A differential pursuit/evasion game of capture between an omnidirectional agent and a differential drive robot, and their winning roles.

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This paper addresses a differential pursuit/evasion game. The players are an omnidirectional agent (OA) and a differential drive robot (DDR). They move in an obstacle free environment, the DDR is faster than the OA but it can only change its motion direction up to a bounded rate. First, we analyze the scenario in which, the OA has as objective to capture a differential drive robot (DDR) in minimum time and the DDR wants to retard the capture as long as possible. We present the time optimal motion primitives of the players to achieve their goals. Later, combining the results obtained in this paper and the ones in Ruiz et al. (2013), we allow the agents to change the roles, namely, the DDR is allowed to play as the pursuer and the OA is allowed to play as the evader. This later analysis allows one to establish which is the winner role for each agent, based only on the initial position of the players and their maximum speed.

Keywords: Differential Games; Optimal Control; Nonholonomic Constraints

1. Introduction

In previous work Jacobo et al. (2015); Ruiz et al. (2013), the authors have addressed, the problem of capturing an omnidirectional agent (OA) using a differential drive robot (DDR) in an obstacle-free environment. At the beginning of this game, the OA is at a distance $L > l$ (the capture distance) from the DDR pursuer. The goal of the OA evader was to keep the DDR pursuer farther than this capture distance for as long as possible. The goal of the DDR pursuer was to capture the OA as soon as possible. In Ruiz et al. (2013), the authors have proposed a partition of the playing space into mutually disjoint regions where time optimal strategies of the players are well established. The time-optimal strategies obtained in Ruiz et al. (2013) are in Nash equilibrium and the proposed strategies are in open loop. Later, in Jacobo et al. (2015), the authors provided a time-optimal control synthesis (state feedback optimal policy) for a DDR pursuer chasing the OA. This later result was achieved by estimating the state of the evader based on images using the 1D trifocal tensor.

In this paper, we addressed first the symmetric problem, in which the agents exchange roles. Now the OA is the pursuer and has as objective to capture the DDR in minimum time and the DDR is the evader and wants to retard the capture as long as possible. Second, combining the results obtained in this paper and the ones in Ruiz et al. (2013), we allow the agents to change the roles, namely, the DDR is allowed to play as the pursuer and the OA is allowed to play as the evader. This later analysis allows one to establish which is the winner role for each agent, based only on the initial position of the players and their maximum speed. The time-optimal motion policies for

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both agents playing each one of the two possible roles are also provided.

The main differences between the work presented in reference Ruiz et al. (2013) and this work are the following:

(1) The DDR and the OA switch their roles compared to Isaacs's (Isaacs (1965)) and Ruiz's (Ruiz et al. (2013)) work. That is, the player that has higher speed and turning constraint (DDR) becomes the evader, and the player that has lower speed but can have an abrupt change on its travel direction (OA) becomes the pursuer.
(2) In this work, the winner role for each agent is found. That is, before the game starts we can establish whether the DDR shall win or loss playing as pursuer (w.r.t. an OA playing as evader) or as evader (w.r.t an OA playing as pursuer). An analogous result is established for the OA. This is achieved based only on the maximum speed of each player and their initial positions over the reduced space. To solve the decision problem of obtaining the winning role for each agent, the results obtained in this paper are combined with the ones obtained in Ruiz et al. (2013). Note that the solution to the decision problem of finding the winner role for each agent has not been presented in Ruiz et al. (2013), since in that work, the case of a DDR playing as evader and an OA playing as pursuer was not considered.

2. Related work

The problem addressed in this paper is related to pursuit/evasion games. A pursuit-evasion game can be defined in several ways. One variant considers one or more pursuers, which are given the task of finding an evader in an environment with obstacles Guibas et al. (1999); Hollinger et al. (2009); Isler et al. (2005); Tovar & LaValle (2008); Vidal et al. (2002). A recent survey of this kind of problem is presented in Chung et al. (2011). Other variant consists in maintaining visibility of a moving evader also in an environment with obstacles Bhattacharya & Hutchinson (2010); Bandyopadhyay et al. (2007); Jung & Sukhatme (2002); Murrieta et al. (2007); LaValle et al. (1997); O'Kane (2008).

A third variant of pursuit-evasion problem consists in giving to the pursuer the goal to capture the evader Isaacs (1965), that is, move to a contact configuration, or closer than a given distance. The work presented in this paper corresponds to this third variant. There is a great deal of work related to this variant, particularly in the dynamics and control area Başar & Olsder (1999); Isaacs (1965), where optimal control is a recurrent used approach address this kind of problems, most of the work takes place in the free space (without obstacles). A classic example of this kind of problems is the homicidal Chauffeur problem Isaacs (1965); Merz (1971). Other related problems are the lady in the lake Başar & Olsder (1999) and the lion and the man Flynn (1974); Karnad et al . (2009).

In the homicidal chauffeur problem Isaacs (1965); Merz (1971), a faster pursuer (w.r.t. the evader) has as its goal to get closer than a given constant distance (the capture condition) from a slower but more agile evader. The evader aims to avoid the capture condition. The pursuer is a nonholonomic system with a minimal turning radius, while the evader is a holonomic (omnidirectional) agent. The game takes place in the Euclidean plane without obstacles. In the lady in the lake problem Başar & Olsder (1999), there is a circular lake where a lady is swimming with a maximum speed v_l, and there is a man that is in the side of the lake and runs along the shore with a maximum speed v_m; the man cannot enter the lake and the lady wants to leave the lake. The man runs with a larger speed than the one of the lady in the lake ($v_l < v_m$). The man needs to capture the lady as soon as she reaches the shore, since on land she runs faster than him. In the lion and the man problem Flynn (1974); Karnad et al . (2009), the players move in a circular arena, both players have the same motion capabilities, the lion wants to capture the man and the man wants to avoid the capture.

The problem of a mobile intruder jamming the communication network in a vehicular formation is addressed in Bhattacharya & Başar (2011). A multi-stage two-player game in which one player represents an attacker with superior dynamic capabilities, and the other player represents a team consisting of a mobile, high-value target and N protective agents is studied in Fuchs & Khargonekar (2011). The problem of steering a team of agents from their initial positions to a predefined end-configuration while avoiding collisions was addressed as a differential game in Mylvaganam et al. (2014). In Bhattacharya et al. (2014), the authors addressed the vision-based target tracking problem between a mobile observer and a target in the presence of a circular obstacle.

3. Problem formulation

A Differential Drive Robot (DDR) and an Omnidirectional Agent (OA) move on a plane without obstacles. The OA tries to capture the DDR while the DDR tries to avoid it. The game is over when the distance between the DDR and the OA is smaller than a critical value l_c. The players have maximum bounded speeds $V_R^{\max}$ (DDR) and $V_A^{\max}$ (OA), respectively. The DDR is faster than the OA, i.e., $V_R^{\max} > V_A^{\max}$, but it can only change its motion direction at a rate that is inversely proportional to its translational speed Balkcom & Mason (2002). We consider here a purely kinematic problem, and neglect any effects due to dynamic constraints (e.g., acceleration bounds). *The OA wants to minimize the time it takes to capture the DDR, while the DDR wants to maximize it. The objective is to find the optimal strategies that are in Nash Equilibrium and may be used by the players to achieve their goals.*

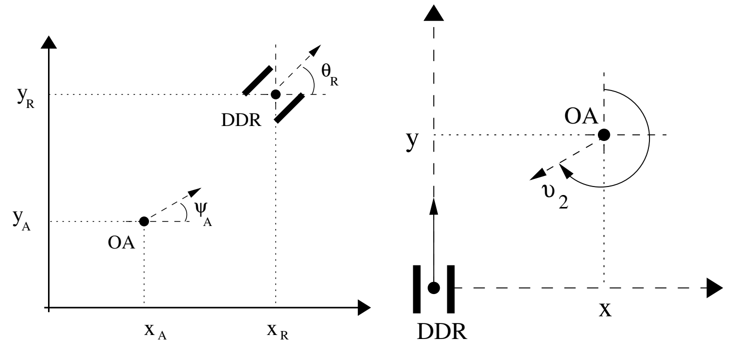
3.1 *Model*

3.1.1 Realistic space

The game can be described in a global coordinate system (refer to Fig. 1(a)). (x_R, y_R, θ_R) represents the pose of the DDR and (x_A, y_A) the position of the OA, both at time t. The state of the system can be expressed as $(x_R, y_R, \theta_R, x_A, y_A) \in \mathbb{R}^2 \times S^1 \times \mathbb{R}^2$. The kinematics of the system are described by the following equations of motion

$$\begin{aligned} &\dot{x}_R = \left(\frac{u_1+u_2}{2}\right)\cos\theta_R,\ \ \dot{y}_R = \left(\frac{u_1+u_2}{2}\right)\sin\theta_R,\ \ \dot{\theta}_R = \left(\frac{u_2-u_1}{2b}\right) \\ &\dot{x}_A = v_A\cos\psi_A,\ \ \dot{y}_A = v_A\sin\psi_A \end{aligned} \tag{1}$$

where $u_1, u_2 \in [-V_R^{\max}/r, V_R^{\max}/r]$ are the controls of the DDR, and they correspond to the angular velocities of its wheels. r is the radius of the wheels, in this problem we assume $r = 1$. u_1 and u_2 are the angular velocities of the left and right wheels respectively. If both controls have the same magnitude and are either positive or negative, respectively, the robot moves forward or backward in a straight line. Using a suitable choice of units Balkcom & Mason (2002), the translational speed is equal to $V_R = \frac{1}{2}(u_1 + u_2)$. If u_1 and u_2 have the same magnitude but opposite signs the robot rotates in place either clockwise or counter-clockwise Balkcom & Mason (2002). The OA's controls are its speed $v_A \in [0, V_A^{\max}]$ and its motion direction $\psi_A \in [0, 2\pi)$. We introduce some useful definitions for the rest of the paper, $\rho_v = V_A^{\max}/V_R^{\max}$ is the ratio between the maximum translational speed of the OA and the DDR, and $\rho_d = b/l_c$ is the ratio of the distance between the center of the robot and the wheel location b and the capture distance l_c. We assume that $l_c \geq b$, i.e., the capture distance is bigger than the robot's radius.



(a) Realistic space (b) Reduced space

Figure 1. System models

3.1.2 *Reduced space*

To simplify the analysis we formulate the problem in a coordinate system that is fixed to the DDR's body (see Fig. 1(b)). The state of the system is expressed as $\mathbf{x}(t) = (x, y) \in \mathbb{R}^2$. All the orientations in this coordinate system are measured with respect to the positive y-axis, in particular, the OA's motion direction v_2. Using the coordinate transformation given by

$$\begin{aligned} x &= (x_A - x_R)\sin\theta_R - (y_A - y_R)\cos\theta_R \\ y &= (x_A - x_R)\cos\theta_R + (y_A - y_R)\sin\theta_R \\ v_2 &= \theta_R - \psi_A \end{aligned} \tag{2}$$

the kinematics in the DDR-fixed coordinate system are described by

$$\begin{aligned} \dot{x} &= \left(\frac{u_2 - u_1}{2b}\right) y + v_1 \sin v_2 \\ \dot{y} &= -\left(\frac{u_2 - u_1}{2b}\right) x - \left(\frac{u_1 + u_2}{2}\right) + v_1 \cos v_2 \end{aligned} \tag{3}$$

where $u_1, u_2 \in [-V_R^{\max}, V_R^{\max}]$ are again the controls of the DDR. For the agent, $v_1 \in [0, V_A^{\max}]$ is the control associated to its speed and $v_2 \in [0, 2\pi)$ the control associated to its motion direction. This set of equations can be expressed in the form $\dot{\mathbf{x}} = f(\mathbf{x}, u, v)$, where $u = (u_1, u_2) \in \widehat{U} = [-V_R^{\max}, V_R^{\max}] \times [-V_R^{\max}, V_R^{\max}]$, and $v = (v_1, v_2) \in \widehat{V} = [0, V_A^{\max}] \times [0, 2\pi)$.

4. Time Optimal Motion Strategies

4.1 *Hamiltonian*

Following the Isaacs' approach Başar & Olsder (1999); Isaacs (1965); Ruiz et al. (2013), we construct the Hamiltonian of our system. Recalling that

$$H(\mathbf{x}, \lambda, \mathbf{u}, \mathbf{v}) = \lambda^T \cdot f(\mathbf{x}, u, v) + L \tag{4}$$

for our problem we have that

$$H(\mathbf{x},\lambda,u_1,u_2,v_1,v_2) = \lambda_x\left(\frac{u_2-u_1}{2b}\right)y + \lambda_x v_1\sin v_2 - \lambda_y\left(\frac{u_2-u_1}{2b}\right)x - \lambda_y\left(\frac{u_1+u_2}{2}\right) \\ + \lambda_y v_1\cos v_2 + 1 \tag{5}$$

4.2 *Optimal Controls*

From Başar & Olsder (1999); Isaacs (1965), we know that along the optimal trajectories,

$$\begin{aligned} &\min_v \max_u H(\mathbf{x},\lambda,\mathbf{u},\mathbf{v}) = 0 \\ &u^* = \arg\max_u H(\mathbf{x},\lambda,\mathbf{u},\mathbf{v}) \\ &v^* = \arg\min_v H(\mathbf{x},\lambda,\mathbf{u},\mathbf{v}) \end{aligned} \tag{6}$$

where u^* and v^* denote the optimal controls for the players. Recall that the OA's goal is minimize the capture time while the DDR's goal is to maximize it. From Eq. (5), Eq. (6), and the separability of the Hamiltonian Başar & Olsder (1999); Isaacs (1965); Ruiz et al. (2013), we obtain the expressions for the players' optimal controls. The controls for the DDR are given by

$$\begin{aligned} u_1^* &= \operatorname{sgn}\left(\frac{-y\lambda_x}{b} + \frac{x\lambda_y}{b} - \lambda_y\right)V_R^{\max} \\ u_2^* &= \operatorname{sgn}\left(\frac{y\lambda_x}{b} - \frac{x\lambda_y}{b} - \lambda_y\right)V_R^{\max} \end{aligned} \tag{7}$$

The controls for the OA are given by

$$v_1^* = V_A^{\max},\ \sin v_2^* = -\frac{\lambda_x}{\gamma},\ \cos v_2^* = -\frac{\lambda_y}{\gamma} \tag{8}$$

where $\gamma = \sqrt{\lambda_x^2 + \lambda y^2}$.

4.3 *Adjoint Equation*

The adjoint equation is found by taking the partial derivative of the Hamiltonian with respect to the state variables. If t_f is the time of termination of the game, we define the retro-time as $\tau = t_f - t$. In this work, we denote the *retro-time derivative* of a variable x as $\mathring{x}$. The adjoint equation in its retro-time form is

$$\mathring{\lambda} = \frac{\partial}{\partial x}H(\mathbf{x},\lambda,u_1^*,u_2^*,v_1^*,v_2^*) \tag{9}$$

In our problem, we have that

$$\mathring{\lambda}_x = -\left(\frac{u_2^*-u_1^*}{2b}\right)\lambda_y,\ \mathring{\lambda}_y = \left(\frac{u_2^*-u_1^*}{2b}\right)\lambda_x \tag{10}$$

4.4 *Terminal conditions*

We need to compute the portions of the playing space where the OA guarantees termination (captures the DDR) regardless of the choice of controls made by the DDR. For this game, the OA captures the DDR when the distance between both players is smaller than the capture distance l_c despite any opposition of the DDR. In the reduced space, ζ is a circle of radius l_c centered at the origin, hence it can be parametrized by the angle s (see Fig. 2), which is the angle between the OA's position and the DDR's heading *at the end of the game* (recall that all orientations in the reduced space are measured with respect to the positive y-axis). At the end of the game

$$x = l_c \sin s, \; y = l_c \cos s \tag{11}$$

From Başar & Olsder (1999); Isaacs (1965), the portion of the terminal surface where the OA guarantees termination is known as the *usable part*, and for this game it is represented by

$$\text{UP} = \left\{ \mathbf{x} \in \zeta : \min_{v} \max_{u} \mathbf{n} \cdot f(\mathbf{x}, u, v) < 0 \right\} \tag{12}$$

where $\mathbf{n}$ is the normal vector to ζ from point $\mathbf{x}$ on ζ and extending into the playing space. From Eq. (11), the outward normal $\mathbf{n}$ to ζ is given by

$$\mathbf{n} = [\sin s \;\; \cos s] \tag{13}$$

Substituting Eq. (13) and Eq. (3) into Eq. (12) we obtain

$$\text{UP} = \left\{ \max_{u_1, u_2} \left[V_A^{\max} - \left(\frac{u_1 + u_2}{2} \right) \cos s < 0 \right] \right\} \tag{14}$$

We have two cases, (1) $\cos s > 0$ and (2) $\cos s < 0$. Note that u_1 and u_2 must be equal and saturated to maximize the inequality in (14), therefore the DDR moves following a straight line when it is captured by the OA. If $\cos s > 0$ then $\left(\frac{u_1+u_2}{2}\right) = -V_R^{\max}$, the DDR is moving backward. If $\cos s < 0$ then $\left(\frac{u_1+u_2}{2}\right) = V_R^{\max}$, the DDR is moving forward. From the definition of the cosine function we have that the DDR's controls take the following values at the end of the game

Interval	Controls
$s \in [\arccos(\rho_v), \frac{\pi}{2})$	$u_1 = -V_R^{\max}$, $u_2 = -V_R^{\max}$
$s \in (\frac{\pi}{2}, \pi - \arccos(\rho_v)]$	$u_1 = V_R^{\max}$, $u_2 = V_R^{\max}$
$s \in [\pi + \arccos(\rho_v), \frac{3\pi}{2})$	$u_1 = V_R^{\max}$, $u_2 = V_R^{\max}$
$s \in (\frac{3\pi}{2}, 2\pi - \arccos(\rho_v)]$	$u_1 = -V_R^{\max}$, $u_2 = -V_R^{\max}$

We must note that when $s = \frac{\pi}{2}$ or $s = \frac{3\pi}{2}$ the motion strategy for the DDR is undefined. As it will be show later in the paper, all configurations having any of those orientations are part of a dispersal surface (DS).

4.5 *Computing the trajectories*

In this section, we will obtain the equilibrium strategies for the players. All the analysis in the following paragraphs is based on the methodology presented in Isaacs (1965). In Ruiz et al. (2013), it has been applied to a similar problem to the one presented in this paper. For more details, we refer the reader to Başar & Olsder (1999); Isaacs (1965).

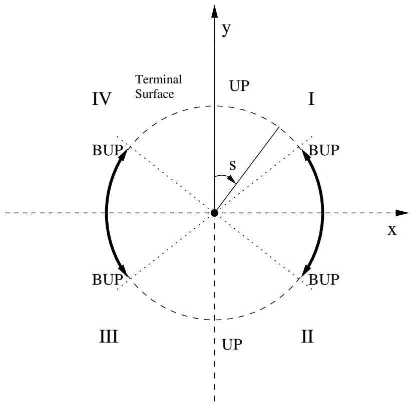


Figure 2. Representation of the terminal surface, usable part and its boundary in the reduced space.

4.5.1 Primary solution

From the UP, we have the values of x and y at the terminal condition. We also need to establish the values of λ_x and λ_y on the UP of ζ. From Eq. (11) we have that

$$\frac{dx}{ds} = l_c \cos s, \; \frac{dy}{ds} = -l_c \sin s \tag{15}$$

Since $\lambda(\mathbf{x}) = 0$ on the UP of ζ it follows that

$$\lambda_s = \frac{d\lambda}{ds} = \frac{\partial \lambda}{\partial x}\frac{dx}{ds} + \frac{\partial \lambda}{\partial y}\frac{dy}{ds} = 0 \tag{16}$$

Substituting Eq. (15) into Eq. (16)

$$\lambda_x \cos s = \lambda_y \sin s \tag{17}$$

From Eq. (17) we have that on the UP

$$\lambda_x = \eta \sin s, \; \lambda_y = \eta \cos s \tag{18}$$

where η is a constant value.

From the analysis in subsection 4.4, we know that at the end of the game the DDR follows a translation. Therefore Eq. (10) takes the form

$$\mathring{\lambda}_x = 0, \; \mathring{\lambda}_y = 0 \tag{19}$$

One can directly verify that

$$\lambda_x = \eta \sin s, \; \lambda_y = \eta \cos s \tag{20}$$

satisfies Eq. (19). This solution for the adjoint equation will be valid at the UP and as long as the DDR controls do not change, which corresponds to a DDR motion following a straight line in the realistic space. Later, we compute the retro-time instant when the DDR switches controls.

Lemma 1: *At the end of game, the time-optimal motion primitives in the realistic space are, for the DDR, moving following a straight line and, for the OA, moving also following a straight line.*

Proof. Since λ_x and λ_y have constant values, the OA's motion direction direction $v_2^* = s$ in the reduced space is also constant. We know that the DDR's is moving following a straight line at the end of the game (subsection 4.4) thus its motion direction θ_R is constant. From Eq. (2), it is straightforward to see that ψ_A, the OA's motion direction in the realistic space is constant. □

From Eq. (3), the retro-time version of the motion equations in the reduced space are

$$\begin{aligned} \mathring{x} &= -\left(\frac{u_2 - u_1}{2b}\right) y - v_1 \sin v_2 \\ \mathring{y} &= \left(\frac{u_2 - u_1}{2b}\right) x + \left(\frac{u_1 + u_2}{2}\right) - v_1 \cos v_2 \end{aligned} \tag{21}$$

Substituting Eq. (20) into the controls expressions in Eq. (7) and Eq. (8), and the resulting expressions into Eq. (21) we obtain

$$\mathring{x} = V_A^{\max} \sin s, \;\; \mathring{y} = V_A^{\max} \cos s + V_R^{\max} \tag{22}$$

when the DDR is translating forward, and

$$\mathring{x} = V_A^{\max} \sin s, \;\; \mathring{y} = V_A^{\max} \cos s - V_R^{\max} \tag{23}$$

when the DDR is translating backward. Integrating Eq. (22) and Eq. (23) with the initial conditions $x = l_c \sin s$ and $y = l_c \cos s$ leads to

$$\begin{aligned} x(\tau) &= \tau V_A^{\max} \sin s + l_c \sin s \\ y(\tau) &= \tau(V_A^{\max} \cos s \pm V_R^{\max}) + l_c \cos s \end{aligned} \tag{24}$$

the sign $+$ is taken if the DDR moves forward in the realistic space when it is captured by the OA and the sign $-$ if it moves backward. Note that Eq. (24) gives the motion of the system in the reduced space, in order to find the corresponding motion in the realistic space we need to apply the transformation given by Eq. (2). Recall that if $s \in (\frac{\pi}{2}, \pi - \arccos(\rho_v)] \cup [\pi + \arccos(\rho_v), \frac{3\pi}{2}])$ then the DDR is moving forward at the end of the game otherwise it is moving backward.

4.5.2 Transition surface

The solutions in Eq. (20), and Eq. (24) are valid as long as the DDR does not switch controls. The place where a control variable abruptly changes in value, is known as a *transition surface*. In this game, after a retro-time interval the DDR switches controls and it starts rotating in place in the realistic space.

Lemma 2: *The DDR switches controls and it starts a rotation in place in the realistic space at* $\tau_s = \left|\frac{b \cos s}{V_p^{\max} \sin s}\right|$. *If* $s \in [0, \pi]$, u_1^* *switches first, otherwise* u_2^* *does.*

Proof. We can compute the time τ_s when the DDR switches controls, substituting Eq. (20) and Eq. (24) into Eq. (7), and verifying which one of the resulting expressions is the first in changing signs. Doing that we find that for $s \in [\arccos(\rho_v), \frac{\pi}{2}]$, u_1^* switches first from $-V_R^{\max}$ to $V_R^{\max}$ (i.e., the DDR's controls after the switch are $u_1^* = V_R^{\max}$ and $u_2^* = -V_R^{\max}$) and it does it at

$$\tau_s = \frac{b \cos s}{V_R^{\max} \sin s} \tag{25}$$

The other cases can be proved using an analogous reasoning. □

We found that for all cases

$$\tau_s = \left| \frac{b \cos s}{V_R^{\max} \sin s} \right| \tag{26}$$

The following table indicates the control switching first based on the initial orientation of the retro-time trajectory

Interval	Switching control
$s \in [\arccos(\rho_v), \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi - \arccos(\rho_v)]$	u_1^*
$s \in [\pi + \arccos(\rho_v), \frac{3\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi - \arccos(\rho_v)]$	u_2^*

At τ_s, we need to start a new integration of the retro-time version of the adjoint equation (10) and the motion equations (21). This integration takes as initial conditions the values of λ_x, λ_y, x, and y at τ_s. We will denote those values as $\lambda_x(\tau_s)$, $\lambda_y(\tau_s)$, $x(\tau_s)$ and $y(\tau_s)$.

Computing the retro-time derivative of Eq. (10), we obtain two ordinary linear differential equations of second order with constant coefficients

$$\frac{d^2\lambda_x}{d\tau^2} = -\left(\frac{u_2^* - u_1^*}{2b}\right)^2 \lambda_x, \quad \frac{d^2\lambda_y}{d\tau^2} = -\left(\frac{u_2^* - u_1^*}{2b}\right)^2 \lambda_y \tag{27}$$

Solving these equations with $\lambda_x(\tau_s)$ and $\lambda_y(\tau_s)$ as initial conditions we obtain the following expressions

$$\begin{aligned} \lambda_x &= \eta \sin \left[s - \left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s) \right] \\ \lambda_y &= \eta \cos \left[s - \left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s) \right] \end{aligned} \tag{28}$$

for $\tau \geq \tau_s$.

Lemma 3: *For $\tau \geq \tau_s$, the time-optimal motion primitives are, for the DDR, rotating in place, and for the OA, continuing following a straight line.*

Proof. Substituting Eq. (28) into Eq. (8) we have that

$$\begin{aligned} \sin v_2^* &= \sin \left[s - \left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s) \right] \\ \cos v_2^* &= \cos \left[s - \left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s) \right] \end{aligned} \tag{29}$$

therefore

$$v_2^* = s - \left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s) \tag{30}$$

From Lemma 2, we have that for $\tau \geq \tau_s$ the DDR is rotating in place, its motion direction is given by

$$\theta_R' = \theta_R^s - \left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s) \tag{31}$$

where θ_R^s is the initial motion direction of the DDR in the realistic space. Substituting Eq. (30) and Eq. (31) into Eq. (2) we obtain $\psi_A = \theta_R^s - s$, the OA's motion direction in the realistic space. Note that ψ_A is a constant value thus the OA is following a straight line in the realistic space. □

To compute the corresponding retro-time trajectories in the reduced space we substitute Eq. (28) into Eq. (8), and the resulting expressions into Eq. (21)

$$\begin{aligned} \mathring{x} &= -\left(\frac{u_2^* - u_1^*}{2b}\right) y + V_A^{\max} \sin\left[s - \left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s)\right] \\ \mathring{y} &= \left(\frac{u_2^* - u_1^*}{2b}\right) x + V_A^{\max} \cos\left[s - \left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s)\right] \end{aligned} \tag{32}$$

Computing the retro-time derivative of Eq. (32) and solving the resulting expressions with the initial conditions $x(\tau_s)$ and $y(\tau_s)$, we obtain

$$\begin{aligned} x(\tau) =& - y(\tau_s) \sin\left[\left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s)\right] + x(\tau_s) \cos\left[\left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s)\right] \\ &+ (\tau - \tau_s) V_A^{\max} \sin\left[s - \left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s)\right] \\ y(\tau) =& x(\tau_s) \sin\left[\left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s)\right] + y(\tau_s) \cos\left[\left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s)\right] \\ &+ (\tau - \tau_s) V_A^{\max} \cos\left[s - \left(\frac{u_2^* - u_1^*}{2b}\right)(\tau - \tau_s)\right] \end{aligned} \tag{33}$$

for $\tau \geq \tau_s$.

From Eq. (33), we have that for $s \in [\arccos(\rho_v), \frac{\pi}{2}) \cup [\pi + \arccos(\rho_v), \frac{3\pi}{2})$,

$$\begin{aligned} x(\tau) =& - y(\tau_s) \sin\left[\left(\frac{-V_R^{\max}}{b}\right)(\tau - \tau_s)\right] + x(\tau_s) \cos\left[\left(\frac{-V_R^{\max}}{b}\right)(\tau - \tau_s)\right] \\ &+ (\tau - \tau_s) V_A^{\max} \sin\left[s - \left(\frac{-V_R^{\max}}{b}\right)(\tau - \tau_s)\right] \\ y(\tau) =& x(\tau_s) \sin\left[\left(\frac{-V_R^{\max}}{b}\right)(\tau - \tau_s)\right] + y(\tau_s) \cos\left[\left(\frac{-V_R^{\max}}{b}\right)(\tau - \tau_s)\right] \\ &+ (\tau - \tau_s) V_A^{\max} \cos\left[s - \left(\frac{-V_R^{\max}}{b}\right)(\tau - \tau_s)\right] \end{aligned} \tag{34}$$

and for $s \in (\frac{\pi}{2}, \pi - \arccos(\rho_v)] \cup (\frac{3\pi}{2}, 2\pi - \arccos(\rho_v)]$ we obtain that

$$\begin{aligned} x(\tau) =& - y(\tau_s) \sin\left[\left(\frac{V_R^{\max}}{b}\right)(\tau - \tau_s)\right] + x(\tau_s) \cos\left[\left(\frac{V_R^{\max}}{b}\right)(\tau - \tau_s)\right] \\ &+ (\tau - \tau_s) V_A^{\max} \sin\left[s - \left(\frac{V_R^{\max}}{b}\right)(\tau - \tau_s)\right] \\ y(\tau) =& x(\tau_s) \sin\left[\left(\frac{V_R^{\max}}{b}\right)(\tau - \tau_s)\right] + y(\tau_s) \cos\left[\left(\frac{V_R^{\max}}{b}\right)(\tau - \tau_s)\right] \\ &+ (\tau - \tau_s) V_A^{\max} \cos\left[s - \left(\frac{V_R^{\max}}{b}\right)(\tau - \tau_s)\right] \end{aligned} \tag{35}$$

Note that Eq. (34) and Eq. (35) give the motion of the system in the reduced space, as it was previously mentioned, to find the corresponding motion in the realistic space we need to apply the transformation given by Eq. (2).

5. Decision problem

We are interested in the conditions that make capture possible for the OA or escape for the DDR. From Isaacs (1965), we have that the *barrier* separates the set of starting positions in those that result in capture and those that result in escape for the DDR. From starting points on the barrier, optimal behavior leads to a contact of the terminal surface without crossing it. The techniques we have used in the calculation of the optimal strategies and their corresponding trajectories, are also applied in the construction of the barrier, which can be interpreted as a *neutral trajectory of the system.* The answer to the capture-escape question relies on whether or not the barrier divides the playing space into two parts.

5.1 *Construction of the barrier*

As we previously mentioned, the usable part (UP) is the portion of the terminal surface where the OA can guarantee termination regardless of the choice of controls of the DDR, its boundary (BUP) is characterized by

$$\text{BUP} = \left\{ \mathbf{x} \in \zeta : \min_{v} \max_{u} \mathbf{n} \cdot f(\mathbf{x}, u, v) = 0 \right\} \tag{36}$$

where $\mathbf{n}$ is the normal vector to ζ from point $\mathbf{x}$ on ζ and extending into the playing space.

For such points, when each player applies its optimal strategies $\mathbf{x}$ moves tangentially to ζ. As the BUP separates the points on ζ where immediate capture occurs from those where it does not, it is used as initial condition for the barrier. The barrier is constructed integrating the adjoint equation (10) and the equations of motion (21), starting at the BUP.

5.2 *Solution to the decision problem*

In the following paragraphs, we show that for this game the barrier always divides the playing space.

First, we prove a useful result that states that the DDR can always reach any desired heading orientation with respect to the segment joining the the OA's position and the DDR's center. A similar result was presented in Jacobo et al. (2015) for a feedback-based policy.

Lemma 4: *If the DDR rotates in place at maximal rotational speed then it can always reach any desired heading orientation with respect to the segment joining the OA's position and the DDR's center regardless of the OA's motion strategy.*

Proof. The Cartesian coordinates of the OA in the reduced space are given by

$$x = r \sin \phi \quad y = r \cos \phi \tag{37}$$

Recall that $r > b$ otherwise the OA will be located inside the robot. Assuming that the DDR does not move, the OA's velocities are given by

$$\dot{x} = v_1 \sin v_2 \quad \dot{y} = v_1 \cos v_2 \tag{38}$$

Using Eqs. (37) and (38), we obtain an expression for $\tan\phi$ which we differentiate to obtain an expression for $\dot{\phi}$. The resulting expression represents the rate of change of ϕ when the DDR does not move

$$\begin{aligned} \frac{d}{dt}\tan\phi &= \frac{x}{y} \\ \dot{\phi}\sec^2\phi &= \frac{y\dot{x} - x\dot{y}}{y^2} \\ \dot{\phi} &= \frac{\cos\phi\dot{x} - \sin\phi\dot{y}}{r} \end{aligned} \tag{39}$$

Substituting Eq. (38) into Eq. (39), we obtain

$$\dot{\phi} = \frac{v_1\sin(v_2 - \phi)}{r} \tag{40}$$

The OA's controls that maximize Eq. (40) are $v_1 = V_A^{\max}$ and $v_2 = \phi + \frac{\pi}{2}$ thus

$$\dot{\phi}^{\max} = \frac{V_A^{\max}}{r} \tag{41}$$

From the DDR control model, one has that if the DDR does not translate and it rotates at maximal angular velocity in the direction that makes ϕ decrease as much as possible, then

$$\dot{\theta}_R^{\max} = \frac{V_R^{\max}}{b} \tag{42}$$

Subtracting the DDR's maximal angular velocity from $\dot{\phi}^{\max}$, we have that $\dot{\phi}$ in the reduced space is given by

$$\dot{\phi} \leq \dot{\phi}^{\max} - \dot{\theta}_R^{\max} \tag{43}$$

Thus, one will have that $\dot{\phi} < 0$ if $\dot{\phi}^{\max} - \dot{\theta}_R^{\max} < 0$, i.e.,

$$\frac{V_R^{\max}}{b} > \frac{V_A^{\max}}{r} \tag{44}$$

but this inequality always holds, since by the definition of this pursuit-evasion game $V_R^{\max} > V_A^{\max}$ and $r > b$. □

Next we prove that the barrier's trajectories in quadrants I and II (see Fig. 2) share the same x coordinate and have opposite y coordinates. A similar result can be obtained for the barrier's trajectories in quadrants III and IV.

Lemma 5: *Let $x_1(\tau)$ and $y_1(\tau)$ be the coordinates of the barrier starting with angle $S = \arctan(\rho_v)$ at the BUP. Similarly, let $x_2(\tau)$ and $y_2(\tau)$ be the coordinates of the barrier starting with angle $\pi - S$ at the BUP. We have that $x_1(\tau) = x_2(\tau)$ and $y_1(\tau) = -y_2(\tau)$ when the DDR moves following a straight line in the realistic space.*

Proof. From Eq. (24), we have that

$$\begin{aligned} x_1(\tau) &= \tau V_A^{\max}\sin S + l_c\sin S \\ y_1(\tau) &= \tau(V_A^{\max}\cos S - V_R^{\max}) + l_c\cos S \end{aligned} \tag{45}$$

and

$$
\begin{aligned}
x_2(\tau) &= \tau V_A^{\max} \sin(\pi - S) + l_c \sin(\pi - S) \\
y_2(\tau) &= \tau (V_A^{\max} \cos(\pi - S) + V_R^{\max}) + l_c \cos(\pi - S)
\end{aligned} \tag{46}
$$

From basic trigonometry, we have that $\sin S = \sin(\pi - S)$ and $\cos S = -\cos(\pi - S)$. Substituting the previous identities into Eq. (45) and Eq. (46) we found that $x_1(\tau) = x_2(\tau)$ and $y_1(\tau) = -y_2(\tau)$ when the DDR moves following a straight line in the realistic space. □

Lemma 6: *Let $x_1'(\tau)$ and $y_1'(\tau)$ be the coordinates of the barrier after the DDR switches controls and it was following the trajectory given by $x_1(\tau)$ and $y_1(\tau)$. Similarly, $x_2'(\tau)$ and $y_2'(\tau)$ be the coordinates of the barrier after the DDR switches controls and it was previously following $x_2(\tau)$ and $y_2(\tau)$. We have that $x_1'(\tau) = x_2'(\tau)$ and $y_1'(\tau) = -y_2'(\tau)$ when the DDR rotates in place in the realistic space.*

Proof. Let $\alpha_1 = -\frac{V_R^{\max}}{b}(\tau - \tau_s)$ and $\tau_n = (\tau - \tau_s)$, substituting into Eq. (35), we have that $x_1'(\tau)$ and $y_1'(\tau)$ are given by

$$
\begin{aligned}
x_1'(\tau) =& - y_1(\tau_s) \sin\alpha_1 + x_1(\tau_s)\cos\alpha_1 + \tau_n V_A^{\max} \sin[S - \alpha_1] \\
y_1'(\tau) =& x_1(\tau_s) \sin\alpha_1 + y_1(\tau_s)\cos\alpha_1 + \tau_n V_A^{\max} \cos[S - \alpha_1]
\end{aligned} \tag{47}
$$

Doing some algebra we have

$$
\begin{aligned}
x_1'(\tau) =& \sin\alpha_1 \, (-y_1(\tau_s) - \tau_n V_A^{\max}\cos S) + \cos\alpha_1 \, (x_1(\tau_s) + \tau_n V_A^{\max} \sin S) \\
y_1'(\tau) =& \sin\alpha_1 \, (x_1(\tau_s) + \tau_n V_A^{\max}\sin S) + \cos\alpha_1 \, (y_1(\tau_s) + \tau_n V_A^{\max} \cos S)
\end{aligned} \tag{48}
$$

Using the trigonometric identity

$$
a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \arctan\frac{b}{a}) \tag{49}
$$

we can rewrite Eq. (48) as

$$
\begin{aligned}
x_1'(\tau) &= -B_1 \sin\alpha_1 + A_1 \cos\alpha_1 = \sqrt{A_1^2 + B_1^2} \sin\left(\alpha_1 + \arctan\frac{A_1}{-B_1}\right) \\
y_1'(\tau) &= A_1 \sin\alpha_1 + B_1 \cos\alpha_1 = \sqrt{A_1^2 + B_1^2} \sin\left(\alpha_1 + \arctan\frac{B_1}{A_1}\right)
\end{aligned} \tag{50}
$$

where $A_1 = x_1(\tau_s) + \tau_n V_A^{\max} \sin S$ and $B_1 = y_1(\tau_s) + \tau_n V_A^{\max} \cos S$. Using a similar approach and recalling that $\sin x = \sin(\pi - x)$ and $\cos x = -\cos(\pi - x)$, from Eq. (34), we have that $x_2'(\tau)$ and $y_2'(\tau)$ are given by

$$
\begin{aligned}
x_2'(\tau) =& - y_2(\tau_s) \sin(-\alpha_1) + x_1(\tau_s)\cos(-\alpha_1) + \tau_n V_A^{\max} \sin[S + (-\alpha_1)] \\
y_2'(\tau) =& x_2(\tau_s) \sin(-\alpha_1) + y_2(\tau_s)\cos(-\alpha_1) - \tau_n V_A^{\max} \cos[S + (-\alpha_1)]
\end{aligned} \tag{51}
$$

Doing some algebra we have

$$
\begin{aligned}
x_2'(\tau) =& \sin(-\alpha_1) \, (-y_2(\tau_s) + \tau_n V_A^{\max}\cos S) + \cos(-\alpha_1) \, (x_2(\tau_s) + \tau_n V_A^{\max} \sin S) \\
y_2'(\tau) =& \sin(-\alpha_1) \, (x_2(\tau_s) + \tau_n V_A^{\max}\sin S) + \cos(-\alpha_1) \, (y_2(\tau_s) - \tau_n V_A^{\max} \cos S)
\end{aligned} \tag{52}
$$

From Lemma 5, we know that $x_1(\tau_s) = x_2(\tau_s)$ and $y_1(\tau_s) = -y_2(\tau_s)$ thus

$$\begin{aligned} x_2'(\tau) &= \sin(-\alpha_1)\left(y_1(\tau_s) + \tau_n V_A^{\max} \cos S\right) + \cos(-\alpha_1)\left(x_1(\tau_s) + \tau_n V_A^{\max} \sin S\right) \\ y_2'(\tau) &= \sin(-\alpha_1)\left(x_1(\tau_s) + \tau_n V_A^{\max} \sin S\right) + \cos(-\alpha_1)\left(-y_1(\tau_s) - \tau_n V_A^{\max} \cos S\right) \end{aligned} \quad (53)$$

Substituting $A_1 = x_1(\tau_s) + \tau_n V_A^{\max} \sin S$ and $B_1 = y_1(\tau_s) + \tau_n V_A^{\max} \cos S$ into Eq. (53) we obtain

$$\begin{aligned} x_2'(\tau) &= -B_1 \sin\alpha_1 + A_1 \cos\alpha_1 = \sqrt{A_1^2 + B_1^2} \sin\left(\alpha_1 + \arctan\frac{A_1}{-B_1}\right) \\ y_2'(\tau) &= -A_1 \sin\alpha_1 - B_1 \cos\alpha_1 = -\sqrt{A_1^2 + B_1^2} \sin\left(\alpha_1 + \arctan\frac{B_1}{A_1}\right) \end{aligned} \quad (54)$$

thus $x_1'(\tau) = x_2'(\tau)$ and $y_1'(\tau) = -y_2'(\tau)$. □

In the following lemma we prove that the barrier's trajectories in quadrants I and II (see Fig. 2) intersect in a point located at the x-axis.

Lemma 7: *The retro-time trajectories (barrier) starting with angle S and $S - \pi$ at the BUP intersect at $y_1'(\tau) = y_2'(\tau) = 0$.*

Proof. From Lemma 6, we know that

$$\begin{aligned} y_1'(\tau) &= \sqrt{A_1^2 + B_1^2} \sin\left(\alpha_1 + \arctan\frac{B_1}{A_1}\right) \\ y_2'(\tau) &= -\sqrt{A_1^2 + B_1^2} \sin\left(\alpha_1 + \arctan\frac{B_1}{A_1}\right) \end{aligned} \quad (55)$$

From Eq. (55), we have that $y_1'(\tau) = y_2'(\tau)$ only when both are equal to zero thus we need to show that

$$\sin\left(\alpha_1 + \arctan\frac{B_1}{A_1}\right) = 0 \quad (56)$$

at some τ_n. Recall that $\alpha_1 = -\frac{V_R^{\max}}{b}\tau_n$ and $\arctan\frac{B_1}{A_1} = \arctan\frac{y_1(\tau_s) + \tau_n V_A^{\max}\cos S}{x_1(\tau_s) + \tau_n V_A^{\max}\sin S}$. We have that $S = \arccos(\frac{V_A^{\max}}{V_R^{\max}}) \in (0, \frac{\pi}{2}]$ thus $\cos S \geq 0$ and $\sin S > 0$. From the last expressions, $\arctan\frac{B_1}{A_1} > 0, \forall \tau_n > 0$. We also have that $\alpha_1 < 0, \forall \tau_n > 0$. From Lemma 4, we have that the DDR can always reach any desired heading orientation with respect to the segment joining the OA's position and the DDR's center regardless of the OA's motion strategy, thus $\alpha_1 + \arctan\frac{B_1}{A_1} = 0$ for some $\tau_n > 0$. From Lemma 6, we know that $x_1'(\tau) = x_2'(\tau)$ thus the trajectories intersect at $y_1'(\tau) = y_2'(\tau)$ for some $\tau_n > 0$. □

The case when the barrier's trajectories start at $\pi + S$ and $2\pi - S$ can be proved in an analogous way.

Theorem 1: *For this game the barrier partitions the space into two regions, one where the OA captures the DDR and one where the DDR avoids capture.*

Proof. From Lemma 7, we have that the barrier's trajectories coming from the upper and bottom parts of the UP intersect at the x-axis. From the definition of the barrier Isaacs (1965) we know that when the players play optimally they cannot cross from one side of the barrier to the other, therefore the OA can only capture the DDR in the region closer to the UP. □

Remark 1: Note that the DDR wants to restrict the configurations where it can be captured by the OA, for that reason the DDR's retro-time strategy tries to intersect the barrier's trajectories. Since the DDR is faster than the OA it will be able to do it at some time τ_n.

We conclude this section with the following theorem.

Theorem 2: *The player's optimal motion primitives in the realistic space correspond, for the DDR, to rotations in place and straight lines, and for the OA, to straight lines.*

Proof. By Lemma 1 at the end of the game the time-optimal motion primitives in the realistic space are, for the Differential Drive Robot (DDR), moving following a straight line and, for the Omnidirectional Agent (OA), moving also following a straight line. By Lemma 3, for $\tau \geq \tau_s$, the time-optimal motion primitives are, for the DDR, rotating in place, and for the OA, continuing following a straight line. By Theorem 1 the barrier partitions the space into two type of regions, one where the OA captures the DDR and one where the DDR avoids capture. For each quadrant, in the reduced space where capture is possible for the OA, there is only a transition surface, which is reached by the OA pursuer at time τ_s. This exhaustively enumerates the motion primitives of both players where capture is possible. They are for the OA, straight lines and for the DDR straight lines and rotations in place. By the definition of the barrier (Isaacs, R. (1965)), when the OA (pursuer) is located in regions where capture is not possible, this holds regardless of the motion primitives used by the OA (pursuer) provided that at the moment where the OA (pursuer) reaches the barrier the DDR follows its time optimal motion primitives. That is either rotating in place or moving in straight line. The result follows. □

6. Partition of the space

In this section, we present a partition of the playing space into two regions. The construction is shown in Figure 3.

- **Region I**. We denote as region I to the set of points that can reach the UP with a single straight line trajectory in the reduced space which corresponds to a straight line motion of both, the DDR and the OA, in the realistic space. Examples of trajectories in Region I are shown in Fig. 3.
- **Region II**. We denote as Region II to the set of points that reach the TS by following a trajectory given in Eq. (33) in the reduced space, which corresponds to a rotation in place for the DDR and a straight line trajectory for the OA, both in the realistic space.

Figure 3 shows that the trajectories coming from the upper and bottom parts of the UP intersect the x-axis defining a dispersal surface (DS). Over the DS both players have two choices for their controls. It is important to note that at the DS, the choice of the control of one player *must correspond* to the choice of the control of the other player. If one player selects the wrong control, the other player will benefit from that decision. In this game, the DS corresponds to configurations where the pursuer's heading (orientation of the wheels) is perpendicular to the pursuer's location, and the DDR has the option to rotate either clockwise or counterclockwise to try to delay the capture. If the DDR fails to initially choose the correct sense of rotation against the OA's decision then feedback will be necessary to correct the decision and it's possible that the OA can capture the DDR in suboptimal time. To avoid the selection problem, the instantaneous velocity vector of both players should be known, but in general (and in particular for this problem) it is assumed that this information is not available. A solution would be to employ an instantaneous mixed strategy (IMS) Isaacs (1965) which means the randomizing of a player's decision in accordance to some probabilistic law until the system is not longer on the DS. The trajectories generated by

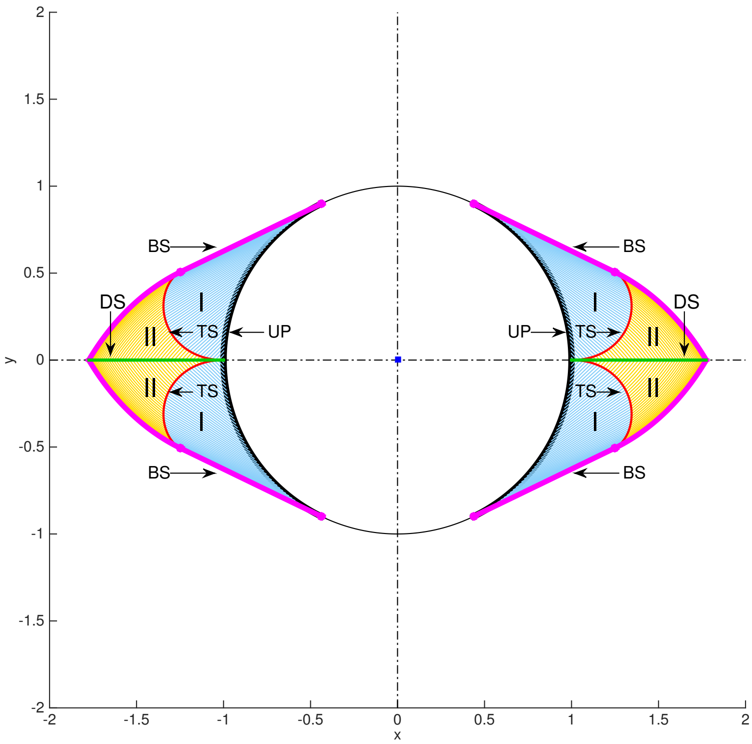


Figure 3. Partition of the reduced space. The OA plays as a pursuer and the OA as an evader. Capture is only possible for initial configurations inside Region I and II. Region I corresponds to configurations where the OA captures the DDR and both players move following a straight line in the realistic space. Region II corresponds to configurations where the OA captures the DDR but in this case the DDR performs a rotation in place and the OA moves following a straight line in the realistic space.

the correct pair of controls will lead to the same optimal time-to-go. In this game, the difference will be that at the end the capture condition will be attained when the DDR is moving forward or backward following a straight line in the realistic space. It is important to remark that this particular situation only occurs when the wheels of the DDR are exactly perpendicular to the line that joins the DDR and the OA, a situation that in practice occurs with probability 0.

7. Finding the winning roles

In this section, we combine the results obtained in this paper and the ones in Ruiz et al. (2013). Using the partitions of the playing space of both works, we found the winning roles for the DDR and the OA. Each player can play either as a pursuer or evader; however, once a role has been assigned for one player then the other can only choose an antagonistic role. Note that each player must remain playing the initial role during the game otherwise he will be cooperating with his adversary. It is also important to mention that the state of the system is given by the relative position of the OA with respect to the DDR regardless of the roles of the players. From Ruiz et al. (2013), we have that when the DDR plays as a pursuer and the OA as an evader, the DDR captures the OA from any initial configuration if $\rho_v^2\rho_d - \sqrt{1-\rho_v^2} < 0$ (see Fig. 4(a)) otherwise the barrier's trajectories intersect at $y = l_c/\rho_v$ and the DDR can only capture the OA in the closed region close to the UP (see Fig. 4(b)). From Theorem 1, we know that when the OA plays as a pursuer and the DDR as an evader, the OA can only capture the DDR in the closed region close to the UP (see Fig. 3). Considering the behavior of the partitions of the playing space for both games, we identify the following two general cases:

- Case I: The barrier's trajectories intersect in both partitions and define the closed regions A and B (see Fig. 5(a)). A DDR pursuer captures an OA evader in Region A and an OA pursuer captures a DDR evader in Region B. Note that A and B are disjoint regions. By the definition of the barrier, one player cannot cross the barrier without help of the other player which means that if the DDR plays as a pursuer in Region A the OA cannot avoid capture unless the DDR applies a suboptimal strategy.
- Case II: The barrier's trajectories intersect only for the game when the OA plays as a pursuer and the DDR as an evader (see Fig. 5(b)). In this case, Region A is open and Region B is closed. Note that Region A contains all configurations in the playing space (reduced space) except for those inside Region B or the capture radius (see Fig. 5(b)).

For Case I, we have the following winning roles for the players.

(1) Assume that the initial configuration of the system in the reduced space is located on Region A (see Fig. 5(a)). We have that if the DDR plays as a pursuer then it captures the OA. Suppose the DDR plays as an evader and the OA plays as a pursuer, the DDR avoids capture since the initial configuration is located outside B (recall that A and B are disjoint regions for case I). Therefore, the DDR wins in Region A either playing as a pursuer or evader.
(2) Analogous, suppose the initial configuration of the system is located on Region B (see Fig. 5(a)). If the DDR plays as a pursuer and the OA plays as an evader then the OA avoids capture (again, A and B are disjoint regions). We have that if the DDR plays as an evader and the OA as a pursuer then the OA captures the DDR. Thus, the OA wins in Region B either playing as a pursuer or evader.
(3) If the initial configuration is located in Region C (see Fig. 5(a)) then capture is not possible for both players, the player playing the evader role wins.

Table 1 summarizes the winning roles described above (see Fig. 5(a)).

Region (OA's location)	**OA's role**	**DDR's role**	**Winner player**	**Winner role**
A	Pursuer	Evader	DDR	Evader
A	Evader	Pursuer	DDR	Pursuer
B	Pursuer	Evader	OA	Pursuer
B	Evader	Pursuer	OA	Evader
C	Pursuer	Evader	DDR	Evader
C	Evader	Pursuer	OA	Evader

Table 1. Winning roles for both players in Case I: The barrier's trajectories intersect in both partitions and define the closed regions A and B (see Fig. 5(a))

For Case II, we have the following winning roles for the players

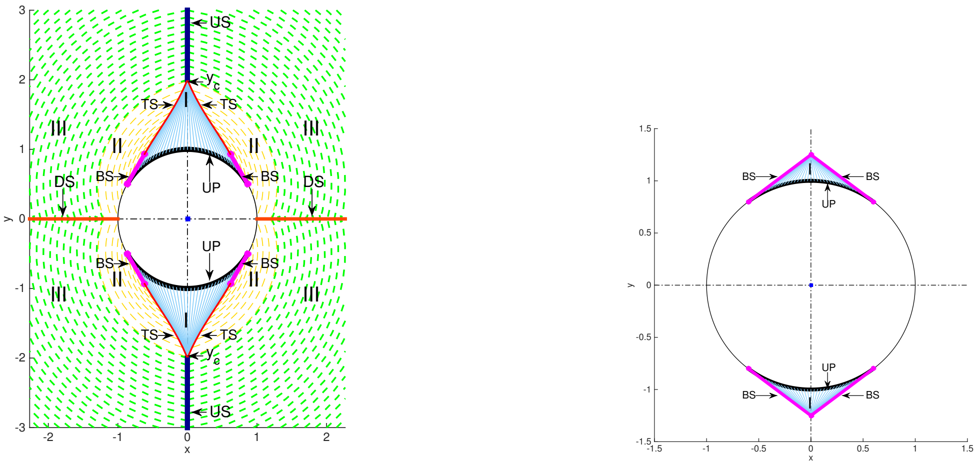
(1) Suppose the initial configuration is located outside B (see Fig. 5(b)). If the DDR plays as a pursuer and the OA as an evader then the DDR captures the OA. Assume the DDR plays as an evader and the OA as a pursuer then the DDR avoids capture since the OA cannot force the system to cross to Region B without help of the DDR. Therefore, the DDR wins in Region A either playing as a pursuer or evader.
(2) Assume the initial configuration is located on B (see Fig. 5(b)). If the DDR plays as a pursuer and the OA as an evader then the DDR captures the OA. Note that, in this case, the DDR can cross the barrier of Region B since the OA is playing the opposite role for which it was defined. Suppose the DDR plays as an evader and the OA as a pursuer then the OA captures

the DDR. Note that the DDR cannot force the system to cross to Region A without help of the OA. We have that for Region B the winning role for the DDR is play as a pursuer and the winning role for the OA is also play as a pursuer.

Table 2 summarizes the winning roles described above (see Fig. 5(b)).

Region (OA's location)	OA's role	DDR's role	Winner player	Winner role
A	Pursuer	Evader	DDR	Evader
A	Evader	Pursuer	DDR	Pursuer
B	Pursuer	Evader	OA	Pursuer
B	Evader	Pursuer	DDR	Pursuer

Table 2. Winning roles for both players in Case II: The barrier's trajectories intersect only for the game when the OA plays as a pursuer and the DDR as an evader (see Fig. 5(b))



(a) $\rho_v^2\rho_d - \sqrt{1-\rho_v^2} < 0$. Capture is possible from any initial configuration. Region II and III correspond to configurations where the DDR needs to rotate in place in order to capture the OA. For initial configurations in Region II, the DDR's heading is not completely aligned with the OA's motion direction in order to capture it. For initial configurations in Region III the DDR's heading is aligned with the OA's motion direction at some time. This image was taken from Ruiz et al. (2013)

(b) $\rho_v^2\rho_d - \sqrt{1-\rho_v^2} \geq 0$ (the barrier's trajectories intersect). Capture is only possible for initial configurations inside region I.

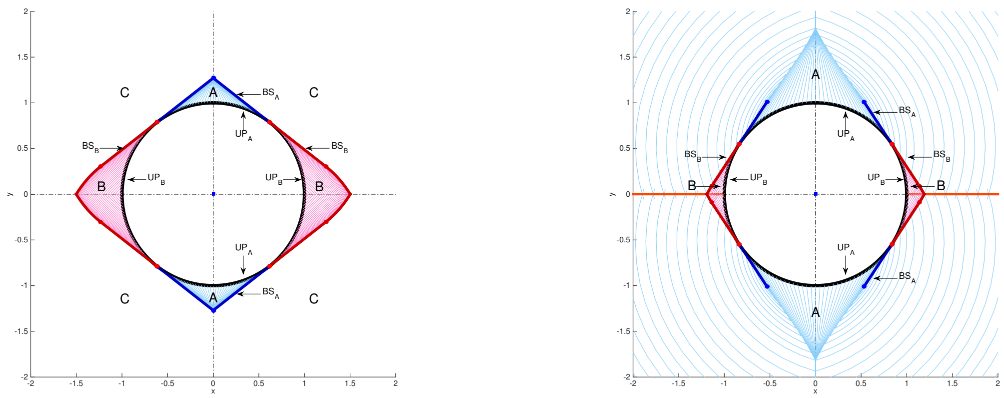
Figure 4. Partition of the reduced space. The DDR plays as a pursuer and the OA as an evader. For more details, we refer the reader to Ruiz et al. (2013).

8. Simulations

In this section, we present some simulations of the pursuit-evasion game where the OA tries to capture the DDR in minimum time. We use m/sec as units for velocities, meters for distance and seconds for time.

8.1 *Capturing the DDR*

First, we present the case where the OA captures the DDR, i.e., the initial configuration is inside the region defined by the barrier trajectory. The parameters of this simulation were $V_R^{\max} = 1$,

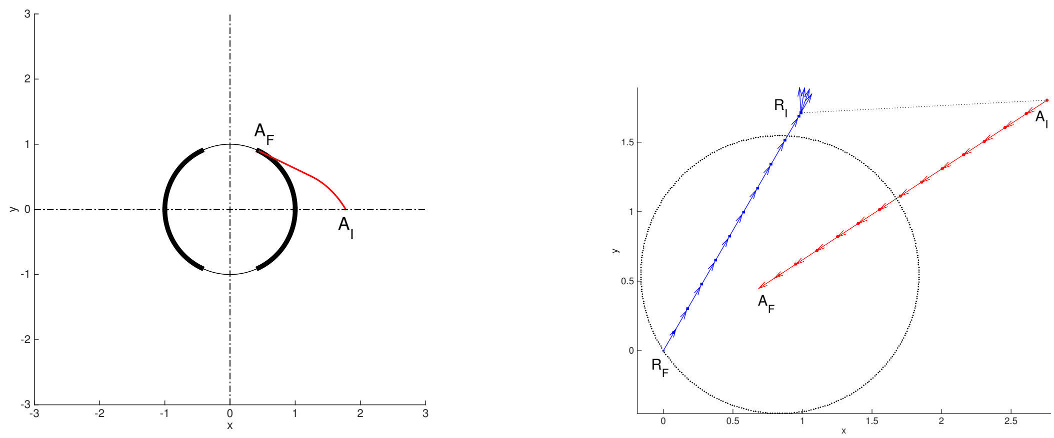


(a) Case I: The barrier's trajectories intersect in both partitions and define the closed regions A and B.

(b) Case II: The barrier's trajectories intersect only for the game when the OA plays as a pursuer and the DDR as an evader.

Figure 5. Partitions of the reduced space given by the two games.

$V_A^{\max} = 0.9$, $b = 1$ and $l_c = 1$. Fig. 6(a) shows the trajectory of the system in the reduced space when the OA captures the DDR. The circle centered at the origin represents the capture distance l_c and the bold arcs the UP. The OA's initial position is denoted by A_I and its final position by A_F. The OA starts in a configuration where the segment joining its position and the DDR's center is perpendicular to the DDR's heading. Figure 6(b) shows the trajectories of the players in the realistic space. The OA's initial and final positions are denoted by A_I and A_F, the DDR's initial and final positions are R_I and R_F. In Fig. 6(b), we can observe that the DDR's was captured while it was moving backwards following a straight line. In this figure, we can also note that the OA follows a straight line trajectory to capture the DDR.

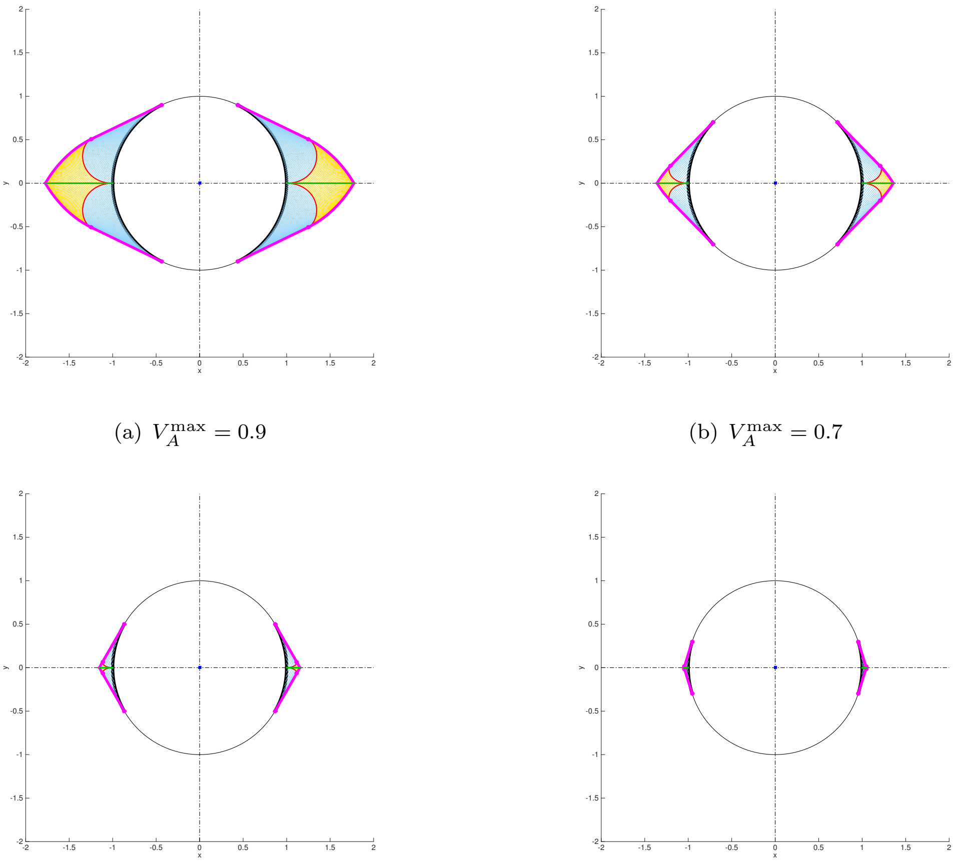


(a) Trajectory in the reduced space.

(b) Trajectory in the realistic space.

Figure 6. Trajectories of the players when the OA captures the DDR.

Fig. 7 shows the evolution of the Region B, where the OA can capture the DDR, as we change the value of $V_A^{\max}$. We can observe that as we decrease $V_A^{\max}$ so does the size of the Region B. This behavior also occurs if we decrease the ratio ρ_v. Note that as Region B decreases its size also implies that the initial position of the OA is closer to the DDR, and the angle between the DDR's heading and the segment joining the OA's position is closer to $\frac{\pi}{2}$. The parameters of this simulation were $V_R^{\max} = 1$, $b = 1$ and $l_c = 1$.



(a) $V_A^{\max} = 0.9$

(b) $V_A^{\max} = 0.7$

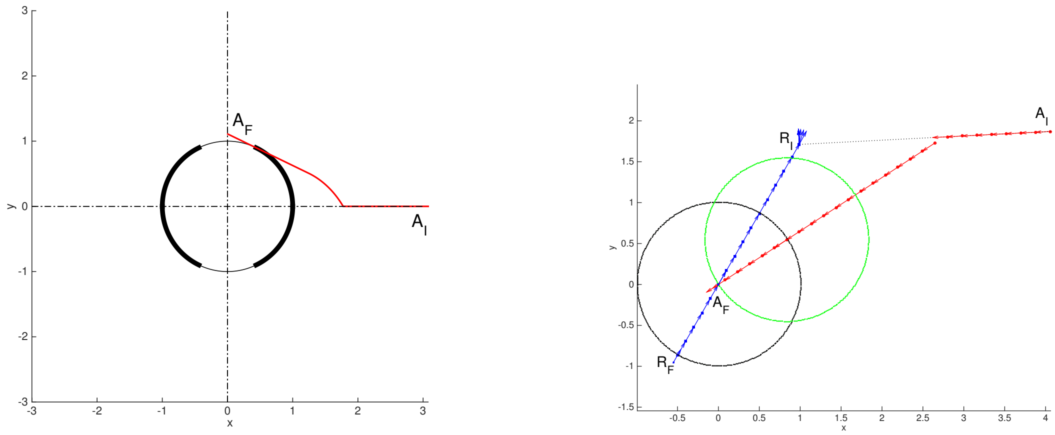
(c) $V_A^{\max} = 0.5$

(d) $V_A^{\max} = 0.3$

Figure 7. Evolution of the region B where the OA captures the DDR as a function of $V_A^{\max}$.

8.2 *The DDR avoids capture*

In the following simulation, we show a case where the DDR avoids capture since the initial configuration is located outside the region defined by the barrier trajectory. The parameters of the simulation where $V_R^{\max} = 1$, $V_A^{\max} = 0.9$, $b = 1$ and $l_c = 1$. Figure 8(a) shows the trajectory of the system in the reduced space. The OA's initial position is denoted by A_I and its final position by A_F. The OA starts in a configuration where the segment joining its position and the DDR's center is perpendicular to the DDR's heading. Figure 8(b) shows the trajectories of the players in the realistic space. The OA's initial and final positions are denoted by A_I and A_F, the DDR's initial and final positions are R_I and R_F. We construct a motion strategy for the players that despite of being initially favorable to the OA shows that the DDR can avoid capture. In the first part of the strategy, the DDR remains in the same position while the OA decreases the distance between both players. This corresponds in the reduced space to the trajectory over the x-axis (see Fig. 8(a)). In realistic space, this strategy is represented by the straight line starting at A_I. Once the position of the OA in the reduced space reaches the beginning of barrier's trajectory (arc trajectory), the DDR starts to rotate in place and the OA starts to follow a new straight line trajectory in the realistic space, as it is described by the time-optimal motion strategies of both players. After some time, the DDR starts to follow a straight line trajectory while the OA remains following its previous motion strategy. At some point, the OA reaches the distance l_c (see Fig. 8(b)) to the DDR but since the system is pointing tangentially to the terminal surface (see Fig. 8(a)), the OA is not able to get closer to DDR than l_c and capture cannot be attained. For this simulation, over the target set the system continues following a straight line in the reduced space which corresponds to straight line trajectories for both players in the realistic space.



(a) Trajectory in the reduced space. The system is initially following a straight line trajectory starting from A_I. After some time, the system follows the barrier's trajectory reaching the capture distance l_c. Since the system reaches that point tangentially to the terminal surface the DDR can avoid capture.

(b) Trajectory in the realistic space. First, the DDR remains at position R_I while the OA moves toward the DDR following a straight line trajectory starting at A_I. At some point, the players begin to follow their time-optimal motion strategies given by the barrier's trajectory, i.e., the DDR rotates in place at R_I while the OA follows a new straight line trajectory. After some time, the DDR also starts to follow a straight line trajectory. The OA is able to reach the distance l_c (green circle) to the DDR but it cannot reduce it thus the DDR can avoid capture. The black circle shows that the DDR is farther than the capture distance l_c after a few time.

Figure 8. Trajectories of the players when the OA captures the DDR.

In Ruiz et al. (2013), the authors have presented simulations for the symmetric game where the DDR plays as a pursuer and the OA as an evader.

9. Conclusions and Future Work

In this paper, we considered the problem of capturing a DDR using an OA in an obstacle free environment. We presented closed-form representations of the motion primitives and time-optimal strategies for each player. In the realistic space, the motion primitives for the OA are straight lines and for the DDR straight lines and rotations in place. We proposed a partition of the playing space into mutually disjoint regions and we showed that the OA only captures the DDR in a closed region of the space. Combining the results obtained in this paper and the ones in Ruiz et al. (2013), we allow the agents to change the roles, namely, the DDR is allowed to play as the pursuer and the OA is allowed to play as the evader. This later analysis permitted to establish which is the winner role for each agent, based only on the initial position of the players and their maximum speed. As future work, we will include acceleration bounds in the solution of this problem. We would also like to consider multiple pursuers and/or multiple evaders and find feedback-based motion policies.

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