Random Matrices and Infinite Divisibility

In Memory of Constantin Tudor (1950-2011) and Mario Wschebor (1939-2011)

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Plan of the Lecture

1. Asymptotic Spectral Distributions of Random Matrices
   1.1 Gaussian random matrices: Wigner law
   1.2 Wishart random matrices: Marchenko-Pastur law
   1.3 Universality
2. Infinitely Divisible Random Matrices
3. Free independence and free central limit theorem
   3.1 Free Gaussian and free Poisson distributions
   3.2 Motivation to study free independence
4. Classical and Free Infinite Divisibility
   4.1 The classical cumulant transform and classical convolution
   4.2 The free cumulant transform and free convolution
   4.3 BP-Bijection between classical and free infinite divisibility
5. Random Matrices Approach to the Bijection
   5.1 General results
   5.2 Concrete realizations
6. Open problems

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$$f_{\lambda_{n,1},\dots,\lambda_{n,n}}(x_1, \dots, x_n) = k_n \left[\prod_{j=1}^{n} \exp\left(-\frac{1}{4t} x_j^2 \right) \right] \left[\prod_{j<k} |x_j - x_k| \right].$$

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- Nondiagonal RM: eigenvalues are strongly dependent.

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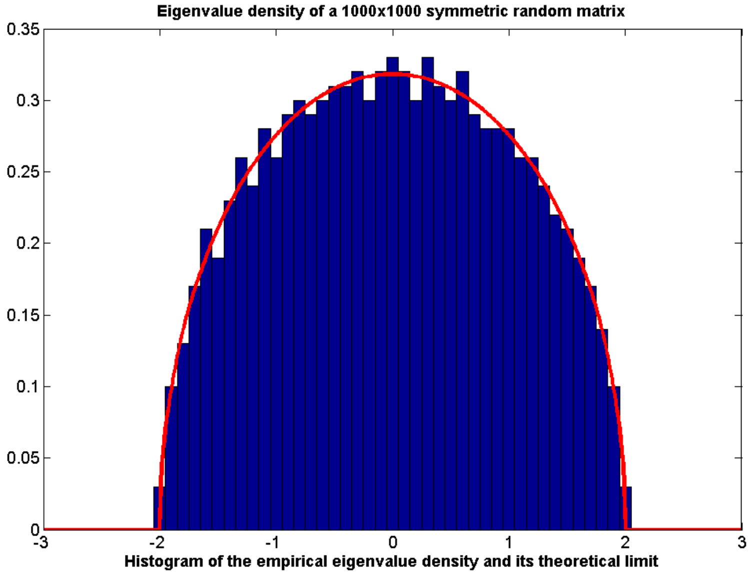
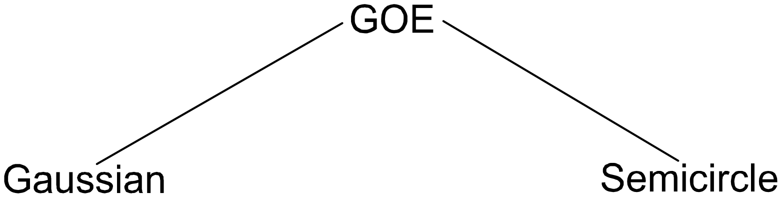
- **Asymptotic spectral distribution (ASD):** $\widehat{F}_n$ converges, as $n \to \infty$, to **semicircle distribution on** $(-2\sqrt{t}, 2\sqrt{t})$

$$\mathrm{w}_t(x) = \frac{1}{2\pi}\sqrt{4t - x^2}, \quad |x| \le 2\sqrt{t}.$$

I. Simulation of Wigner law




GOE
Gaussian
Semicircle

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- **Theorem:** $t = 1$, $\forall\ f \in C_b(\mathbb{R})$ and $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left| \int f(x) \mathrm{d}\widehat{F}_n(x) - \int f(x) \mathrm{w}(x) \mathrm{d}x \right| > \varepsilon \right) = 0.$$

where $\mathrm{w}(x)\mathrm{d}x$ is the *semicircle distribution on* $(-2, 2)$

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- **Wigner random matrices***:*

$$X_n(k,j) = X_n(j,k) = \frac{1}{\sqrt{n}} \begin{cases} Z_{j,k}, & \text{if } j < k \\ Y_j, & \text{if } j = k \end{cases}$$

$\{Z_{j,k}\}_{j\leq k}, \{Y_j\}_{j\geq 1}$ independent sequences of i.i.d. r.v.

$$\mathbb{E}Z_{1,2} = \mathbb{E}Y_1 = 0, \quad \mathbb{E}Z_{1,2}^2 = 1.$$

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- Complex Hermitian case can also be considered.

I. Marchenko-Pastur law (1967)

- $X = X_{p\times n} = (Z_{j,k} : j = 1, .., p, k = 1, ..., n)$ complex i.i.d.
$$\mathbb{E}(Z_{1,1}) = 0, \mathbb{E}(|Z_{1,1}|^2) = 1.$$
- $W_n = XX^*$ is **Wishart matrix** if X has Gaussian entries.
- Covariance matrix $S_n = \frac{1}{n}XX^*$, eigenvalues $0 \le \lambda_{p,1} \le ... \le \lambda_{p,p}$ and ESD
$$\widehat{F}_p(\lambda) = \frac{1}{p}\sum_{j=1}^{p} \mathbf{1}_{\{\lambda_{p,j}\le x\}}.$$
- If $p/n \to c > 0$, $\widehat{F}_n$ converges weakly in probability to Marchenko-Pastur (MP) distribution
$$\mu_c(\mathrm{d}x) = \begin{cases} f_c(x)\mathrm{d}x, & \textit{if} \quad c \ge 1 \\ (1-c)\delta_0(\mathrm{d}x) + f_c(x)\mathrm{d}x, & \textit{if} \quad 0 < c < 1, \end{cases}$$
$$f_c(x) = \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\mathbf{1}_{[a,b]}(x)$$
$$a = (1-\sqrt{c})^2, \quad b = (1+\sqrt{c})^2.$$

I. Marchenko-Pastur law: comments

1. **Applications: Large Dimensional RM (LDRM):**

- Data dimension of same magnitude order than sample size.
- Wireless communication, MIMO channels.

Recent books:

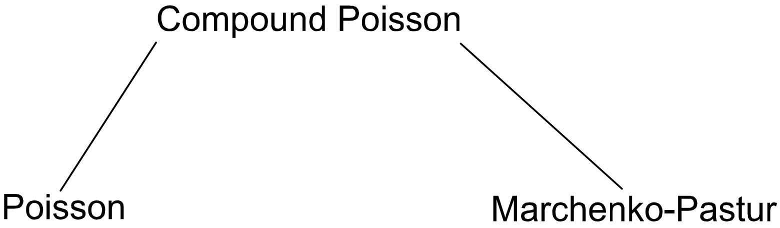
- Bai & Silverstein (2010). Spectral Analysis of LDRM.
- Couillet & Debbah (2011). RM Methods for Wireless Comm.

2. **Today:**

- $(N_t)_{t\geq 0}$ Poisson process of mean n, $(u_j)_{j\geq 1}$ i.i.d. random vectors with uniform distribution on unit sphere of $\mathbb{C}^p$.
- $p \times p$ matrix compound Poisson process

$$X_t = \sum_{j=1}^{N_t} u_j u_j^*.$$

- Distribution of X_t is invariant under unitary conjugations.
- ASD of X_t, when $n/p \to c$, is M-P with parameter c.


Compound Poisson
Poisson
Marchenko-Pastur

II. Infinitely divisible random matrices

- $\mathbb{M}_d$ space of $d \times d$ matrices (real or complex entries).
- A random matrix M in $\mathbb{M}_d$ is **Infinitely Divisible** (ID) iff $\forall$ $n \geq 1$ $\exists_n$ i.i.d. random matrices $M_1, ..., M_n$ in $\mathbb{M}_d$ such that

$$M_1 + ... + M_n \stackrel{\mathcal{L}}{=} M.$$

- Gaussian random matrices GOE and GUE are ID.
- Wishart random matrix is not ID.
- Compound Poisson matrix process $X_t = \sum_{j=1}^{N_t} u_j u_j^*$ is ID.
- **Open problem**: ASD for other Hermitian infinitely divisible random matrices.
- Partial answer today.

II. Why infinitely divisible random matrices?

Applied and theoretical reasons

1. Stochastic modelling (fixed dimension):

- There exists a matrix Lévy process $(M_t)_{t \geq 0}$ such that

$$M_1 \stackrel{\mathcal{L}}{=} M.$$

- Multivariate financial modelling via Lévy and non Gaussian Ornstein-Uhlenbeck matrix processes: Barndorff-Nielsen & Stelzer (09, 11), Pigorsch & Stelzer (09), Stelzer (10).
- ID random matrix models alternative to Wishart random matrix: Barndorff-Nielsen & PA (08), PA & Stelzer (12).

2. **Today**: (asymptotic spectral distribution)

- Random matrices approach to the relation between classical and free infinite divisibility.
- Benaych-Georges (05), Cabanal-Duvillard (05), PA & Sakuma (08), Molina & Rocha-Arteaga (12), Molina, PA, Rocha-Arteaga (in progress).

III. Free Central Limit Theorem

Very roughly speaking

- The concept of free independence is defined for noncommutative random variables: LDRM, operators, etc.
- Distribution is the spectral distribution of an operator or ASD of an ensemble of random matrices.
- Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be a sequence of freely independent random variables with the same distribution with all moments, with mean zero and variance one. Then the distribution of

$$\mathbf{Z}_n = \frac{1}{\sqrt{n}}(\mathbf{X}_1 + \ldots + \mathbf{X}_n)$$

  converges in distribution to the semicircle distribution.
- **Free Gaussian distribution**: the semicircle distribution plays in free probability the role Gaussian distribution does in classical probability.
- **Free Poisson distribution**: The Marchenko-Pastur distribution plays in free probability the role the Poisson distribution does in classical probability.

III. Motivation to study RMT and Free Probability

From the Blog of Terence Tao (Free Probability, 2010):

1. The significance of free probability to random matrix theory lies in the fundamental observation that *random matrices which are independent in the classical sense, also tend to be independent in the free probability sense*, in the large limit.

2. This is only possible because of the highly *non-commutative nature* of these matrices; it is not possible for non-trivial commuting independent random variables to be *freely independent.*

3. Because of this, many tedious computations in random matrix theory, particularly those of an algebraic or enumerative combinatorial nature, can be done more quickly and systematically by using the *framework of free probability.*

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C) *Something similar for the distribution of the product XY (multiplicative convolution).*

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Analytic tools

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- Free cumulant transform

$$C_\mu(z) = zG_\mu^{-1}(z) - 1, \quad z \in \Gamma_\mu$$

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$$\mu_1 \boxplus \mu_2 = \mathcal{L}\left(\mathbf{X}_1 + \mathbf{X}_2\right)$$

- Free *multiplicative* convolution $\mu_1 \boxtimes \mu_2$ can also be defined.

IV. Example: free convolution of Wigners

- *Semicircle distribution* w_{m,σ^2} *on* $(m-2\sigma, m+2\sigma)$ *centered at m*

$$w_{m,\sigma^2}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-(x-m)^2}\,1_{[m-2\sigma,m+2\sigma]}(x).$$

- Cauchy transform*:*

$$G_{\mathrm{w}_{m,\sigma^2}}(z) = \frac{1}{2\sigma^2}\left(z-\sqrt{(z-m)^2-4\sigma^2}\right),$$

- Free cumulant transform:

$$C_{\mathrm{w}_{m,\sigma^2}}(z) = mz + \sigma^2 z.$$

- $\boxplus$*-convolution of Wigner distributions is a Wigner distribution*:

$$\mathrm{w}_{m_1,\sigma_1^2} \boxplus \mathrm{w}_{m_2,\sigma_2^2} = \mathrm{w}_{m_1+m_2,\sigma_1^2+\sigma_2^2}.$$

IV. Example: free convolutions of MPs

- MP distribution of parameter $c > 0$

$$\mathrm{m}_c(\mathrm{d}x) = (1-c)_+\delta_0 + \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\,1_{[a,b]}(x)\mathrm{d}x.$$

- Cauchy transform

$$G_{\mathrm{m}_c}(z) = \frac{1}{2} - \frac{\sqrt{(z-a)(z-b)}}{2z} + \frac{1-c}{2z}$$

- Free cumulant transform

$$C_{\mathrm{m}_c}(z) = \frac{cz}{1-z}.$$

- $\boxplus$-*convolution of M-P distributions is a MP distribution*:

$$\mathrm{m}_{c_1} \boxplus \mathrm{m}_{c_2} = \mathrm{m}_{c_1+c_2}$$

IV. Example: free convolution of Cauchy distributions

- Cauchy distribution of parameter $\theta > 0$

$$\mathrm{c}_\theta(\mathrm{d}x) = \frac{1}{\pi} \frac{\theta}{\theta^2 + x^2} \mathrm{d}x$$

- Cauchy transform

$$G_{\mathrm{c}_\theta}(z) = \frac{1}{z + \theta i}$$

- Free cumulant transform

$$C_{\mathrm{c}_\lambda}(z) = -i\theta z$$

- $\boxplus$-*convolution of Cauchy distributions is a Cauchy distribution*

$$\mathrm{c}_{\theta_1} \boxplus \mathrm{c}_{\theta_2} = \mathrm{c}_{\theta_1+\theta_2}.$$

IV. Classical and free infinite divisibility

- Let μ be a probability distribution on $\mathbb{R}$.
- μ is **infinitely divisible** w.r.t. $\star$ iff $\forall n \geq 1$, $\exists\ \mu_{1/n}$ and

$$\mu = \mu_{1/n} \star \mu_{1/n} \star \cdots \star \mu_{1/n}.$$

- μ is **infinitely divisible** w.r.t. $\boxplus$ iff $\forall n \geq 1$, $\exists\ \mu_{1/n}$ and

$$\mu = \mu_{1/n} \boxplus \mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}.$$

- Notation: $I^{\boxplus}$ (I^{*}) class of all free (classical) ID distributions.
- **Problem:** characterize the class $I^{\boxplus}$ similar to I^{*}.

IV. Classical and free infinite divisibility

Lévy-Khintchine representations

- Classical Lévy-Khintchine representation $\mu \in I^*$

$$c_\mu(s) = \eta s - \frac{1}{2} a s^2 + \int_{\mathbb{R}} \left(e^{isx} - 1 - sx 1_{[-1,1]}(x) \right) \rho(\mathrm{d}x), \ \ s \in \mathbb{R}.$$

- Free Lévy-Khintchine representation $\nu \in I^{\boxplus}$

$$C_\nu(z) = \eta z + a z^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz 1_{[-1,1]}(x) \right) \rho(\mathrm{d}x), \ \ z \in \mathbb{C}^-$$

- In both cases (η, a, ρ) is the unique *Lévy triplet*: $\eta \in \mathbb{R}$, $a \geq 0$, $\rho(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \min(1, x^2) \rho(\mathrm{d}x) < \infty.$$

IV. Relation between classical and free infinite divisibility

Bercovici, Pata (Biane), Ann. Math. (1999)

- Classical Lévy-Khintchine representation $\mu \in I^*$

$$c_\mu(s) = \eta s - \frac{1}{2} a s^2 + \int_{\mathbb{R}} \left(e^{isx} - 1 - sx 1_{[-1,1]}(x) \right) \rho(\mathrm{d}x).$$

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$$C_\nu(z) = \eta z + a z^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz 1_{[-1,1]}(x) \right) \rho(\mathrm{d}x).$$

- *Bercovici-Pata bijection*: $\Lambda : I^* \to I^{\boxplus}$, $\Lambda(\mu) = \nu$

$$I^* \ni \mu \sim (\eta, a, \rho) \leftrightarrow \Lambda(\mu) \sim (\eta, a, \rho)$$

- Λ preserves convolutions (and weak convergence)

$$\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$$

IV. Examples of free infinitely divisible distributions

Images of classical i.d. distributions under Bercovici-Pata bijection

- *Free Gaussian:* For classical Gaussian distribution γ_{m,σ^2},

$$\mathrm{w}_{m,\sigma^2} = \Lambda(\gamma_{m,\sigma^2})$$

is Wigner distribution on $(m - 2\sigma, m + 2\sigma)$ with

$$C_{\mathrm{w}_{\eta,\sigma^2}}(z) = mz + \sigma^2 z^2.$$

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- *Free Poisson:* For classical Poisson distribution p_c, $c > 0$,

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- Bellinschi, Bozejko, Lehner & Speicher (11): γ_{m,σ^2} is free ID.
- **Open problem:** $\gamma_{m,\sigma^2} = \Lambda(?)$.

IV. Examples of free infinitely divisible distributions

Images of classical i.d. distributions under Bercovici-Pata bijection

- *Free Cauchy:* $\Lambda(\mathrm{c}_\lambda) = \mathrm{c}_\lambda$ for the Cauchy distribution

$$\mathrm{c}_\lambda(\mathrm{d}x) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 + x^2}\mathrm{d}x$$

  with free cumulant transform

$$C_{\mathrm{c}}(z) = -i\lambda z.$$

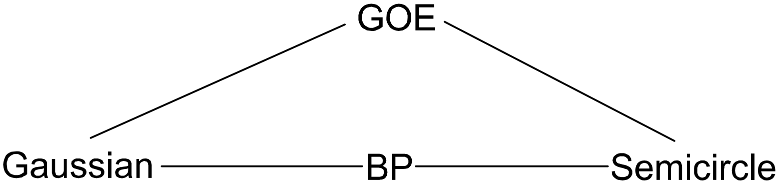
- *Free stable*

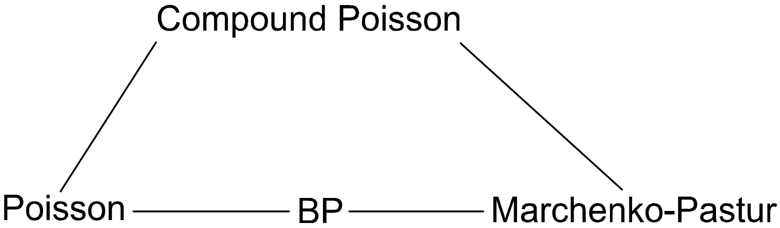
$$S^{\boxplus} = \{\Lambda(\mu); \mu \text{ is classical stable}\}.$$

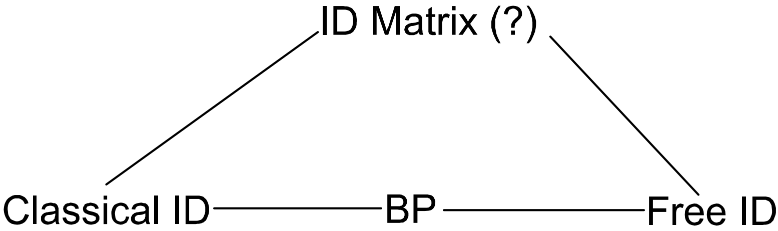
- *Free Generalized Gamma Convolutions (GGC)*

$$GGC^{\boxplus} = \{\Lambda(\mu); \mu \text{ is classical } GGC\}$$

Classical ID —— BP —— Free ID


GOE
Gaussian
BP
Semicircle


Compound Poisson
Poisson
BP
Marchenko-Pastur


ID Matrix (?)
Classical ID
BP
Free ID

V. Random matrix approach to BP bijection

- Benachy-Georges (2005, AP), Cavanal-Duvillard (2005, EJP): For $\mu \in I^*$ there is an ensemble of unitary invariant random matrices $(M_d)_{d \geq 1}$, such that with probability one its ESD converges in distribution to $\Lambda(\mu) \in I^{\boxplus}$.
- M_d is infinitely divisible in the space of matrices $\mathbb{M}_d$.
- The existence of $(M_d)_{d \geq 1}$ is not constructive.
- How are the random matrix $(M_d)_{d \geq 1}$ realized?
- How are the corresponding matrix Lévy processes $\{M_d(t)\}_{t \geq 0}$ realized?
- The jump $\Delta M_d(t) = M_d(t) - M_d(t^-)$ has rank one!
- *Open problem:* $\Delta M_d(t)$ has rank $k \geq 2$.

V. Concrete realization for RM models to BP bijection

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  - If μ is Gaussian, Z_d *GUE* independent of $g \stackrel{\mathcal{L}}{=} N(0,1)$

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- Molina & Rocha-Arteaga (12): If for some 1-dim Lévy process $\{X_t\}_{t\geq 0}$ and for a non random function h
$$\mu = \mathcal{L}\left(\int_0^\infty h(t)\mathrm{d}X_t\right),$$
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- PA-Sakuma (08): $X_t, \mathbf{X}_t$ 1-dim and matrix Gamma processes.

V. Matrix Lévy processes for BP bijection

Molina, PA, Rocha-Arteaga:

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- Simple case: μ $CP(\nu, \psi)$, ν p.m. on $\mathbb{R}$, $\psi \in \mathbb{R}$

$$M_1(t) = t\psi + \sum_{j=1}^{N_t} R_j$$

N_t PP independent of $(R_j)_{j \geq 1}$, independent $\mathcal{L}(R_j) = \nu$.

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- For each $d \geq 2$

$$M_d(t) = \psi t \mathrm{I}_d + \sum_{j=1}^{N_t} R_j u_j u_j^*$$

  $(u_j)_{j\geq 1}$ independent d-vectors uniform on unit sphere of $\mathbb{C}^d$, independent of (N_t) and $(R_j)_{j\geq 1}$.

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- Realization as quadratic covariation $M_d(t) = [X_d, Y_d]_t$.
- $\{X_d(t)\}_{t\geq 0}$, $\{Y_d(t)\}_{t\geq 0}$ are $\mathbb{C}_d$-Lévy processes

$$X_d(t) = \sqrt{|\psi|} B_t + \sum_{j=1}^{N_t} \sqrt{|R_j|} u_j, \quad t \geq 0,$$

$$Y_d(t) = \mathrm{sign}(\psi) \sqrt{|\psi|} B_t + \sum_{j=1}^{N_t} \mathrm{sign}\left(R_j\right) \sqrt{|R_j|} u_j, \quad t \geq 0,$$

$\{B_t\}$ is $\mathbb{C}_d$-Brownian motion independent of (R_j), (u_j), $\{N_t\}$.

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$\{B_t\}$ is $\mathbb{C}_d$-Brownian motion independent of (R_j), (u_j), $\{N_t\}$.

- For general bounded variation μ, "explicit" quadratic covariation realization is possible for $M_d(t)$.

VI. Open problems

Corresponding Dyson process?

Matrix Brownian process:

$$B_n(t) = (b_{ij}(t)), t \geq 0$$

- $b_{ij}, 1 \leq i \leq j \leq n$ 1-dim independent Brownian motions,
- $\forall\ t > 0\ B_n(t)$ is a *GOE* of parameter t.
- $(\lambda_1(t), \cdots, \lambda_n(t))$ eigenvalue process of $B_n(t)$.

Dyson Brownian motion:

- $\exists_n$ n independent 1-dimensional Brownian motions $\widetilde{b}_1, \ldots, \widetilde{b}_n$
- If $\lambda_1(0) < \cdots < \lambda_n(0)$ a.s.

$$\lambda_i(t) = \lambda_i(0) + \widetilde{b}_i(t) + \sum_{j \neq i} \int_0^t \frac{1}{\lambda_j(s) - \lambda_i(s)} \mathrm{d}s, \quad i = 1, \ldots, n.$$

- *What is the Dyson process associated to the matrix Lévy process $M_d(t)$?*

Constantin Tudor

- Matrix valued and corresponding Dyson processes:
  - Functional limit theorems for trace processes in a Dyson Brownian motion, vpa & C. Tudor, COSA (2007).
  - Traces of Laguerre Processes, vpa & C. Tudor, EJP (2009).
  - Eigenvalues of operator Wishart and Laguerre processes.

Mario Wschebor

- Conditioning number of random matrices
  - Remarks on the condition number of a real random square matrix. J.A. Cuesta & M. Wschebor. J. Complexity (2003).
  - Upper and lower bounds for the tails of the distribution of the condition number of a Gaussian matrix. Azaïs, JM & M. Wschebor. SIAM J. Matrix Anal. Appl. (2005).
  - Some work of Mario Wschebor on condition number of random matrices. Jean-Marc Azaïs, Clapem Caracas 2009.

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