Random Matrices
(Some basic ideas and results)

Pre-School for Young Researchers, VIII World Congress in Probability and Statistics, Istanbul, Turkey

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Log-Gases and Random Matrices, P. Forrester (2010):

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4. *Depth* comes via the linking together of multiple ideas and topics, often seemingly removed from the original context.
5. And *fertility* means that with a reasonable effort there are new results, some useful, some with beauty, and a few maybe with depth, still waiting to be found.

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An Introduction to Random Matrices, G. W. Anderson, A. Guionnet & O. Zeitouni (2010):

1. The study of random matrices, and in particular the properties of their eigenvalues, has emerged from the applications, first in data analysis (*Wishart, 1928*) and later on as statistical models for heavy-nuclei atoms (*Wigner, 1955*).
2. Thus, the field of random matrices owes its existence to *applications*.
3. Over the years, however, it became clear that models related to random matrices play an important role in areas of *pure mathematics.*
4. Moreover, the tools used in the study of random matrices came themselves from different and *seemingly unrelated branches of mathematics (combinatorics, graphs, functional analysis, orthogonal polynomials, probability, operator algebras, free probability, number theory, complex analysis, compact groups).*

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Random Matrices, 3rd ed, M. L. Mehta (2004):

1. In the last decade following the publication of the second edition of this book (1967, 1991) the subject of random matrices found *applications in many new fields of knowledge:*
2. *Physics*: In heterogeneous conductors (mesoscopy systems) where the passage of electric current may be studied by transfer matrices, quantum chromo dynamics characterized by some Dirac operator, quantum gravity modeled by some random triangulation of surfaces.
3. *Traffic and communication networks.*
4. Zeta function and L-series in *number theory*,
5. Even stock movements in *financial markets*,
6. **Wherever imprecise matrices occurred, people dreamed of random matrices.**

Applications of Large Dimensional Random Matrices:

1. Data dimension of same magnitude order than sample size.
2. Wireless communication (channel capacity of MIMO channels)
3. *Some recent books:*
   3.1 Bai & Silverstein (2010). Spectral Analysis of Large Dimensional Random Matrices.
   3.2 Couillet & Debbah (2011). Random Matrix Methods for Wireless Communications.

Random matrices in Istanbul World Congress Program

Invited talks: statistical applications and methodology

1. Invited Session: **Random Matrices and Applications**
   1.1 IS 21, Tuesday 10 July, 14:00-15.30
   1.2 J. Najim, *Enigenvalue Estimation of covariance matrices of large dimensional data.*
   1.3 W. Wu, *Covariance matrix estimation in time series-data*
   1.4 M. Krhishnapur, *Nodal length of random eigenfunctions of the Laplacian on the 2-d tours.*
2. **IMS Medallion Lecture 1**
   2.1 Tuesday 10 July, 9.00-10.00,
   2.2 Van Vu, *Recent progresses in random matrix theory.*
3. A talk, Invited Session **Extremes for Complex Phenomena**
   3.1 IS-7, Friday 13 July, 16.15-16-45,
   3.2 Richard Davis *Limit theory for the largest eigenvalue of a sample covariance matrix from high-dimensional observations with heavy tails.*

Random matrices in Istanbul World Congress Program

Contributed talks

1. A talk in CS- 18 **Theory 1**
   1.1 Monday July 9, 16-55-17-15, O. Pfaffel, *Asymptotic spectrum of large sample covariance matrices of linear processes*
2. CS 78, **Random Matrices**, Friday 13 July, 14.00-15.40.
   2.1 H. Osada, Infinite-dimensional *stochastic differential equations related to Airy random point fields-soft edge scaling limits.*
   2.2 A. Rohde, *Accuracy of empirical projections of high-dimensional Gaussian matrices.*
   2.3 K. Glombek, *A Jarque-Beran test for sphericity of a Large dimensional covariance matrix*
   2.4 J. Hu, *Convergence of the empirical spectral distribution of eigenvalues of Beta-type matrices.*
   2.5 G. Pan, *Independence test for high dimensional data based on regularized canonical correlation coefficients.*

Plan of the Lecture

1. Why random matrices?
2. Wigner Law
   - 2.1 Gaussian ensembles
   - 2.2 Asymptotic spectral distribution: Wigner distribution
   - 2.3 Universality
   - 2.4 Idea of proof
3. Marchenko-Pastur law
   - 3.1 Ensemble of sample covariance matrices
   - 3.2 Asymptotic spectral distribution
   - 3.3 Application to Wireless communication
4. Random matrices and free probability
   - 4.1 Free Gaussian and free Poisson distributions
   - 4.2 Motivation to study free independence
   - 4.3 The classical cumulant transform and classical convolution
   - 4.4 The free cumulant transform and free convolution

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- Ensemble: $\mathbf{Z} = (Z_n)$, Z_n is $n \times n$ matrix with random entries.

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Z_n &= (Z_n(j,k)) \\
Z_n(j,k) &= Z_n(k,j) \sim N(0,t), \quad j \neq k, \\
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- Density of eigenvalues of $\lambda_{n,1} < \dots < \lambda_{n,n}$ of Z_n:

$$f_{\lambda_{n,1},\dots,\lambda_{n,n}}(x_1,\dots,x_n) = k_n \left[\prod_{j=1}^{n} \exp\left(-\frac{1}{4t}x_j^2\right)\right]\left[\prod_{j<k}|x_j - x_k|\right].$$

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- Nondiagonal RM: eigenvalues are strongly dependent due to Vandermont determinant: $x = (x_1,\ldots,x_n) \in \mathbb{C}^n$

$$\Delta(x) = \det\left(\left\{x_j^{k-1}\right\}_{j,k=1}^{n}\right) = \prod_{j<k} (x_j - x_k).$$

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$$\widehat{Ft}_n(x) = \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{\{\lambda_{n,j} \leq x\}}.$$

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- **Asymptotic spectral distribution (ASD):** $\widehat{F^t}_n$ converges, as $n \to \infty$, to **semicircle distribution on** $(-2\sqrt{t}, 2\sqrt{t})$

$$w_t(x) = \frac{1}{2\pi}\sqrt{4t - x^2}, \quad |x| \leq 2\sqrt{t}.$$

II. Universality of Wigner law.

Wigner (Ann Math. 1955, 1957, 1958)

Theorem

$t > 0$. $\forall\ f \in C_b(\mathbb{R})$ and $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\int f(x)\mathrm{d}\widehat{F^t}_n(x) - \int f(x)\mathrm{w}_t(\mathrm{d}x)\right| > \varepsilon\right) = 0.$$

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- **Universality**. Law holds for **Wigner random matrices**:

$$X_n(k,j) = X_n(j,k) = \frac{1}{\sqrt{n}}\begin{cases} Z_{j,k}, & \text{if } j < k \\ Y_j, & \text{if } j = k \end{cases}$$

$\{Z_{j,k}\}_{j\leq k}, \{Y_j\}_{j\geq 1}$ independent sequences of i.i.d. r.v.
$\mathbb{E}Z_{1,2} = \mathbb{E}Y_1 = 0, \mathbb{E}Z_{1,2}^2 = 1$.

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- Convergence of extreme eigenvalues as $n \to \infty$

$$\mathbb{P}(\lambda_{\max} \to 2\sqrt{t}) = \mathbb{P}(\lambda_{\min} \to -2\sqrt{t}) = 1$$

II. Idea of a proof of Wigner theorem

- Basic observation

$$\widehat{m}_k(t) = \int x^k \widehat{F^t}_n(x) = \frac{1}{n}(\lambda_{n,1}^k + \ldots + \lambda_{n,n}^k) = \frac{1}{n}\mathrm{tr}(X_n^k).$$

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- Use method of moments to show that $\overline{m}_k \underset{n\to\infty}{\to} m_k$, $\forall k \geq 1$.
- *Catalan numbers* and non crossing partitions

$$C_k = \frac{1}{k+1}\binom{2k}{k},\ k \geq 1.$$

III. Marchenko-Pastur law (1967)

- $X = X_{p\times n} = (Z_{j,k} : j = 1, .., p, k = 1, ..., n)$ complex i.i.d.
$$\mathbb{E}(Z_{1,1}) = 0, \mathbb{E}(\,|Z_{1,1}|^2) = 1.$$
- $W_n = XX^*$ is **Wishart matrix** if X has Gaussian entries.
- Sample covariance matrix $S_n = \frac{1}{n}XX^*$, with eigenvalues $0 \le \lambda_{p,1} \le ... \le \lambda_{p,p}$ and ESD
$$\widehat{F}_p(\lambda) = \frac{1}{p}\sum_{j=1}^{p} \mathbf{1}_{\{\lambda_{p,j}\le x\}}.$$
- If $p/n \to c > 0$, $\widehat{F}_n$ converges weakly in probability to Marchenko-Pastur (MP) distribution
$$\mu_c(\mathrm{d}x) = \begin{cases} f_c(x)\mathrm{d}x, & \text{if} \quad c \ge 1 \\ (1-c)\delta_0(\mathrm{d}x) + f_c(x)\mathrm{d}x, & \text{if} \quad 0 < c < 1, \end{cases}$$
$$f_c(x) = \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\mathbf{1}_{[a,b]}(x)$$
$$a = (1-\sqrt{c})^2, \ \ b = (1+\sqrt{c})^2.$$

III. Example: Communication Channel Capacity

Circularly symmetric complex Gaussian random matrices

A $p \times 1$ complex random vector $\mathbf{u}$ has a *Q-circularly symmetric complex Gaussian distribution* if

$$\mathbb{E}[(\mathbf{u} - \mathbb{E}[\mathbf{u}])(\mathbf{u} - \mathbb{E}[\mathbf{u}])^*] = \frac{1}{2} \begin{bmatrix} \mathrm{Re}[Q] & -\mathrm{Im}[Q] \\ \mathrm{Im}[Q] & \mathrm{Re}[Q] \end{bmatrix},$$

for some nonnegative definite Hermitian $p \times p$ matrix Q.

$$\mathbf{u} = \left[\mathrm{Re}(u_1) + i\,\mathrm{Im}(u_1), \ldots, \mathrm{Re}(u_p) + i\,\mathrm{Im}(u_p)\right]^\top .$$

III. Example: Communication Channel Capacity

A Model for MIMO antenna systems

- n_T antennas at transmitter and n_R antennas at receiver
- Linear vector channel with Gaussian noise

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

- $\mathbf{x}$ is the n_T-dimensional *input vector*.
- $\mathbf{y}$ is the n_R-dimensional *output vector*.
- $\mathbf{n}$ is the received Gaussian *noise*, zero mean and $\mathbb{E}\left(\mathbf{n}\mathbf{n}^*\right) = \mathrm{I}_{n_T}$.
- The $n_R \times n_T$ random matrix $\mathbf{H}$ is the *channel matrix.*
- $\mathbf{H} = \{h_{jk}\}$ is a random matrix, it models the propagation coefficients between each pair of trasmitter-receiver antennas.
- $\mathbf{x}, \mathbf{H}$ and $\mathbf{n}$ are independent.

III. Example: Communication Channel Capacity

Raleigh fading channel

- h_{jk} are i.i.d. complex random variables with mean zero and variance one ($\mathrm{Re}(h_{jk}) \sim N(0, \frac{1}{2})$ independent of $\mathrm{Im}(h_{jk}) \sim N(0, \frac{1}{2})$).

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- Signal to Noise Ratio

$$SNR = \frac{\mathbb{E}||\mathbf{x}||^2/n_T}{\mathbb{E}||\mathbf{n}||^2/n_R} = \frac{P}{n_T}.$$

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- The capacity of this MIMO system channel is

$$C(n_R, n_T) = \max_{Q} \mathbb{E}_{\mathbf{H}} \left[\log_2 \det \left(\mathrm{I}_{n_R} + \mathbf{H} Q \mathbf{H}^* \right) \right]$$

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- $C(n_R, n_T)$ in terms of ESD $\widehat{F}_{n_R}$ of the random covariance $\frac{1}{n_R}\mathbf{H}\mathbf{H}^*$

$$C(n_R, n_T) = \int_0^\infty \log_2\left(1 + \frac{n_R}{n_T}Px\right) n_R \mathrm{d}\widehat{F}_{n_R}(x).$$

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$$C(n_R, n_T) = \mathbb{E}_{\mathbf{H}}\left[\log_2 \det\left(\mathrm{I}_{n_R} + \frac{P}{n_T}\mathbf{H}\mathbf{H}^*\right)\right]$$

- $C(n_R, n_T)$ in terms of ESD $\widehat{F}_{n_R}$ of the random covariance $\frac{1}{n_R}\mathbf{H}\mathbf{H}^*$

$$C(n_R, n_T) = \int_0^\infty \log_2\left(1 + \frac{n_R}{n_T}Px\right) n_R \mathrm{d}\widehat{F}_{n_R}(x).$$

- By Marchenko-Pastur theorem, if $n_R/n_T \to c$,

$$\frac{C(n_R, n_T)}{n_R} \to \int_a^b \log_2(1 + cPx)\,\mathrm{d}\mu_c(x) = K(c, P).$$

III. Example: Communication Channel Capacity

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- *Increase capacity with more transmitter and receiver antennas without increasing the total power constraint P.*

IV. Free Central Limit Theorem

Very roughly speaking

- The concept of free independence is defined for noncommutative random variables: Large dimensional random matrices.
- Distribution is the spectral distribution of an operator or asymptotic spectral distribution of an ensemble of random matrices.
- Let $\mathbf{X}_1, \mathbf{X}_2, ...$ be a sequence of freely independent random variables with the same distribution with all moments, with mean zero and variance one. Then the distribution of

$$\mathbf{Z}_n = \frac{1}{\sqrt{n}}(\mathbf{X}_1 + ... + \mathbf{X}_n)$$

  converges in distribution to the semicircle distribution.
- **Free Gaussian distribution**: the semicircle distribution plays in free probability the role Gaussian distribution does in classical probability.
- **Free Poisson distribution**: The Marchenko-Pastur distribution plays in free probability the role the Poisson

IV. Motivation to study RMT and Free Probability

From the Blog of Terence Tao (Free Probability, 2010):

1. The significance of free probability to random matrix theory lies in the fundamental observation that *random matrices which are independent in the classical sense, also tend to be independent in the free probability sense*, in the large limit.

2. This is only possible because of the highly *non-commutative nature* of these matrices; it is not possible for non-trivial commuting independent random variables to be *freely independent.*

3. Because of this, many tedious computations in random matrix theory, particularly those of an algebraic or enumerative combinatorial nature, can be done more quickly and systematically by using the *framework of free probability.*

IV. Motivation to study RMT and Free Probability

Independence and free Independence

A) **Basic question**: knowing eigenvalues of $n \times n$ random matrices X_n & Y_n, what are the eigenvalues of $X_n + Y_n$?

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- Distribution of $X + Y$ is the *classical convolution*

$$\mu_{X+Y} = \mu_X * \mu_Y .$$

C) *Something similar for the distribution of the product XY (multiplicative convolution).*

IV. Classical and free convolutions

Analytic tools

- Fourier transform of probability measure μ on $\mathbb{R}$

$$\widehat{\mu}(s) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sx} \mu(\mathrm{d}x), \quad s \in \mathbb{R},$$

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- Classical cumulant transform

$$c_\mu(s) = \log \widehat{\mu}(s), \quad s \in S.$$

- Free cumulant transform

$$C_\mu(z) = zG_\mu^{-1}(z) - 1, \quad z \in \Gamma_\mu$$

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- Classical convolution $\mu_1 * \mu_2$ is defined by

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- $\mathbf{X}_1$ & $\mathbf{X}_2$ free independent, $\mu_i = \mathcal{L}(\mathbf{X}_i)$,

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- $\mathbf{X}_1$ & $\mathbf{X}_2$ free independent, $\mu_i = \mathcal{L}(\mathbf{X}_i)$,

$$\mu_1 \boxplus \mu_2 = \mathcal{L}(\mathbf{X}_1 + \mathbf{X}_2)$$

- Free *multiplicative* convolution $\mu_1 \boxtimes \mu_2$ can also be defined.

IV. Example: free convolution of Wigners

- *Semicircle distribution* w_{m,σ^2} *on* $(m-2\sigma, m+2\sigma)$ *centered at* m

$$w_{m,\sigma^2}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-(x-m)^2}\,1_{[m-2\sigma,m+2\sigma]}(x).$$

- Cauchy transform*:*

$$G_{\mathrm{w}_{m,\sigma^2}}(z) = \frac{1}{2\sigma^2}\left(z-\sqrt{(z-m)^2-4\sigma^2}\right),$$

- Free cumulant transform:

$$C_{\mathrm{w}_{m,\sigma^2}}(z) = mz+\sigma^2 z.$$

- $\boxplus$*-convolution of Wigner distributions is a Wigner distribution*:

$$\mathrm{w}_{m_1,\sigma_1^2}\boxplus \mathrm{w}_{m_2,\sigma_2^2} = \mathrm{w}_{m_1+m_2,\sigma_1^2+\sigma_2^2}.$$

IV. Example: free convolutions of MPs

- MP distribution of parameter $c > 0$

$$\mathrm{m}_c(\mathrm{d}x) = (1-c)_+\delta_0 + \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\ 1_{[a,b]}(x)\mathrm{d}x.$$

- Cauchy transform

$$G_{\mathrm{m}_c}(z) = \frac{1}{2} - \frac{\sqrt{(z-a)(z-b)}}{2z} + \frac{1-c}{2z}$$

- Free cumulant transform

$$C_{\mathrm{m}_c}(z) = \frac{cz}{1-z}.$$

- $\boxplus$-*convolution of M-P distributions is a MP distribution*:

$$\mathrm{m}_{c_1} \boxplus \mathrm{m}_{c_2} = \mathrm{m}_{c_1+c_2}$$

IV. Example: free convolution of Cauchy distributions

- Cauchy distribution of parameter $\theta > 0$

$$\mathrm{c}_\theta(\mathrm{d}x) = \frac{1}{\pi}\frac{\theta}{\theta^2 + x^2}\mathrm{d}x$$

- Cauchy transform

$$G_{\mathrm{c}_\theta}(z) = \frac{1}{z + \theta i}$$

- Free cumulant transform

$$C_{\mathrm{c}_\lambda}(z) = -i\theta z$$

- $\boxplus$-*convolution of Cauchy distributions is a Cauchy distribution*

$$\mathrm{c}_{\theta_1} \boxplus \mathrm{c}_{\theta_2} = \mathrm{c}_{\theta_1+\theta_2}.$$

IV. Classical and free infinite divisibility

- Let μ be a probability distribution on $\mathbb{R}$.
- μ is **infinitely divisible** w.r.t. $\star$ iff $\forall n \geq 1$, $\exists$ $\mu_{1/n}$ and

$$\mu = \mu_{1/n} \star \mu_{1/n} \star \cdots \star \mu_{1/n}.$$

- μ is **infinitely divisible** w.r.t. $\boxplus$ iff $\forall n \geq 1$, $\exists$ $\mu_{1/n}$ and

$$\mu = \mu_{1/n} \boxplus \mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}.$$

- Notation: $I^{\boxplus}$ (I^{*}) class of all free (classical) ID distributions.
- **Problem:** characterize the class $I^{\boxplus}$ similar to I^{*}.

- Classical Lévy-Khintchine representation $\mu \in I^*$

$$c_\mu(s) = \eta s - \frac{1}{2}as^2 + \int_{\mathbb{R}} \left(e^{isx} - 1 - sx1_{[-1,1]}(x)\right) \rho(\mathrm{d}x), \ \ s \in \mathbb{R}.$$

- Free Lévy-Khintchine representation $\nu \in I^{\boxplus}$

$$C_\nu(z) = \eta z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz1_{[-1,1]}(x)\right) \rho(\mathrm{d}x), \ \ z \in \mathbb{C}^-$$

- In both cases (η, a, ρ) is the unique *Lévy triplet*: $\eta \in \mathbb{R}$, $a \geq 0$, $\rho(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \min(1, x^2)\rho(\mathrm{d}x) < \infty.$$

IV. Relation between classical and free infinite divisibility

Bercovici, Pata (Biane), Ann. Math. (1999)

- Classical Lévy-Khintchine representation $\mu \in I^*$

$$c_\mu(s) = \eta s - \frac{1}{2}as^2 + \int_{\mathbb{R}} \left(e^{isx} - 1 - sx1_{[-1,1]}(x) \right) \rho(\mathrm{d}x).$$

- Free Lévy-Khintchine representation $\nu \in I^{\boxplus}$

$$C_\nu(z) = \eta z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz1_{[-1,1]}(x) \right) \rho(\mathrm{d}x).$$

- *Bercovici-Pata bijection*: $\Lambda : I^* \to I^{\boxplus}$, $\Lambda(\mu) = \nu$

$$I^* \ni \mu \sim (\eta, a, \rho) \leftrightarrow \Lambda(\mu) \sim (\eta, a, \rho)$$

- Λ preserves convolutions (and weak convergence)

$$\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$$

IV. Examples of free infinitely divisible distributions

Images of classical i.d. distributions under Bercovici-Pata bijection

- *Free Gaussian:* For classical Gaussian distribution γ_{m,σ^2},

$$\mathrm{w}_{m,\sigma^2} = \Lambda(\gamma_{m,\sigma^2})$$

is Wigner distribution on $(m - 2\sigma, m + 2\sigma)$ with

$$C_{\mathrm{w}_{\eta,\sigma^2}}(z) = mz + \sigma^2 z^2.$$

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- *Free Poisson:* For classical Poisson distribution p_c, $c > 0$,

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- Bellinschi, Bozejko, Lehner & Speicher (11): γ_{m,σ^2} is free ID.
- **Open problem:** $\gamma_{m,\sigma^2} = \Lambda(?)$.

Random matrices in Istanbul World Congress Program

Invited talks: statistical applications and methodology

1. Invited Session: **Random Matrices and Applications**
   1.1 IS 21, Tuesday 10 July, 14:00-15.30
   1.2 J. Najim, *Enigenvalue Estimation of covariance matrices of large dimensional data.*
   1.3 W. Wu, *Covariance matrix estimation in time series-data*
   1.4 M. Krhishnapur, *Nodal length of random eigenfunctions of the Laplacian on the 2-d tours.*
2. **IMS Medallion Lecture 1**
   2.1 Tuesday 10 July, 9.00-10.00,
   2.2 Van Vu, *Recent progresses in random matrix theory.*
3. A talk, Invited Session **Extremes for Complex Phenomena**
   3.1 IS-7, Friday 13 July, 16.15-16-45,
   3.2 Richard Davis *Limit theory for the largest eigenvalue of a sample covariance matrix from high-dimensional observations with heavy tails.*

Random matrices in Istanbul World Congress Program

Contributed talks

1. A talk in CS- 18 **Theory 1**
   1.1 Monday July 9, 16-55-17-15, O. Pfaffel, *Asymptotic spectrum of large sample covariance matrices of linear processes*
2. CS 78, **Random Matrices**, Friday 13 July, 14.00-15.40.
   2.1 H. Osada, Infinite-dimensional *stochastic differential equations related to Airy random point fields-soft edge scaling limits.*
   2.2 A. Rohde, *Accuracy of empirical projections of high-dimensional Gaussian matrices.*
   2.3 K. Glombek, *A Jarque-Beran test for sphericity of a Large dimensional covariance matrix*
   2.4 J. Hu, *Convergence of the empirical spectral distribution of eigenvalues of Beta-type matrices.*
   2.5 G. Pan, *Independence test for high dimensional data based on regularized canonical correlation coefficients.*

IV. Examples of free infinitely divisible distributions

Images of classical i.d. distributions under Bercovici-Pata bijection

- *Free Cauchy:* $\Lambda(\mathrm{c}_\lambda) = \mathrm{c}_\lambda$ for the Cauchy distribution

$$\mathrm{c}_\lambda(\mathrm{d}x) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 + x^2}\mathrm{d}x$$

  with free cumulant transform

$$C_{\mathrm{c}}(z) = -i\lambda z.$$

- *Free stable*

$$S^{\boxplus} = \{\Lambda(\mu); \mu \text{ is classical stable}\}.$$

- *Free Generalized Gamma Convolutions (GGC)*

$$GGC^{\boxplus} = \{\Lambda(\mu); \mu \text{ is classical } GGC\}$$

V. Random matrix approach to BP bijection

- Benachy-Georges (2005, AP), Cavanal-Duvillard (2005, EJP): For $\mu \in I^*$ there is an ensemble of unitary invariant random matrices $(M_d)_{d \geq 1}$, such that with probability one its ESD converges in distribution to $\Lambda(\mu) \in I^{\boxplus}$.
- M_d is infinitely divisible in the space of matrices $\mathbb{M}_d$.
- The existence of $(M_d)_{d \geq 1}$ is not constructive.
- How are the random matrix $(M_d)_{d \geq 1}$ realized?
- How are the corresponding matrix Lévy processes $\{M_d(t)\}_{t \geq 0}$ realized?
- The jump $\Delta M_d(t) = M_d(t) - M_d(t^-)$ has rank one!
- *Open problem:* $\Delta M_d(t)$ has rank $k \geq 2$.

References

Free Probability, Random matrices and Asymptotic Freeness

- Voiculescu, D (1991). Limit Laws for random matrices and free products. *Inventiones Mathematica* **104**, 201-220.
- Nica, A. & R. Speicher (2006). *Lectures on the Combinatorics of Free Probability.* London Mathematical Society Lecture Notes Series **335,** Cambridge University Press, Cambridge.
- D. Voiculescu, J.K Dykema & A. Nica (1992). *Free Random Variables.* American Mathematical Society.
- Hiai, F. & D. Petz (2000). *The Semicircle Law, Free Random Variables and Entropy*. Mathematical Surveys and Monographs **77,** American Mathematical Society, Providence.
- G.W. Anderson, A. Guionnet and O- Zeitouni (2010). *An Introduction to Random Matrices.* Cambridge University Press. (Chapter 5).

References for Bercovici-Pata bijection

- H. Bercovici & V. Pata with an appendix by P. Biane (1999). Stable laws and domains of attraction in free probability theory. *Ann. Math.*
- O. E. Barndorff-Nielsen & S. Thorbjørnsen (2004). A connection between free and classical infinite divisibility. *Inf. Dim. Anal. Quantum Probab.*
- O. E. Barndorff-Nielsen and S. Thorbjørnsen (2006). Classical and free infinite divisibility and Lévy processes. *LNM* 1866.
- F. Benaych-Georges, F. (2005). Classical and free i.d. distributions and random matrices. *Ann. Probab.*
- T. Cabanal-Duvillard (2005): A matrix representation of the Bercovici-Pata bijection. *Electron. J. Probab.*
- A. Dominguez & A. Rocha Arteaga (2012). Random matrix models of stochastic integral type for free infinitely divisible distributions. *Period. Math. Hung.*

References for free infinite divisibility

Analytic approach

- H. Bercovici & D. Voiculescu (1993). Free convolution of measures with unbounded supports. *Indiana Univ. Math. J.*
- O. Arizmendi, O.E. Barndorff-Nielsen & VPA (2009). On free and classical type G distributions. *Rev. Braz. Probab. Statist.*
- VPA & Sakuma Noriyoshi (2008). Free generalized gamma convolutions. *Elect. Comm. Probab.*
- O. Arizmendi and VPA (2010). On the non-classical infinite divisibility of power semicircle distributions. *Comm. Stochastic Analysis.*
- VPA & Sakuma Noriyoshi (2012). Free multiplicative convolutions of free multiplicative mixtures of the Wigner distribution. *J. Theoretical Probab.*
- O. Arizmendi, T. Hasebe & N. Sakuma (2012). On free regular infinitely divisible distributions.

References for free multiplicative convolutions

- D. Voiculescu (1987). Multiplication of certain non-commuting random variables. *J. Operator Theory.*
- H. Bercovici & D. Voiculescu (1993). Free convolution of measures with unbounded supports. *Indiana Univ. Math. J.*
- H. Bercovici & J.C. Wang (2008). Limit theorems for free multiplicative convolutions. *Trans. Amer. Math. Soc.*
- N. Raj Rao & R. Speicher (2007). Multiplication of free random variables and the S-transform: The case of vanishing mean. *Elect. Comm. Probab.*
- O. Arizmendi & VPA (2009). *The S-transform of symmetric probability measures with unbounded supports. Proc. Amer. Math. Soc.*

References for Lévy matrix modelling

- O.E. Barndorff-Nielsen & VPA (2008). Matrix subordinators and related Upsilon transformations. *Theory Probab. Appl.*
- O.E. Barndorff-Nielsen & R. Stelzer (2011). The multivariate supOU stochastic volatility model. *Math. Finance.*
- O.E. Barndorff-Nielsen & R. Stelzer (2011): Multivariate supOU processes. *Ann. Appl. Probab.*
- VPA & R. Stelzer (2012). A class of ID multivariate and matrix Gamma distributions and cone-valued GGC.
- C. Pigorsch & R. Stelzer (2009). A multivariate Ornstein-Uhlenbeck type stochastic volatility model.
- R. Stelzer (2010). Multivariate COGARCH(1, 1) processes. *Bernoulli.*