Random Matrices and Free Probability

Short Course IAS-TUM
Victor Pérez-Abreu
CIMAT, Guanajuato, Mexico

pabreu@cimat.mx, www.cimat.mx/~pabreu

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Goals

1. Introduction to Random Matrix Theory (RMT)
2. Emphasis on Asymptotic Spectrum of Random Matrices
3. Role of Free Probability in RMT
4. Examples: Information Theory and Statistics
5. Idea of mathematics used to study RMT
6. Open problems

General Outline

Talk 1: Introduction to Random Matrices

1. References and motivation to study RMT
2. Classic results on the asymptotic spectrum of random matrices

Talk 2: Examples and the Cauchy transform

1. Examples: Wireless communications & high-dimensional data
2. Role of Stieltjes transform in RMT and free probability.

Talk 3: Introduction to Free Probability

1. Asymptotically free random matrices.
2. Free convolution of measures.
3. Examples.

Part 1 *Motivation to study Random Matrix Theory (RMT)*

1. Why to study RMT.
2. "From the Introduction of some recommended books and blogs"
3. Role of free probability in RMT

Part 2 *Classic results on the Asymptotic Spectrum of Random Matrices*

1. Classical Gaussian Ensembles GOE, GUE and Semicircle Law
2. Wishart Ensemble and deformed and Quarter Semicircle Laws
3. Other Ensembles: Circular Law
4. Asymptotics for maximum eigenvalue: Tracy Widom distribution

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4. *Depth* comes via the linking together of multiple ideas and topics, often seemingly removed from the original context.
5. And *fertility* means that with a reasonable effort there are new results, some useful, some with beauty, and a few maybe with depth, still waiting to be found.

Motivation to study RMT

An Introduction to Random Matrices, G. W. Anderson, A. Guionnet & O. Zeitouni (2010):

1. The study of random matrices, and in particular the properties of their eigenvalues, has emerged from the applications, first in data analysis (*Wishart, 1928*) and later on as statistical models for heavy-nuclei atoms (*Wigner, 1955*).
2. Thus, the field of random matrices owes its existence to *applications*.
3. Over the years, however, it became clear that models related to random matrices play an important role in areas of *pure mathematics.*
4. Moreover, the tools used in the study of random matrices came themselves from different and *seemingly unrelated branches of mathematics (combinatorics, graphs, functional analysis, orthogonal polynomials, probability, operator algebras, free probability, number theory, complex analysis, compact groups).*

Motivation to study RMT

Random Matrices, 3rd ed, M. L. Mehta (2004):

1. In the last decade following the publication of the second edition of this book (1967, 1991) the subject of random matrices found *applications in many new fields of knowledge:*
2. *Physics*: In heterogeneous conductors (mesoscopy systems) where the passage of electric current may be studied by transfer matrices, quantum chromo dynamics characterized by some Dirac operator, quantum gravity modeled by some random triangulation of surfaces.
3. *Traffic and communication networks.*
4. Zeta function and L-series in *number theory*,
5. Even stock movements in *financial markets*,
6. **Wherever imprecise matrices occurred, people dreamed of random matrices.**

Motivation to study RMT

Random Matrix Theory: Invariant Ensembles and Universality, P. Deift & D. Gioev (2009):

- There has been a great upsurge of interest in RMT in recent years.
- This upsurge has been fueled primary by the fact that an extraordinary variety of problems in *physics, pure mathematics*, and *applied mathematics* are now known to be modeled by RMT. *By this we mean the following:*
- Suppose we are investigating some *statistical quantities* $\{a_k\}$ in a neighborhood of some point A, say.
- The a_ḱs are to be compared with the *eigenvalues* $\{\lambda_k\}$, in a neighborhood of some point L, of a matrix taken from some *random matrix ensemble.*
- If the statistics of the $\{a_k\}$, *appropriately scaled*, are described by the statistics of the $\{\lambda_k\}$, appropriately scaled, then we say that the $\{a_k\}$ are *modeled by random matrix theory*.

From the Blog of Terence Tao (Free Probability, 2010):

1. The significance of free probability to random matrix theory lies in the fundamental observation that *random matrices which are independent in the classical sense, also tend to be independent in the free probability sense*, in the large limit.
2. This is only possible because of the highly *non-commutative nature* of these matrices; it is not possible for non-trivial commuting independent random variables to be *freely independent.*
3. Because of this, many tedious computations in random matrix theory, particularly those of an algebraic or enumerative combinatorial nature, can be done more quickly and systematically by using the *framework of free probability*, which by design is optimized for algebraic tasks rather than analytical ones.

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- **RMT is a link between classical probability and free probability**

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- *Recent book*: Bai & Silverstein (2010), "Spectral analysis of large dimensional random matrices".

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$$Z_n = \begin{bmatrix} Z_n(1,1) & \cdots & Z_n(1,n) \\ \vdots & & \vdots \\ Z_n(n,1) & \cdots & Z_n(n,n) \end{bmatrix}$$

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- $Z_n \in GOE(n)$

Characterization of GOE

Theorem

An $n \times n$ symmetric matrix Z_n belongs to $GOE(n)$ if and only if the following two conditions hold
a) Z_n is invariant under orthogonal conjugations (symmetry)

$$OZ_nO^T \sim Z_n \quad \forall O \in \mathbb{O}(n)$$

b) Z^n has independent entries in diagonal and upper diagonal (Wigner matrix)

Proof: Density of Z_n with respect to Lebesgue measure on $\mathbb{S}_n(\mathbb{R})$

$$f_{Z_n}(A) = c_n \exp(-\frac{1}{4}\mathrm{tr}(A^2)), \quad A \in \mathbb{S}_n(\mathbb{R}).$$

$\mathbb{S}_n(\mathbb{R}) \approx \mathbb{R}^{n(n+1)/2}$, $n \times n$ symmetric matrices with real entries

Joint eigenvalues distribution

Lemma

Joint density of eigenvalues $\lambda_{n,1} \leq \cdots \leq \lambda_{n,n}$ *of* $Z_n \in GOE(n)$

$$f_{\lambda_{n,1},\ldots,\lambda_{n,n}}(x_1, \ldots, x_n) = k_n \prod_{j<k} |x_j - x_k| \exp(-\frac{1}{4}\sum_{j=1}^{n} x_j^2).$$

- $\lambda_{n,1}, \ldots, \lambda_{n,n}$ are highly dependent.
- This is a general phenomena for several random matrices.
- Vandermont determinant: $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$

$$\Delta(x) = \det\left(\left\{x_j^{k-1}\right\}_{j,k=1}^{n}\right) = \prod_{j<k}(x_j - x_k)$$

Empirical distributions of scaled eigenvalues

- $\lambda_{n,1} \leq \dots \leq \lambda_{n,n}$ eigenvalues of $X_n = \frac{1}{\sqrt{n}} Z_n$

$$X_n(j,j) \sim N(0, \frac{2}{n}), \quad X_n(j,k) \sim N(0, \frac{1}{n}), \quad j \neq k.$$

- *Empirical spectral distribution*

$$\widehat{F}_n(x) = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_{n,j}}(x).$$

- *Mean empirical spectral distribution*

$$\overline{F}_n(x) = \mathbb{E}\left(\widehat{F}_n(x)\right).$$

- **Study of asymptotics of** $\widehat{F}_n$ and $\overline{F}_n$ **is not easy due to highly dependent eigenvalues**

Wigner Semicircle Law

Theorem (Wigner, 1958)

For any $f \in C_b(\mathbb{R})$ *and any* $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\int f(x)\mathrm{d}\widehat{F}_n(x) - \int f(x)\mathrm{w}(x)\mathrm{d}x\right| > \varepsilon\right) = 0.$$

$\mathrm{w}(x)\mathrm{d}x$ *is the semicircle distribution on* $(-2, 2)$ *with density*

$$\mathrm{w}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}, \quad |x| \le 2.$$

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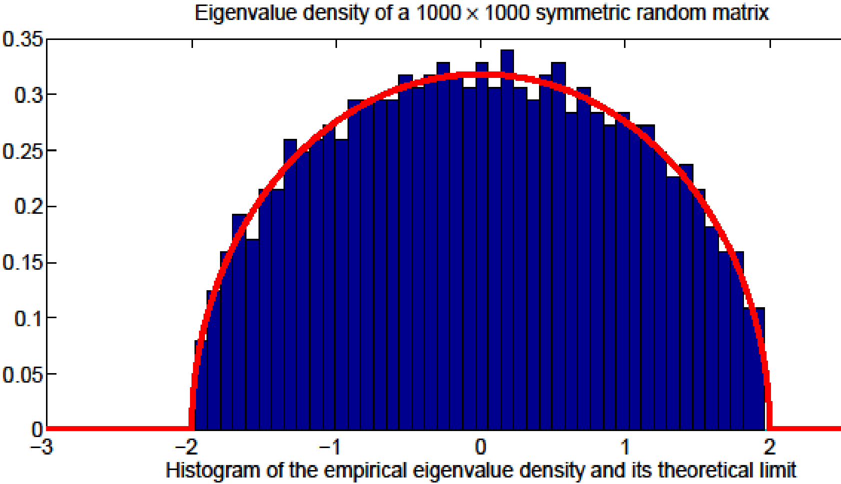
- $Y \sim \mathrm{w}$, $\mathbb{E}(Y) = 0$ and $\mathbb{E}Y^2 = 1$.

Simulation Wigner theorem

$$w(x) = \frac{1}{2\pi}\sqrt{4 - x^2},\ x \in (-2, 2)$$



Very general idea of the proof

- Basic observation: $\lambda_{n,1} < \ldots < \lambda_{n,n}$ eigenvalues of X_n

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- Catalan numbers appear in combinatorics.

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- Wigner (1967): Random Matrices in Physics, SIAM Review, vol. 6, No.1, 1-23 (7th Von Neumann Lecture).

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- $r_2 < \infty$ is only needed.

Universality of Wigner Semicircle Law

- Wigner (1958): Gaussian Unitary Ensembles (GUE) of Hermitian RMs whose upper-triangle entries are i.i.d. complex Gaussian.
- Wigner (1955): X_n is $n \times n$ real symmetric matrix, entries are zero in diagonal and in upper-triangle are i.i.d. Bernoulli$(1/2)$.
- Wigner (1959): *Wigner random matrices*

$$X_n(k,j) = X_n(j,k) = \frac{1}{\sqrt{n}} \begin{cases} Z_{j,k}, & \text{if } j < k \\ Y_j, & \text{if } j = k \end{cases}$$

$\{Z_{j,k}\}_{j \le k}, \{Y_j\}_{j \ge 1}$ i.i.d. $\mathbb{E} Z_{1,2} = \mathbb{E} Y_1 = 0$, $\mathbb{E} Z_{1,2}^2 = 1$ and $\forall \ q \ge 1$

$$r_q := \max(\mathbb{E} Z_{1,2}^q, \mathbb{E} Y_1^q) < \infty.$$

- $r_2 < \infty$ is only needed.
- a.s. versions (see book by Bai and Silverstein (2010)).

Wishart Ensemble

- $Z_n = Z_{p\times n} = (Z_n(j,k))$ is $p \times n$ Gaussian *rectangular* RM.
- $\{Z_n(j,k); j = 1, ..., p, k = 1, ..., n\}$ i.i.d. standard Gaussian.
- **Wishart random matrix:** $W_p = Z_{p\times n} Z_{p\times n}^\top$.
- W_p is $p \times p$ symmetric nonnegative definite random matrix

$$W_p \sim \mathrm{W}_p(n, \mathrm{I}_p).$$

- Entries of W_p are dependent.
- Distribution of W_p is invariant under orthogonal conjugations:

$$O W_p O^\top \sim W_p, \forall O \in \mathbb{O}(p)$$

- **Wishart ensemble:** $W = (W_p; p \geq 1)$.

- Density of W_p when $n \geq p$

$$f_{W_p}(A) = c_{n,p} \det(A)^{\frac{1}{2}(n-p-1)} \exp(-\mathrm{tr}(A)), \quad A > 0.$$

- Characteristic function

$$\mathbb{E}(\exp(i\mathrm{tr}(W_p\Theta)) = \det(\mathrm{I}_p - 2i\Theta)^{-p/2}, \quad \Theta \geq 0.$$

- Joint density of eigenvalues $\lambda_{p,1} < ... < \lambda_{p,p}$ of W_p when $n \geq p$

$$f_\lambda(x) = k_{n,p} \, |\Delta(x)| \prod_{j=1}^{p} x_j^{(n-p-1)/2} \exp(-\frac{1}{2}\sum_{j=1}^{p} x_j)$$

$x_1 < \cdots < x_p$, $x = (x_1, ..., x_p)$, $\lambda = (\lambda_{p,1}, ..., \lambda_{p,p})$.

Theorem (Marchenko-Pastur, 1967)

$X_{p\times n} = (Z_{j,k} : j = 1, .., p, k = 1, ..., n)$ *i.i.d.* $\mathbb{E}(Z_{1,1}) = 0$, $\mathbb{E}(Z_{1,1}^2) = 1$

$S_n = \frac{1}{n} X_{p\times n} X_{p\times n}^\top$ *with eigenvalues* $0 \leq \lambda_{p,1} \leq ... \leq \lambda_{p,p}$ *and*

$$\widehat{F}_n(\lambda) = \frac{1}{p} \# \{ j = 1, ..., p; \quad \lambda_{p,j} \leq \lambda \} .$$

If $n/p \to c > 0$, $\widehat{F}_n$ *converges weakly in probability to M-P*

$$\mu_c(\mathrm{d}x) = \begin{cases} f_c(x)\mathrm{d}x, & \text{if } \; c \geq 1 \\ (1-c)\delta_0(\mathrm{d}x) + f_c(x)\mathrm{d}x, & \text{if } \; 0 < c < 1, \end{cases}$$

where

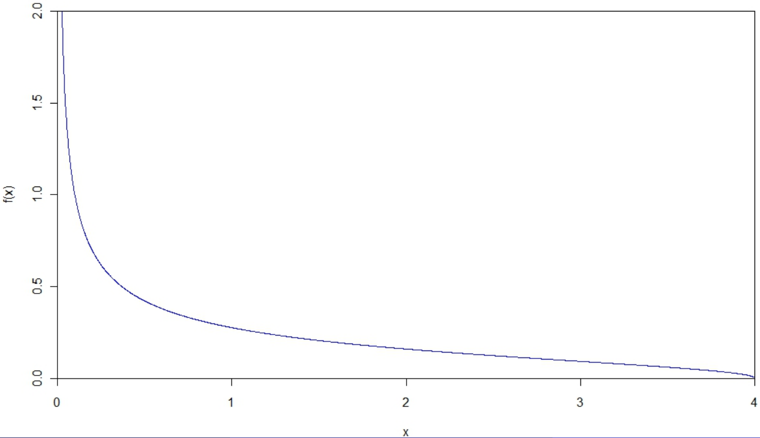
$$f_c(x) = \frac{c}{2\pi x} \sqrt{(x-a)(b-x)} \mathbf{1}_{[a,b]}(x)$$

$$a = (1 - \sqrt{c})^2, \;\; b = (1 + \sqrt{c})^2$$

M-P distribution c=1

Deformed semicircle distribution

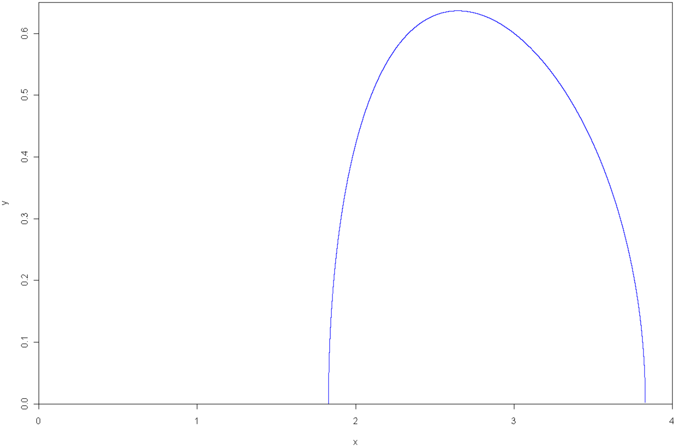
No atom at 0, zero mean and standard deviation 2,
density $f_1(x) = \frac{1}{2\pi x}\sqrt{x(4-x)}\mathbf{1}_{[0,4]}(x)$



M-P distribution c<1

Deformed semicircle distribution

Dirac mass (1-c) at 0



M-P distribution c>1

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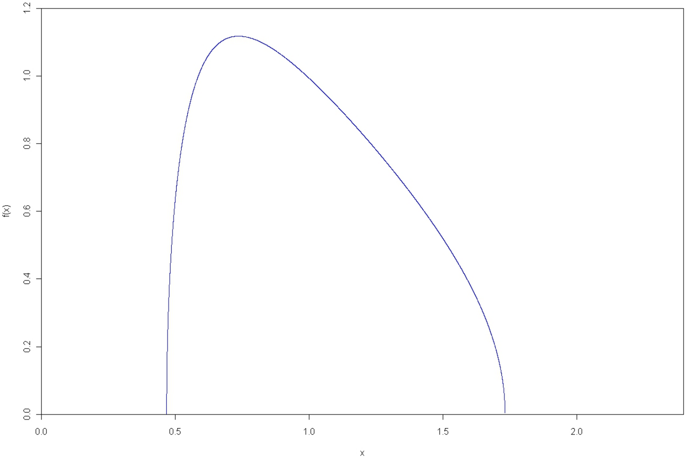

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- Find asymptotic spectrum of $X_{p\times n}^\top X_{p\times n}$ exchanging role of p & n.

Semicircle, MP and quarter circular distributions

- $Y \sim$ semicircle on $(-2, 2)$, $Y^2 \sim$ M-P f_1 $(n/p \to c = 1)$

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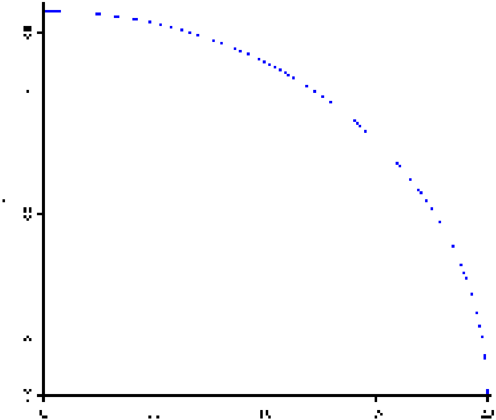

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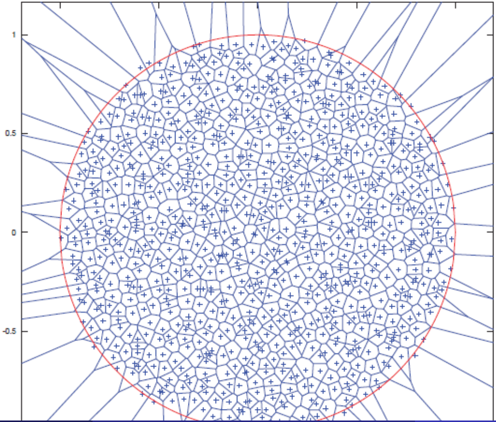
- Bordenave & Chafai (Sep 2011 arXive): "Around the Circular Law"

Around the Circular Law

Simulation Gaussian complex matrices dimension 800

From Bordenave & Chafai (2011)

$$f(z) = \frac{1}{\pi}, z \in \{z \in \mathbb{C}; |z| \leq 1\}$$



Unitary Ensemble

- $Z_n = \{Z_n(j,k); j,k = 1,\ldots n\}$ are i.i.d. r.v. with complex Gaussian distribution with real and imaginary parts independent.

$$Z_n V^* \sim V Z_n \sim Z_n \quad \forall V \in \mathbb{U}(n)$$

- $U_n = Z_n (Z_n Z_n^*)^{-1/2}$ is a unitary matrix: $U_n U_n^* = \mathrm{I}_n$.
- U_n has a Haar distribution

$$U_n V^* \sim V U_n \sim U_n \quad \forall U \in \mathbb{U}(n)$$

- Empirical spectral measure

$$\widehat{\mu}_n(A) = \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_{k,n}}(A), \quad A \in \mathcal{B}(\mathbb{S}_1).$$

- $\widehat{\mu}_n$ converges to uniform measure on $\mathbb{S}_1 = \{z \in \mathbb{C}; \|z\| = 1\}$:

$$f(z) = \frac{1}{2\pi}, z \in \mathbb{S}_1.$$

Asymptotics of extreme eigenvalues

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$$\lim_n \mathbb{P}\left[n^{2/3}(\frac{\lambda_{n,n}}{\sqrt{n}} - 2) \leq t \right] = \exp\left(- \int_t^\infty \frac{q(x)}{2} \mathrm{d}x \right) (F_2(t))^{1/2}$$

$$F_2(t) = \exp\left(- \int_t^\infty (x-t) q(y)^2 \mathrm{d}x \right)$$

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