

A BERRY-ESSEEN TYPE THEOREM FOR FINITE FREE CONVOLUTION

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ABSTRACT. We prove that the rate of convergence for the central limit theorem in finite free convolution is of order $n^{-1/2}$.

1. INTRODUCTION

In recent years, the convolution of polynomials, first studied by Walsh [8], was revisited by Marcus, Spielman and Srivastava [6], in order to exhibit bounds for the eigenvalues of expected characteristic polynomials of certain d -regular graphs, in their aim to construct bipartite Ramanujan graphs of all sizes [7]. The authors refer to this convolution as finite free additive convolution because of its relation to free convolution, see [1, 5, 7].

In [5], Marcus showed that the Central Limit Theorem for this convolution is given by the polynomial $d^{-d/2}H_d(\sqrt{d}x)$, where $H_d(x)$ is an Hermite polynomial and may be written as

$$H_d(x) = d! \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \frac{(-1)^i}{i!(d-2i)!} \frac{x^{d-2i}}{2^i}.$$

In this note we prove that the rate of convergence in the Central Limit Theorem is of order $n^{-1/2}$ as for the free case [2, 3]. However, we use the Lévy distance instead of Kolmogorov distance, the reason being that we are dealing with measures supported in d atoms with size $1/d$ and thus we cannot expect better.

Thus, for two polynomials of degree d , p and q , let us define the distance between them to be $L(p, q) := d_L(\mu_p, \mu_q)$, where d_L is the Lévy distance and the measures μ_p and μ_q are the distributions of p and q , respectively.

In this language we can state our contribution as follows.

Theorem 1.1. *Let $d \in \mathbb{N}$ and let p be a real polynomial with $m_1 = 0$ and $m_2 = 1$. Then, there exists an **absolute constant** C_d , only depending on d , such that for all $n > 0$,*

$$L\left(D_{1/\sqrt{n}}(p^{\boxplus n}), d^{-d/2}H_d(\sqrt{d}x)\right) < \frac{C_d}{\sqrt{n}}.$$

The main tool to prove the above rate of convergence are the cumulants for finite free convolution, as we defined in [1]. These cumulants give a combinatorial



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approach to investigate this convolution and its relation to free probability. In particular we showed that finite free cumulants approach free cumulants and thus as finite free convolution approaches free convolution in the limit. Using these cumulants we were able to show that some properties of free convolution are valid already in the finite free case. The above theorem is another instance of the fact that many properties in free probability already appear in the finite level.

Apart from this introduction this note consists of two sections. Section 2 gives the preliminaries for the theory of finite free probability and in Section 3 we give the proof of the main theorem, Theorem 1.1.

2. PRELIMINARIES

We give very basic preliminaries on finite free convolution we refer to [1, 5] for details.

2.1. Finite Free Convolution. For two polynomials, $p(x) = \sum_{i=0}^d x^{d-i}(-1)^i a_i^p$ and $q(x) = \sum_{i=0}^d x^{d-i}(-1)^i a_i^q$, the finite free additive convolution of p and q is given by

$$p(x) \boxplus_d q(x) = \sum_{k=0}^d x^{d-k}(-1)^k \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-i-j)!} a_i^p a_j^q.$$

The finite R -transform of a polynomial is defined by

$$(2.1) \quad \mathcal{R}_p^d(s) \equiv -\frac{1}{d} \frac{\partial}{\partial s} \ln \left(\sum_{i=0}^d \frac{(-d)^i a_i^p}{(d)_i} s^i \right) \quad \text{mod } [s^d],$$

when p is the monic polynomial $p(x) = \sum_{i=0}^d x^{d-i}(-1)^i a_i^p$.

We consider the truncated R -transform given by the sum of the first d terms in the series expansion of \mathcal{R}_p^d , which will have the cumulants as coefficients.

Definition 2.1 ([1]). Let p be a monic polynomial of degree d , and suppose the $\mathcal{R}_p^d(s)$ satisfies

$$\mathcal{R}_p^d(s) \equiv \sum_{j=0}^{d-1} \kappa_{j+1}^p s^j \quad \text{mod } [s^d].$$

- (1) We call the sum of the first d terms in the series expansion of \mathcal{R}^d the *truncated R -transform* and denote by $\tilde{\mathcal{R}}_p^d(s)$, i.e.

$$\tilde{\mathcal{R}}_p^d(s) := \sum_{j=0}^{d-1} \kappa_{j+1}^p s^j.$$

- (2) The numbers $\kappa_1^p, \kappa_2^p, \dots, \kappa_d^p$ will be called the finite free cumulants.

We want to use the combinatorial framework in terms of moments for these cumulants. Hence, for a polynomial p with roots $\lambda_1, \dots, \lambda_n$ we define the moments of p , by the formula $m_n(p) = \frac{1}{d} \sum_{i=1}^d \lambda_i^n$.

These finite free cumulants satisfy the following properties which are the analog of the properties in the the axiomatization of cumulants by Lehner [4], in non-commutative probability.

- (1) **Polynomial in the first n moments:** $k_n(p)$ is a polynomial in the first n moments of p with leading term

$$\frac{d^n}{(d)_n} m_n(p).$$

- (2) **Homogeneity:** for all monic polynomial $p(x)$ we have

$$k_n(D_\lambda p(x)) = \lambda^n k_n(p(x)),$$

where $D_\lambda p(x) := \lambda^d p(x/\lambda)$ is the *dilation by $\lambda \neq 0$ of the polynomial p (of degree d)* and $D_0(p(x)) = x^d$.

- (3) **Additivity:** for all monic polynomials p and q , we have

$$k_n(p \boxplus_d q) = k_n(p) + k_n(q).$$

2.2. Moment-cumulant formula. Moment-cumulant formulas involve summing over partitions on the set $[n]$. Let us introduce this definition and some notation.

Definition 2.2. We call $\pi = \{V_1, \dots, V_r\}$ a **partition** of the set $[n] := \{1, 2, \dots, n\}$ if V_i ($1 \leq i \leq r$) are pairwise disjoint, non-void subsets of $[n]$, such that $V_1 \cup V_2 \dots \cup V_r = \{1, 2, \dots, n\}$. We call V_1, V_2, \dots, V_r the **blocks** of π . The number of blocks of π is denoted by $|\pi|$. We will denote the set of partitions of $[n]$ by $\mathcal{P}(n)$.

The set $\mathcal{P}(n)$ can be equipped with the partial order \leq of reverse refinement ($\pi \leq \sigma$ if and only if every block of π is completely contained in a block of σ). With this order the minimum is given by the partition with n blocks, $0_n = \{\{1\}, \{2\}, \dots, \{n\}\}$, and the maximum is given by the partition with 1 block, $1_n = \{\{1, 2, \dots, n\}\}$.

Thus one can consider the incidence algebra of $\mathcal{P}(n)$. For two partitions σ, ρ in the set of partitions $\mathcal{P}(n)$ the Möbius function is given by

$$\mu(\sigma, \rho) = (-1)^{|\sigma| - |\rho|} (2!)^{r_3} (3!)^{r_4} \dots ((n-1)!)^{r_n},$$

where r_i is the number of blocks of ρ that contain exactly i blocks of σ . In particular, for $\sigma = 0_n$ we have

$$\mu(0_n, \rho) = (-1)^{n - |\rho|} (2!)^{t_3} (3!)^{t_4} \dots ((n-1)!)^{t_n},$$

where t_i is the number of blocks of ρ of size i .

Given a sequence of complex numbers $f = \{f_n\}_{n \in \mathbb{N}}$ we may extend f to partitions in a multiplicative way by the formula

$$f_\pi = f_{|V_1|} f_{|V_2|} \dots f_{|V_n|},$$

where V_1, \dots, V_n are the blocks of π . In this note we will frequently use the multiplicative extensions of the Pochhammer sequence $(d)_n = (d)(d-1) \dots (d-n+1)$ and the factorial sequence $n!$, whose extensions will be denoted by $(d)_\pi$ and $N!_\pi$, respectively.

In [1], we gave formulas that relate the moments and coefficients of a polynomial and its finite free cumulants. First, we have a formula that writes coefficients in terms of cumulants.

Proposition 2.3 (Coefficient-cumulant formula). *Let $p(x) = \sum_{i=0}^d x^{d-i} (-1)^i a_i^p$ be a polynomial of degree d and let $(\kappa_n^p)_{n=1}^d$ be its finite free cumulants. The following*

formulas hold.

$$(2.2) \quad a_n^p = \frac{(d)_n}{d^n n!} \sum_{\pi \in \mathcal{P}(n)} d^{|\pi|} \mu(0_n, \pi) \kappa_\pi^p, \quad n \in \mathbb{N}.$$

We also have a moment-cumulant formula for finite free cumulants:

Proposition 2.4. *Let p be a monic polynomial of degree d and let $(m_n)_{n=1}^\infty$ and $(\kappa_n)_{n=1}^d$, be the moments and cumulants of p , respectively. Then*

$$\kappa_n = \frac{(-d)^{n-1}}{(n-1)!} \sum_{\sigma \in \mathcal{P}(n)} d^{|\sigma|} \mu(0, \sigma) m_\sigma \sum_{\pi \geq \sigma} \frac{\mu(\pi, 1_n)}{(d)_\pi},$$

for $n = 1, \dots, d$ and

$$m_n = \frac{(-1)^n}{d^{n+1} (n-1)!} \sum_{\sigma \in \mathcal{P}(n)} d^{|\sigma|} \mu(0, \sigma) \kappa_\sigma \sum_{\pi \geq \sigma} -\mu(\pi, 1_n) (d)_\pi,$$

for $n \in \mathbb{N}$.

Remark 2.5. The explicit moment-cumulant formulas of the first three finite cumulants are

$$\begin{aligned} \kappa_1 &= m_1, & \kappa_2 &= \frac{d}{d-1} (m_2 - m_1^2), \\ \kappa_3 &= \frac{d^2}{(d-1)(d-2)} (2m_1^3 - 3m_1 m_2 + m_3), \end{aligned}$$

and the explicit moment-cumulant formulas of the first three finite moments are

$$\begin{aligned} m_1 &= \kappa_1, & m_2 &= \frac{d-1}{d} \kappa_2 + \kappa_1^2, \\ m_3 &= \frac{(d-1)(d-2)}{d^2} \kappa_3 + \frac{3(d-1)}{d} \kappa_2 \kappa_1 + \kappa_1^3. \end{aligned}$$

2.3. Convergence of polynomials and Levy distance. In this setting of [1, 5] convergence of polynomials is pointwise convergence of the coefficients. We prefer to consider the weak convergence of the induced measures since it is common with the free probability setting. Thus, for a polynomial p , with roots $\lambda_1, \lambda_2, \dots, \lambda_n$, we define its distribution μ_p as the uniform measure on the roots of p , $\mu_p = 1/d \sum_i \delta_{\lambda_i}$.

To quantify this convergence we use the Lévy distance

$$d_L(\mu, \nu) := \inf\{\epsilon > 0 \mid F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon \text{ for all } x \in \mathbb{R}\},$$

where F and G are the cumulative distribution functions of μ and ν respectively.

3. PROOF OF THEOREM 1.1

Before going in to the proof of the main theorem we prove a couple of lemmas about the support and cumulants of polynomials with mean 0 and variance 1.

Lemma 3.1. *Let p be a real polynomial of degree d with $k_1 = 0$ and $k_2 = 1$. Then the support of p is contained in $(-\sqrt{d-1}, \sqrt{d-1})$.*

Proof. If $k_1 = 0$ and $k_2 = 1$ then

$$1 = k_2 = \frac{d}{d-1} m_2 = \frac{1}{d-1} \sum_{i=1}^d \lambda_i^2.$$

This means that $\lambda_i^2 < d - 1$ (strict because there is at least another non-zero λ) and thus $|\lambda_i| < \sqrt{d - 1}$ for all $i = 1, \dots, d$. \square

Lemma 3.2. *Let p be a real polynomial of degree d with $k_1 = 0$ and $k_2 = 1$. Then there exists a constant c_d , depending exclusively of d , such that $\max_{2 \leq s \leq d} |\kappa_s(p)| < c_d$.*

Proof. By the previous lemma $m_n \leq (d - 1)^n$ and then $\max_{2 \leq s \leq d} |\kappa_s(p)| < (d - 1)^d$, so we can bound uniformly k_n by the moment-cumulant formulas. \square

Now we are able to prove the main theorem which we state again for convenience of the reader.

Proposition 3.3. *Let p be a real polynomial with $k_1 = 0$ and $k_2 = 1$. Then, there exists C_d such that for all $n > 0$*

$$L\left(D_{1/\sqrt{n}}(p^{\boxplus d n}), d^{-d/2} H_d(\sqrt{dx})\right) < \frac{C_d}{\sqrt{n}}.$$

Proof. Let us denote $h = d^{-d/2} H_d(\sqrt{dx})$, $p_n = p^{\boxplus d n}$ and $q_n = D_{1/\sqrt{n}}(p_n)$. By the coefficient-cumulant formula, we know that

$$\begin{aligned} a_m^{q_n} &= \frac{(d)_m}{d^m m!} \sum_{\pi \in \mathcal{P}(m)} d^{|\pi|} \mu(0_m, \pi) \kappa_\pi^{q_n} \\ &= a_m^h + \frac{(d)_m}{d^m m!} \sum_{\pi \in \mathcal{P}(m) \setminus \mathcal{P}_{12}(m)} d^{|\pi|} \mu(0_m, \pi) \kappa_\pi^{q_n}, \end{aligned}$$

where $\mathcal{P}_{12}(m)$ is the set of partitions $\pi = (V_1, \dots, V_r) \in \mathcal{P}(m)$ such that $|V_i| \leq 2$ for all $i \in \{1, \dots, r\}$ (i.e., $\pi = (V_1, \dots, V_r) \in \mathcal{P}(m) \setminus \mathcal{P}_{12}(m)$, if $|V_i| > 2$ for some $i \in \{1, \dots, r\}$).

Recall that

$$|\kappa_s(q_n)| = |\kappa_s(D_{1/\sqrt{n}}(p_n))| = \frac{n}{n^{s/2}} |\kappa_s(p)| \leq n^{1-s/2} c,$$

for $s = 3, \dots, d$, where $c = \max_{2 \leq s \leq d} |\kappa_s(p)|$. Thus, for any $3 \leq m \leq d$ and $\pi = (V_1, \dots, V_r) \in \mathcal{P}(m) \setminus \mathcal{P}_{12}(m)$ we get

$$(3.1) \quad |\kappa_\pi| \leq c^r \cdot n^r \cdot n^{-\frac{|V_1| + \dots + |V_r|}{2}} = c^r n^{r - \frac{m}{2}} \leq c^r n^{\frac{m}{3} - \frac{m}{2}} = c^r n^{-\frac{m}{6}} \leq c^d n^{-\frac{1}{2}}.$$

Then,

$$|a_m^{q_n} - a_m^h| \leq \frac{c^d K_1(d)}{\sqrt{n}}, \quad \forall m \in \{1, \dots, d\}$$

where

$$K_1(d) = \max_{1 \leq m \leq d} \frac{(d)_m}{d^m m!} \sum_{\pi \in \mathcal{P}(m) \setminus \mathcal{P}_{12}(m)} d^{|\pi|} |\mu(0_m, \pi)|.$$

Let's denote z_1, z_2, \dots, z_d the d distinct roots of h and $\delta = \frac{1}{2} \min_{1 \leq i < j \leq d} |z_i - z_j|$. For $0 < \varepsilon < \delta$ we define $B_i = \{z \in \mathbb{C} : |z - z_i| \leq \varepsilon\}$ and $\bar{D}_i = \partial B_i = \{z \in \mathbb{C} : |z - z_i| = \varepsilon\}$. For a fixed root i , using the previous bound we can see that for any $z \in \bar{D}_i$ we have that

$$|q_n(z) - h(z)| \leq \left| \sum_{m=0}^d z^{d-m} (-1)^m (a_m^{q_n} - a_m^h) \right| \leq \sum_{m=1}^d |z|^{d-m} |a_m^{q_n} - a_m^h|$$

$$\leq \frac{c^d K_1(d)}{\sqrt{n}} \sum_{m=1}^d (|z_i| + |\varepsilon|)^{d-m} \leq \frac{c^d K_1(d) K_2(d)}{\sqrt{n}}$$

where

$$K_2(d) = \max_{1 \leq i \leq d} \sum_{m=1}^d (|z_i| + |\varepsilon|)^{d-m}.$$

On the other hand, if $z \in D_i$, we know that

$$|h(z)| = |(z - z_0) \cdots (z - z_{n-1})| = |z - z_1| \cdots |z - z_n| \geq |z - z_i| \delta^{d-1} = \varepsilon \delta^{d-1}.$$

Finally, if we take

$$n > \frac{C(p)K(d)}{\varepsilon^2},$$

where $K(d) = \frac{K_1^2(d)K_2^2(d)}{\delta^{2d-2}}$ and $C(p) = c^{2d}$. Since $c^{2d}K(d)$ does not depend on i , we get that for any $i = 1, \dots, n$, if $z \in D_i$, then

$$|q_n(z) - h(z)| \leq \frac{c^d K_1(d) K_2(d)}{\sqrt{n}} < \varepsilon \delta^{d-1} \leq |h(z)| \leq |h(z)| + |q_n(z)|.$$

Thus, Rouché's theorem implies that q_n and h have the same number of roots (counting multiplicity) in B_i for $i = 1, \dots, n$. By the definition of the B_i we know that they are pairwise disjoint and each one contains exactly one of the d roots of h . Thus, each B_i contains exactly one of the d roots of q_n implying that distance between the roots of q_n and h , (and therefore de Lévy distance) is less than ε . \square

Observe that Theorem 1.1 directly gives a bound for T in the next proposition.

Proposition 3.4 ([1]). *Let $p \neq x^d$ be a real polynomial, then there exists $T > 0$ such that for all $t > T$ the polynomial $p^{\boxplus at}$ has d different real roots.*

Finally, we show that one cannot do better than $O(\sqrt{n})$ as long as $m_3(p) \neq 0$.

Proposition 3.5. *Let p be a real polynomial with $\kappa_1 = 0$ and $\kappa_2 = 1$ and $|m_3| = \lambda \neq 0$. Then, there exists C_2 depending on λ such that for all $n > 0$*

$$L\left(D_{1/\sqrt{n}}(p^{\boxplus an}), d^{-d/2} H_d(\sqrt{dx})\right) \geq \frac{\lambda}{3d\sqrt{n}}.$$

Proof. We use again the notation $h = d^{-d/2} H_d(\sqrt{dx})$, $p_n = p^{\boxplus an}$ and $q_n = D_{1/\sqrt{n}}(p_n)$ and suppose that $L(q_n, h) = L\left(p^{\boxplus an}, d^{-d/2} H_d(\sqrt{dx})\right) < \frac{\lambda}{3d\sqrt{n}}$. Since $\kappa_1(q_n) = 0$, from the moment cumulant formulas we have $m_3(q_n) = \frac{(d-1)(d-2)}{d^2} \kappa_3(q_n)$ and then

$$|m_3(q_n)| = \frac{(d-1)(d-2)}{d^2} |\kappa_3(q_n)| = \frac{(d-1)(d-2)}{d^2} \frac{n}{n^{3/2}} |\kappa_3(p)| = \frac{|m_3(p)|}{\sqrt{n}} = \frac{\lambda}{\sqrt{n}}.$$

Since $m_3(h) = 0$, we can compute

$$m_3(q_n) = \frac{1}{d} \sum_{i=1}^d \lambda_i^3(q_n) = \frac{1}{d} \sum_{i=1}^d \lambda_i^3(q_n) - \frac{1}{d} \sum_{i=1}^d \lambda_i^3(h),$$

and thus

$$\begin{aligned}
|m_3(q_n)| &\leq \frac{1}{d} \sum_i |\lambda_i^3(q_n) - \lambda_i^3(h)| \\
&= \frac{1}{d} \sum_{i=1}^d |\lambda_i(q_n) - \lambda_i(h)| |\lambda^2(q_n) + \lambda_i(q_n)\lambda_i(h) + \lambda^2(h)| \\
&< \frac{1}{d} \sum_{i=1}^d \left(\frac{\lambda}{3d\sqrt{n}} \right) (d + d + d) = \frac{\lambda}{\sqrt{n}}.
\end{aligned}$$

This is a contradiction since the inequality is strict. \square

Remark 3.6. A specific example with $k_3 \neq 0$ is the finite free Poisson distribution which has cumulants $\kappa_n = \lambda$ for all n . If λd is a positive integer we obtain a valid polynomial. This is a modification of a Laguerre polynomial, thus we obtain a precise estimate for the difference between the roots of Laguerre and Hermite polynomials.

Remark 3.7. A closer look at (3.1) shows that if $m_3(p) = 0$ then the convergence rate is of order $1/n$. Indeed, $m_3(p) = 0$ implies $\kappa_3(q_n) = \kappa_3(p) = 0$. So in (3.1) we only need to consider partitions with $|V_i| \geq 4$. In this case, for any $4 \leq m \leq d$ we have

$$|\kappa_\pi| \leq c^r n^{r - \frac{m}{2}} \leq c^r n^{\frac{m}{4} - \frac{m}{2}} = c^r n^{-\frac{m}{4}} \leq c^d n^{-1}.$$

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