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Rational maps with generalized Sierpinski gasket Julia sets

Robert L. Devaney [a,1], Mónica Moreno Rocha [b,*], Stefan Siegmund [c,2]

[a] *Department of Mathematics, Boston University, Boston, MA 02215, USA*
[b] *The Fields Institute, University of Toronto, Toronto, ON M5T 3J1, Canada*
[c] *Department of Mathematics, University of Frankfurt, Frankfurt, 60325, Germany*

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Abstract

We study a family of rational maps acting on the Riemann sphere with a single preperiodic critical orbit. Using a generalization of the well-known Sierpinski gasket, we provide a complete topological description of their Julia sets. In addition, we present a combinatorial algorithm that allows us to show when two such Julia sets are not topologically equivalent.

MSC: primary 37F10, 37B10; secondary 37B45

Keywords: Rational maps; Julia sets; Sierpinski gaskets

1. Introduction

In this paper we study a class of postcritically finite rational maps whose Julia sets are given by *generalized Sierpinski gaskets*. Briefly, a generalized gasket is the limit set obtained by a similar recursive process defined for the Sierpinski triangle, but applied instead to the closed unit disk as starting set and by removing polygons of N sides.

The class of rational maps considered here have the form:

$$R_{\lambda,n,m}(z) = z^n + \frac{\lambda}{z^m},$$

with $n \geqslant 2, m \geqslant 1$ and $\lambda \in \mathbb{C}$. This collection of rational maps is special in the sense that each possesses a single critical orbit up to symmetry. Indeed, each map in the family $F_\lambda = R_{\lambda,2,2}$ possesses four critical points (excluding the superattracting fixed point at ∞ and the pole at 0), but each of them lands on the same orbit after the second iteration. For the family $G_\lambda = R_{\lambda,2,1}$, there are three free critical points but the behavior of one of the critical orbits determines the behavior of the other two by symmetry.

* Corresponding author.
E-mail addresses: bob@math.bu.edu (R.L. Devaney), mmorenor@fields.utoronto.ca (M.M. Rocha), siegmund@math.uni-frankfurt.de (S. Siegmund).
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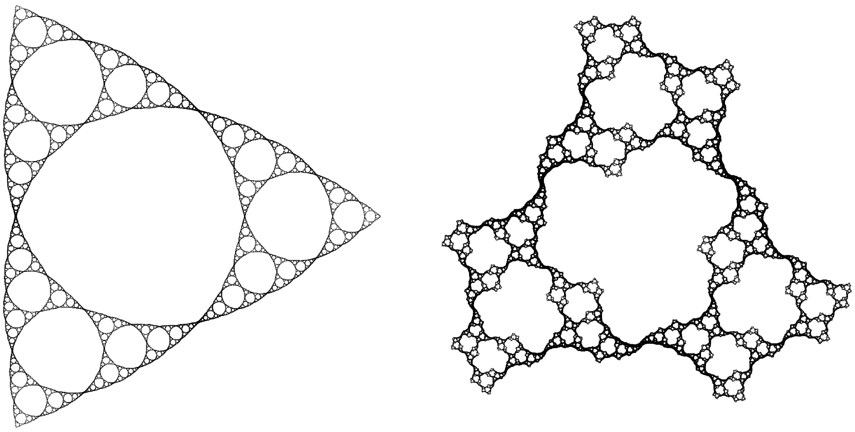


Fig. 1. $J(G_\lambda)$ for $\lambda \approx -0.59257$ and $-0.03804 + i0.42622$.

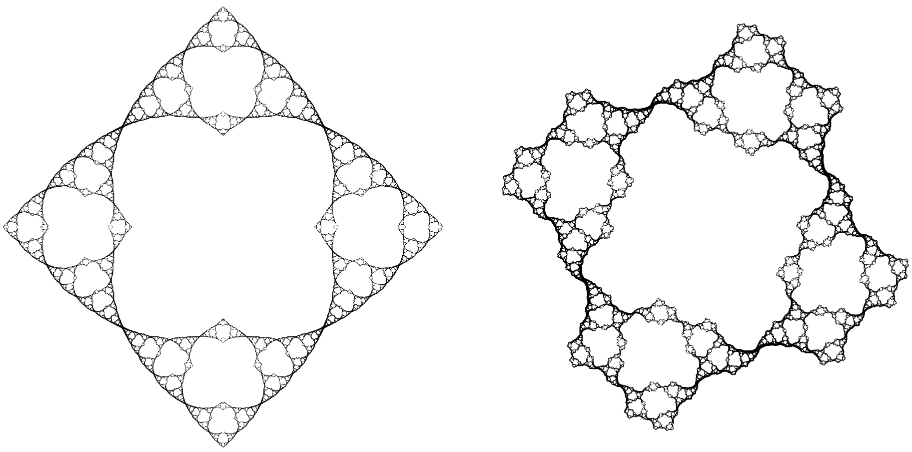


Fig. 2. $J(F_\lambda)$ for $\lambda \approx -0.36428$ and $\lambda \approx -0.01965 + i0.2754$.

The Julia set of a rational map f is the set of points at which the family of iterates fails to be a normal family in the sense of Montel. We denote the Julia set by $J(f)$. In Fig. 1 we display the Julia set of G_λ when $\lambda \approx -0.59257$ and $-0.03804 + i0.42622$. In the first case, the Julia set is homeomorphic to the usual Sierpinski triangle whereas, in the second case, note that the removed regions have different configurations in terms of how their vertices lie along the outer boundary of the Julia set. In Fig. 2 we display a similar phenomenon for Julia sets drawn from the family F_λ. Again note how the removed "squares" assume different configurations in terms of how their vertices meet the outer boundary.

In this paper we shall consider the special case where the critical orbits for the maps $R_{\lambda,n,m}$ are all *strictly* preperiodic to a repelling cycle. Such maps are often called Misiurewicz rational maps. Our main goal is to show that, when these critical orbits also lie on the boundary of the basin of ∞, the Julia set of any such map is a generalized Sierpinski gasket. In addition, we provide an algorithm to determine when two such Julia sets are topologically distinct (excluding the obvious symmetric cases).

For simplicity, we restrict most of our analysis to the degree four family $F_\lambda(z) = z^2 + \lambda/z^2$. Section 2 consists of basic definitions, a review of the fundamental results for this family and the main assumptions. In Section 3 we show that if λ satisfies certain conditions, then the associated Julia set of F_λ is a generalized gasket. Some other technical results are included in this section. Section 4 contains the description of the algorithm and proofs of the main results. The general case for families of higher order and further remarks appear in Section 5.

2. Preliminaries

2.1. Gasket-like sets

Recall that the familiar Sierpinski gasket (sometimes called the Sierpinski triangle) is obtained by the following iterative process. Starting with a triangle in the plane, remove the open middle triangular region, leaving three congruent triangular regions behind. Then remove the middle triangular regions from each of these remaining triangles, leaving nine triangular regions behind. When this process is carried to the limit, the resulting set is the Sierpinski gasket.

A *generalized Sierpinski gasket* is obtained by a similar process performed on the closed unit disk Λ and removing homeomorphic copies of a polygon of N sides with straight edges. Let P denote the interior of such homeomorphic copy, so the boundary of P is a simple closed curve with N distinguished points (or *corners*) that correspond to the vertices of the original polygon.

At the first stage of the construction, we remove from Λ the region P having only its corners lying in the boundary of Λ. Thus we are left with a connected set composed by the union of N homeomorphic copies of Λ which we denote by $\Lambda_1, \ldots, \Lambda_N$. At the second stage, we remove from each Λ_k a smaller copy of P with only its corners lying in the boundary of Λ_k, so we are left with N^2 copies of Λ. If we continue this process to the limit, we obtain a set X which is compact, connected and locally connected. To complete the definition of a generalized gasket, we add the following conditions to the above construction. See [4] for related constructions.

Definition 2.1. A set X is a generalized Sierpinski gasket if it is obtained by the recursive process described above in such a way that

(1) X has N-fold symmetry, and
(2) from the second stage and onward, m corners of a removed region P lie in the boundary of one of the removed regions in the previous stage, with $1 \leqslant m < N$.

For later use, we call condition 2 above the *m-corners condition*.

2.2. Basic properties of F_λ

For clarity, we shall deal mainly with the family

$$F_\lambda(z) = z^2 + \lambda/z^2$$

for the remainder of this paper. The results below are easily modified to apply to the family $G_\lambda(z) = z^2 + \lambda/z$ and, in general, to $R_{\lambda,n,m}$.

First note that $F_\lambda(-z) = F_\lambda(z)$ and $F_\lambda(iz) = -F_\lambda(z)$. It follows that $J(F_\lambda)$ is symmetric under $z \mapsto iz$. Also, we have

$$F_{\bar\lambda}(\bar z) = \overline{F_\lambda(z)}.$$

Hence $J(F_\lambda)$ is homeomorphic to $J(F_{\bar\lambda})$ and so we restrict from now on to the case where the imaginary part of λ is nonnegative.

Note that F_λ has critical points at the fourth roots of λ. We denote by c_1 the critical point of F_λ that lies in the first quadrant and set $c_2 = ic_1$, $c_3 = -c_1$, and $c_0 = -ic_1$.

There are two other symmetries for F_λ, namely the involutions $H_\lambda^\pm(z) = \pm\sqrt{\lambda}/z$. We have $F_\lambda(H_\lambda^\pm(z)) = F_\lambda(z)$ for all z. Note that each of these involutions fixes two of the critical points and reflects the plane through the circle of radius $|\lambda^{1/4}|$ centered at the origin.

The following facts are known about $J(F_\lambda)$ (see [1]):

(1) The point at infinity is a superattracting fixed point for F_λ and F_λ is conjugate in a neighborhood of ∞ to $z \mapsto z^2$. Let B_λ denote the immediate basin of attraction of ∞.
(2) If the critical points lie in B_λ, then $J(F_\lambda)$ is a Cantor set; otherwise, $J(F_\lambda)$ is a connected set. In particular, in the Misiurewicz case, $J(F_\lambda)$ is locally connected.

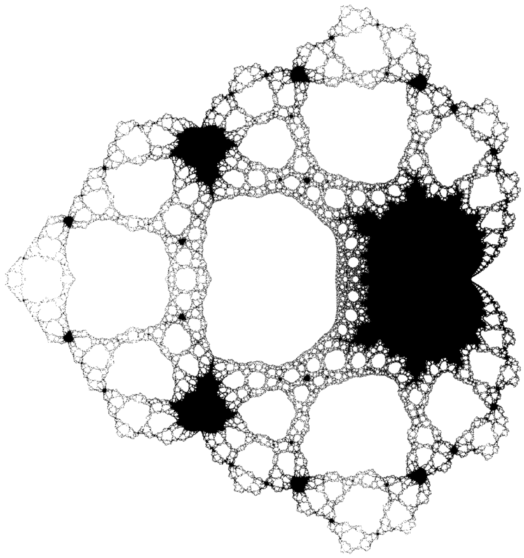


Fig. 3. The locus of connectedness $\mathcal{D}$ of the family F_λ.

(3) In the connected Julia set case, there is an open neighborhood of 0 that is mapped in two-to-one fashion onto B_λ. This region is called the trap door and is denoted by T_λ. We have $T_\lambda \cap B_\lambda = \emptyset$.

We denote the boundary of B_λ by β_λ and the boundary of T_λ by τ_λ. Note that the involution $H_\lambda^{\pm}$ interchanges B_λ and T_λ as well as their boundaries. For the remainder of this paper, we work with the following type of maps.

Definition 2.2. If the critical points of F_λ satisfy the following conditions,

(1) each critical point lies in β_λ, and
(2) each of the critical points are strictly preperiodic,

then we say that F_λ is a *Misiurewicz–Sierpinski map*, or, a little more succinctly, an *MS map*.

Note that the symmetry $H_\lambda^{\pm}$ implies that the critical points of an MS map also lie in τ_λ. In general, $R_{\lambda,n,m}$ is an MS map if its $n+m$ finite, nonzero critical points satisfy the above conditions.

In Fig. 3 we display the parameter plane for the family F_λ. The unbounded region consists of those parameter values for which the Julia set is a Cantor set; its complement represents the locus of connectedness of the family F_λ, which we denote by $\mathcal{D}$. If a parameter value is drawn from any of the small copies of the Mandelbrot set in $\mathcal{D}$ (regions in black), then the critical orbit is bounded. On the other hand, if λ belongs to any of the white bounded regions, then the critical orbit escapes to infinity and the Julia set is homeomorphic to the Sierpinski carpet, [2,5]. We call the white regions Sierpinski holes.

Let $\mathcal{M}$ denote the subset of parameter values associated to MS maps. In general, Misiurewicz parameters form a dense subset in the unstable locus of families of rational-like maps, [8]. For the family F_λ (and in general, for every family $R_{\lambda,n,m}$), the bifurcation locus contains not only the boundary of $\mathcal{D}$, but also every boundary component of the Sierpinki holes and boundary points of the black regions. We claim that the set $\mathcal{M}$ is a dense subset in the boundary of the connectedness locus $\mathcal{D}$.

3. Homeomorphisms of MS maps

3.1. Topological description of $J(F_\lambda)$

Our main goal here is to prove the following theorem.

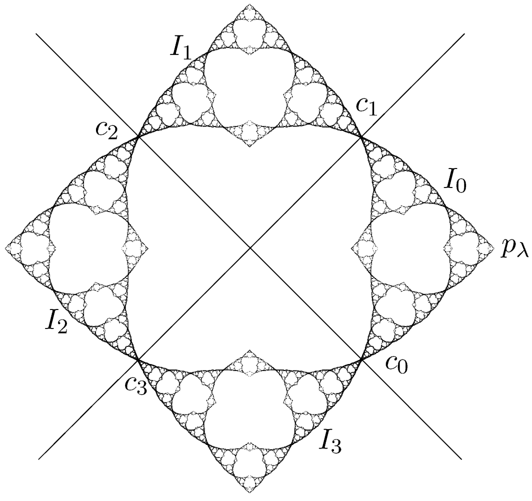


Fig. 4. Partition of the Julia set of F_λ with $\lambda \approx -0.36428$.

Theorem 3.1. *If F_λ is an MS map, then $J(F_\lambda)$ is a generalized Sierpinski gasket.*

We begin with the following result proved in [2]:

Theorem. *If the critical points of F_λ are preperiodic, then β_λ is a simple closed curve. If $\beta_\lambda \cap \tau_\lambda$ is nonempty, then the critical points of F_λ are the only four points in this intersection.*

We call the critical points the *corners* of the trap door. The four corners separate τ_λ into four *edges*. Using the fact that F_λ is conjugate to $z \mapsto z^2$ in B_λ, we may construct four disjoint smooth curves, γ_j for $j = 0, 1, 2, 3$, connecting c_j to ∞ in B_λ. Let ν_j denote the image of γ_j under the involution that fixes c_j. Then the curve $\eta_j = \gamma_j \cup \nu_j$ connects 0 to ∞ and meets $J(F_\lambda)$ only at c_j. Moreover, the η_j are pairwise disjoint (except at 0 and ∞). Hence these four curves divide the Julia set into four symmetric pieces $I_0, \dots, I_3$ where we assume that $c_j \in I_j$ but c_j does not lie in the other three regions. Hence the I_j are neither open nor closed subsets of $J(F_\lambda)$. Later on we will use the I_j to describe the symbolic dynamics generated by F_λ on its Julia set.

For $j = 0, 1, 2, 3$, let I_j denote the connected component of the Julia set that lies in one of the four "quadrants" defined by the corners of τ_λ and the four curves η_j (see Fig. 4). Let I_0 be the component that contains the repelling fixed point p_λ, which lies in β_λ.

Since there are no critical points in any of the preimages of the trap door, it follows that each of its preimages is mapped in one-to-one fashion onto the trap door by F_λ. Hence each component of $F_\lambda^{-k}(\tau_\lambda)$ also has four corners and edges, and each of these corners is mapped by F_λ^k onto a distinct critical point in τ_λ.

We may now show that the Julia set of an MS map is a gasket-like set. Let $K_0 = \overline{\mathbb{C}} \setminus B_\lambda$ and $K_1 = K_0 \setminus T_\lambda$. Then $K_1 = I_0 \cup I_1 \cup I_2 \cup I_3$. Let $K_{n+1} = K_n \setminus F_\lambda^{-n}(T_\lambda)$. It follows that each K_n is a nested collection of closed and connected subsets of the Riemann sphere with exactly 4^n homeomorphic copies of a rectangle removed at each nth step [7]. Moreover,

$$J(F_\lambda) = \bigcap_{n=0}^{\infty} K_n,$$

so $J(F_\lambda)$ is a compact and connected set with 4-fold symmetry. Local connectivity follows from subhyperbolicity of the map. To see that the Julia set satisfies the m-corners condition, we have the following result.

Lemma 3.2. *Let τ_λ^k be the union of all of the components of $F_\lambda^{-k}(\tau_\lambda)$ and let A be a particular component in τ_λ^k with $k \geqslant 1$. Then exactly two of the corner points of A lie in a particular edge of a single component of τ_λ^{k-1}.*

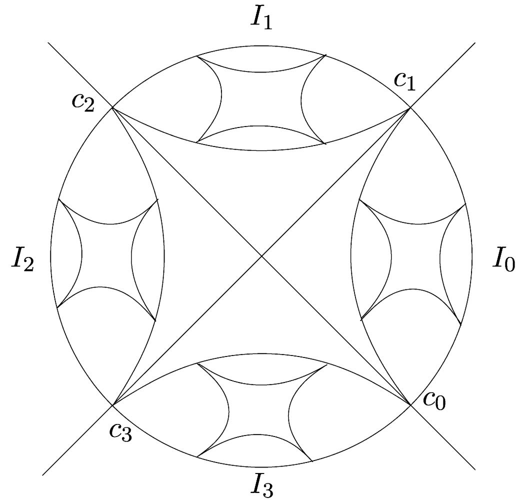


Fig. 5. A topological representation of the boundaries β_λ, τ_λ and the four components of τ_λ^1. Compare with Fig. 4.

Proof. The case $k = 1$ is seen as follows. We have that F_λ maps each I_j for $j = 0, \dots, 3$ in one-to-one fashion onto all of $J(F_\lambda)$, with $F_\lambda(I_j \cap \beta_\lambda)$ mapped onto one of the two halves of β_λ lying between two critical values (which, by assumption, are not equal to any of the critical points). Hence $F_\lambda(I_j \cap \beta_\lambda)$ contains exactly two critical points. Similarly, $F_\lambda(I_j \cap \tau_\lambda)$ maps onto the other half of β_λ and so also meets two critical points. The preimages of these latter two critical points in τ_λ are precisely the corners of the component of τ_λ^1 that lies in I_j. Thus we see that each component in τ_λ^1 meets the boundary of one of the I_j's in two points lying in β_λ and two points lying in τ_λ. In particular, two of the corners lie in the edge of τ_λ that meets I_j.

Now consider a component in τ_λ^k with $k > 1$. F_λ^k maps each component in τ_λ^k onto τ_λ and therefore F_λ^{k-1} maps the components in τ_λ^k onto one of the four components of τ_λ^1. Since each of these four components meets a particular edge of τ_λ in exactly two corner points, it follows that each component of τ_λ^k meets an edge of one of the components of τ_λ^{k-1} in exactly two corner points as claimed. □

As a consequence of the above lemma, we display in Fig. 5 a schematic representation of β_λ, τ_λ, and τ_λ^1 which holds true for any MS λ-value.

3.2. Local cut points of $J(F_\lambda)$

The next Theorem provides a topological characterization of the critical points and it will allow us to show that any homeomorphism between two Julia sets of MS maps must send critical points to critical points.

Theorem 3.3. *The four corners of the trap door are the only set of four points in the Julia set whose removal disconnects $J(F_\lambda)$ into exactly four components. Any other set of four points removed from $J(F_\lambda)$ will yield at most three components.*

The proof of this result will follow from Propositions 3.9–3.11. First we introduce several definitions.

Definition 3.4. Define the 0-*disk* to be the whole Julia set $J(F_\lambda)$ and let $\tau_\lambda^0 = \tau_\lambda$ denote the boundary of the trap door T_λ. For any $k \geqslant 1$, we define a *k-disk* to be a compact and connected subset of a $(k-1)$-disk such that

- the k-disk is mapped in one-to-one fashion onto $J(F_\lambda)$ by F_λ^k,
- if A is the component of τ_λ^{k-1} that lies in the $(k-1)$-disk, then two adjacent corner points of A lie in the k-disk, and
- a component B of τ_λ^k lies completely in the k-disk.

It follows that each k-disk is the union of four $(k+1)$-disks. Moreover, Lemma 3.2 implies that the outside boundary of any $(k+1)$-disk must contain an edge and two corner points of a single component A in τ_λ^k. Through the rest of this section D_k will denote a k-disk.

Definition 3.5. The *skeleton* of the Julia set, denoted by $\mathbb{J}$, is defined by

$$\mathbb{J} := \beta_\lambda \cup \bigcup_{k\geqslant 0} \tau_\lambda^k.$$

Note that $\mathbb{J}$ is an arcwise connected subset of the Julia set whose closure is the whole Julia set. Moreover, a path connecting two points lying in different 1-disks must pass by at least one critical point. So in this sense, the critical points are *local cut points* of $\mathbb{J}$. Clearly, the skeleton of any k-disk is also an arcwise connected set given by $\mathbb{D}_k := \mathbb{J} \cap D_k$ and the corner points of the component $A \subset \tau_\lambda^k$ that lies in $\mathbb{D}_k$ are local cut points of the k-disk.

Clearly, the Julia set of an MS map does not contain parabolic periodic points or recurrent critical points. Indeed, the ω-limit set of the critical points c_λ is a repelling periodic cycle that lies in β_λ, and by assumption, the critical points are disjoint from the limit set. Then we may apply the following result due to Mañé (see [6]).

Theorem. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. If a point $z \in J(f)$ is not a parabolic periodic point and is not contained in the ω-limit set of a recurrent critical point, then for all $\varepsilon > 0$ there exists a neighborhood U of z such that for all $n > 0$, every component V of $f^{-n}(U)$ satisfies $\operatorname{diam} V \leqslant \varepsilon$.*

Proposition 3.6. $\operatorname{diam} D_k \to 0$ *as* $k \to \infty$.

Proof. Let $z \in J(F_\lambda)$, $\varepsilon > 0$, U and V as above. By Montel's theorem, there exists an $N \gg 1$ so $J(F_\lambda) \subset F_\lambda^N(U)$. Then U contains at least one N-disk and consequently, for all $j > 0$,

$$\operatorname{diam} D_{N+j} \leqslant \operatorname{diam} V \leqslant \varepsilon. \qquad \square$$

As a consequence, we have

Corollary 3.7. *For any two points in the Julia set there exists an integer $k > 0$ such that each point lies in distinct k-disks.*

Now, let z be any point in β_λ and for any small $\varepsilon > 0$, let l_ε denote an arc of β_λ that contains z and has length ε. By the invariance of β_λ and conjugacy of F_λ with $z \mapsto z^2$, there exists an integer $N > 0$ such that $\beta_\lambda \subset F_\lambda^N(l_\varepsilon)$. This implies that l_ε contains corner points of components of τ^{N+j} for $j \geqslant 0$. We can easily extend this result to components in τ_λ^k, so we have proved,

Lemma 3.8. *Any point in $\mathbb{J}$ is an accumulation point of corner points.*

The proof of Theorem 3.3 follows by combining the next three propositions.

Proposition 3.9. *Let $\{p_1, \ldots, p_n\}$ be a finite collection of non-corner points in $J(F_\lambda)$. Then $J(F_\lambda) - \{p_0, \ldots, p_n\}$ is connected.*

Proof. Assume first that each p_i is a point in the skeleton $\mathbb{J}$. Then each p_i lies in an edge l_i of some component of $\tau_\lambda^{n_i}$. By Lemma 3.8, any small neighborhood of p_i contains corner points of higher order accumulating on p_i, and in particular, along its edge. This implies that l_i is pathwise connected to infinitely many k-disks of higher order that, in turn, are pathwise connected to lower level disks all the way to β_λ.

Assume now each p_i is not in $\mathbb{J}$. Let N and M be two open sets in the relative topology of $J(F_\lambda)$ so $J(F_\lambda) - \{p_0, \ldots, p_n\} = N \cup M$ and $N \cap M = \emptyset$. Without loss of generality assume $\mathbb{J}$ is contained in N. If M is not empty, it must contain at least an accumulation point of $\mathbb{J}$. But arbitrarily close to such a point, there must exist points that belong to $\mathbb{J}$. This implies a non-empty intersection of M and N, yielding a contradiction. Thus, M is empty and $J(F_\lambda) - \{p_0, \ldots, p_n\}$ is connected.

The general statement now follows easily. □

Next we show that removing the critical points from $J(F_\lambda)$ yields exactly four disjoint components.

Proposition 3.10. *Let G be the set obtained by removing the two critical points lying in a 1-disk in the Julia set. Let $\mathbb{G} = G \cap \mathbb{J}$. Then G is a connected set and $\mathbb{G}$ is arcwise connected.*

Proof. Since F_λ is a MS map, all the corners of every preimage of the trap door are distinct from the critical points that were removed to obtain G. Hence the arcs of β_λ and τ_λ that lie in G are arcwise connected by the edges of the component of τ_λ^1 that lies in G. The same is true for any other component: if A is a component of any τ_λ^j then, from Lemma 3.2, A is arcwise connected to a component of τ_λ^{j-1}. Similarly, this component is arcwise connected to a component of τ_λ^{j-2} and so on. It follows that A must be arcwise connected to τ_λ and β_λ and hence $\mathbb{G}$ is the largest arcwise connected subset in G. Similar arguments as in the previous proposition show that G is connected. □

The proof of Theorem 3.3 reduces now to consider only $\mathbb{J}$ and the removal of corner points that are not the critical points.

Proposition 3.11. *The removal of four corner points (not all critical points) from $\mathbb{J}$ results in at most three components.*

Proof. To establish the claim, we consider the remaining four possible cases, namely when the quartet of points contain one, two, three or four non-critical corner points.

One non-critical point: Assume without loss of generality that we remove the critical points c_0, c_1 and c_2 from $\mathbb{J}$. This yields three connected components, $\mathbb{G}_1$ and $\mathbb{G}_2$ (containing no critical points) and $\mathbb{G}_3$ (containing c_3). But since $\mathbb{G}_3$ is the union of two 1-disks, then c_3 is the only local cut point of $\mathbb{G}_3$, so we are done.

Two non-critical points: Here we may remove two adjacent or non-adjacent critical points. Removing two non-adjacent critical points gives two components with local cut points at the remaining critical points, so again we are done. Assume then that c_0 and c_1 are two adjacent critical points removed from $\mathbb{J}$. This yields two disjoint components $\mathbb{G}_1$ (with no critical points) and $\mathbb{G}_2$ (containing c_2 and c_3). From the previous case, we only need to consider removing p and q from either $\mathbb{G}_1$. By Corollary 3.7, p and q have to be corner points of some n-disk and some m-disk respectively. Clearly, if $n > m$ then $\mathbb{G}_1$ remains connected. If $n = m$, then $\mathbb{D}_m - \{p, q\}$ disconnects into two disjoint components. By the structure of k-disks, at least one of these components contains a lower level corner in its boundary, hence it is arcwise connected to $\mathbb{G}_1$. Thus $\mathbb{G}_1 - \{p, q\}$ becomes at most two disjoint components and $\mathbb{J} - \{p, q, c_0, c_1\}$ has at most three components.

Three non-critical points: Assume without loss of generality that we remove c_1 from $\mathbb{J}$. This yields a single component with four 1-disks (up to a point). From the previous cases, we only need to consider removing the three remaining non-critical points p, q and r from a single 1-disk. Assume all three points lie in the same k-disk, since the other cases are trivial. Then $\mathbb{D}_k - \{p, q, r\}$ becomes three disjoint components and one of them must contain a lower level corner. This yields again three components.

Four non-critical points: Assume p_1, p_2, p_3 and p_4 are non-critical points. We consider the general case of removing the four points from the same k-disk (here $k \geqslant 0$). In the worst case scenario, all the points are corner points of a single n-disk in $\mathbb{D}_k$. Then $\mathbb{D}_n - \{p_1, \ldots, p_4\}$ yields four disjoint components with at least two of them containing lower level corners in their outside boundaries and hence, they must be connected to β_λ, yielding at most three components. The other cases are trivial. □

3.3. Homeomorphisms between Julia sets

We are now able to state a key result.

Proposition 3.12. *Suppose F_λ and F_μ are MS maps. If there exists an orientation preserving homeomorphism $h : J(F_\lambda) \to J(F_\mu)$, then*

(1) *For each $k \geqslant 0$, h maps the corners of $F_\lambda^{-k}(\tau_\lambda)$ to the corners of $F_\mu^{-k}(\tau_\mu)$.*
(2) *For $k \geqslant 1$, each component of $F_\lambda^{-k}(\tau_\lambda)$ is mapped to a unique component of $F_\mu^{-k}(\tau_\mu)$.*

Proof. Theorem 3.3 establishes that the removal of the corners of τ_λ disconnects $J(F_\lambda)$ into exactly four components and no other component of τ_λ^k for $k \geqslant 1$ has this property.

Hence, the homeomorphism h cannot take τ_λ to some component of τ_μ^k when $k \geqslant 1$, since τ_λ and τ_μ are the only components of all the preimages of the respective trap door whose corner-removal separates their respective Julia set into four disjoint regions. As a consequence, we have that h maps each of the pieces $I_0, \dots, I_3$ of $J(F_\lambda)$ to one of the corresponding I_j's in $J(F_\mu)$. As in the proof of Theorem 3.3, the only set of points in the components of τ_λ^k for $k \geqslant 1$ that may separate one of the I_{s_j} (that is, a 1-disk) into four pieces is the set of corner points of the four components of τ_λ^1, and each of these components lies in a distinct I_{s_j}. Hence h maps each of the four preimages of τ_λ to a distinct preimage of τ_μ under F_μ^{-1}. Continuing inductively, we see that h must map each component of τ_λ^k to a distinct component of τ_μ^k. □

4. Model

Recall that $\mathcal{M}$ denotes the set of parameter values corresponding to MS maps. Our goal in this section is to construct a geometric model for $J(F_\lambda)$ for each $\lambda \in \mathcal{M}$.

For each $\lambda \in \mathcal{M}$ define a partition of the Julia set $J(F_\lambda)$ in four half-open regions I_0, I_1, I_2 and I_3 as defined in Section 2. Each point z in the Julia set has an address $s_0 s_1 s_2 \ldots \in \{0,1,2,3\}^{\mathbb{N}}$ defined in the natural way by its orbit in the regions I_k. For example, the fixed point contained in I_0 has itinerary $\bar{0}$; the preimage of the fixed point lying in I_2 has itinerary $2\bar{0}$ and so on.

Recall that the map F_λ satisfies the following symmetries

$$F_\lambda(-z) = F_\lambda(z), \qquad F_\lambda(iz) = F_\lambda(-iz) = -F_\lambda(z),$$

so if $c_1 = \lambda^{1/4}$, then $c_0 = -ic_1$, $c_2 = ic_1$ and $c_3 = -c_1$. If $S(z) = s_0 s_1 s_2 \cdots$ gives the address of the point z in the Julia set, then

$$\begin{aligned} S(-z) &= (s_0+2)s_1 s_2 \cdots, \\ S(iz) &= (s_0+1)(s_1+2)s_2 \cdots, \\ S(-iz) &= (s_0-1)(s_1+2)s_2 \cdots, \end{aligned}$$

where addition is taken mod 4, and

$$\begin{aligned} S\big(F_\lambda(-z)\big) &= s_1 s_2 \cdots, \\ S\big(F_\lambda(iz)\big) &= S\big(F_\lambda(-iz)\big) = (s_1+2)s_2 \cdots. \end{aligned}$$

As an example, assume $\lambda \approx -0.36428$. Then the address of c_1 is $112\bar{0}$ and $S(c_2) = 232\bar{0}$, $S(c_3) = 312\bar{0}$ and $S(c_0) = 032\bar{0}$. On the other hand, for $\lambda \approx -0.01965 + i0.2754$, $S(c_1) = 1112\bar{0}$ and thus $S(c_2) = 2312\bar{0}$, $S(c_3) = 3112\bar{0}$ and $S(c_0) = 0312\bar{0}$. The Julia sets of these examples are shown in Fig. 2.

For given $\lambda \in \mathcal{M}$, $k \geqslant 2$, we construct a homeomorphism between the skeleton $\mathbb{J}(F_\lambda)$ and a so-called model $M_k(F_\lambda) = M(F_\lambda, k)$ in the following way. Since β_λ is a simple closed curve and F_λ is conjugate to $z \mapsto z^2$ on β_λ, there exists a homeomorphism h_0 between $J(F_\lambda)$ and a set $M_0(F_\lambda)$ such that

$$h_0(\beta_\lambda) = S^1 \subset M_0(F_\lambda)$$

and h_0 is a conjugacy between F restricted to $\beta_\lambda \subset \mathbb{J}(F_\lambda)$ and the angle doubling map $z \mapsto z^2$ restricted to $S^1 \subset M_0(F_\lambda)$. We may assume that $M_0(F_\lambda)$ satisfies the same symmetry relations as F_λ and that the four half-open regions I_0, I_1, I_2 and I_3 are mapped to corresponding regions in $M_0(F_\lambda)$. We say that h_0 "straightens" β_λ to a circle with the conjugate dynamics on the circle given by angle doubling. In the next step we construct a homeomorphism h_1 between $M_0(F_\lambda)$ and a set $M_1(F_\lambda)$ which is the identity on $S^1 \subset M_0(F_\lambda)$, straightens $h_0(\tau_\lambda) \subset M_0(F_\lambda)$ to any given "nice" homeomorphic image of the (boundary of the) trap door $\tau_\lambda = \tau_\lambda^0$ (as in Fig. 5) and keeps the symmetry. Here "nice" means that the image of τ_λ is a smooth curve except at the four critical points. Successively straightening out $\tau_\lambda^1, \dots, \tau_\lambda^k$ with homeomorphisms $h_2, \dots, h_{k+1}$ which do not alter the preceding changes, we get a homeomorphism

$$h = h_{k+1} \circ \dots \circ h_0 : \mathbb{J}(F_\lambda) \to M(F_\lambda)$$

between the skeleton $\mathbb{J}(F_\lambda)$ and the *model* $M(F_\lambda) = M(F_\lambda, k)$ (we can even construct h to be a conjugacy between F_λ restricted to $\beta_\lambda \cup \tau_\lambda \cup \tau_\lambda^1 \cup \dots \cup \tau_\lambda^k$ and the induced map on $S^1 \cup h(\tau_\lambda) \cup h(\tau_\lambda^1) \cup \dots \cup h(\tau_\lambda^k)$). In pictures of $M(F_\lambda)$ we typically visualize only

$$h\big(\beta_\lambda \cup \tau_\lambda \cup \tau_\lambda^1 \cup \dots \cup \tau_\lambda^k\big) = S^1 \cup h(\tau_\lambda) \cup h\big(\tau_\lambda^1\big) \cup \dots \cup h\big(\tau_\lambda^k\big) \subset M(F_\lambda).$$

To each $z \in \beta_\lambda \subset \mathbb{J}(F_\lambda)$ we assign the angle θ of $h(z) \in S^1 \subset M(F_\lambda)$

$$\theta(z) := \frac{\angle h(z)}{2\pi} \in [0, 1].$$

Note that $\theta(z)$ is well-defined, since $h|_{\beta_\lambda} : \beta_\lambda \to S^1$ is the unique orientation preserving conjugacy (up to complex conjugation) with the angle doubling map on S^1. Indeed, the itineraries of z under F_λ and $e^{2i\pi\theta(z)}$ under $z \mapsto z^2$ (with respect to the partition $\{h(I_0), h(I_1), h(I_2), h(I_3)\}$ of $M(F_\lambda)$) are the same. Note that, since the imaginary part of λ is positive, $\theta(c_1) \in [0, \frac{1}{8}]$.

Since F_λ is an MS map, the forward orbit of $e^{2i\pi\theta(z)}$ under $z \mapsto z^2$ does not intersect the other critical points, i.e., $\theta = \theta(c_1(\lambda))$ satisfies

$$2^j\theta \bmod 1 \notin \left\{\theta, \theta + \frac{1}{4}, \theta + \frac{1}{2}, \theta + \frac{3}{4}\right\} \quad \text{for every } j \geqslant 1. \tag{4.1}$$

We define $P^1 = (0, \frac{1}{8})$ and use condition (4.1) to define for each $k \geqslant 2$ the set

$$P^k = \left\{\theta \in \left(0, \frac{1}{8}\right) \colon \text{condition (4.1) holds for } 2 \leqslant j \leqslant k\right\}.$$

Note that P^k is a union of open intervals $P_1^k, \dots, P_{i_k}^k$ for some $i_k \geqslant 2$, and we may assume that $\sup P_i^k \leqslant \inf P_j^k$ if $i < j$. For example,

$$P^2 = P_1^2 \cup P_2^2 = \left(0, \frac{1}{12}\right) \cup \left(\frac{1}{12}, \frac{1}{8}\right)$$

since the only 2-periodic orbit under $z \mapsto z^2$ is $\frac{1}{3}, \frac{2}{3}$. Hence, for (4.1) to be violated with $j = 2$, we get the relations

$$\begin{aligned}
&\theta \in \left[0, \frac{1}{8}\right],\\
&\theta + \frac{1}{4} \in \left[\frac{1}{4}, \frac{3}{8}\right] \ni \frac{1}{3},\\
&\theta + \frac{1}{2} \in \left[\frac{1}{2}, \frac{5}{8}\right],\\
&\theta + \frac{3}{4} \in \left[\frac{3}{4}, \frac{7}{8}\right]
\end{aligned}$$

which yields $\theta = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ and $2^2\theta = \theta + \frac{1}{4}$.

The only 3-periodic orbits under $z \mapsto z^2$ are $\frac{1}{7}, \frac{2}{7}, \frac{4}{7}$ and $\frac{3}{7}, \frac{6}{7}, \frac{5}{7}$, hence for (4.1) to be violated with $j = 3$ we get the relations

$$\begin{aligned}
&\theta \in \left[0, \frac{1}{8}\right],\\
&\theta + \frac{1}{4} \in \left[\frac{1}{4}, \frac{3}{8}\right] \ni \frac{2}{7},\\
&\theta + \frac{1}{2} \in \left[\frac{1}{2}, \frac{5}{8}\right] \ni \frac{4}{7},
\end{aligned}$$

$$\theta + \frac{3}{4} \in \left[\frac{3}{4}, \frac{7}{8}\right] \ni \frac{6}{7}$$

which yields the angles $\frac{1}{28}, \frac{1}{14}, \frac{3}{28}$ with $2^3\theta = \theta + \frac{1}{4}, 2^3\theta = \theta + \frac{1}{2}, 2^3\theta = \theta + \frac{3}{4}$. Hence

$$P^3 = P_1^3 \cup \cdots \cup P_5^3 = \left(0, \frac{1}{28}\right) \cup \left(\frac{1}{28}, \frac{1}{14}\right) \cup \left(\frac{1}{14}, \frac{1}{12}\right) \cup \left(\frac{1}{12}, \frac{3}{28}\right) \cup \left(\frac{3}{28}, \frac{1}{8}\right).$$

Note that $P^j \subsetneq P^k$ if $j > k$ and $\bigcap_{j=2}^{\infty} P^j$ consists exactly of those angles in $(0, \frac{1}{8})$ which do not lie on a j-periodic orbit after j-times angle doubling for any $j \geqslant 2$. Now for each $k \geqslant 2$ the sets $P_1^k, \ldots, P_{i_k}^k$ define a partition of the set $\mathcal{M}$ of MS parameter values

$$\mathcal{M} = \bigcup_{i=1}^{i_k} \mathcal{M}_i^k, \quad \mathcal{M}_i^k := \{\lambda \in \mathcal{M}\colon \theta(c_1(\lambda)) \in P_i^k\}.$$

Choose $\lambda \in \mathcal{M}_1^2$ and $\mu \in \mathcal{M}_2^2$. Now we display $M(F_\lambda)$ as follows:

(1) Draw $h(c_0), h(c_1), h(c_2), h(c_3)$ and the (homeomorphic image of the) trap door $h(\tau_\lambda)$. Note that $0 < \theta = \theta(c_1) < \frac{1}{12}$.
(2) Draw the component of $h(\tau_\lambda^1)$ in $h(I_0)$, starting with the corners x_0, x_1, x_2 and x_3 (see Fig. 6). Note that, due to Lemma 3.2, two corners are in $h(\tau_\lambda)$, say x_0, x_1, and then $F_\lambda(h^{-1}(x_0)) = c_2$, $F_\lambda(h^{-1}(x_1)) = c_3$. The other two corners x_2, x_3 are on $S^1 \subset M(F_\lambda)$ and as preimages under angle doubling they have the angles $\frac{\theta}{2}$ and $\theta + \frac{3}{4} + \frac{1}{2}(1 - (\theta + \frac{3}{4})) = \frac{\theta}{2} + \frac{7}{8}$.
(3) Draw the other components of $h(\tau_\lambda^1)$ in $h(I_1), h(I_2), h(I_3)$ using the symmetry.
(4) Draw the components of $h(\tau_\lambda^2)$ in the "triangle" in $h(I_0)$ with corners $x_0, x_3, h(c_1)$, starting with the corners y_0, y_1, y_2, y_3. Note that due to Lemma 3.2 two corners are in $h(\tau_\lambda^2) \cap h(I_0)$, say y_0, y_1, and $F_\lambda(h^{-1}(y_0)) = h(ix_0)$, $F_\lambda(h^{-1}(y_1)) = h(ix_1)$. To find the location of y_2, y_3, note that the (short) arc in S^1 from x_3 to $h(c_1)$ maps under angle doubling to the (short) arc from $h(c_1) = e^{2\pi i\theta}$ to $e^{2\pi i 2\theta}$. The angle of the corner ix_2 is $\frac{\theta}{2} + \frac{7}{8} + \frac{1}{4} \bmod 1 = \frac{\theta}{2} + \frac{1}{8}$ which is larger than 2θ (since $\theta < \frac{1}{12}$), showing that no preimage of ix_2 and ix_3 is contained in the arc from x_3 to $h(c_1)$. This fact or a similar direct argument shows that y_2, y_3 lie on $h(\tau_\lambda)$ and $F_\lambda(h^{-1}(y_2)) = h(ix_2)$, $F_\lambda(h^{-1}(y_3)) = h(ix_3)$.
(5) Draw the remaining three components of $h(\tau_\lambda^2)$ in $h(I_0)$ using similar arguments. Draw the other components of $h(\tau_\lambda^2)$ in $h(I_1), h(I_2), h(I_3)$ using the symmetry.

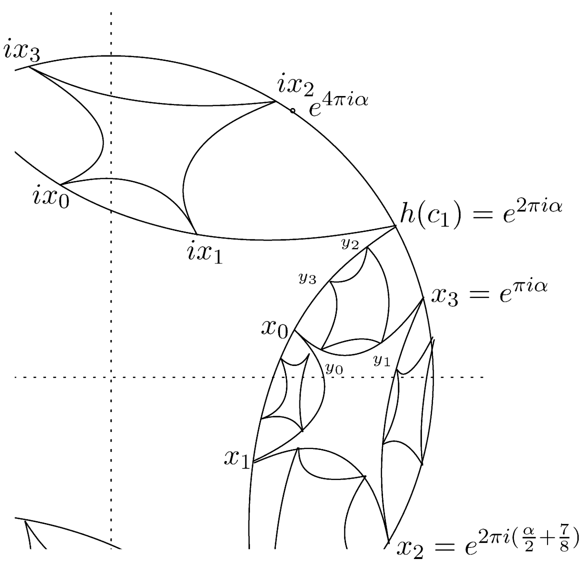


Fig. 6. The model $M(F_\lambda)$ of Julia set for $\theta(c_1(\lambda)) = 5/64$.

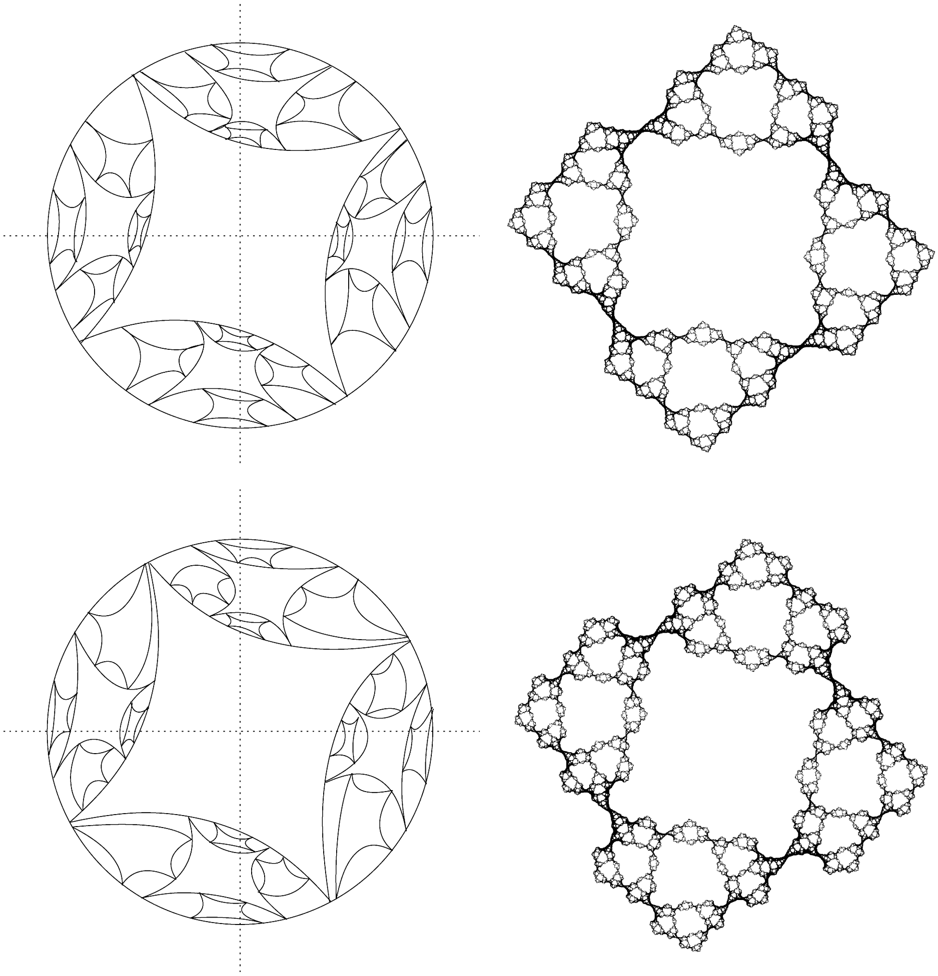


Fig. 7. The topological model and the Julia set are displayed on top when $\mu \approx -0.246 + i0.15913$. The address of $c_1(\mu)$ is $11232\bar{0}$ and $\theta(c_1(\mu)) = 3/32 > 1/12$. The bottom images show the model and the Julia set when $\lambda \approx -0.12713 + i0.21384$. The address of $c_1(\lambda)$ is $111132\bar{0}$ and $\theta(c_1(\lambda)) = 5/64 < 1/12$.

Next we display $M(F_\mu)$. For simplicity let h denote again the homeomorphism. The only change is step (4) when the locations of y_2, y_3 are determined. We replace it with

(4′) To find the location of y_2, y_3, again note that the (short) arc in S^1 from x_3 to $h(c_1)$ maps under angle doubling to the (short) arc from $h(c_1) = e^{2\pi i\theta}$ to $e^{2\pi i 2\theta}$. The angle of the corner ix_2 is $\frac{\theta}{2} + \frac{1}{8}$ which is less than 2θ (since $\theta > \frac{1}{12}$), showing that the preimage of ix_2 is contained in the arc from x_3 to $h(c_1)$ and hence y_2 lies on β_μ. This fact or a similar direct argument show that y_3 lies on $h(\tau_\mu)$.

Two examples of the models and their Julia sets corresponding to parameters $\lambda \in \mathcal{M}_1^2$ and $\mu \in \mathcal{M}_2^2$ are displayed in Fig. 7.

Angles of bifurcation

Before we continue showing that $J(F_\lambda)$ and $J(F_\mu)$ are not homeomorphic, we observe that our model makes sense even for the angle $\theta = \frac{1}{12}$, although the parameter associated to this angle does not correspond to a Misiurewicz parameter value. Indeed, numerical evidence shows that for this parameter there exists a parabolic cycle of period two and the critical orbit lies in the basin of attraction of the cycle. Therefore, the Julia set is not homeomorphic to our model. Nevertheless, if we identify each Fatou component of the parabolic cycle to a point, we conjecture that the resulting set is homeomorphic to the corresponding model.

For $\theta = \frac{1}{12}$, the point with angle θ (which we still denote by $h(c_1)$) is a preimage (under angle-doubling) of ix_2 with angle $\frac{\theta}{2} + \frac{1}{8} = 2\theta$, hence $y_2 = h(c_1)$ (cp. with Fig. 6). We will see that $J(F_\lambda)$ and $J(F_\mu)$ are not homeomorphic, hence we have a (topological) bifurcation of the model at $\theta = \frac{1}{12}$, more precisely we call $\theta \in (0, \frac{1}{8})$ a

level-k bifurcation, if $\quad \theta \in P^{k-1} \quad$ and $\quad \theta \notin P^k$

At every level-k bifurcation, $k \geqslant 2$, the point $h(c_1)$ in the model (and therefore also $h(c_0), h(c_2), h(c_3)$ by symmetry) equals a corner point in a component of $h(\tau_\lambda^k)$ by definition of P^k. Now for the model $M(F_\lambda)$

the points $h\big(F_\lambda^{-i}\big(\{c_0, \ldots, c_3\}\big)\big) \cap h\big(\tau_\lambda^i\big)$ are called level-i corners

e.g. $h(c_1)$ in Fig. 6 is a level-0 corner, and $x_0, \ldots, x_3 \in h(\tau_\lambda^1) \cap h(I_0)$ are level-1 corners. Then at a level-2 bifurcation a 0-corner equals a 2-corner. In fact more can be said. Since each level-i corner, $i \geqslant 1$, is a preimage of a level-$(i-1)$ corner, we immediately get the following lemma:

Lemma 4.1. *At each level-k bifurcation, $k \geqslant 2$, infinitely many corners pairwise meet. For each $i, j \in \mathbb{N}_0$ with*

$$|i - j| = k$$

a level-i and a level-j corner coincide.

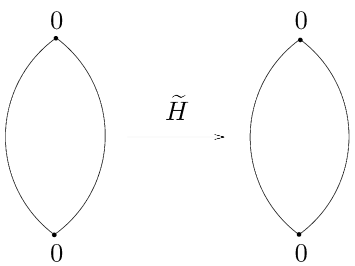
Now we can formulate the first case of our main result.

Theorem 4.2. *$J(F_\lambda)$ and $J(F_\mu)$ are not homeomorphic for $\lambda \in \mathcal{M}_1^2$ and $\mu \in \mathcal{M}_2^2$.*

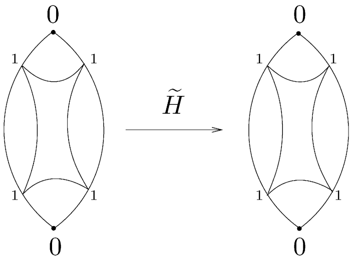
Proof. Assume that H is a homeomorphism between $J(F_\lambda)$ and $J(F_\mu)$, let $\widetilde{H} = h_\mu \circ H \circ h_\lambda^{-1}$ denote the induced homeomorphism between $M(F_\lambda)$ and $M(F_\mu)$ with the homeomorphisms h_λ and h_μ between the corresponding Julia sets and models.

$$\begin{tikzcd}
J(F_\lambda) \arrow[r, "h_\lambda"] \arrow[d, "H"'] & M(F_\lambda) \arrow[d, "\widetilde{H}"] \\
J(F_\mu) \arrow[r, "h_\mu"] & M(F_\mu)
\end{tikzcd}$$

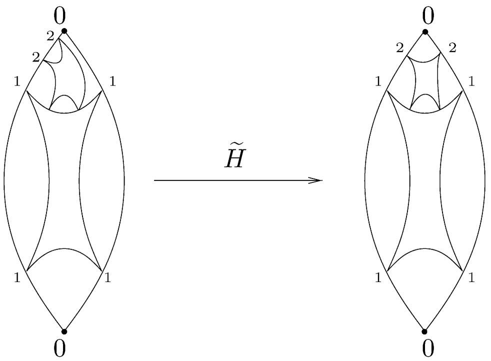
Since H maps corners of τ_λ to corners of τ_μ by Lemma 3.2, it follows that two adjacent level-0 corners $h_\lambda(c_i)$, $h_\lambda(c_{i+1 \bmod 4})$ in $M(F_\lambda)$ are mapped by $\widetilde{H}$ onto two level-0 corners in $M(F_\mu)$ and they also have to be adjacent because otherwise a path in $M(F_\lambda)$ connecting $h_\lambda(c_i)$, $h_\lambda(c_{i+1 \bmod 4})$ with no other level-0 corner on it would be mapped to a path in $M(F_\mu)$ with at least one other level-0 corner on it, a contradiction. So far we have the following picture for $\widetilde{H}$ restricted to $h_\lambda(\tau_\lambda \cap \beta_\lambda)$ within one of the four sectors.



The 0's indicate level-0 corners. Now Lemma 3.2 implies the following picture for the level-1 corners and their connecting paths in $h_\lambda(\tau_\lambda^1)$



where the unknown homeomorphism $\widetilde{H}$ is restricted only by the fact that it maps level-i corners on level-i corners. Without loss of generality we assume that $\widetilde{H}$ maps the upper level-0 corner to the upper level-0 corner. From the construction of the model $M(F_\lambda)$ we get



a contradiction, since in $M(F_\lambda)$ there exists a path from a level-0 corner to a level-1 corner with exactly two level-2 corners on it whereas the $\widetilde{H}$ image of this path contains only one level-2 corner. This proves that $J(F_\lambda)$ and $J(F_\mu)$ are not homeomorphic. □

We now state our main result.

Theorem 4.3. *For any two MS maps F_λ and F_μ with $\lambda \neq \overline{\mu}$, their respective Julia sets are not topologically equivalent.*

The existence of a conjugacy between two MS maps implies the existence of a homeomorphism of the Julia sets. Thus it follows from the above Theorem,

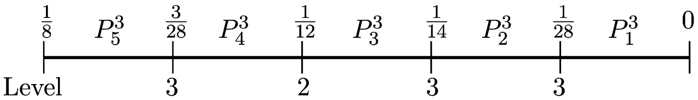
Corollary 4.4. *If F_λ and F_μ are two conjugated MS maps, then either $\lambda = \overline{\mu}$ or $\lambda = \mu$.*

Proving that any two MS Julia sets are not topologically equivalent reduces to combinatorics for computation of the sets P_i^k and uses the fact that for any two MS parameter values $\lambda, \mu \in \mathcal{M}$ there exists k, i, j such that

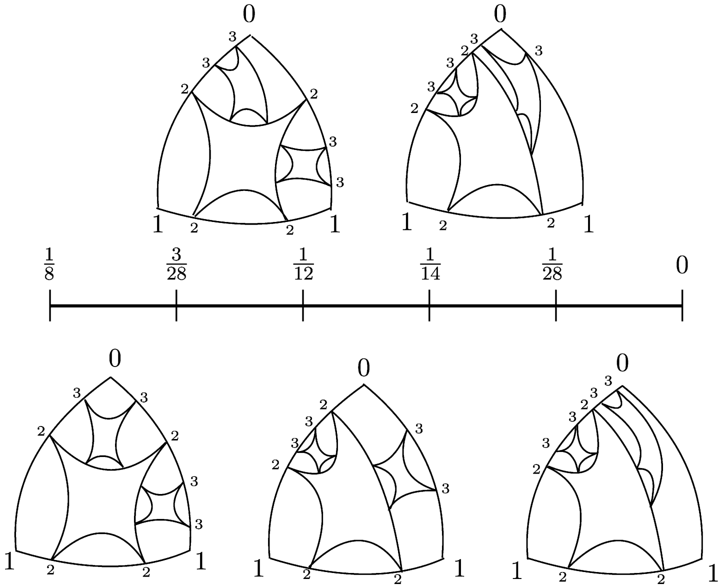
$$\lambda \in P_i^k \quad \text{and} \quad \mu \in P_j^k.$$

It is clear that there always exists a $k \geqslant 2$ such that λ and μ are not in the same $\mathcal{M}_i^k$, and therefore the model undergoes at least one bifurcation which changes the topology when passing from λ to μ. The same argument then shows that the level-k models $M(F_\lambda, k)$ and $M(F_\mu, k)$ are not homeomorphic.

To explain the ideas for the general case, we do it explicitly for all bifurcations up to level 3, the angles at which the bifurcations occur are displayed in the following figure from $\frac{1}{8}$ down to 0 because the angle $\theta(c_1(\lambda))$ is decreasing if the real part of $\lambda \in \mathcal{M}$ is increasing.



For $\lambda \in \mathcal{M}_i^3$ the angle $\theta(c_1(\lambda))$ is in P_i^3. To see that $J(F_\lambda)$ and $J(F_\mu)$ are not homeomorphic for $\lambda \in \mathcal{M}_i^3$, $\mu \in \mathcal{M}_j^3$, $i \neq j$, we can apply Theorem 4.2 if $(i, j) \in \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$ because then $\lambda \in \mathcal{M}_1^2$, $\mu \in \mathcal{M}_2^2$. If $(i, j) \in \{(1,2), (1,3), (2,3), (4,5)\}$ then it is enough to look at the following subsets of the models which are displayed in the next figure.



In each case we get a contradiction, since either in $M(F_\lambda)$ or $M(F_\mu)$ there is a path with a sequence of corners of specific levels which cannot be found in the other model. More precisely

(i,j)	unique path
$(1,2)$	0 • —— • 1
$(1,3)$	0 • —— • 1
$(2,3)$	0 • —— • 3 —— • 1
$(4,5)$	0 • —— • 2

We just proved the following extension of Theorem 4.2.

Theorem 4.5. *$J(F_\lambda)$ and $J(F_\mu)$ are not homeomorphic for $\lambda \in \mathcal{M}_i^3$ and $\mu \in \mathcal{M}_j^3$ with $i \neq j$.*

5. The general case

As it was mentioned in the introduction, the family $G_\lambda(z) = z^2 + \lambda/z$ is a special family of rational maps that besides sharing many of the dynamical properties of the family F_λ, it also possesses a unique critical orbit up to symmetry. To see this, note that G_λ has three finite, nonzero critical points given by the cubic roots of $\lambda/2$. Let $\omega = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})$, so $\omega^3 = 1$. As before, denote by c_1 the critical point that lies in the first quadrant and set $c_2 = \omega c_1$ and $c_0 = \omega^2 c_1$. It is not difficult to see that

$$G_\lambda(\omega z) = \omega^2 G_\lambda(z) \quad \text{and thus} \quad G_\lambda(\omega^2 z) = \omega G_\lambda(z).$$

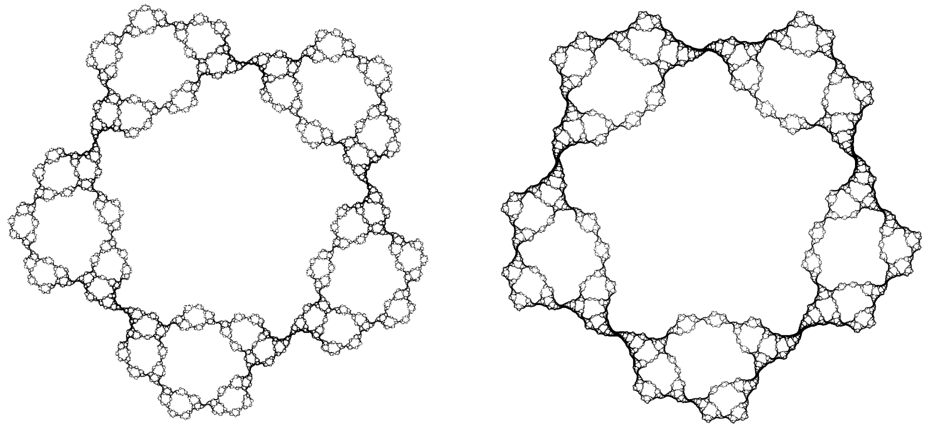


Fig. 8. The Julia sets of $R_{\lambda,3,2}$ with $\lambda \approx 0.221166 + i0.224453$ is to the left. To the right is the Julia set of $R_{\lambda,2,3}$ with $\lambda \approx -0.01965 + i0.2754$.

Due to these symmetries, the behavior of one critical point determines the behavior of the other two. As an example, consider the parameter $\lambda \approx -0.59257$ corresponding to the Julia set homeomorphic to the Sierpinski triangle (see Fig. 1). G_λ maps c_1 into the fixed point that lies in the boundary of the immediate basin of infinity exactly after two iterations. On the other hand, c_0 and c_2 land in a two periodic cycle after the same number of iterations.

A similar analysis can be performed to a more general class of rational maps of the form $R_{\lambda,n,m}(z) = z^n + \lambda/z^m$, with $m, n > 1$. Straightforward computations show the point at infinity is again a superattracting fixed point, the origin is a pole of order $n+m$ and there exist $n+m$ finite, simple, nonzero critical points with a unique critical orbit up to symmetry. These symmetries are given by the equation

$$R_{\lambda,n,m}\left(\omega^k z\right) = \omega^{nk} R_{\lambda,n,m}(z),$$

for $k = 1, 2, \ldots, n+m-1$ and ω the $(n+m)$th root of unity.

As before, let β_λ denote the boundary of the immediate basin at infinity, τ_λ denote the boundary of the trap door and τ_λ^k denote the union of all of the components of $R_{\lambda,n,m}^{-k}(\tau_\lambda)$. It is known that all these boundaries are simple closed curves whenever λ is a Misiurewicz value (see [3,2]).

Note that $R_{\lambda,n,m}$ is conjugate to the map $z \mapsto z^n$ in a neighborhood of infinity, and again, this conjugacy can be extended to β_λ. Hence, we may extend the results of the previous sections in order to conclude the existence of MS maps for the $R_{\lambda,n,m}$ families.

In Fig. 8 we display two Julia sets corresponding to MS maps for the families $R_{\lambda,3,2}$ and $R_{\lambda,2,3}$. Although both rational maps have degree five, the reader can clearly distinguish different configurations of the five corners of τ_λ^1 along the boundary of β_λ and τ_λ. Indeed, while the Julia set of $R_{\lambda,3,2}$ has two corners lying in the boundary of its trap door, $R_{\lambda,2,3}$ has only three corners. In general, we may characterize the associated generalized Sierpinski gaskets of the families $R_{\lambda,n,m}$ by the following lemma, which is a generalization of Lemma 3.1 and we state it without proof.

Lemma 5.1. *If A is a component in τ_λ^k with $k \geqslant 1$, then exactly m corners of the $n+m$ corners of A lie in an edge of a single component of τ_λ^{k-1}.*

In the same fashion, a generalization of the algorithm described for the family $F_\lambda(z) = z^2 + \lambda/z^2$ in Section 4 can be performed to distinguish when two Julia sets of MS maps in the family $R_{\lambda,n,m}$ are not homeomorphic.

References

[1] P. Blanchard, R.L. Devaney, D. Look, P. Seal, Y. Shapiro, Sierpinski-curve Julia sets and singular perturbations of complex polynomials, Ergodic Theory Dynam. Systems 25 (4) (2005) 1047–1055.

[2] R.L. Devaney, D. Look, A criterion for Sierpinski curve Julia sets for rational maps, Topology Proc., in press.

[3] R.L. Devaney, D. Look, D. Uminsky, The escape trichotomy for singularly perturbed rational maps, Indiana Univ. Math. J. 54 (2005) 1621–1634.

[4] A. Kameyama, Julia sets of postcritically finite rational maps and topological self-similar sets, Nonlinearity 13 (2000) 165–188.
[5] K. Kuratowski, Topology, vol. II, Academic Press, New York, 1968.
[6] R. Mañé, On a theorem of Fatou, Bol. Soc. Brasil. Mat. (N.S.) 24 (1) (1993) 1–11.
[7] J. Milnor, Dynamics in One Complex Variable, Vieweg, New York, 1999.
[8] L. Tan, Hausdorff dimension of subsets of the parameter space for families of rational maps, Nonlinearity 11 (2) (1998) 233–246.