Medidas Convexas

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Probabilidad y Estadisticas
CIMAT

Charlas Cortas – 18 Marzo 2021

Outline

Background

- Convexidad y concentracion (PhD Sergey Bobkov)
- Entropia Power desigualdades (PostDoc Mokshay Madiman)
- Aplicaciones en Ingenieria Eléctrica (PostDoc Murti Salapaka)

Proyectos

Mathematically motivated projects:

- Desigualdades: geometria convexa, probabilidad, y teoria de informacion
- f-divergencias - Informacion geometry and Wasserstein geometry, statistical applications
- Convexidad discreta especialmente medidas log-concavas (and heavy tailed generalizations)
- Concentracion y Anti-concentracion (Littlewood-Offord Problems)
- Reordenamiento de la medida, Schur convexidad, Ordenaciones estocÃ¡sticas
- Rényi entopy power inequalities

Proyectos

Application motivated:

- (Overdamped) Langevin Dynamics - sharpenings of the second law of thermodynamics.
- Markov processes
    - Mixing times - Convergence of consensus algorithms.
    - Molecular motors and intracellular transport
- Mixture distributions, f-divergences

Dilation and Rearrangement

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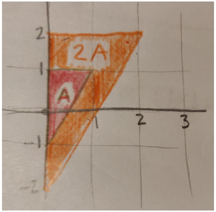

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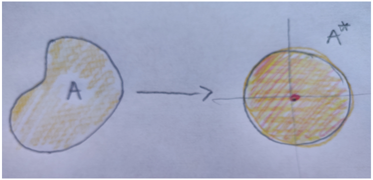
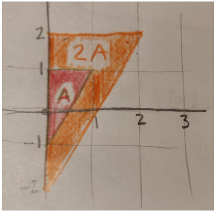




Rearrangement

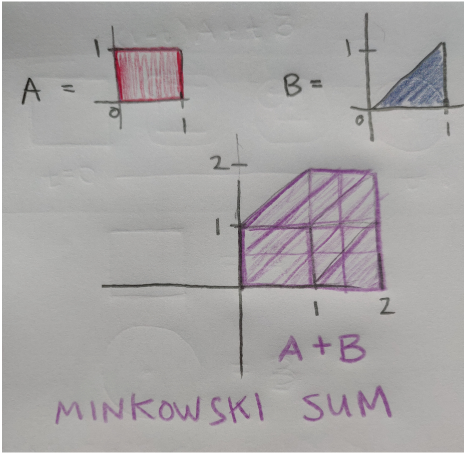
For $A \subseteq \mathbb{R}^n$

$$A^* = \{|x| < s\}, |A^*| = |A|.$$

Minkowski Sum

For $A, B \subseteq \mathbb{R}^n$,

$$A + B = \{x \in \mathbb{R}^n : x = a + b, a \in A, b \in B\}$$



Brunn-Minkowski Inequality

Theorem:

For $A, B \subseteq \mathbb{R}^n$, and $t \in (0,1)$,

$$|(1-t)A + tB|^{\frac{1}{n}} \geq (1-t)|A|^{\frac{1}{n}} + t|B|^{\frac{1}{n}}$$
$$|A+B| \geq |A^* + B^*|$$

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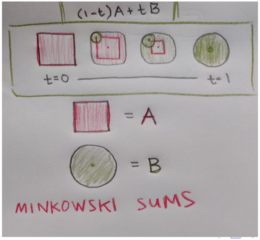
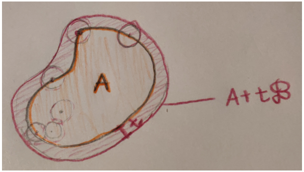
- $A \mapsto |A|^{\frac{1}{n}}$ is concave
- Volume of Minkowski Sum decreases on rearrangement



Isoperimetric Inequality



A general, $\mathcal{B} = \{x \in \mathbb{R}^n : |x| < 1\}$

$$S(A) = \limsup_{t \downarrow 0} \frac{|A + t\mathcal{B}| - |A|}{t}$$

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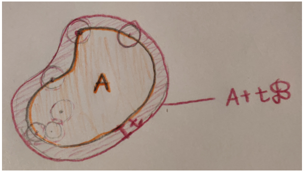

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$$\frac{|A + t\mathcal{B}| - |A|}{t} \geq \frac{|A^* + t\mathcal{B}| - |A^*|}{t}$$

$$S(A) \geq S(A^*). \tag{1}$$

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- Convex Geometry - Foundational Inequality
- Information Theory - Rényi Entropy Power Inequality

Alta-dimension

$n \to \infty$

$$|(1-t)A + tB| \geq \left((1-t)|A|^{\frac{1}{n}} + t|B|^{\frac{1}{n}}\right)^n \approx |A|^{1-t}|B|^t$$

Thus, the dimension-free statement of Brunn-Minkowski is that $|\cdot|$ is log-concave.

Medidads log-concava

Definition

(Radon) espacio medida $(\mathcal{X}, \mu)$, log-cóncava cuando $t \in (0,1)$, y A, B compacto:

$$\mu((1-t)A + tB) \geq \mu^{1-t}(A)\mu^t(B).$$

- $\mathcal{X} = \mathbb{R}^d$, $\frac{d\mu}{dx} = e^{-V(x)}$, V convexa.
- s-concavo cuando.

$$\mu((1-t)A + tB) \geq ((1-t)\mu^s(A) + t\mu^s(B))^{\frac{1}{s}}$$

  - $s < 0$ "heavy tails"
  - $s > 0$, $s \sim d$, $\kappa = 0$ curvatura-dimension desigualdades

Ejemplos

Prekopa-Leindler Desigualdad

$f, g, h : \mathbb{R}^d \to [0, \infty)$, $t \in (0,1)$ que

$$f((1-t)x + ty) \geq g^{1-t}(x) h^t(y)$$

para $x, y \in \mathbb{R}^d$,

$$\int f(z)dz \geq \left(\int g(z)dz \right)^{1-t} \left(\int h(z)dz \right)^t$$

- Para $\phi(x) = e^{-V(x)}$, V convexa.

$$f = \phi \mathbb{1}_{(1-t)A+tB}, \quad g = \phi \mathbb{1}_A, \quad h = \phi \mathbb{1}_B$$

- Medidad con densidad con ϕ es log-concava. Exponencial, Gaussiano, χ^2, Laplace, uniformo sobre un conjunto convexo

Processes

- $(\mathcal{X}, \mu)$ is log-concave iff $T_*\mu$ is log-concave for every continuous linear map $T : \mathcal{X} \to \mathbb{R}^d$
- Gaussian measures are log-concave

Localization

Localization Lemma

Para f semicontinua superior, K compacto y convexa,

$$\mathcal{L}_f(K) = \{\mu : \text{log-concava}, \, supp(\mu) \subseteq K, \, \int f d\mu = 0\}$$

$$\mathcal{E}(\mathcal{L}_f(K)) = \{\nu : \text{log-affine}, \, supp(\mu) \subseteq [a,b], \, \int f d\mu = 0\}$$

- '60 Payne-Weinberger '93 Lovasz-Simonovits '06 Fradelizi-Guedon,
- Stochastic version: '12 Eldan, '19 Lee-Vempala, '20 Chen
- Riemannian version: Klartag '15
- Infinite dimensional: Bobkov-M. '16
- Discrete: Marsiglietti-M '20

Aplicaciones

- Poincare inequality (log-Sobolev), Volume algorithm, geometric inequalities
- Thin Shell estimates, slicing conjecture of Bourgain
- Small ball inequalities, large deviations

Small and Large Deviations

Theorem: Bobkov-M. '16

Given a symmetric, Borel measurable, convex set $B \subsetneq \mathcal{X}$ $\mu(B) \leq \frac{1}{2}$,

$$\mu(\varepsilon B) \leq \varepsilon 2 \log 2$$

Theorem: Bobkov-M. '16

Given a symmetric convex set $B \subseteq \mathcal{X}$ for $r > 1$ and $\mu(rB) < 1$,

$$1 - \mu(rB) \leq (1 - \mu(B))^{(r+1)/2}$$

Small and large deviation

Discussion:

- Inequalities are sharp for Lebesgue measure in Euclidean space.
- Thus cannot improve for in general for Gaussians
- Any improvement for Gaussians, letting variance to ∞ would improve bounds for the Lebesgue measure.

Prekopa-Leindler revisted

Prekopa-Leindler Desigualdad

$f((1-t)x+ty) \geq g^{1-t}(x)h^{t}(y)$ entonces

$$\int f(z)dz \geq \left(\int g(z)dz\right)^{1-t}\left(\int h(z)dz\right)^{t}$$

When $\phi(x) = e^{-|x|^2/2}$,
$\phi((1-t)x+ty) \geq e^{t(1-t)|x-y|^2/2}\phi^{1-t}(x)\phi^{t}(y)$

Applying

$f = \phi \mathbb{1}_{(1-t)A+tB} e^{-t(1-t)d^2(A,B)/2}, g = \phi \mathbb{1}_A, h = \phi \mathbb{1}_B,$

$$\mathbb{P}(Z \in (1-t)A+tB) \geq \kappa_t \mathbb{P}^{1-t}(Z \in A)\mathbb{P}^{t}(Z \in B),$$

$\kappa_t = e^{-t(1-t)d^2(A,B)/2}$

Question 1

Strongly log-concave measures

μ is strongly log-concave when

$$\mu((1-t)A+tB) \geq \kappa_t \mu^{1-t}(A)\mu^t(B)$$

holds with $\kappa_t = e^{-t(1-t)d^2(A,B)/2}$.

Strongly log-concave

For f upper semicontinuous $\mathcal{L}_f(1,K)$ be the strongly log-concave measures supported on a convex, compact set K, such that $\int f d\mu = 0$, What is $\mathcal{E}(\mathcal{L}_f(1,K))$?

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- However, to a probabilist $co(\mathcal{L}_f(K))$ has a natural interpretation. Convex combinations of log-concave measures, are mixture distributions.
- For example $co(\mathcal{L}(1, \mathbb{R}^d))$ would contain all Gaussian mixtures

Question 2

Question:

How much (if any) of known results derived from localization, functional inequalities, small and large deviations etc, can be derived for mixtures?

$$\mathcal{E}(co(\mathcal{L}_f(K))) = \mathcal{E}(\mathcal{L}_f(K)))$$

The End

Thank you!

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