A New Look to Type G Distributions

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Notation and Review on Infinite Divisibility

- Recall **Fourier transform** of r.v. X (or distribution μ_X)

$$\widehat{\mu}_X(t) = \mathbb{E}(\exp(itX)) = \int_{\mathbb{R}} \exp(itx)\mu_X(\mathrm{d}x), \quad \forall t \in \mathbb{R}.$$

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- μ is **infinitely divisible** ($\mu \in ID(\mathbb{R})$) iff $\forall n \geq 1$, $\exists$ pm $\mu_{1/n}$ and

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- **iff Lévy-Khintchine representation**

$$\log \widehat{\mu}(t) = \eta t - \frac{1}{2}at^2 + \int_{\mathbb{R}} \left(e^{itx} - 1 - tx1_{[-1,1]}(x) \right) \nu(\mathrm{d}x), \ t \in \mathbb{R}$$

Lévy Processes and Infinite Divisibility

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- Given a LP $X = \{X_t : t \geq 0\}$ there is a unique $\mu \in ID(\mathbb{R})$ s.t. $X_1 \stackrel{L}{=} \mu$

$$\mathcal{L}(X_1) = \mu.$$

More notation

- $\mathrm{a}(x,s)$ density of **arcsine distribution** $\mathrm{a}(x,s)\mathrm{d}x$

$$\mathrm{a}(x,s) = \begin{cases} \frac{1}{\pi}(s-x^2)^{-1/2}, & |x| < \sqrt{s} \\ 0 & |x| \geq \sqrt{s}. \end{cases} \tag{1}$$

A_s random variable with density $\mathrm{a}(x,s)$ on $(-\sqrt{s},\sqrt{s})$. $(A = A_1)$.

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$$\varphi(x;\tau) = (2\pi\tau)^{-1/2}\mathrm{e}^{-x^2/(2\tau)},\ x \in \mathbb{R}. \quad (2)$$

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- Gaussian and exponential distributions are ID, but arcsine is not.

Representation of the Gaussian distribution

Fact

$$\varphi(x;\tau) = \frac{1}{2\tau}\int_0^\infty \mathrm{e}^{-s/(2\tau)}\mathrm{a}(x;s)\mathrm{d}s,\ \tau > 0,\ x \in \mathbb{R}. \tag{4}$$

Equivalently: If E_τ and A are independent random variables, then

$$Z_\tau \stackrel{L}{=} \sqrt{E_\tau} A.$$

Gaussian distribution is an exponential mixture of the arcsine distribution.

Goal of the Talk

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- **Goal: show some implications of this representation in the construction of infinitely divisible distributions.**
- **Motivation comes from free infinite divisibility: construction of free ID distributions (not today)**

Content of the talk

I. **Gaussian representation and infinite divisibility**

1. Simple consequences.
2. Power semicircle distributions

II. **A new look to type G distributions**

1. Lévy measure characterization (known).
2. New Lévy measure characterization using the Gaussian representation.

III. **Distributions of class A**

1. Lévy measure characterization.
2. Integral representation of type G distributions w.r.t. LP
3. Integral representation of distributions of class A w.r.t to LP.

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Corollary

If $X \stackrel{L}{=} \sqrt{V}Z$ is variance mixture of Gaussians, $V > 0$ arbitrary independent of Z, then X^2 is infinitely divisible.

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- **Examples:** X^2 is infinitely divisible if X is stable symmetric, normal inverse Gaussian, normal variance gamma, $t-$student.

I. A characterization of Exponential Distribution

- $G(\alpha, \beta)$, $\alpha > 0, \beta > 0$, gamma distribution with density

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- Y_{α}, $\alpha > 0$, random variable with gamma distribution $G(\alpha, \beta)$ independent of A. Let

$$X = \sqrt{Y_{\alpha}} A.$$

  Then X has an ID distribution if and only if $\alpha = 1$, in which case Y_1 has exponential distribution and X has Gaussian distribution.

I. Extension: Power semicircle distributions

Similar representations of the Gaussian distribution

- **PSD** (Kingman (63)) $PS(\theta, \sigma)$: $\theta \geq -3/2$, $\sigma > 0$

$$f_\theta(x\,;\sigma) = c_{\theta,\sigma}\left(\sigma^2 - x^2\right)^{\theta+1/2} \quad -\sigma < x < \sigma \tag{6}$$

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- $\theta = -1/2$ is uniform distribution
- $\theta = \infty$ is classical Gaussian distribution: *Poincaré´s theorem*: $(\theta \to \infty)$

$$f_\theta(x\,;\sqrt{(\theta+2)/2}\sigma) \to \frac{1}{\sqrt{2\pi}\sigma}\exp(-x^2/(2\sigma^2)).$$

I. Other Gaussian representations

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Theorem (Kingman (63), Arizmendi- PA (10))

Let Y_α, $\alpha > 0$, r.v. with gamma distribution $G(\alpha, \beta)$ independent of r.v. S_θ with distribution $PS(\theta, 1)$. Let ,

$$X \stackrel{L}{=} \sqrt{Y_\alpha} S_\theta \tag{8}$$

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Moreover, *the distribution of X is infinitely divisible iff $\alpha = \theta + 2$ in which case X has a classical Gaussian distribution.*

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- Proof uses a very simple kurtosis criteria.

I. Recursive representations

- S_θ is r.v. with distribution $PS(\theta, 1)$. For $\theta > -1/2$ it holds that

$$S_\theta \stackrel{L}{=} U^{1/(2(\theta+1))} S_{\theta-1} \tag{9}$$

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- This fact and the Gaussian representation suggest that the arcsine distribution is a "nice small" distribution to mixture with.

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- $X_t^2 = (B_{V_t})^2$ is always infinitely divisible.

II. Type G distributions: Lévy measure characterization

- If $V > 0$ is ID with Lévy measure ρ, then $\mu \stackrel{L}{=} \sqrt{V}Z$ is ID with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$

$$l(x) = \int_{\mathbb{R}_+} \varphi(x; s)\rho(\mathrm{d}s), \quad x \in \mathbb{R}. \tag{11}$$

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Theorem (Rosinski (91))

A symmetric distribution μ on $\mathbb{R}$ is type G iff is infinitely divisible and its Lévy measure is zero or $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where $l(x)$ is representable as

$$l(r) = g(r^2), \tag{12}$$

g is completely monotone on $(0,\infty)$ and $\int_0^\infty \min(1, r^2)g(r^2)\mathrm{d}r < \infty$.

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- In general $G(\mathbb{R})$ is the class of generalized type G distributions with Lévy measure (12).

II. Type G distributions: new characterization

- Using Gaussian representation in $I(x) = \int_{\mathbb{R}_+} \varphi(x;s)\rho(\mathrm{d}s)$:

$$I(x) = \int_0^\infty \mathrm{a}(x;s)\eta(s)\mathrm{d}s. \tag{13}$$

where $\eta(s) := \eta(s;\rho)$ is the completely monotone function

$$\eta(s;\rho) = \int_{\mathbb{R}_+} (2r)^{-1} e^{-s(2r)^{-1}} \rho(\mathrm{d}r). \tag{14}$$

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Theorem (Arizmendi, Barndorff-Nielsen, PA (2010))

A symmetric distribution μ on $\mathbb{R}$ is type G iff it is infinitely divisible with Lévy measure ν zero or $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where
1) $l(x)$ is representable as (13),
2) η is a completely monotone function with $\int_0^\infty \min(1,s)\eta(s)\mathrm{d}s < \infty$.

Lemma

Let g be a real function. The following statements are equivalent:
(a) g is completely monotone on $(0,\infty)$ with

$$\int_0^\infty (1 \wedge r^2) g(r^2) \mathrm{d}r < \infty. \tag{15}$$

(b) There is a function $h(s)$ completely monotone on $(0,\infty)$, with $\int_0^\infty (1 \wedge s) h(s) \mathrm{d}s < \infty$ and $g(r^2)$ has the arcsine transform

$$g(r^2) = \int_0^\infty \mathrm{a}^+(r;s) h(s) \mathrm{d}s, \quad r > 0, \tag{16}$$

where

$$\mathrm{a}^+(r;s) = \begin{cases} 2\pi^{-1}(s-r^2)^{-1/2}, & 0 < r < s^{1/2}, \\ 0, & \textit{otherwise}. \end{cases} \tag{17}$$

- Lévy measure is a (special) mixture of arcsine measure: There is a completely monotone function $\eta(s)$ on $(0,\infty)$ such that

$$l(x) = \int_0^\infty \mathrm{a}(x;s)\eta(s)\mathrm{d}s. \tag{18}$$

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- **Next problem**: Characterization of ID distributions which Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$ **is the arcsine transform**

$$l(x) = \int_0^\infty \mathrm{a}(x; s)\lambda(\mathrm{d}s). \tag{19}$$

Definition

$A(\mathbb{R})$ is the class of A of distributions on $\mathbb{R}$: ID distributions with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where

$$l(x) = \int_{\mathbb{R}_+} \mathrm{a}(x; s)\lambda(\mathrm{d}s) \tag{20}$$

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- **How large is the class $A(\mathbb{R})$?**

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$$\nu(B) = \int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi) h_\xi(r) \mathrm{d}r, \quad B \in \mathcal{B}(\mathbb{B}^d). \tag{21}$$

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III. Recall some known classes of ID distributions

Characterization via Lévy measure

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- Observation: Arcsine density $a(x; s)$ is increasing in $r \in (0, \sqrt{s})$

III. Relation between type G and type A distributions

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Theorem (A, BN, PA (10); Maejima, PA, Sato (11).)

Let $\Psi : ID(\mathbb{R}^d) \to ID(\mathbb{R}^d)$ *be the mapping given by*

$$\Psi(\mu) = \mathcal{L}\left(\int_0^{1/2} \left(\log \frac{1}{s} \right)^{1/2} dX_s^{(\mu)} \right). \tag{22}$$

An ID distribution $\widetilde{\mu}$ *belongs to* $G(\mathbb{R}^d)$ *iff there exists a type A distribution* μ *such that* $\widetilde{\mu} = \Psi(\mu)$. *That is*

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- **Next problem: integral representation for type A distributions?**

- Jurek (85): $U(\mathbb{R}^d) = \mathcal{U}(ID(\mathbb{R}^d))$,

$$\mathcal{U}(\mu) = \mathcal{L}\left(\int_0^1 s\mathrm{d}X_s^{(\mu)}\right).$$

III. Stochastic integral representations for some ID classes

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- Jurek, Vervaat (83), Sato, Yamazato (83): $L(\mathbb{R}^d) = \Phi(ID_{\log}(\mathbb{R}^d))$

$$\Phi(\mu) = \mathcal{L}\left(\int_0^\infty \mathrm{e}^{-s} \mathrm{d}X_s^{(\mu)}\right),$$

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- Barndorff-Nielsen, Maejima, Sato (06): $B(\mathbb{R}^d) = \Upsilon(I(\mathbb{R}^d))$ and $T(\mathbb{R}^d) = \Upsilon(L(\mathbb{R}^d))$

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 \log\frac{1}{s} \mathrm{d}X_s^{(\mu)}\right).$$

Theorem (Maejima, PA, Sato (11))

Let $\Phi_{\cos} : ID(\mathbb{R}^d) \rightarrow ID(\mathbb{R}^d)$ *be the mapping*

$$\Phi_{\cos}(\mu) = \mathcal{L}\left(\int_0^1 \cos(\frac{\pi}{2}s)\mathrm{d}X_s^{(\mu)}\right), \quad \mu \in ID(\mathbb{R}^d). \tag{24}$$

Then

$$A(\mathbb{R}^d) = \Phi_{\cos}(ID(\mathbb{R}^d)). \tag{25}$$

.

IV. General framework

- **Upsilon transformations of Lévy measures:**

$$\Upsilon_\sigma(\rho)(B) = \int_0^\infty \rho(u^{-1}B)\sigma(\mathrm{d}u), \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{26}$$

[Barndorff-Nielsen, Rosinski, Thorbjørnsen (08)].

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- **Fractional transformations of Lévy measures:**

$$(\mathcal{A}^{\alpha,\beta}_{q,p}\nu)(C) = \frac{1}{\Gamma(p)}\int_0^\infty r^{-q-1}dr\int_{\mathbb{R}^d} 1_C(r\frac{x}{|x|})(|x|^\beta - r^\alpha)_+^{p-1}\nu(\mathrm{d}x),$$

$p, \alpha, \beta \in \mathbb{R}_+$, $q \in \mathbb{R}$ [Maejima, PA, Sato (11), Sato (11)].

GRACIAS ONESIMO
FOR YOUR ADVICES IN THE BEGINNING
OF MY ACADEMIC CARREAR IN 1985

Talk based on parts of joint works

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