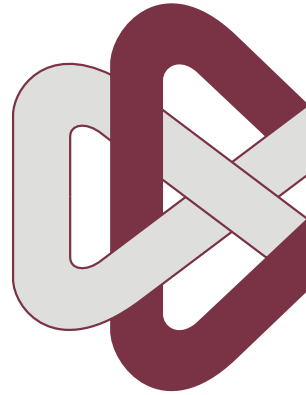


A motion planner for nonholonomic mobile robots



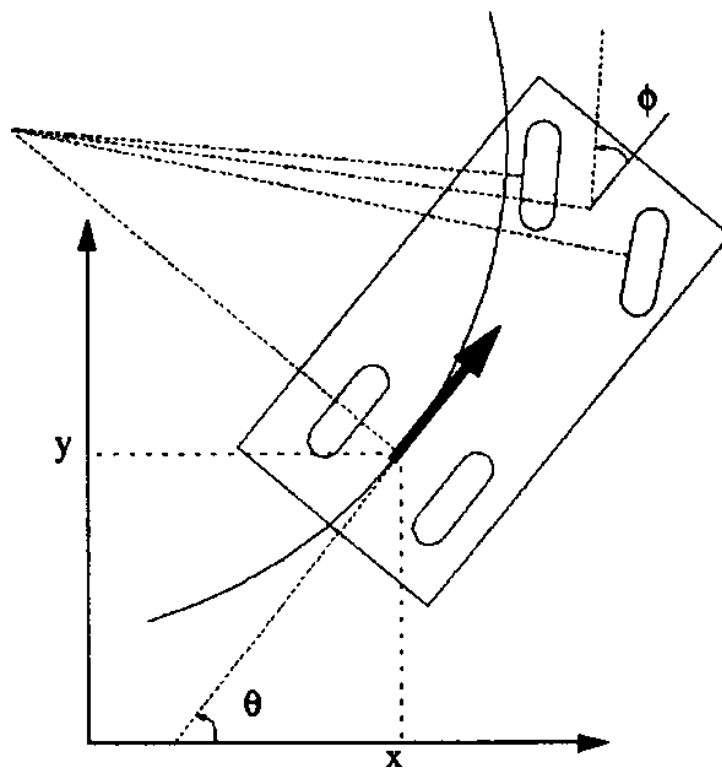
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Material taken from: J. P. Laumond, P. E. Jacobs, M. Taix, R. M. Murray. A motion planner for nonholonomic mobile robots. IEEE Transactions on Robotics and Automation. Vol. 10-5. Pages 577- 593. Oct 1994.

The car like robot

From the driver's point of view, a car has two degrees of freedom: the accelerator and the steering wheel. The reference point is the midpoint of the rear wheels. The distance between front and rear axes is 1.



Velocity $\dot{q} = (\dot{x} \quad \dot{y} \quad \dot{\theta})^T$, can not assume arbitrary values, it has the nonholonomic constraint

$$(-\sin \theta \quad \cos \theta \quad 0) (\dot{x} \quad \dot{y} \quad \dot{\theta})^T = 0,$$

that means that the car can not slip on the surface (zero lateral velocity).

Let be v the speed of the front wheels ($v \leq 1$) and ϕ the angle between the front wheels and the main direction of the car ($|\phi| \leq \frac{\phi}{4}$).

The control system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} v \cos \phi \cos \theta \\ v \cos \phi \sin \theta \\ v \sin \theta \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v_2.$$

For this problem, the position of the front wheels and the vehicle speed are not relevant. Thus we will work only with the simplified system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} v \cos \phi + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v \sin \phi.$$

Defining the control as $u_1 = v \cos \phi$ and $u_2 = v \sin \phi$, with $|u_2(t)| \leq |u_1(t)| \leq 1$ the system can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Property 1: The Car-like system is controllable

It suffices to consider two constant admissible controls that respect the curvature bounds. The straight-line motion (corresponding to the vector field X_1)

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the arc of circle of minimal radius (corresponding to the vector field $X_1 + X_2$)

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The coordinates of $[X_1, X_1 + X_2]$ are:

$$X_1 = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \quad X_1 + X_2 = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix}; \quad [X_1, X_1 + X_2] = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}.$$

Definition: A system is **locally controllable** from some point c , if there is a neighborhood of c all of whose points are reachable from c by an admissible path.

A system is controllable if it is locally controllable at every point.

Proof using Campbell-Baker-Hausdorff-Dynkin formula

It is easy to see that

$$\{X_1, [X_1, X_1 + X_2], X_1 + X_2\} \quad (1)$$

spans the tangent space at every point.

Proving that the system is locally controllable from the origin would hold the proof for every point.

Let X be a vector field, following X for a time a is the same as taking e^{aX} . The exponential describes a motion from a point to another on a given path. Following aX for a given time and then bY , leaves us at the point $e^{aX} \cdot e^{bY}$. The exponential of a vector field appears as an operation on the manifold.

Let $c = (t_1, t_2, t_3)$, a point near the origin, in the coordinate basis induced by (1). c is reachable from the origin by the following flow

$$\phi(t_1, t_2, t_3) = e^{t_1 X_1} e^{t_2 [X_1, X_1 + X_2]} e^{t_3 (X_1 + X_2)}. \quad (2)$$

The first and the third flows obey the constraints on the controls. For the second the following approximation is taken (for $t > 0$) given by the Campbell-Baker-Hausdorff-Dynkin formula

$$e^{t[X_1, X_1+X_2]+o(t^{3/2})} = e^{t^{1/2}X_1} e^{t^{1/2}(X_1+X_2)} e^{-t^{1/2}X_1} e^{-t^{1/2}(X_1+X_2)}.$$

This shows that any configuration obtained by (2) can be approximated by the flow

$$\tilde{\phi}(t_1, t_2, t_3) = e^{t_1 X_1} e^{t_2^{1/2} X_1} e^{t_2^{1/2}(X_1+X_2)} e^{-t_2^{1/2} X_1} e^{-t_2^{1/2}(X_1+X_2)} e^{t_3(X_1+X_2)},$$

which obey the constraints.

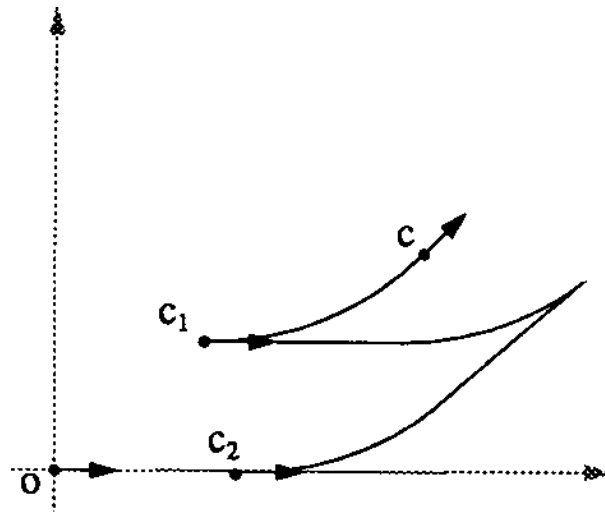
The mapping $\tilde{\phi}$ is a local homeomorphism, the inverse image of a neighborhood of the origin in the configuration is a neighborhood of $(0, 0, 0)$ in \mathbb{R}^3 .

A choice exists for t_1 , t_2 and t_3 that exactly attains any given configuration in a neighborhood of the origin. Hence, the system is controllable from the origin.

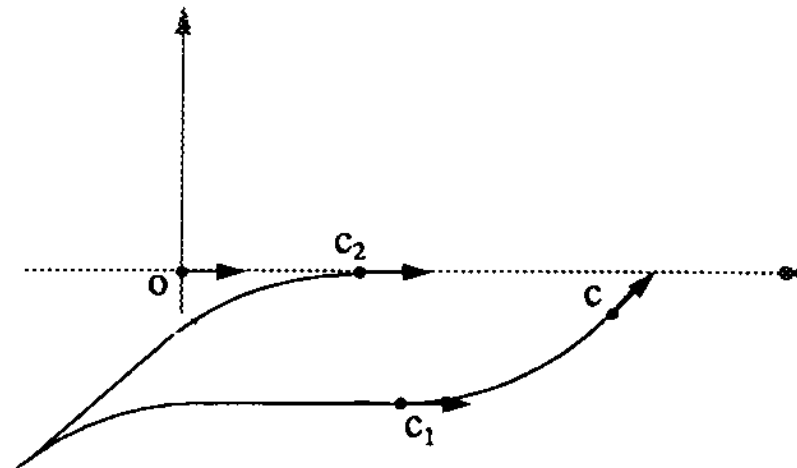
The flow $\tilde{\phi}$ corresponds exactly to the path provided by the direct proof (next section).

Direct proof

It suffices to prove that the system is locally controllable from the origin. Let $c = (x, y, \theta)$, a point near the origin.

Case : $\theta > 0$ and $y - (1 - \cos \theta) > 0$

(a)

Case : $\theta > 0$ and $y - (1 - \cos \theta) < 0$

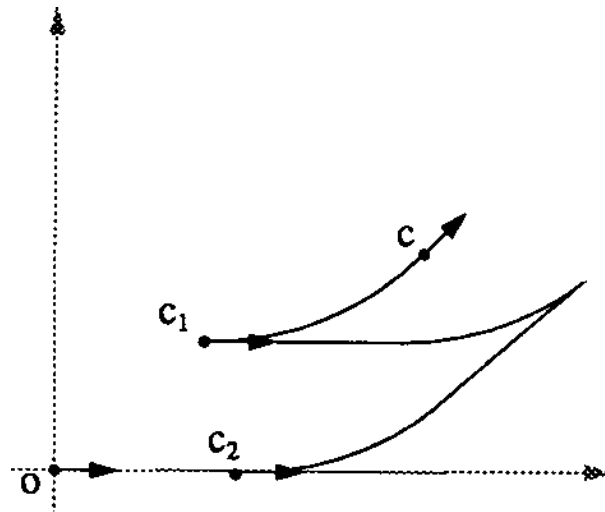
(b)

Let γ_1 the arc of circle tangent to c with length θ (assuming $\theta \geq 0$ W.L.O.G.). Moving with direction -1 , it attains the point

$$c_1 = (x - \sin \theta, y - (1 - \cos \theta), 0),$$

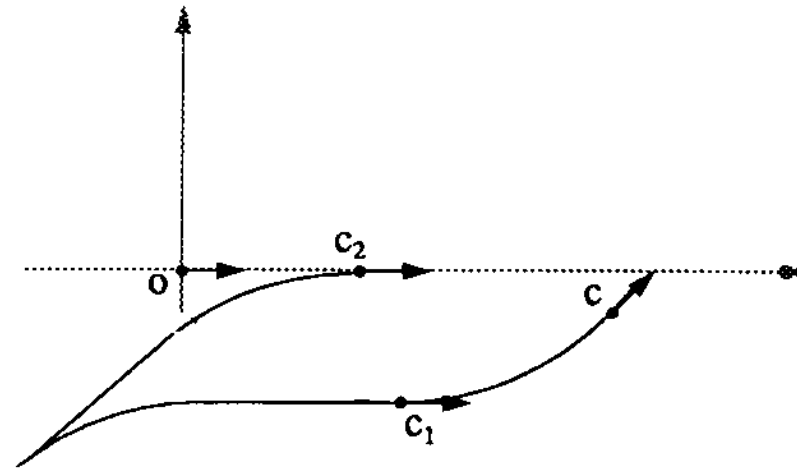
assuming $y - (1 - \cos \theta) > 0$.

Other assumptions are processed in the same way (see figure).



Case : $\theta > 0$ and $y - (1 - \cos \theta) > 0$

(a)



Case : $\theta > 0$ and $y - (1 - \cos \theta) < 0$

(b)

Let $\mathcal{Y}_2(\tau)$ be the path consisting of four pieces of same length τ :

- a forward motion on a straight line segment
- a forward motion on an arc of circle
- a backward motion on a straight line segment
- a backward motion on an arc of a circle

The coordinates of the point attained by this sequence are

$$(x_1 + \tau - \tau \cos \tau, y_1 - \tau \sin \tau, 0),$$

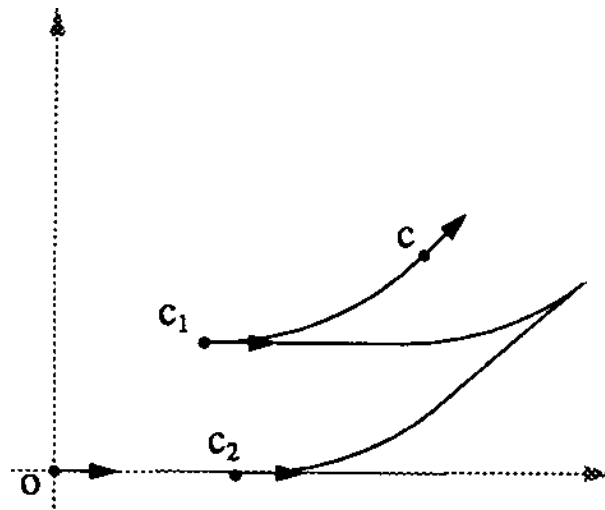
choosing τ_c such that

$$|y - (1 - \cos \theta)| = \tau_c \sin \tau_c,$$

this τ_c always exists and is unique for any c sufficiently near the origin.

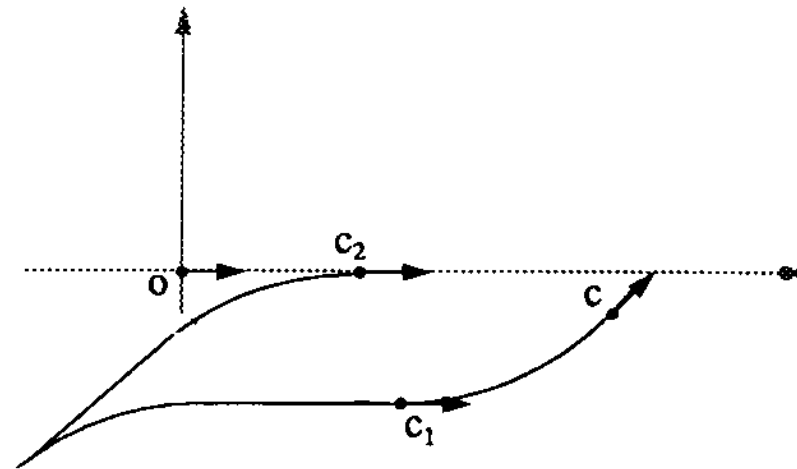
The coordinates of point c_2 attained by $\gamma_2(\tau_c)$ are

$$c_2 = \left(x - \sin \theta + \tau_c (1 - \cos \tau_c), 0, 0 \right).$$



Case : $\theta > 0$ and $y - (1 - \cos \theta) > 0$

(a)



Case : $\theta > 0$ and $y - (1 - \cos \theta) < 0$

(b)

Finally, let γ_3 be the straight line motion from c_2 to the origin.

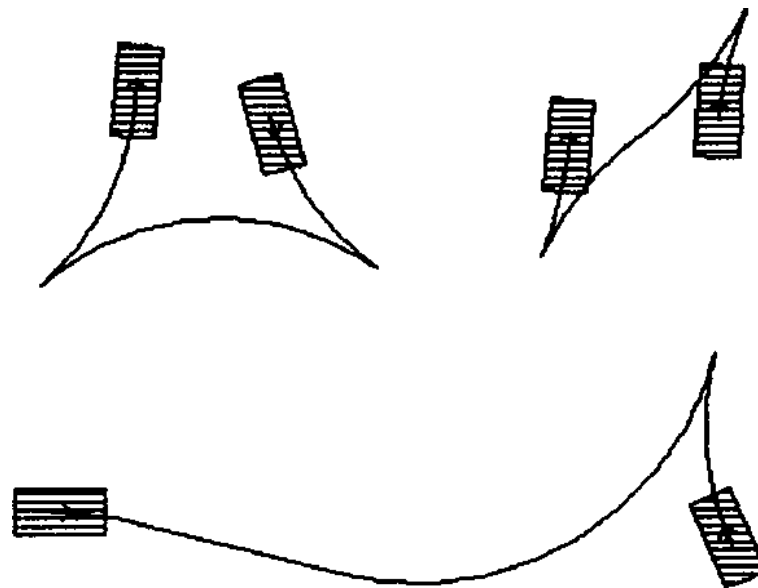
The path formed by the sequence γ_1 , γ_2 and γ_3 , followed in reverse direction, goes from the origin to c . Thus the car-like system is locally controllable.

Shortest paths for a car-like robot

For car-like robots in absence of obstacles and linear velocity control $u_1=1$, Dubins proved that shortest paths are curves of class C^1 composed of

- Arcs of circle with radius 1
- Straight line segments

Reeds and Sheep extended Dubin's work for car-like systems where u_1 can take positive and negative values, this allow maneuvers, or cusps, along the path. Between cusps the paths follow the form given by Dubins.



Any path with more than two cusps can be reduced to a path with at most two cusps.

This model assume linear velocity to be constant and equal to 1.

For more general systems where the linear velocity is upper bounded by 1, Sussmann and Tang proved that the shortest paths are the same founded for Reed and Sheep. In this systems the constains are $|u_2(t)| \leq |u_1(t)| \leq 1$.

The shortest path metric and sub-Riemannian geometry

Having the exact form of the shortest paths for the car-like system, it can be algorithmically compute the arclength in the plane of the shortest path connecting any two configurations.

There is one-to-one correspondece between the paths in the Euclidean plane \mathbb{R}^2 and the paths in the configuration space $\mathbb{R}^2 \times S^1$ that satisfy the nonholonomic constrain. This distance denotes a metric in the configuration space, lets call it d_{RS} .

In other words, d_{RS} is the metric induced by the lenght of the shortest paths between two configurations.

Also, d_{RS} allows the determination of the reachable set in the presence of obstacles.

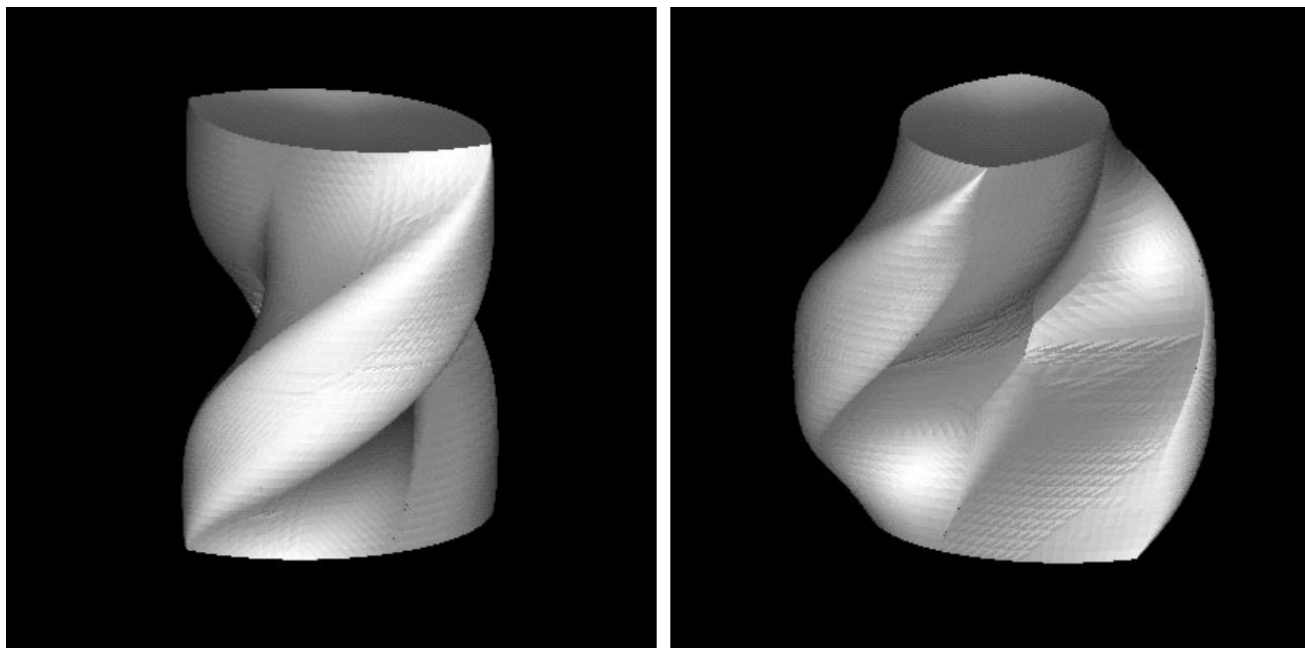
Property 2

For any point $c = (x, y, \theta)$ sufficiently near to the origin $o = (0, 0, 0)$,

$$\frac{1}{3}(|x| + |y|^{1/2} + |\theta|) \leq d_{\text{RS}}(c, o) \leq 12(|x| + |y|^{1/2} + |\theta|).$$

The shape of d_{RS} implies that the associated topology and the Euclidean are the same.

Corollary: For each neighborhood (in the Euclidean topology) $N(c)$ of a configuration c , there exists a neighborhood (in the Euclidean topology) $N'(c)$ such that for any configuration $c' \in N'(c)$ the path corresponding to the shortest path between c and c' is included in $N(c)$.



The comparison of the two topologies is done using the methods of sub-Riemannian geometry.

Metrics can be defined by minimizing the length of all trajectories linking two given points.

For nonholonomic systems, these metrics are said to be sub-Riemannian, or singular.

For the car-like system this metric is obtained by minimizing $\int (u_1^2 + u_2^2)^{1/2} dt$. The shortest path metric consists in minimizing the integral of the linear speed $\int |u_1| dt$. These are equivalent.

Computational consequences

Any path for the holonomic system (included in an open set of the admissible configuration space) can be discretized into a finite number of points such that, if one joins two consecutive points of the path by a Reeds-Shepp curve, one obtains a new path that constitutes a feasible collision-free path for the nonholonomic system.

Property 2 provides an **upper bound** of the number of subdividing points required by the method.

Algorithm analysis

Convergence and completeness

Given any $\epsilon > 0$, there exists $\delta > 0$ such that for any two configurations that are separated by less than δ , all the configurations along the shortest path connecting them will lie in some neighborhood of diameter ϵ of the two configurations.

This is consequence of the corollary of property 2.

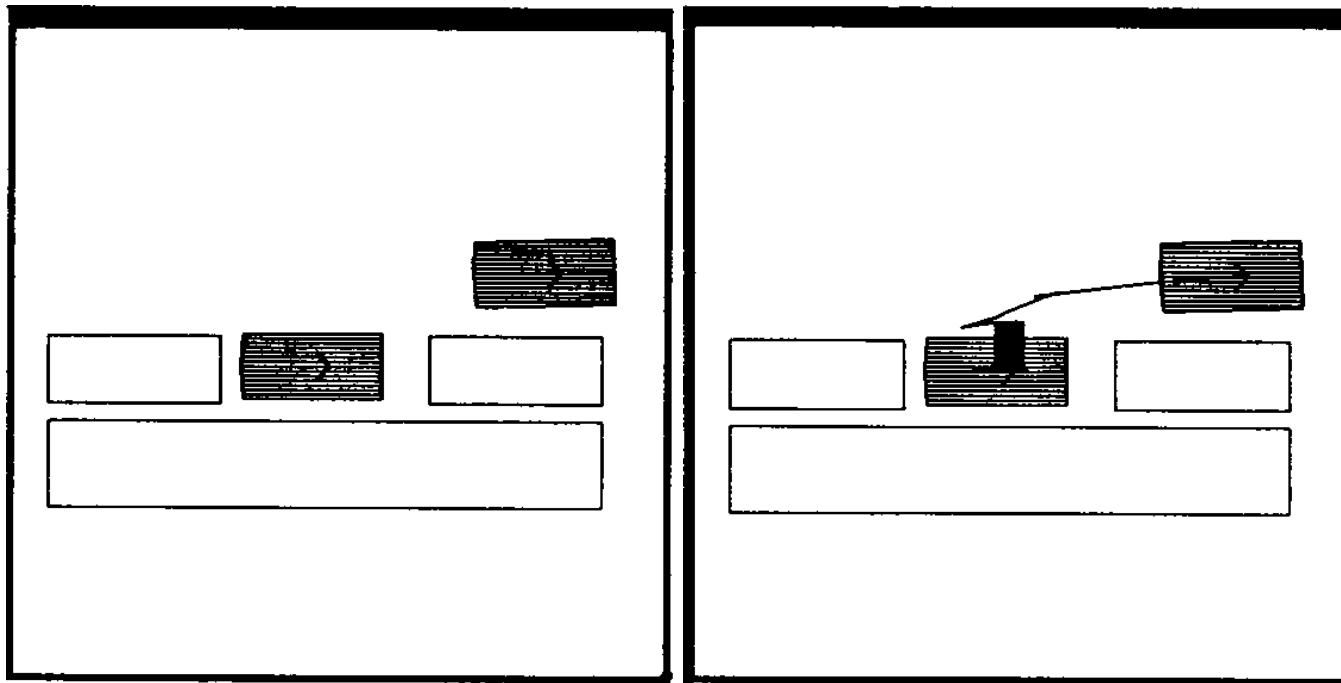
This is enough to prove both convergence and completeness of the algorithm, because if one continue dividing the path, at some point a sequence of configurations will be generated that are sufficiently close that the shortest paths linking each of these configurations must lie in the free configuration space.

Complexity

Property 2 says that in the worst case the length of the feasible path connecting two configurations is of the order of the square root of their separation measured in the Euclidean norm, i.e. δ is in $O(\epsilon^{1/2})$. Near the origin $|x| + |y|^{1/2} + |\theta|$ is dominated by $|y|^{1/2}$.

If the geometric planner computes a path for which all the configurations are contained in the free space, then the geometric path must be cut in pieces of length at most $O(\epsilon^2)$ in order to guaranteed that the feasible path joining them does not leave the free configuration space.

The worst case is reached in the case of the parking task.

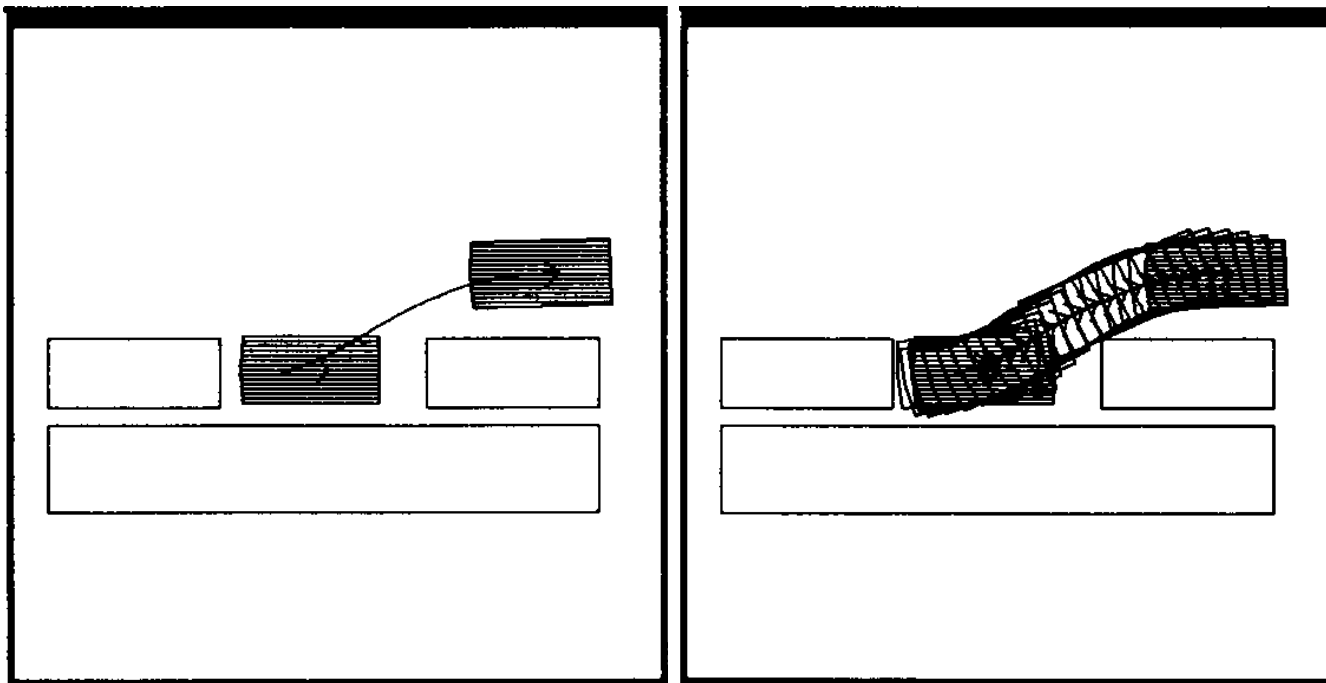


If L is the length of the walls, and ϵ is the difference between the width of the corridor and the length of the car, then the algorithm runs in $O(L/\epsilon^2)$.

The algorithm is more efficient as the geometric path is farther from the obstacles.

The complexity also depends on the lower bound ρ of the turning radius. The algorithm runs in $O(\rho/\epsilon^2)$.

It is possible to reduce the number of pieces of the shortest path.



The number of maneuvers for the classical car-parking increases as the square of the decreasing free space ϵ .

The algorithm does not find an optimal length path (it requires an open set to move in). To do so the algorithm would need to work with contact with the obstacles.

Complexity of the complete problem

The following quantities are needed:

n Geometric complexity of the obstacles

m Geometric complexity of the robot

ϵ Minimum “size” of the free space

ρ Minimum turning radius

σ “Complexity” of the output path

n and m are the classical parameters used for evaluating the complexity of the methods that solve the piano-mover problem.

Let c be any point in the free configuration space and $B(c, \epsilon)$ the biggest Riemannian ball containing c , with radius ϵ . Then ϵ is defined as $\text{Min}_c(\epsilon)$. By property 2, the number of Reeds and Shepp balls of radius ϵ required to cover a Riemannian ball of radius ϵ is $O(\epsilon^{-2})$.

The complexity of σ can be characterized by the number of elementary pieces of the solution path

Experimental results

