mathrm{d}x.$$

Cauchy transform

$$G_{\mathrm{m}_c} = \frac{1}{2} - \frac{\sqrt{(z-a)(z-b)}}{2z} + \frac{1-c}{2z}$$

Free cumulant transform

$$C_{\mathrm{m}_c}(z) = \frac{cz}{1-z}.$$

$\boxplus$-*convolution of MP distributions is a MP distribution*:

$$\mathrm{m}_{c_1} \boxplus \mathrm{m}_{c_2} = \mathrm{m}_{c_1+c_2}$$

III. Free additive convolutions: Examples

Cauchy distribution

$\sigma > 0$, Cauchy distribution

$$\mathrm{c}_\sigma(\mathrm{d}x) = \frac{1}{\pi}\frac{\sigma}{\sigma^2 + x^2}\mathrm{d}x$$

Cauchy transform

$$G_{\mathrm{c}_\sigma}(z) = \frac{1}{z + \sigma i}$$

Free cumulant transform

$$C_{\mathrm{c}_\sigma}(z) = -i\sigma z$$

$\boxplus$-*convolution of Cauchy distributions is a Cauchy distribution*

$$\mathrm{c}_{\sigma_1} \boxplus \mathrm{c}_{\sigma_2} = \mathrm{c}_{\sigma_1+\sigma_2}.$$

III. Free additive convolutions: Examples

Pathological example

What is $\mathrm{b} \boxplus \mathrm{b}$ when b is symmetric Bernoulli distribution

$$\mathrm{b}(\mathrm{d}x) = \frac{1}{2}\left(\delta_{\{-1\}}(\mathrm{d}x) + \delta_{\{1\}}(\mathrm{d}x)\right)?.$$

Cauchy transform:

$$G_{\mathrm{b}}(z) = \frac{z}{z^2 - 1}.$$

Free cumulant transform:

$$C_{\mathrm{b}}(z) = \frac{1}{2}(\sqrt{1+4z^2} - 1).$$

Then

$$C_{\mathrm{b}\boxplus\mathrm{b}}(z) = \sqrt{1+4z^2} - 1.$$

Solving for $\mu = \mathrm{b} \boxplus \mathrm{b}$

$$G_\mu\left(\frac{1}{z}(C_\mu(z) + 1)\right) = z.$$

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Pathological example

Solving for $\mathrm{b} \boxplus \mathrm{b}$

$$G_{\mathrm{b}\boxplus\mathrm{b}}(\frac{1}{z}(\sqrt{1+4z^2}) = z$$

$$G_{\mathrm{b}\boxplus\mathrm{b}}(z) = \frac{1}{\sqrt{z^2-4}},$$

which is the Cauchy transform of the **arcsine distribution**

$$\mathrm{a}(\mathrm{d}x) = \frac{1}{\pi\sqrt{1-x^2}} 1_{(-1,1)}(x)\mathrm{d}x.$$

Then

$$\mathrm{b} \boxplus \mathrm{b} = \mathrm{a}.$$

Free additive convolution of atomic distributions may be absolutely continuous!

IV. Free multiplicative convolution

Classical multiplicative convolution of random variables

:

- Given independent classical r.v. $X > 0$, $Y > 0$, with distribution μ_X, μ_Y, what is the distribution μ_{XY} of XY?

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- Analogous in free probability?

IV. Free multiplicative convolution: The S-transform

For distributions with nonnegative support: Bercovici & Voiculescu (93)

- Ψ_μ-transform of a general probability distribution μ on $\mathbb{R}$

$$\Psi_\mu(z) = \frac{1}{z} G_\mu(\frac{1}{z}) - 1.$$

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- For μ_1, μ_2 in $\mathcal{P}^+(\neq \delta_0)$, $\mu_1 \boxtimes \mu_2$ is unique p.m. in $\mathcal{P}^+$

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- If $(X_n)_{n\geq 1}, (Y_n)_{n\geq 1}$ are asymptotically free nonnegative definite random matrices with ASD μ_1 and μ_2, then the product $(X_n^{1/2} Y_n X_n^{1/2})_{n\geq 1}$ has ASD $\mu_1 \boxtimes \mu_2$.

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- Arizmendi and PA (2008): *Analytic approach*, μ_1, μ_2 with *unbounded support*, $\mu_1 \in \mathcal{P}^+$, μ_2 *symmetric.*

IV. Free multiplicative convolution: The S-transform

For symmetric distributions: Arizmendi-PA (2009).

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- If $\mu \neq \delta_0$, Ψ_μ, there are disjoint sets H, $\widetilde{H}$ in $\mathbb{C}$, Ψ_μ has unique inverses $\chi_\mu : \Psi_\mu(H) \to H$ and $\widetilde{\chi}_\mu : \Psi_\mu(\widetilde{H}) \to \widetilde{H}$.

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- If μ_1 in $\mathcal{P}^+$ and μ_2 in $\mathcal{P}_s$

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z)S_{\mu_2}(z) = S_{\mu_1}(z)\widetilde{S}_{\mu_2}(z).$$

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- There is no $\lambda \in \mathcal{P}^+$ such that $\mathrm{a} = \lambda \boxtimes \mathrm{w}$.

Overview Talks 2 and 3

- Talk 2: Random matrices.
  - More on asymptotic spectrum of random matrices.
  - Some applications.
  - Dyson Brownian motion and other eigenvalues processes.
- Talk 3: Infinite divisibility (ID).
  - Infinitely divisible random matrices.
  - Free ID.
  - A bijection between free and classical ID.
  - Random matrices: bridge between classical & free ID.

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