Some Roles of the Arcsine Distribution in Infinite Divisibility

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Notation

- $\mathrm{a}(x,s)$ density of **arcsine distribution** $\mathrm{a}(x,s)\mathrm{d}x$

$$\mathrm{a}(x,s) = \begin{cases} \frac{1}{\pi}(s-x^2)^{-1/2}, & |x| < \sqrt{s} \\ 0 & |x| \geq \sqrt{s}. \end{cases} \tag{1}$$

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- $\varphi(x;\tau)$ density of the **Gaussian distribution** $\varphi(x;\tau)\mathrm{d}x$ **zero mean and variance** $\tau > 0$

$$\varphi(x;\tau) = (2\pi\tau)^{-1/2}\mathrm{e}^{-x^2/(2\tau)}, \ x \in \mathbb{R}. \tag{2}$$

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  E_τ random variable with exponential density $f_\tau(x)$. $(E = E_1)$.
- Gaussian and exponential distributions are ID, but arcsine is not.

Simple representation of the Gaussian distribution

Fact

$$\varphi(x;\tau) = \frac{1}{2\tau}\int_0^\infty \mathrm{e}^{-s/(2\tau)}\mathrm{a}(x;s)\mathrm{d}s, \ \tau > 0, \ x \in \mathbb{R}. \tag{4}$$

Equivalently: If E_τ and A are independent random variables, then

$$Z_\tau \stackrel{L}{=} \sqrt{E_\tau}A. \tag{5}$$

Gaussian distribution is an exponential mixture of the arcsine distribution.

Goal of the Talk

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- **<u>Goal:</u> show some implications of this representation in the construction of infinitely divisible distributions.**
- Motivation comes from free infinite divisibility: construction of free ID distributions (not today).

Overview of the talk

I. **Gaussian representation and infinite divisibility**

1. Simple consequences.
2. Extensions: Kingman power semicircle distributions.

II. **Type G distributions again: a new look.**

1. Lévy measure characterization (known).
2. New Lévy measure characterization using the Gaussian representation.

III. **Type A distributions**

1. Introduction.
2. Lévy measure characterization.
3. Integral representation of a type G distributions w.r.t. to a Lévy process.
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I. Simple consequences, for example

- **Variance mixture of Gaussians:** V positive random variable

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- **Examples:** X^2 is infinitely divisible if X is stable symmetric, normal inverse Gaussian, normal variance gamma, $t-$student.

I. A characterization of Exponential Distribution

- $G(\alpha, \beta)$, $\alpha > 0, \beta > 0$, gamma distribution with density

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- Y_α, $\alpha > 0$, random variable with gamma distribution $G(\alpha, \beta)$ independent of A. Let

$$X = \sqrt{Y_\alpha} A.$$

  Then X has an ID distribution if and only if $\alpha = 1$, in which case Y_1 has exponential distribution and X has Gaussian distribution.

- **PSD** (Kingman (63)) $PS(\theta, \sigma)$: $\theta \geq -3/2$, $\sigma > 0$

$$f_\theta(x\,;\sigma) = c_{\theta,\sigma}\left(\sigma^2 - x^2\right)^{\theta+1/2} \quad -\sigma < x < \sigma \tag{8}$$

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I. Extension: Power semicircle distribution

Similar representations of the Gaussian distribution

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Theorem (Kingman (63), Arizmendi- PA (10))

Let Y_α, $\alpha > 0$, r.v. with gamma distribution $G(\alpha, \beta)$ independent of r.v. S_θ with distribution $PS(\theta, 1)$. Let ,

$$X \stackrel{L}{=} \sqrt{Y_\alpha} S_\theta \tag{9}$$

When $\alpha = \theta + 2$, X has a Gaussian distribution.
Moreover, *the distribution of X is infinitely divisible iff $\alpha = \theta + 2$ in which case X has a Gaussian distribution.*

I. Part of proof uses a simple kurtosis criteria

- *Kurtosis* of a probability measure μ with finite fourth moment

$$Kurt(\mu) = \frac{c_4(\mu)}{(c_2(\mu))^2} = \frac{m_4(\mu)}{(m_2(\mu))^2} - 3, \tag{10}$$

$c_2(\mu)$, $c_4(\mu)$ are second and fourth cumulants, $m_2(\mu)$, $m_4(\mu)$ second and fourth centered moments. $(Kurt(\mu) \geq -2)$.

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- **Useful simple criteria**: If μ is infinitely divisible, then $Kurt(\mu) \geq 0$.
- Proof: If $Y = X_1 + \cdots + X_n$, X_i independent with distribution of X, then $nKurt[Y] = Kurt[X]$. That is

$$Kurt(\mu) = nKurt(\underbrace{\mu * \cdots * \mu}_{n \text{ times}}).$$

  If μ is ID and $Kurt(\mu) = \alpha < 0$, let μ_n be such that $\underbrace{\mu_n * \cdots * \mu_n}_{n \text{ times}} = \mu$. Since $Kurt(\mu_n) = nKurt(\mu) = n\alpha$, choose n large enough such that $n\alpha < -2$, which is not possible since $Kurt \geq -2$.

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- S_θ is r.v. with distribution $PS(\theta, 1)$. For $\theta > -1/2$ it holds that

$$S_\theta \stackrel{L}{=} U^{1/(2(\theta+1))} S_{\theta-1} \tag{11}$$

where U is r.v. with uniform distribution $U(0, 1)$ independent of r.v. $S_{\theta-1}$ with distribution $PS(\theta - 1, 1)$.

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- This fact and the Gaussian representation suggest that the arcsine distribution is a "nice" distribution to mixture with.

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II. Type G distributions

Recall: Definition and relevance

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  - $X_t = B_{V_t}$ has type G distribution

- $[X_t^2 = (B_{V_t})^2$ is always infinitely divisible].

II. Type G distributions: Lévy measure characterization

- If $V > 0$ is ID with Lévy measure ρ, then $\mu \stackrel{L}{=} \sqrt{V}Z$ is ID with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$

$$l(x) = \int_{\mathbb{R}_+} \varphi(x;s)\rho(\mathrm{d}s), \quad x \in \mathbb{R}. \tag{13}$$

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Theorem (Rosinski (91))

A symmetric distribution μ on $\mathbb{R}$ is type G iff is infinitely divisible and its Lévy measure is zero or $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where $l(x)$ is representable as

$$l(r) = g(r^2), \tag{14}$$

g is completely monotone on $(0, \infty)$ and $\int_0^\infty \min(1, r^2)g(r^2)\mathrm{d}r < \infty$.

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- In general $G(\mathbb{R})$ is the class of generalized type G distributions with Lévy measure (14).(Maejima-Sato (09)).

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- $G_{sym}(\mathbb{R})$ denotes type G distributions.

II. Type G distributions: new characterization

- Using Gaussian representation in $I(x) = \int_{\mathbb{R}_+} \varphi(x;s)\rho(\mathrm{d}s)$:

$$I(x) = \int_0^\infty \mathrm{a}(x;s)\eta(s)\mathrm{d}s. \tag{15}$$

where $\eta(s) := \eta(s;\rho)$ is the completely monotone function

$$\eta(s;\rho) = \int_{\mathbb{R}_+} (2r)^{-1} e^{-s(2r)^{-1}} \rho(\mathrm{d}r). \tag{16}$$

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A symmetric distribution μ on $\mathbb{R}$ is type G iff it is infinitely divisible with Lévy measure ν zero or $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where $l(x)$ is representable as (15), η is a completely monotone function with $\int_0^\infty \min(1,s)\eta(s)\mathrm{d}s < \infty$.

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- Maejima, PA, Sato (10): Multivariate case, $G(\mathbb{R}^d)$, relation to Upsilon transformations, stochastic integral representation.

Lemma

Let g be a real function. The following statements are equivalent:
(a) g is completely monotone on $(0, \infty)$ *with*

$$\int_0^\infty (1 \wedge r^2) g(r^2) \mathrm{d}r < \infty. \tag{17}$$

(b) There is a function $h(s)$ *completely monotone on* $(0, \infty)$*, with* $\int_0^\infty (1 \wedge s) h(s) \mathrm{d}s < \infty$ *and* $g(r^2)$ *has the arcsine transform*

$$g(r^2) = \int_0^\infty \mathrm{a}^+(r; s) h(s) \mathrm{d}s, \quad r > 0, \tag{18}$$

where

$$\mathrm{a}^+(r; s) = \begin{cases} 2\pi^{-1}(s - r^2)^{-1/2}, & 0 < r < s^{1/2}, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

II. Type G distributions: Summary new representation

- Lévy measure is a (special) mixture of arcsine measure: There is a completely monotone function $\eta(s)$ on $(0,\infty)$ such that

$$l(x) = \int_0^\infty \mathrm{a}(x;s)\eta(s)\mathrm{d}s. \tag{20}$$

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- In free probability the (free) ID distribution with arcsine Lévy measure plays a key role (motivation).
- **Next problem**: Characterization of ID distributions with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$

$$l(x) = \int_0^\infty \mathrm{a}(x;s)\lambda(\mathrm{d}s). \tag{21}$$

III. Type A distributions

Definition

$A(\mathbb{R})$ is the class of type A distributions on $\mathbb{R}$: ID distributions with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where

$$l(x) = \int_{\mathbb{R}_+} \mathrm{a}(x; s)\lambda(\mathrm{d}s) \tag{22}$$

and λ is a Lévy measure on $\mathbb{R}_+ = (0, \infty)$.

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- Introduced by Arizmendi, Barndorff-Nielsen, PA (10): Univariate and symmetric case (also in context of free ID).

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$A(\mathbb{R})$ is the class of type A distributions on $\mathbb{R}$: ID distributions with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where

$$l(x) = \int_{\mathbb{R}_+} \mathrm{a}(x;s)\lambda(\mathrm{d}s) \tag{22}$$

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- **How large is the class** $A(\mathbb{R})$?

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III. Recall some known classes of ID distributions

Characterization via Lévy measure

- $ID(\mathbb{R}^d)$ **class of infinitely divisible distributions on** $\mathbb{R}^d$.
- Class of polar decomposition of Lévy measure ν (univariate case $\xi = -1, 1$)

$$\nu(B) = \int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi) h_\xi(r) \mathrm{d}r, \quad B \in \mathcal{B}(\mathbb{B}^d). \tag{23}$$

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- $A(\mathbb{R}^d)$, **Type A class,** $h_\xi(r)$ is an arcsine transform.

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- Observation: Arcsine density $a(x; s)$ is increasing in $r \in (0, \sqrt{s})$

III. Relation between type G and type A distributions

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Let $\Psi : ID(\mathbb{R}^d) \to ID(\mathbb{R}^d)$ *be the mapping given by*

$$\Psi(\mu) = \mathcal{L}\left(\int_0^{1/2}\left(\log\frac{1}{s}\right)^{1/2} dX_s^{(\mu)}\right). \tag{24}$$

An ID distribution $\widetilde{\mu}$ *belongs to* $G(\mathbb{R}^d)$ *iff there exists a type A distribution* μ *such that* $\widetilde{\mu} = \Psi(\mu)$. *That is*

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- **Next problem: integral representation for type A distributions?**

III. Stochastic integral representations for some ID classes

- Jurek (85): $U(\mathbb{R}^d) = \mathcal{U}(ID(\mathbb{R}^d))$,

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- Maejima, Sato and coauthors: Several classes of distributions and integral representations.

III. Type A distributions: stochastic integral representation

Theorem (Maejima, PA, Sato (10))

Let $\Phi_{\cos} : ID(\mathbb{R}^d) \rightarrow ID(\mathbb{R}^d)$ *be the mapping*

$$\Phi_{\cos}(\mu) = \mathcal{L}\left(\int_0^1 \cos(\frac{\pi}{2}s)\mathrm{d}X_s^{(\mu)}\right), \quad \mu \in ID(\mathbb{R}^d). \tag{26}$$

Then

$$A(\mathbb{R}^d) = \Phi_{\cos}(ID(\mathbb{R}^d)). \tag{27}$$

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General framework

- **Upsilon transformations of Lévy measures:**

$$\Upsilon_\sigma(\rho)(B) = \int_0^\infty \rho(u^{-1}B)\sigma(\mathrm{d}u), \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{28}$$

[Jurek (85); Barndorff-Nielsen, Thorbjørnsen (04,05,06); Barndorff-Nielsen, PA (07); Barndorff-Nielsen, Rosinski, Thorbjørnsen (08); Sato (06, 07, 10); Maejima, Nakahara (09)].

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$$(\mathcal{A}_{q,p}^{\alpha,\beta}\nu)(C) = \frac{1}{\Gamma(p)}\int_0^\infty r^{-q-1}dr \int_{\mathbb{R}^d} 1_C\left(r\frac{x}{|x|}\right)(|x|^\beta - r^\alpha)_+^{p-1}\nu(\mathrm{d}x),$$

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- **(Improper) Stochastic integrals:** Sato (2006, 2007, 2010).

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Theorem (Maejima, PA, Sato (10b))

$$U(\mathbb{R}^d) \subset A_{q,p}^{\alpha}(\mathbb{R}^d) \qquad \textit{for } 0 < p \le 1 \textit{ and } q \le -1,$$
$$A_{q,p}^{\alpha}(\mathbb{R}^d) \subset U(\mathbb{R}^d) \qquad \textit{for } p \ge 1 \textit{ and } -1 \le q < 2.$$

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- **Special case:** $\alpha > 0, p > 0$, $q = -\alpha$. Let $\Phi_{\alpha,p} : ID(\mathbb{R}^d) \longrightarrow ID(\mathbb{R}^d)$

$$\Phi_{\alpha,p}(\mu) = \mathcal{L}\left(c_{p+1}^{-1/(\alpha p)} \int_0^{c_{p+1}} \left(c_{p+1}^{1/p} - s^{1/p} \right)^{1/\alpha} dX_s^{(\mu)} \right), \qquad (29)$$

with $c_p = 1/\Gamma(p)$. Then $A^{\alpha}_{-\alpha,p}(\mathbb{R}^d) = \Phi_{\alpha,p}(ID(\mathbb{R}^d))$.

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If $p = 1/2$, $\alpha = 1$, $(q = -1)$

$$A^{1}_{-1,1/2}(\mathbb{R}) = \Phi_{1,1/2}(ID(\mathbb{R}^d)),$$

$$\Phi_{1,1/2}(\mu) = \frac{\pi}{4} \int_0^{2/\sqrt{\pi}} \left(\frac{4}{\pi} - s^2 \right) dX_s^{(\mu)}, \quad \mu \in I(\mathbb{R}^d).$$

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