CONICAL REPRESENTATIONS FOR DIRECT LIMITS OF RIEMANNIAN SYMMETRIC SPACES

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Table of Contents

Abstract

We extend the definition of conical representations for Riemannian symmetric space to a certain class of infinite-dimensional Riemannian symmetric spaces. Using an infinite-dimensional version of Weyl's Unitary Trick, there is a correspondence between smooth representations of infinite-dimensional noncompact-type Riemannian symmetric spaces and smooth representations of infinite-dimensional compact-type symmetric spaces. We classify all smooth conical representations which are unitary on the compact-type side. Finally, a new class of non-smooth unitary conical representations appears on the compact-type side which has no analogue in the finite-dimensional case. We classify these representations and show how to decompose them into direct integrals of irreducible conical representations.

Chapter 1
Introduction

Harmonic analysis and representation theory of topological groups have been very well-studied over the past century and have produced many fruitful applications in areas such as PDEs and quantum physics. Two broad developments in the theory are brought together in this thesis: first, Helgason's theory of horocycle spaces and conical representations for noncompact-type Riemannian symmetric spaces and second, the more recent study of representation theory and harmonic analysis on infinite-dimensional Lie groups.

In the theory of Riemannian symmetric spaces, there are two crucially important dualities. One is the duality between compact-type and noncompact-type Riemannian symmetric spaces. The other is the duality between a noncompact-type Riemannian symmetric space and its horocycle space. These dualities are intimately connected to the representation theory of their corresponding isometry groups (see [20], [22], and [23], for instance). For instance, Weyl's unitary trick sets up a correspondence between finite-dimensional spherical representations for a compact-type symmetric space and finite-dimensional spherical representations for its corresponding noncompact-type symmetric space. In turn, the finite-dimensional spherical representations for a noncompact-type symmetric space are identical to the conical representations for its corresponding horocycle space.

More recently, researchers have turned their attention to the study of infinite-dimensional Lie groups. These are groups which are modeled by locally convex topological vector spaces in the same way that finite-dimensional Lie groups are modeled on finite-dimensional vector spaces. The simplest and "smallest" infinite-dimensional groups are the *direct-limit groups*, which are constructed by taking unions of increasing chains of finite-dimensional Lie groups. In a similar way, one can form an infinite-dimensional symmetric space by forming a direct limit of finite-dimensional symmetric spaces. Representation theory and even harmonic analysis questions for direct-limit groups and direct-limit symmetric spaces have been studied in some depth (e.g., see [2], [4], [43], [44], [39], [40], [54], [55], and[56] for just a few examples). A good overview of the field may be found in [45].

In particular, spherical representations for infinite-dimensional symmetric spaces are well-studied in the literature (e.g., see [7] and [44]). On the other hand, the theory of conical representations for infinite-dimensional Riemannian symmetric spaces appears to have been largely neglected up to this point. In this thesis, we begin to rectify this situation by classifying all of the smooth conical representations for direct limits of noncompact-type Riemannian symmetric spaces that satisfy certain technical conditions. Combined with the results of [7], we see that for infinite-dimensional symmetric spaces of infinite rank, none of the smooth conical representations are spherical, a situation which is in stark contrast with the classical result of Helgason that all finite-dimensional representations are spherical if and only if they are conical. We further demonstrate the existence, in certain cases, of nonsmooth unitary conical representations for direct limits of compact-type Riemannian symmetric spaces. This is a phenomenon which has no analogue for finite-dimensional symmetric spaces. We also show how these conical representations decompose into direct integrals of irreducible representations.

The arrangement of this thesis is as follows. Chapter 2 reviews relevant theorems from elementary representation theory and harmonic analysis. Chapter 3 reviews the relevant structure theory for Riemannian symmetric spaces and their associated horocycle spaces. It also reviews the basic results about spherical representations and conical representations and their role in harmonic analysis on Riemannian symmetric spaces and horocycle spaces. Much of the theory of spherical representations are due to Harish-Chandra, and the corresponding results for conical representations are mostly due to Helgason. Chapter 4 introduces the concept of direct-limit Lie groups. It also introduces the necessary technical machinery for studying direct limits of symmetric spaces and horocycle spaces. We define what we call *admissible direct limits* of Riemannian symmetric spaces and show that the classical examples of direct limits of Riemannian symmetric spaces meet this definition. Chapter 5 contains several useful results about representations of direct-limit groups, including an infinite-dimensional generalization of Weyl's Unitary Trick. Finally, Chapter 6 contains the main results of the thesis. We provide natural definitions of conical representations for infinite-dimensional Riemannian symmetric spaces. We construct and classify all unitary conical representations for direct limits of compact-type symmetric spaces. Finally, in Chapter 7 we end by describing some interesting questions which remain unanswered.

1.1 Notational Preliminaries

If A is a set, then its cardinality is denoted by $\#A$. If G is a group, then e denotes the identity element. If H and K are subgroups of G, then $Z_H(K)$ and $N_H(K)$ denote the centralizer and normalizer, respectively, of K in H. Similar notation is used for centralizers and normalizers of Lie algebras.

All vector spaces, except for Lie algebras, are assumed to be over the field of complex numbers unless otherwise stated. We denote by $\langle A \rangle$ the algebraic linear span of a subset A of a topological vector space V. The closed linear span of A is denoted by $\overline{\langle A \rangle}$. The space of continuous linear functionals on V is denoted by V^*, and the space of continuous conjugate-linear functionals on V is denoted by V'. If $\mathcal{H}$ is a Hilbert space, then the inner product of two vectors $u, v \in \mathcal{H}$ is denoted by $\langle u, v \rangle_{\mathcal{H}}$, or if the choice of Hilbert space is understood, by $\langle u, v \rangle$. We consider inner products to be linear in the first variable and conjugate-linear in the second variable. The space of bounded linear operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})$.

If M is a manifold, then the space of smooth, compactly supported functions on M is denoted by $\mathcal{D}(M)$. The usual topology given to $\mathcal{D}(M)$ gives it the structure of a lim-Fréchet space (i.e., it is a direct limit of Fréchet spaces). The space of distributions on M is denoted by $\mathcal{D}'(M)$ and is defined to be the space of continuous conjugate-linear functionals on $\mathcal{D}(M)$ (we choose to think of $\mathcal{D}'(M)$ as the anti-dual of $\mathcal{D}(M)$ so that there is a continuous linear embedding $\mathcal{D}(M) \hookrightarrow \mathcal{D}'(M)$). We give $\mathcal{D}'(M)$ the weak-* topology. We denote by $C^\infty(M)$ the space of smooth functions on M.

Chapter 2

A Brief Review of Harmonic Analysis and Representation Theory

Representation theory is the study of linear actions of groups on vector spaces, which are called *representations*. Of particular interest are the *unitary representations*, in which a group acts on a Hilbert space by isometries. Given a group, representation theory seeks to explicitly construct and classify the representations of that group to the broadest extent possible. It thus fits naturally into the broader theory of groups, which has been traditionally motivated primarily by the study of symmetry.

Fundamental insights in representation theory often come from relating group representations to representations of other related objects. For instance, unitary representations of locally compact groups may be integrated to yield representations of a group C^*-algebra, which allows the application of powerful tools from operator theory. Similarly, unitary representations of a Lie group may be differentiated to yield representations of the group's Lie algebra, which allows the representation to be studied using basic linear algebra techniques instead of more difficult tools from differential geometry and analysis. Finally, through the beautiful and classical construction of Gelfand-Naimark-Segal, the theory of unitary representations may be connected with the theory of positive-definite functions. A distributional variant of this construction uses positive-definite distributions on Lie groups to embed unitary representations into spaces of distributions on homogeneous spaces.

The foundational task of the field of harmonic analysis, on the other hand, is to use the information provided by the action of a group to decompose a space of functions into simpler pieces. Such exploitations of symmetry, to borrow a phrase of Mackey, have many applications, particularly in the study of linear PDEs and in quantum physics. Because it is concerned with symmetries of vector spaces of functions, representation theory naturally plays a very important role, although harmonic analysis may be distinguished from the study of representation theory as an end in itself.

The material in this chapter is entirely classical and may be found in standard references on abstract harmonic analysis, such as [8], [14], and [31]. See also the survey article [30] for an excellent and concise introduction to the theory.

2.1 Unitary Representations

We begin by defining the basic terms.

Definition 2.1. *Let G be a topological group and let V be a locally convex topological vector space. A* ***representation*** *of G on V is a continuous homomorphism:*

$$\pi : G \to \mathrm{GL}(V),$$

where $\mathrm{GL}(V)$ *is given the strong operator topology. (We say that* π *is a* ***norm-continuous representation*** *if it is continuous when* $\mathrm{GL}(V)$ *is given the operator norm topology.) If* V *is a Hilbert space, then* π *is said to be a* ***unitary representation*** *if* $\pi(g)$ *is a unitary operator for all* $g \in G$*.*

Given two representations (π, V) *and* (σ, W) *of* G*, we say that a bounded linear operator* $T : V \to W$ *is an* ***intertwining operator*** *if* $T\pi(g) = \sigma(g)T$ *for all* $g \in G$*. If* π *and* σ *possess a continuously-invertible intertwining operator between them, then we say that they are* ***equivalent representations****. We write* $\mathrm{Hom}(\pi, \sigma)$ *for the space of all intertwining operators between* π *and* σ*.*

Among more general continuous representations, unitary representations in particular possess the important property that they may be decomposed into smaller representations. In fact, suppose that $(\pi, \mathcal{H})$ is a unitary representation of a group G on a Hilbert space $\mathcal{H}$ and that V is a closed subspace of $\mathcal{H}$ such that $\pi(g)v \in V$ for all $v \in V$. Then we say that V is an **invariant subspace** of $\mathcal{H}$. One may form a representation π_V of G on V simply by restricting the action of π on $\mathcal{H}$ to the subspace V. We say that π_V is a **subrepresentation** of π.

Now consider the closed subspace

$$V^{\perp} = \{w \in \mathcal{H} | \langle w, v\rangle = 0 \text{ for all } v \in V\}.$$

Note that if $w \in V^{\perp}$, then

$$\langle \pi(g)w, v\rangle = \langle w, \pi(g^{-1})v\rangle = 0$$

for all v in V. It follows that $V^{\perp}$ is also an invariant subspace of $\mathcal{H}$.

In fact, we see that $\mathcal{H} = V \oplus V^{\perp}$ and that

$$\pi(g)(v + w) = \pi_V(g)v + \pi_{V^{\perp}}(g)w$$

for all $v \in V$ and $w \in V^{\perp}$. In this case, we write $\pi = \pi_V \oplus \pi_{V^{\perp}}$ and say that π decomposes into a **direct sum of representations**.

If a representation $(\pi, \mathcal{H})$ of G possesses no invariant subspaces besides $\mathcal{H}$ and $\{0\}$, then we say that π is an **irreducible** representation. Let $\widehat{G}$ denote the set of equivalence classes of irreducible representations of G.

Theorem 2.2. *(Schur's Lemma; see [14, p. 71]). Suppose that* $(\pi, \mathcal{H})$ *is an irreducible unitary representation of a group* G*. Then every intertwining operator* $T \in \mathrm{Hom}(\pi, \pi)$ *for the representation* π *may be written* $T = \lambda \mathrm{Id}$ *for some* $\lambda \in \mathbb{C}$*.*

Now suppose that π is a unitary representation of G on a finite-dimensional Hilbert space $\mathcal{H}$. If π is not irreducible, then we can repeatedly follow the process outlined above of decomposing it into sums of subrepresentations. Because $\mathcal{H}$ is finite-dimensional, the process must terminate at some point, which will occur when π has been decomposed into a direct sum of irreducible subrepresentations. In this way, irreducible representations play a role similar to that of prime numbers in arithmetic.

More generally, we say that a representation $(\pi, \mathcal{H})$ of G is **cyclic** if there is a vector $v \in \mathcal{H}$ such that the span $\langle \pi(G)v \rangle$ is dense in $\mathcal{H}$. In that case, we say that v is a **cyclic vector** for π. The following powerful and broad-ranging result may be proven using Zorn's Lemma.

Theorem 2.3. *([14, p. 70]). Every unitary representation $(\pi, \mathcal{H})$ of a group G may be decomposed into an orthogonal direct sum of cyclic subrepresentations.*

It is a classical result that all unitary representations of compact groups may be decomposed into a direct sum of irreducible subrepresentations. On the other hand, there are many interesting examples of representations of noncompact groups on infinite-dimensional Hilbert spaces which do not possess any irreducible subrepresentations (and thus cannot be decomposed into a direct sum of irreducible subrepresentations). However, it is possible to write such representations as a sort of "continuous" direct sum of irreducible representations in a matter which we now describe, roughly following the construction in [14, p. 219–232].

Suppose that μ is a Borel measure on a topological space X, and that for each $x \in X$ we are given a unitary representation $(\pi_x, \mathcal{H}_x)$ of a group G. Suppose we are also given a collection of maps $s_i : X \to \dot{\cup}_{x \in X} \mathcal{H}_x$ for i in some countable index set I such that:

1. $s_i(x) \in \mathcal{H}_x$ for each $x \in X$ and $i \in I$.

2. $\langle s_i(x) | i \in I \rangle$ is dense in $\mathcal{H}_x$ for all $x \in X$.

3. $x \mapsto \langle s_i(x), s_j(x) \rangle_{\mathcal{H}_x}$ is a Borel-measurable function on X for all $i, j \in I$.

The set $\{s_i\}_{i \in I}$ is called a **measurable frame**. We then say that a map $s : X \to \dot{\cup}_{x \in X} \mathcal{H}_x$ is a **measurable section** if

1. $s(x) \in \mathcal{H}_x$ for each $x \in X$.

2. $x \mapsto \langle s(x), s_i(x) \rangle_{\mathcal{H}_x}$ is a Borel-measurable function on X for all $i \in I$.

Finally, we define a **direct-integral Hilbert space** by

$$\mathcal{H} \equiv \int_X^{\oplus} \mathcal{H}_x d\mu(x) = \left\{ \text{measurable sections } s \,\middle|\, \int_X ||s(x)||^2_{\mathcal{H}_x} d\mu(x) < \infty \right\}$$

where the inner product is given by

$$\langle u, v \rangle = \int_X \langle u(x), v(x) \rangle_{\mathcal{H}_x} d\mu(x)$$

for $u, v \in \mathcal{H}$. We can also define a continuous unitary representation $\pi \equiv \int_X^{\oplus} \pi_x d\mu(x)$ of G on $\mathcal{H}$ by

$$(\pi(g)s)(x) = \pi_x(g)(s(x))$$

for all $s \in \mathcal{H}$ and $g \in G$. We say that π is a **direct integral of the representations $\mathcal{H}_x$** for $x \in X$.

It is easy to see that orthogonal direct sums of Hilbert spaces and representations are a special case of direct integrals in which the measure is discrete. Moreover, every continuous unitary representation of a group G may be decomposed as a direct integral of irreducible representations, although there are some subtleties surrounding the uniqueness of such decompositions for certain groups.

We end this section by defining two very important classes of representations.

Definition 2.4. *A unitary representation $(\pi, \mathcal{H})$ of a topological group G is said to be* ***multiplicity free*** *if every decomposition $\pi = \pi_1 \oplus \pi_2$ of π into a direct sum of subrepresentations has the property that no subrepresentation of π_1 is equivalent to a subrepresentation of π_2.*

One can show that a unitary representation π is multiplicity-free if and only if its ring $\mathrm{Hom}(\pi, \pi)$ of intertwining operators is commutative. The term "multiplicity free" comes from the face that a direct sum $\pi = \oplus_{i \in I} \pi_i$ of irreducible representations of a group G is multiplicity free if and only if each equivalence class in $\widehat{G}$ appears at most once in the collection of π_i's. This basic result is a corollary of Schur's lemma (see [9, p. 123]).

Definition 2.5. *A unitary representation $(\pi, \mathcal{H})$ of a topological group G is said to be* ***primary*** *if the center of its ring of intertwining operators is trivial—that is, if*

$$Z(\mathrm{Hom}(\pi, \pi)) = \{\lambda \mathrm{Id} | \lambda \in \mathbb{C}\}.$$

One can show (see [9, p. 122]) that a direct sum $\pi = \oplus_{i \in I} \pi_i$ of irreducible representations of a group G is primary if and only if all the irreducible components π_i are equivalent to each other. However, for some groups it is possible to construct primary representations which cannot be decomposed into a direct sum of irreducible representations.

2.2 Invariant Measures

It is well-known that every locally-compact topological group G possesses a Radon measure μ_G which is left-invariant under left translations of the group and such that every open subset of G has positive measure. That is,

$$\int_G f(gx) d\mu_G(x) = \int_G f(x) d\mu_G(x) \tag{2.1}$$

for all $f \in C_c(G)$ and $g \in G$. Such measures, called **Haar measures**, are unique up to multiplication by a constant. If G is a compact group, then μ_G is a finite measure, which we will always normalize so that $\mu_G(G) = 1$.

The existence of Haar measures has several important and useful consequences. For example, the fact that compact groups have invariant probability measures makes it possible to construct many arguments in which one averages some object over the group:

Theorem 2.6. *(See also [27, Proposition 4.6]). If G is a compact topological group, then every norm-continuous representation $(\pi, \mathcal{H})$ of G on a Hilbert space is equivalent to a unitary representation.*

Proof. We denote the inner product on $\mathcal{H}$ by $\langle , \rangle_{\mathcal{H}}$ and construct a new inner product $\langle , \rangle_\pi$ on $\mathcal{H}$ by defining:

$$\langle v, w\rangle_\pi = \int_G \langle \pi(g)v, \pi(g)w\rangle_{\mathcal{H}} d\mu_G(g)$$

for all $v, w \in \mathcal{H}$.

Now define

$$M = \sup_{g\in G} ||\pi(g)||_{\mathcal{H}}$$

and note that $M < \infty$ because π is norm-continuous and G is compact. We then have $||\pi(g)^{-1}||_{\mathcal{H}} < M$ for all $g \in G$. Thus

$$M^{-2}||v||^2_{\mathcal{H}} \leq ||v||^2_\pi = \int_G ||\pi(g)v||^2_{\mathcal{H}} d\mu_G(g) \leq M^2||v||^2_{\mathcal{H}}$$

for all $v \in \mathcal{H}$. Hence the identity map on $\mathcal{H}$ forms a homeorphism between $\mathcal{H}$ under $\langle , \rangle_{\mathcal{H}}$ and $\mathcal{H}$ under $\langle , \rangle_\pi$.

Finally, for all $h \in G$ and $u, v \in \mathcal{H}$, we have that

$$\begin{aligned}\langle \pi(h)u, \pi(h)v\rangle_\pi &= \int_G \langle \pi(gh)v, \pi(gh)w\rangle_{\mathcal{H}} d\mu_G(g)\\ &= \int_G \langle \pi(g)v, \pi(g)w\rangle_{\mathcal{H}} d\mu_G(g)\\ &= \langle u, v\rangle_\pi.\end{aligned}$$

Thus, we see that π is a unitary representation of G on $\mathcal{H}$ under the inner product $\langle , \rangle_\pi$. ☐

For certain groups, the left-invariant Haar measure is also right invariant. That is, the Haar measure μ_G satisfies the property that

$$\int_G f(g_1 x g_2) d\mu_G(x) = \int_G f(x) d\mu_G(x) \tag{2.2}$$

for all $f \in C_c(G)$ and $g_1, g_2 \in G$. In this case, we say that G is **unimodular**. Many basic results in harmonic analysis can be formulated most cleanly when the group under consideration to be unimodular; fortunately several broad classes of groups are known to be unimodular, including (see [12, p. 88]) all compact groups, abelian groups, semisimple Lie groups, and connected nilpotent Lie groups (in contrast, not all solvable Lie groups are unimodular).

At any rate, with a Haar measure μ_G on G, we may consider the Hilbert space $L^2(G) \equiv L^2(G, \mu_G)$ of square-integrable functions on G. It is easy to show that the action given by

$$(g \cdot f)(x) = f(g^{-1}x) \tag{2.3}$$

for $g \in G$ and $f \in L^2(G)$ gives a continuous representation of G on $L^2(G)$ that is unitary by (2.1). This representation is called the **(left) regular representation** of G.

The foundational problem of harmonic analysis is to provide, for a particular group G, a decomposition of the regular representation into irreducible components. A general result states that this is possible for a very broad class of locally-compact groups, called **Type I groups**.

Definition 2.7. *A topological group G is said to be of **Type I** if every primary representation of G decomposes into a direct sum of copies of the same irreducible representation.*

This class includes all compact groups (see [12, p. 206] and all semisimple Lie groups (see [19, p. 230]), for example.

Theorem 2.8. *(The Abstract Plancherel Theorem; see [9, p. 368]). Let G be a Type I separable, locally-compact topological group. For each $\lambda \in \widehat{G}$, choose a representative irreducible representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G. Then there is a measure μ on $\widehat{G}$ (whose measure class is uniquely determined) such that*

$$L^2(G) \cong_G \int_{\widehat{G}}^{\oplus} \mathcal{H}_\lambda \otimes \overline{\mathcal{H}_\lambda} d\sigma(\lambda).$$

*Such a decomposition is called a **Plancherel formula** for G.*

One of the basic tasks of harmonic analysis is to make the Plancherel formula as explicit as possible for particular groups.

There are several variants of the regular representation that will be useful to us later. We can define continuous representations L and R of G on the space $\mathcal{D}(G)$ of smooth, compactly supported functions as follows:

$$L(g)f(x) = f(g^{-1}x)$$
$$R(g)f(x) = f(xg)$$

for $g, x \in G$ and $f \in \mathcal{D}(G)$. These representation may be dualized to produce continuous representations L and R on the space $\mathcal{D}'(G)$ of distributions on G.

Similarly, one can define continuous left- and right-regular representations of G on the space $C^\infty(G)$ of smooth functions on G. We say that a function $f \in C^\infty(G)$ is G-finite if the subspace $\langle L(G)f \rangle \subseteq C^\infty(G)$ generated by all G-translations of f is finite-dimensional. We denote the space of all G-finite smooth functions by $C^\infty_{\text{fin}}(G)$ and note that $C^\infty_{\text{fin}}(G)$ is an invariant subspace of $C^\infty(G)$.

2.3 Homogeneous Spaces

More generally, we wish to study not only functions on a group G but also functions on spaces on which G acts. To that end, suppose that G is a Lie group which acts smoothly and transitively on a manifold X. Let $x_o \in X$ and consider the stabilizer subgroup of G given by

$$G^{x_o} = \{g \in G | g \cdot x_o = x_o\}.$$

Note that G^{x_o} is a closed subgroup of G.

One can then form the space G/G^{x_0} of left cosets. There is a transitive action of G on G/G^{x_0} given by

$$g \cdot hG^{x_0} = ghG^{x_0}.$$

In fact, one can show (see [5, Proposition 4.6]) that there is a G-equivariant diffeomorphism

$$X \to G/G^{x_0}.$$

In other words, we have an identification of transitive G-actions with quotient spaces of the form G/H, where H is a closed subgroup of G. Such spaces are called **homogeneous spaces**, because the transitive group action forces them to have the same local behavior around each point. We refer to G as the **translation group** of G/H and to H as the **isotropic subgroup** of G.

We would like to study harmonic analysis on homogeneous spaces. Just as for harmonic analysis on groups, the natural place to start is to construct an invariant measure. Unfortunately, not every homogeneous space G/H, where G and H are locally compact groups, possesses a Radon measure that is invariant under the action of G. However, as long as both G and H are unimodular, then such a measure always exists:

Theorem 2.9. *(See [8, p. 41–44]). If G and H are locally compact unimodular topological groups, then there is a Radon measure $\mu_{G/H}$ on G/H, unique up to multiplication by a constant, such that*

$$\int_{G/H} f(g \cdot x) d\mu_{G/H}(x) = \int_{G/H} f(x) d\mu_{G/H}(x)$$

for all $g \in G$ and $f \in C_c(G/H)$.

Furthermore, $\mu_{G/H}$ satisfies the functional equation

$$\int_G f(g) d\mu_G(g) = \int_{G/H} \int_H f(gh) d\mu_H(h) d\mu_{G/H}(g)$$

for all $f \in C_c(G)$.

As before, we construct the Hilbert space $L^2(G/H) \equiv L^2(G/H, \mu_{G/H})$ and note that a continuous unitary representation of G on $L^2(G/H)$ may be constructed using the action given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x)$$

for $f \in L^2(G/H)$, $g \in G$, and $x \in G/H$. This representation is also called a **regular representation** of G for the homogeneous space G/H. Just as for $L^2(G)$, it is a basic problem of harmonic analysis to explicitly decompose $L^2(G)$ into a direct integral of irreducible representations.

In the interest of brevity, from this point forward we will use the simplified notations $dg = d\mu_G(g)$ and $dx = d\mu_{G/H}(x)$ to denote integration against a Haar measure on G and against a G-invariant measure on G/H, respectively.

2.4 Gelfand-Naimark-Segal

In this section we explore the connection between unitary representations and positive-definite functions. We begin with a basic definition:

Definition 2.10. *Let G be a group. We say that a function $\phi : G \to \mathbb{C}$ is* ***positive-definite*** *if*

$$\sum_{i,j=1}^{n} \phi(g_i^{-1} g_j) c_i \overline{c_j} \geq 0$$

where $g_i \in G$ and $c_i \in \mathbb{C}$ for $1 \leq i \leq n$.

Positive-definite functions have several basic properties which may be proved directly from the definition (see [8, Lemma 5.1.8]):

1. $\phi(e) \geq 0$
2. $|\phi(g)| \leq \phi(e)$ for all $g \in G$
3. $\phi(g^{-1}) = \overline{\phi(g)}$ for all $g \in G$

The canonical examples of positive-definite functions are provided by matrix coefficients of unitary representations. That is, if $(\pi, \mathcal{H})$ is a unitary representation of a group G and $v \in \mathcal{H} \backslash \{0\}$, then the function $\phi_{\pi,v} : G \to \mathbb{C}$ given by

$$\phi_{\pi,v}(g) = \langle v, \pi(g) v \rangle \tag{2.4}$$

is continuous and positive-definite, as may be shown straightforwardly using the unitarity of π and the definition of positive-definite functions.

The key insight of Gelfand-Naimark-Segal is that every continuous positive-definite function arises in this way from a unitary representation. In particular, given a continuous positive-definite function $\phi : G \to \mathbb{C}$, one can define a representation. We now show how this may be done.

For each $g \in G$, define the function $g \cdot \phi : G \to \mathbb{C}$ by

$$g \cdot \phi(x) = \phi(g^{-1} x)$$

for each $x \in G$. We can then define the vector space

$$V_\phi = \langle \{ g \cdot \phi | g \in G \} \rangle,$$

which is the algebraic span of all G-translates of ϕ. We define a pre-Hilbert space structure on V_ϕ:

$$\left\langle \sum_{i=1}^{n} c_i (g_i \cdot \phi), \sum_{j=1}^{n} d_j (h_i \cdot \phi) \right\rangle = \sum_{i,j=1}^{n} \phi(g_i^{-1} h_j) c_i \overline{d_j} \tag{2.5}$$

where $c_i, d_j \in \mathbb{C}$ and $g_i, h_j \in G$. It can be shown that this bilinear form is well-defined on V_ϕ and turns it into a pre-Hilbert space.

We can then define a representation π_ϕ of G on V_ϕ by

$$\pi_\phi(g)v(h) = v(g^{-1}h)$$

for all $v \in V_f$ and $g, h \in G$. It is clear from (2.5) that π_ϕ extends to a unitary representation on the the Hilbert-space completion $\mathcal{H}_\phi$ of V_ϕ. Then one has

$$\phi(g) = \langle \phi, \pi(g)\phi \rangle_{\mathcal{H}_\phi}.$$

Thus every positive-definite function may be given the form (2.4). In fact, a stronger result may be proven:

Theorem 2.11. *(Gelfand-Naimark-Segal; see [8, p. 54, 61]). The map*

$$(\pi, v) \mapsto \phi_{\pi,v}$$

is a surjection from the set of all pairs (π, v) of cyclic representations $(\pi, \mathcal{H})$ of G and cyclic vectors $v \in \mathcal{H}\backslash\{0\}$ to the set of all continuous positive-definite functions on G.

Furthermore, suppose that $(\pi, \mathcal{H})$ and $(\sigma, \mathcal{K})$ are unitary representations of G such that $v \in \mathcal{H}$ and $w \in \mathcal{K}$ are cyclic vectors. Then one has

$$\phi_{\pi,v} = \phi_{\sigma,w}$$

if and only if there is a unitary intertwining operator $T : \mathcal{H} \to \mathcal{K}$ such that $T(v) = w$.

Let G be a locally-compact topological group. We write $\mathcal{P}(G)$ for the space of all positive-definite functions ϕ on G such that $\phi(e) = 1$. One can show that $\mathcal{P}(G)$ is a closed convex subset of the space $L^\infty(G)$ of almost-everywhere-bounded measurable functions on G. The convexity may be shown by noticing that

$$\lambda\phi_{\pi,v} + (1-\lambda)\phi_{\sigma,w} = \phi_{\pi\oplus\sigma,\sqrt{\lambda}v+\sqrt{1-\lambda}w}, \tag{2.6}$$

where $(\pi, \mathcal{H})$ and $(\sigma, \mathcal{K})$ are unitary representations of G with cyclic vectors $v \in \mathcal{H}$ and $w \in \mathcal{K}$.

In fact, $L^\infty(G)$ is the dual of the Banach space $L^1(G)$ by the Riesz Representation Theorem. One can show that $\mathcal{P}(G)$ is closed in the weak-$*$ topology on $L^\infty(G)$. Since $|\phi(g)| \leq \phi(e) = 1$ for all ϕ in $\mathcal{P}(G)$ and $g \in G$, we see that $\mathcal{P}(G)$ is contained in the unit ball $B_1(L^\infty(G))$. It follows from the Banach-Alaoglu theorem that $\mathcal{P}(G)$ is a compact convex subset of $L^\infty(G)$ in the weak-$*$ topology. Thus, the Krein-Milman theorem may be applied to $\mathcal{P}(G)$:

Theorem 2.12. *(Krein-Milman [8, Theorem 5.2.7]) If K is a compact, convex subset of a locally convex topological vector space V, then*

$$K = \overline{\mathrm{co}(\mathrm{ex}(K))},$$

where co *denotes the convex hull and* $\mathrm{ex}(K)$ *denotes the set of extremal points of* K*.*

In other words, all normalized positive-definite functions may be formed by taking a limit of convex combinations of normalized positive-definite functions. In fact, by exploiting the identity in (2.6), one has the following result:

Theorem 2.13. *Let G be a locally compact topological group. Then the extremal points of $\mathcal{P}(G)$ are given by functions of the form $\phi_{\pi,v}$, where $(\pi, \mathcal{H})$ is an irreducible representation of G and v is a cyclic unit vector in $\mathcal{H}$.*

Thus, positive-definite functions are generated in some sense by the ones coming from irreducible representations. These are just a few examples of how powerful theorems from functional analysis may be applied to provide insight into the decomposition of unitary representations.

2.5 Smooth Vectors and Distribution Vectors

Suppose now that G is a Lie group with Lie algebra $\mathfrak{g}$. In a certain sense, $\mathfrak{g}$ is a "linearization" of G that encapsulates all of the local aspects of its structure. This is exemplified best by the famous Campbell-Baker-Hausdorff Theorem, which shows how the group product on a Lie group may be recovered, in a neighborhood of the identity, from the Lie bracket on its Lie algebra.

Similarly, it is desirable to recover information about a representation of a group G by first passing to a representation of $\mathfrak{g}$. If (π, V) is a continuous finite-dimensional representation of G (not necessarily unitary), then one can show that the map $g \to \pi(g)v$ is a smooth (in fact analytic) function from G to V. Thus, π induces a representation $\mathrm{d}\pi$ of $\mathfrak{g}$ on V by:

$$\mathrm{d}\pi(X)v = \left.\frac{d}{dt}\right|_{t=0} (\pi(\exp(tX))v)$$

for all $X \in \mathfrak{g}$ and $v \in V$. One can show that two finite-dimensional representations π and ρ of G are equivalent if and only if $\mathrm{d}\pi$ and $\mathrm{d}\rho$ are equivalent.

However, the situation is more delicate for infinite-dimensional representations. Let $(\pi, \mathcal{H})$ be a continuous representation of G on a Hilbert space. We say that $v \in \mathcal{H}$ is a **smooth vector** in $\mathcal{H}$ if the map $g \mapsto \pi(g)v$ is smooth. We denote the space of all smooth vectors by $\mathcal{H}^\infty$. Similarly, we say that a vector is **G-finite** if the G-invariant subspace $\langle \pi(G)v \rangle$ generated by v is finite-dimensional. We denote the space of G-finite vectors by $\mathcal{H}^{\text{fin}}$. It is not difficult to show that $\mathcal{H}^\infty$ and $\mathcal{H}^{\text{fin}}$ are linear subspaces of $\mathcal{H}$ and that $\mathcal{H}^{\text{fin}} \subseteq \mathcal{H}^\infty$.

Unfortunately, there are many interesting examples of infinite-dimensional representations $(\pi, \mathcal{H})$ for which not every vector is a smooth vector. Nevertheless, a classical result of Gårding uses the integrated representation of π to show that $\mathcal{H}^\infty$ is a dense subspace of $\mathcal{H}$.

Theorem 2.14. *(Gårding; see [8, p. 131–133]). Let $(\pi, \mathcal{H})$ be a continuous representation of a locally compact group G on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}^\infty$ is a dense subspace of $\mathcal{H}$. In fact, for each $f \in \mathcal{D}(G)$ and $v \in \mathcal{H}$, the vector*

$$\pi(f)v \equiv \int_G f(g)\pi(g)vdg \tag{2.7}$$

is in $\mathcal{H}^\infty$.

In fact, a beautiful theorem of Dixmier and Malliavin shows that the vectors Gårding constructed generate all of the smooth vectors:

Theorem 2.15. *(The Decomposition Lemma; see [10]) If $(\pi, \mathcal{H})$ is a continuous representation of a locally compact group G on a Hilbert space $\mathcal{H}$, then every element of $\mathcal{H}^\infty$ may be written as a finite linear combination of vectors of the form (2.7).*

Theorem 2.14 allows us to define a representation of $\mathfrak{g}$ on the space $\mathcal{H}^\infty$ in the same way as before, namely

$$\mathrm{d}\pi(X)v = \lim_{t\to 0} \frac{\pi(\exp(tX))v - v}{t}$$

for all $X \in \mathfrak{g}$ and $v \in \mathcal{H}^\infty$. Furthermore,

$$\mathrm{d}\pi(X)\mathrm{d}\pi(Y)v - \mathrm{d}\pi(Y)\mathrm{d}\pi(X)v = \mathrm{d}\pi([X,Y])v$$

for all $X, Y \in \mathfrak{g}$ and $v \in \mathcal{H}^\infty$ (see [22, p. 387]). This representation of $\mathfrak{g}$ on $\mathcal{H}^\infty$ extends to a representation $\mathrm{d}\pi$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ in the natural way.

Finally, $\mathcal{H}^\infty$ may be given a Fréchet topology under the family of seminorms given by

$$||v||_D = ||\mathrm{d}\pi(D)v||_{\mathcal{H}}$$

for each $D \in \mathcal{U}(\mathfrak{g})$ and $v \in \mathcal{H}$. Under this topology, the inclusion map

$$\mathcal{H}^\infty \hookrightarrow \mathcal{H}$$

is a continuous dense embedding of a Fréchet space into a Hilbert space ([8, p. 132]).

Now consider the anti-dual $\mathcal{H}^{-\infty}$ of $\mathcal{H}^\infty$–that is, the space of all conjugate-linear continuous functionals on $\mathcal{H}^\infty$. Elements of $\mathcal{H}^{-\infty}$ are called **distribution vectors** for the representation π. We give $\mathcal{H}^{-\infty}$ the weak-* topology. Then there is a continuous embedding

$$\mathcal{H} \hookrightarrow \mathcal{H}^{-\infty}$$

given by mapping a vector $v \in \mathcal{H}$ to the conjugate-linear functional on $\mathcal{H}^\infty$ given by

$$w \mapsto \langle v, w\rangle_{\mathcal{H}}$$

for all $w \in \mathcal{H}^\infty$.

There are several ways in which the space of distribution vectors is well-behaved. Just as distributions on a manifold are infinitely differentiable in a weak sense, the derived representations of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$ on $\mathcal{H}^\infty$ extend by dualization to representations on $\mathcal{H}^{-\infty}$. Furthermore, distribution vectors can be "smoothed out" by integration against smooth functions on G:

Lemma 2.16. *([8, p. 136]). For each $v \in \mathcal{H}^{-\infty}$ and $\phi \in \mathcal{D}(G)$, the distribution vector*

$$\pi(\phi)v = \int_G \phi(g)v dg$$

is an element of $\mathcal{H}^\infty$.

As a corollary of this result, one has that $\mathcal{H}^\infty$ is densely contained in $\mathcal{H}^{-\infty}$. Putting everything together, we have continuous, dense embeddings

$$\mathcal{H}^\infty \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\infty}.$$

2.6 Invariance and Harmonic Analysis on Homogeneous Spaces

Suppose that G is a compact group with a closed subgroup K, and consider the space G/K and the regular representation of G on $L^2(G/K)$. The basic task of harmonic analysis on G/K is to decompose $L^2(G/K)$ into a direct sum of irreducible representations. We now show how to determine which equivalence classes of unitary representations of G appear in this decomposition, as well as how many times they appear.

Suppose that $(\sigma, \mathcal{H})$ is a unitary representation of G. We consider the space

$$\mathcal{H}^K \equiv \{v \in V | \pi(k)v = v \text{ for all } k \in K\}$$

of K-invariant vectors in $\mathcal{H}$. One then has the following theorem.

Theorem 2.17. *For each irreducible unitary representation $(\sigma, \mathcal{H})$ of G, we have that*

$$\dim \mathcal{H}^K = \dim \mathrm{Hom}(\sigma, L^2(G/K)).$$

That is, the multiplicity of σ in $L^2(G/K)$ is equal to the dimension of the space of K-invariant vectors in $\mathcal{H}$.

This result is a special case of the Frobenius Reciprocity Theorem for unitary representations of compact groups (see [14, p. 160]).

Corollary 2.18. *Let G be a compact group. For each $(\pi, \mathcal{H}_\pi) \in \widehat{G}$. Then*

$$L^2(G/K) \cong_G \bigoplus_{\pi \in \widehat{G}} m_\pi \mathcal{H}_\pi,$$

where $m_\pi = \dim \mathcal{H}_\pi^K$ and $m_\pi \mathcal{H}_\pi = \mathcal{H}_\pi \oplus \cdots \oplus \mathcal{H}_\pi$ refers to the direct sum of m_π copies of $\mathcal{H}_\pi$.

Unfortunately, the analysis just described is not applicable to homogeneous spaces G/H where either G or H is non-compact. If G is locally compact but not compact, then $L^2(G/H)$ is no longer guaranteed to decompose into a direct sum of irreducible subrepresentations; a direct integral decomposition is necessary. Furthermore, if H is non compact, then $L^2(G)$ may not possess any nontrivial H-invariant functions, so that $L^2(G/H)$ cannot be embedded as a subrepresentation of $L^2(G)$. The solution to this problem is to move to the theory of distributions and distribution vectors; one attempts to decompose $L^2(G/H)$ into a direct integral of irreducible representations $(\pi, \mathcal{H})$ which possess $\mathcal{H}$-invariant distribution vectors (i.e., $(\mathcal{H}^{-\infty})^{\mathcal{H}} \neq 0$).

Chapter 3

Finite-Dimensional Riemannian Symmetric Spaces

Riemannian symmetric spaces form a class of particularly well-behaved homogeneous spaces with a rich structure theory and relatively well-understood harmonic analysis. Among other important properties, they possess a Riemannian metric that is invariant under the action of the translation group. Furthermore, the isotropic subgroup is fixed under an involution on the translation group, which essentially forces the regular representations on Riemannian symmetric spaces to have multiplicity-free direct integral decompositions. We shall also see that there is a beautiful duality between *compact-type* and *noncompact-type* Riemannian symmetric spaces.

In addition, the noncompact-type Riemannian symmetric spaces possess an associated homogeneous space called a *horocycle space*. The relationship between a Riemannian symmetric space and its horocycle space is analogous to, for instance, the relationship between points and hyperplanes in $\mathbb{R}^n$, or the relationship between points and horocycles of hyperbolic space (it is for this reason that the terminology *horocycle space* was originally chosen).

In the late 1950s, Gelfand and Graev developed a "horospherical method" which relates harmonic analysis on the noncompact-type Riemannian symmetric space $\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$ and harmonic analysis on its corresponding horocycle space (see [31, p. 283–287]). These ideas were generalized to all noncompact-type Riemannian symmetric spaces and developed quite completely in the pioneering work of Helgason (see [20], for instance). The relationship between symmetric spaces and horocycle spaces, together with its implications for representation theory, provides the primary context for this thesis.

See [21] for a comprehensive overview of the structure theory for Riemannian symmetric spaces. See also [22] and [23] for applications of representation theory to analysis on Riemmanian symmetric spaces and horocycle spaces, respectively. A good concise overview of this theory from the perspective of unitary group representations may be found in [38].

3.1 Basic Definitions

Suppose that G is a semisimple Lie group with finite center and that K is a closed subgroup. Furthermore, we suppose that there is an involutive automorphism $\theta : G \to G$ such that

$$(G^\theta)_0 \leq K \leq G^\theta, \tag{3.1}$$

where G^θ is the fixed-point subgroup for θ and $(G^\theta)_0$ is the connected component of the identity for G^θ. Then G/K is said to be a **symmetric space**.

The involution θ differentiates to an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of G. By (3.1), the $+1$-eigenspace for θ is just $\mathfrak{k}$ (i.e., the Lie algebra for K). We denote the -1-eigenspace of θ by $\mathfrak{p}$. Just as $\mathfrak{k}$ may be naturally identified with the tangent space T_eK, there is a natural identification of $\mathfrak{p}$ with the tangent space $T_{eK}G/K$ (see [21, p. 214]). We may write down the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Due to the fact that θ is also a Lie algebra involution, one easily computes that $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

Let G/K be a symmetric space with involution θ, and recall that the Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ provides an $\mathrm{Ad}(G)$-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$. If B restricts to a positive-definite or negative-definite symmetric bilinear form on $\mathfrak{p}$, then G/K is said to be a **Riemannian symmetric space**. This terminology comes from the fact that an $\mathrm{Ad}(G)$-invariant positive-definite bilinear form on $\mathfrak{p}$ may be translated by the action of g to produce a G-invariant Riemannian metric on G/K.

If U/K is a Riemannian symmetric space with U compact, then B restricts to a negative-definite form on $\mathfrak{p}$ and U/K is said to be a **compact-type Riemannian symmetric space**. On the other hand, if G/K is a Riemannian symmetric space with G noncompact, then K is compact and B restricts to a positive-definite form on $\mathfrak{p}$ and G/K is said to be a **noncompact-type Riemmanian symmetric space**.

There is a beautiful duality between compact-type and noncompact-type Riemannian symmetric spaces. Suppose that U/K is a compact-type symmetric space with involution θ. We make the further simplifying assumption that G is simply-connected. As before, we consider the θ-eigenspace decomposition $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$. Recall that $\mathfrak{g}$ may be embedded in the complexified Lie algebra $\mathfrak{u}_{\mathbb{C}} = \mathfrak{u} \otimes_{\mathbb{R}} \mathbb{C}$. Furthermore, θ extends to a complex Lie algebra involution on $\mathfrak{u}_{\mathbb{C}}$, which we also denote by θ. Furthermore, the Killing form B on $\mathfrak{u}$ extends to a complex bilinear form on $\mathfrak{g}_{\mathbb{C}}$. We can then consider the real vector space $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ defined by

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}.$$

It can be shown that $\mathfrak{g}$ is a real semisimple Lie algebra that is invariant under θ. In fact, $\mathfrak{k}$ and $i\mathfrak{p}$ are the $+1$- and -1-eigenspaces for $\theta : \mathfrak{g} \to \mathfrak{g}$. Also, since U/K is a compact-type Riemannian symmetric space, we see that $B(X, X) < 0$ for all $X \in \mathfrak{p}$. But then $B(iX, iX) > 0$ for all $X \in \mathfrak{p}$ and hence B is positive-definite on $i\mathfrak{p}$. [1]

We now consider the unique connected complex Lie group $U_{\mathbb{C}}$ with Lie algebra $\mathfrak{u}_{\mathbb{C}}$ such that U is the analytic subgroup of $U_{\mathbb{C}}$ corresponding to the Lie algebra $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$. The Lie algebra involution θ on $\mathfrak{u}_{\mathbb{C}}$ integrates to an involution on $U_{\mathbb{C}}$ by Proposition 7.5 in [27]. We then consider the analytic subgroup $G \leq U_{\mathbb{C}}$ corresponding to the Lie algebra $\mathfrak{g} \subseteq \mathfrak{u}_{\mathbb{C}}$. By Proposition 7.9 in [27], we see that G is a closed subgroup of $U_{\mathbb{C}}$ and has a finite center. Putting everything together, we see that G/K is a *noncompact-type* Riemannian symmetric space, called the **c-dual** of U/K.

[1] Here we have used the fact (see [27, p.37]) that $B_{\mathfrak{g}_{\mathbb{C}}}|_{\mathfrak{g} \times \mathfrak{g}} = B_{\mathfrak{g}}$ together with the fact that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}_{\mathbb{C}}$

3.2 The Structure of Noncompact-Type Riemannian Symmetric Spaces

In this section we review the basic structure theory for noncompact-type Riemannian symmetric spaces. All of the material is entirely classical and may be found in standard references, such as Chapters VI and VII in [27] or in Chapter VI of [21].

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$. It can be shown (see [27, p. 355–358]) that there is an involution θ on $\mathfrak{g}$ such that the symmetric bilinear form given by

$$(X, Y) \mapsto -B(X, \theta Y)$$

is positive-definite. Such an involution is called a **Cartan involution** and is unique up to inner automorphisms. One shows that a Cartan involution on $\mathfrak{g}$ integrates to an involution on G (see [27, p. 362]). Furthermore, if G has a finite center, then $K = G^\theta$ is a maximal compact subgroup of G. For a subgroup $G \leq \mathrm{GL}(n, \mathbb{C})$ which is stabilized by the taking of adjoints, then one may define Cartan involutions on G and $\mathfrak{g}$ by setting $\theta(g) = (g^{-1})^*$ and $\theta(X) = -X^*$, respectively.

Now suppose that G/K is a noncompact-type Riemannian symmetric space with involution θ and that G has finite center. It can be shown that θ is a Cartan involution on G and thus that K is a maximal compact subgroup of $\mathfrak{g}$. Thus, the classification of real semisimple Lie groups may be used to provide a classification of noncompact-type Riemannian symmetric spaces.

As before, we write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Now let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Denote the real linear dual of $\mathfrak{a}$ by $\mathfrak{a}^*$, whose elements are called **weights**. For each $\alpha \in \mathfrak{a}^*$, write

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$$

Note that because $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}$, we have

$$\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a},$$

where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The set of all $\alpha \neq 0$ in $\mathfrak{a}^*$ such that $\mathfrak{g}_\alpha \neq 0$ is denoted by $\Sigma(\mathfrak{g}, \mathfrak{a}) = \Sigma$. Elements of this set are called **restricted roots**.

Fix $H \in \mathfrak{a}$. Because $\mathrm{ad}(H)$ is skew-adjoint under B and $\theta(H) = -H$ (since $H \in \mathfrak{p}$), we see that $\mathrm{ad}(H)$ is self-adjoint under the inner product $(\cdot, \cdot) = -B(\cdot, \theta \cdot)$. Thus, $\mathfrak{g}$ decomposes into joint eigenspaces under the action of $\mathrm{ad}(\mathfrak{a})$:

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \tag{3.2}$$

(Note that all of the restricted roots in $\Sigma(\mathfrak{g}, \mathfrak{a})$ are real-valued weights on $\mathfrak{a}$, in contrast with the roots of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$, which are in general complex-valued.)

The Jacobi identity shows that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \tag{3.3}$$

for all $\alpha, \beta \in \Sigma$. In this way the restricted root spaces provide a great deal of information about the Lie algebra structure of $\mathfrak{g}$.

Furthermore, we claim that

$$\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}. \tag{3.4}$$

In fact, suppose that $X \in \mathfrak{g}_\alpha$ and $H \in \mathfrak{a}$. Then

$$\begin{aligned}[H, \theta(X)] &= \theta([\theta(H), X]) \\ &= \theta(-[H, X]) \\ &= -\alpha(H)\theta(X),\end{aligned}$$

where we use the fact that θ is a Lie algebra involution and that $H \in \mathfrak{p}$. Thus $X \in \mathfrak{g}_{-\alpha}$.

An element $H \in \mathfrak{a}^*$ is said to be **regular** if $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$. The set of all regular elements of $\mathfrak{a}$ will be denoted by $\widetilde{\mathfrak{a}}$. The connected components of $\widetilde{\mathfrak{a}}$ are called **Weyl chambers**. We choose a Weyl chamber $\mathfrak{a}^+ \subseteq \widetilde{\mathfrak{a}}$. Under this choice, a weight $\lambda \in \mathfrak{a}^*$ is said to be **positive** if $\lambda(H) > 0$ for all $H \in \mathfrak{a}^+$. We let $\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \Sigma^+$ denote the set of all positive restricted roots. Since the negative of any restricted root is again a restricted root (see 3.4), one obtains a decomposition

$$\Sigma = \Sigma^+ \,\dot{\cup}\, (-\Sigma^+) \tag{3.5}$$

We denote by $\Sigma_0(\mathfrak{g}, \mathfrak{a}) = \Sigma_0$ the set of nonmultiplicable restricted roots (that is, roots $\alpha \in \Sigma$ such that $c\alpha \notin \Sigma$ for all $c \neq 1$ in $\mathbb{R}$).[2] We set $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$. Finally, it is possible to choose a set $\Psi = \{\alpha_1, \ldots, \alpha_r\} \subset \Sigma_0^+$, where $r = \dim \mathfrak{a}$, such that Ψ is a basis for $\mathfrak{a}^*$. Each root $\alpha \in \Sigma_0^+$ may then be written $\alpha = n_1\alpha_1 + \cdots + n_r\alpha_r$ where $n_1, \ldots, n_r \in \mathbb{Z}^+$. Roots in Ψ are called **simple roots**.

Now consider the normalizer M' of $\mathfrak{a}$ in K (that is, M' consists of all $k \in K$ such that $\mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}$). Similarly, let $M = Z_K(\mathfrak{a})$ denote the centralizer of $\mathfrak{a}$ in K (that is, M consists of all $k \in K$ such that $\mathrm{Ad}(k)X = X$ for all $X \in \mathfrak{a}$). Note that $M \trianglelefteq M'$. The quotient group $W = M'/M$ is called the **restricted Weyl group** for $(\mathfrak{g}, \mathfrak{a})$. In fact, one may show that elements in W, acting by conjugation on A, permute the Weyl chambers. Furthermore, there is a unique element $w^* \in W$ whose action on A sends the Weyl chamber $\mathfrak{a}^+$ to the Weyl chamber $-\mathfrak{a}^+$. We refer to w^* as the **longest element** of the Weyl group.

As a word of caution to the reader, we note that $M = Z_K(\mathfrak{a})$ is generally not connected (even when G is connected), in which case $M \neq \exp \mathfrak{m}$. We define $M_0 = \exp \mathfrak{m}$ and note that M_0 is the connected component of the identity for M. We will recall some well-known results about the structure of the component group M/M_0 when the need arises later.

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[2] It is standard in the literature to define Σ_0 to be the set of all indivisible roots, but here we follow the notation of [7].

Consider the nilpotent Lie algebras

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$$
$$\overline{\mathfrak{n}} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}.$$

We note that $\theta(\mathfrak{n}) = \theta(\overline{\mathfrak{n}})$ by 3.4. By combining (3.2) and (3.5), we then have a **triangular decomposition** of $\mathfrak{g}$:

$$\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}. \tag{3.6}$$

There is a triangular decomposition on the level of the group G, as well. Consider the subgroups $N = \exp \mathfrak{n}$, $\overline{N} = \exp \overline{n}$, and $A = \exp \mathfrak{a}$ of G. One can show that $\overline{N}$, M, A, and N are closed Lie subgroups of G with Lie algebras $\overline{\mathfrak{n}}$, $\mathfrak{m}$, $\mathfrak{a}$, and $\mathfrak{n}$, respectively. Furthermore, one shows that $\theta(N) = \overline{N}$. Then the map

$$\begin{aligned} \overline{N} \times M \times A \times N &\to G \\ (\overline{n}, m, a, n) &\mapsto \overline{n}man \end{aligned} \tag{3.7}$$

is a smooth embedding of the manifold $\overline{N} \times M \times A \times N$ into an open dense subset of G.

There are other decompositions of G that are useful to consider. The **Iwasawa decomposition** states that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}. \tag{3.8}$$

We will prove this result in the next subsection, when we discuss the Killing form more deeply. This Lie algebra decomposition integrates nicely to the group level; in fact, the map

$$\begin{aligned} K \times A \times N &\to G \\ (k, a, n) &\mapsto kan \end{aligned}$$

is a diffeomorphism.

Recall the choice of positive Weyl chamber C in $\mathfrak{a}$. The set

$$\widetilde{A} = \exp \widetilde{\mathfrak{a}}$$

is called the **regular set** of A. Similarly, we define

$$A^+ = \exp C \subseteq \widetilde{A}.$$

In fact, because the elements of W permute the Weyl chambers of A, there is a natural identification $A^+ \cong \widetilde{A}/W$.

One can show (see Theorem 7.39 in [27]) that $G = KAK$; that is, each $g \in G$ may be written $g = k_1 a k_2$ where $a \in A$ and $k_1, k_2 \in K$. More strongly, one has the decomposition

$$G = K\overline{A^+}K. \tag{3.9}$$

In fact, if $g = k_1 a k_2$ where $a \in A^+$ and $k_1, k_2 \in K$, then $a \in A^+$ is uniquely determined by g. In other words, there is a natural identification

$$K\backslash G/K \cong A/W, \tag{3.10}$$

where $K\backslash G/K$ denotes the space of double-cosets of G over K.

It follows easily from (3.3) that $[\mathfrak{a}, \mathfrak{n}] \subseteq \mathfrak{n}$ and $[\mathfrak{m}, \mathfrak{n}] \subseteq \mathfrak{n}$. One can in fact show that both A and M normalize N in G. Since M normalizes N, we see that MN is a closed Lie subgroup of G with Lie algebra $\mathfrak{m} \oplus \mathfrak{n}$. Also, since A centralizes M and also normalizes N, it follows that MAN is a closed Lie subgroup of G with Lie algebra $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Furthermore, $MN \trianglelefteq MAN$ and $MAN/MN \cong A$. One refers to MAN as a **minimal parabolic subgroup** of G (more generally, a **parabolic subgroup** of G is a group H such that $MAN \leq H \leq G$). Note that each choice of maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ and Weyl chamber in $\mathfrak{a}$ produces a minimal parabolic subgroup in this way.

We move now to the final decomposition of this section. For each $w \in W$, we choose a representative $m_w \in M'$. We then consider double cosets of the form

$$MANm_wMAN = Nm_wMAN,$$

called **Bruhat cells**. The **Bruhat decomposition** (see [27, Theorem 7.40]) states that G decomposes into a disjoint union of Bruhat cells. That is,

$$G = \dot{\bigcup_{w \in W}} Nm_wMAN. \tag{3.11}$$

From (3.7) we see that one of the Bruhat cells is an open and dense subset of G, namely the cell corresponding to the longest Weyl group element w^*.

The Bruhat decomposition should be viewed as analogous to the decomposition in (3.9). Helgason exploited this analogy and many others to relate analysis on the symmetric space G/K to analysis on the associated horocycle space, which we discuss in Section 3.3.

3.2.1 More on the Killing Form

In this section, we review some more facts about the Killing form on $\mathfrak{g}$. Because it is neither positive- nor negative-definite (only compact-type Lie groups produce definite Killing forms), there are some subtleties with which to be careful. For instance, there are many nonzero vectors $X \in \mathfrak{g}$ such that $B(X, X) = 0$.

First, we note that $\mathfrak{k} \perp \mathfrak{p}$ in $\mathfrak{g}$ under both B and the inner product $(\cdot, \cdot) = -B(\cdot, \theta\cdot)$. Orthogonality under the latter inner product follows immediately from orthogonality under B, since $\theta(\mathfrak{k}) = \mathfrak{k}$ and $\theta(\mathfrak{p}) = \mathfrak{p}$. Orthogonality under B follows from the relations $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. In fact, they imply that if $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, then $\mathrm{ad}(X)$ and $\mathrm{ad}(Y)$ may be given in two-by-two block form under the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as

$$\mathrm{ad}(X) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$
$$\mathrm{ad}(Y) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

Thus we have the product

$$\mathrm{ad}(X)\mathrm{ad}(Y) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},$$

and it follows that $B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) = 0$.

Since B is negative-definite on $\mathfrak{k}$ and positive-definite on $\mathfrak{p}$, it follows that both $\mathfrak{k}$ and $\mathfrak{p}$ are non-degenerate subspaces of $\mathfrak{g}$. In other words, $\mathfrak{p}^\perp = \mathfrak{k}$ and $\mathfrak{k}^\perp = \mathfrak{p}$ under B. It is also not difficult to explicitly write down the orthogonal projections onto $\mathfrak{k}$ and $\mathfrak{p}$. In fact, we note that $X + \theta(X) \in \mathfrak{k}$ and $X_\theta(X) \in \mathfrak{p}$ for all $X \in \mathfrak{g}$. Since $X = \frac{1}{2}(X + \theta(X)) + \frac{1}{2}(X - \theta(X))$ for all $X \in \mathfrak{g}$, we have the orthogonal projections $p_\mathfrak{k}(X) = \frac{1}{2}(X + \theta(X))$ and $p_\mathfrak{p}(X) = \frac{1}{2}(X - \theta(X))$.

Note that the orthogonal projections $p_\mathfrak{k}$ and $p_\mathfrak{p}$ commute with θ because $\mathfrak{k}$ and $\mathfrak{p}$ are both stable under θ. Since $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$, we see that $p_\mathfrak{k}(\mathfrak{n}) = p_\mathfrak{k}(\overline{\mathfrak{n}})$ and $p_\mathfrak{p}(\mathfrak{n}) = p_\mathfrak{p}(\overline{\mathfrak{n}})$. Furthermore, we easily see that $p_\mathfrak{k}(\mathfrak{n}) \subseteq \mathfrak{n} \oplus \overline{\mathfrak{n}}$ and $p_\mathfrak{p}(\mathfrak{n}) \subseteq \mathfrak{n} \oplus \overline{\mathfrak{n}}$. Hence, we obtain the decomposition

$$\mathfrak{g} = (p_\mathfrak{k}(\mathfrak{n}) \oplus \mathfrak{m}) \oplus (\mathfrak{a} \oplus p_\mathfrak{p}(\mathfrak{n})). \tag{3.12}$$

Furthermore, we will soon see that $B(\mathfrak{n}, \mathfrak{m}) = B(\mathfrak{n}, \mathfrak{a}) = 0$, from which it will quickly follow that this decomposition is orthogonal under both B and $(\cdot, \cdot)$ (hint: use the fact that the orthogonal projections $p_\mathfrak{k}$ and $p_\mathfrak{p}$ are self-adjoint and act as the identity on $\mathfrak{m} \subset \mathfrak{k}$ and $\mathfrak{a} \subset \mathfrak{p}$, respectively).

In fact, 3.12 can be used to prove the Iwasawa decomposition on the Lie algebra level. One begins by showing that $p_\mathfrak{p}|_\mathfrak{n} : \mathfrak{n} \to \mathfrak{p}$ and $p_\mathfrak{k}|_\mathfrak{n} : \mathfrak{n} \to \mathfrak{k}$ are linear isomorphisms onto their images. This comes from the fact that $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ are linearly independent subspaces and thus, if $X + \theta(X) = 0$ for some $X \in \mathfrak{n}$, then $X = 0 \in \mathfrak{n}$ and $\theta(X) = 0 \in \overline{\mathfrak{n}}$. It follows from this and 3.12 that each $X \in \mathfrak{g}$ may be written as $X = p_\mathfrak{k}(Y_1) + Z + H + p_\mathfrak{p}(Y_2)$, where $Y_1, Y_2 \in \mathfrak{n}$, $Z \in \mathfrak{m}$, and $H \in \mathfrak{a}$ are uniquely determined. But then we have

$$\begin{aligned} X &= p_\mathfrak{k}(Y_1) + Z + H + p_\mathfrak{p}(Y_2) \\ &= p_\mathfrak{k}(Y_1 - Y_2) + Z + H + p_\mathfrak{p}(Y_2) + p_\mathfrak{k}(Y_2) \\ &= p_\mathfrak{k}(Y_1 - Y_2) + Z + H + Y_2, \end{aligned}$$

where we note that $p_\mathfrak{k}(Y_1 - Y_2) + Z \in \mathfrak{k}$, $H \in \mathfrak{a}$, and $Y_2 \in \mathfrak{n}$. Because $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$ have pairwise trivial intersections, the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ follows.

Next, we note that the decomposition in 3.2 is an orthogonal decomposition of $\mathfrak{g}$ under the inner product $(\cdot, \cdot)$ but is *not* an orthogonal decomposition under

B. This is because $\mathrm{ad}(\mathfrak{a})$ consists of operators that are self-adjoint for $(\cdot,\cdot)$ and skew-adjoint under B. To prove this, we need the following elementary fact about skew-adjoint operators on real scalar product spaces:

Lemma 3.1. *Suppose that $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ is a symmetric bilinear form for a vector space V over $\mathbb{R}$. Suppose that $A : V \to V$ is skew-adjoint with respect to Q. If V_λ and V_μ are eigenspaces of A with eigenvalues $\lambda, \mu \in \mathbb{R}$ such that $\lambda \neq -\mu$, then $V_\lambda \perp V_\mu$.*

Proof. Fix $x \in V_\lambda$ and $y \in V_\mu$. Then $\lambda\langle x, y\rangle = \langle Ax, y\rangle = -\langle x, Ay\rangle = -\mu\langle x, y\rangle$. Thus $\langle x, y\rangle = 0$ since $\lambda \neq -\mu$. □

Corollary 3.2. *Since $\mathrm{ad}(\mathfrak{a})$ acts by skew-adjoint operators on $\mathfrak{g}$ under B, we have that $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ for all $\alpha \neq -\beta$. Furthermore, because B is nondegenerate and $\mathfrak{g} = \oplus_{\alpha\in\Sigma\cup\{0\}}\mathfrak{g}_\alpha$, this implies that the restriction $B|_{\mathfrak{g}_\alpha\times\mathfrak{g}_{-\alpha}} : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{R}$ is a non-degenerate pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ for any $\alpha \in \Sigma \cup \{0\}$.*

In particular, since $\mathfrak{m}\oplus\mathfrak{a} = \mathfrak{g}_0$ and $\mathfrak{n} = \oplus_{\alpha\in\Sigma^+\mathfrak{a}_\alpha}$, we have that $\mathfrak{n} \perp \mathfrak{a}$ and $\mathfrak{n} \perp \mathfrak{m}$. It also follows that $\mathfrak{g}_\alpha \perp \mathfrak{g}_\alpha$ for all $\alpha \in \Sigma$. In fact, $\mathfrak{n} \perp \mathfrak{n}$ and one can show that $B|_{\mathfrak{n}\times\overline{n}} : \mathfrak{n} \times \overline{\mathfrak{n}} \to \mathbb{R}$ is a nondegenerate pairing of $\mathfrak{n}$ and $\overline{\mathfrak{n}}$. Thus $\mathfrak{n}$ is far away from being a nondegenerate subspace of $\mathfrak{g}$ under B. Finally, we note that $\mathfrak{n}^\perp = \mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n}$.

Since $\mathfrak{g}_\alpha$ is not orthogonal to $\mathfrak{g}_{-\alpha}$ for all $\alpha \in \Sigma$, it follows that neither the root-space decomposition in 3.2, nor the related triangular decomposition in 3.6, nor the Iwasawa decomposition in 3.8 are orthogonal under B.

3.3 The Horocycle Space

In this section, we introduce and explore the geometric aspects of horocycles for Riemannian symmetric spaces. We begin by briefly reviewing the analogous geometric considerations for Euclidean space in order to motivate the definition of a horocycle.

3.3.1 Motivation from Euclidean Space

In the early twentieth century, the famous Radon transform was introduced. Formally, the idea is as follows. Consider $\mathbb{R}^n$ and denote the space of all $n-1$-dimensional planes in $\mathbb{R}^n$ by $\mathbb{P}^n$. For each function $f : \mathbb{R}^n \to \mathbb{C}$ with sufficient decay characteristics (for example, Schwartz or compact support), we can define the **Radon Transform** of f as a function $Rf : \mathbb{P}^n \to \mathbb{C}$ as follows:

$$(Rf)(\xi) = \int_{x\in\xi} f(x)dx.$$

If we could put a suitable measure on $\mathbb{P}^n$, then it would be possible to construct a **dual Radon Transform** which would take a function $g : \mathbb{P}^n \to \mathbb{C}$ and construct a new function $R^*g : \mathbb{P}^n \to \mathbb{C}$ by

$$(Rg)(x) = \int_{\xi\ni x} g(\xi)d\xi.$$

What properties should this measure have? For one thing, we would like it to have suitable invariance characteristics, like the ones Lebesgue measure on $\mathbb{R}^n$ has.

In particular, the Euclidean motion group $M(n) = \mathrm{O}(n) \rtimes \mathbb{R}^n$ acts transitively on $\mathbb{R}^n$. In fact, the subgroup of $M(n)$ which stabilizes 0 is equal to $\mathrm{O}(n)$. In other words, we can make the identification

$$\mathbb{R}^n \cong \mathrm{O}(n) \rtimes \mathbb{R}^n/\mathrm{O}(n).$$

Because $\mathrm{O}(n)$ and $\mathrm{O}(n) \rtimes \mathbb{R}^n$ are both unimodular, it follows that $\mathbb{R}^n$ possesses a unique (up to constant multiple) measure invariant under $M(n)$. This measure, of course, is Lebesgue measure.

Similarly, note that applying $M(n) = \mathrm{O}(n) \rtimes \mathbb{R}^n$ pointwise to hyperplanes in $\mathbb{P}^n$ produces a natural action of $M(n)$ on $\mathbb{P}^n$. The action is readily seen to be transitive. It is clear that the base hyperplane $\{(x, 0) \in \mathbb{R}^n | x \in \mathbb{R}^{n-1} \cong \mathbb{R}^{n-1}\}$ is stabilized by the subgroup $\mathrm{M}(n-1) \subseteq \mathrm{M}(n)$. Furthermore, we see that $\mathbb{R}^{n-1}$ is also stabilized by the reflection group $L = \{\mathrm{Id}, -\mathrm{Id}\}$ on $\mathbb{R}^n$. In fact, the full stabilizer subgroup of our base hyperplane is $\mathrm{M}(n-1) \times L$. Thus, we have the identification

$$\mathbb{P}^n \cong \mathrm{M}(n)/(\mathrm{M}(n-1) \times L).$$

It is thus possible to give $\mathbb{P}^n$ a smooth manifold structure and a measure which is invariant under M(n), which in turn makes it possible to define a dual Radon transform.

Next we examine one more way to look at hyperplanes in $\mathbb{R}^{n-1}$: namely, each hyperplane is an orbit of a subgroup of $\mathbb{R}^n$. In particular, one way to construct a hyperplane ξ is to fix a point $x \in \mathbb{R}^n$ and consider a subgroup $g\mathrm{M}(n-1)g^{-1} \leq \mathrm{O}(n) \rtimes \mathbb{R}^n$, where $g \in \mathrm{O}(n)$. The orbit

$$g\mathrm{M}(n-1)g^{-1} \cdot x$$

is then a hyperplane. In fact, every hyperplane may be constructed in this fashion. In particular, $x \in \mathbb{R}^n$ serves to fix a point in the hyperplane and $g \in \mathrm{O}(n)$ serves to give the hyperplane its "tilt."

Next we need to find a property of hyperplanes that we can readily generalize to Riemannian symmetric spaces. One important property of hyperplanes is that each hyperplane in $\mathbb{R}^n$ is orthogonal to a maximal collection of parallel lines in $\mathbb{R}^{n-1}$. For this reason, we expect that an appropriate generalization of a hyperplane should be orthogonal to a family of parallel geodesics.

3.3.2 Horocycles for Noncompact-Type Riemannian Symmetric Spaces

We continue with the same notation as in Section 3.2. A **horocycle** on a noncompact-type Riemannian symmetric space G/K is an orbit in G/K of a subgroup of G that is conjugate to N. In other words, it takes the form

$$gNg^{-1} \cdot g_0K \subseteq G/K,$$

where g, g_0 are arbitrary elements G. We denote the space of all Horocycles by Ξ.

Horocycles on Riemannian symmetric spaces were studied in detail by Helgason in the 1970s. The relationship between horocycles and points in a Riemannian symmetric space was intended to be analogous to the relationship between points and hyperplanes in $\mathbb{R}^n$ (see [23, p. 59]). Most of the results in this section may be found in either [20] and [23].

It is not difficult to see that left translations of horocycles by elements of G are also horocycles. In fact,

$$h \cdot (gNg^{-1} \cdot g_0K) = (hg)N(hg)^{-1} \cdot hg_0K$$

where $h \in G$ and $gNg^{-1} \cdot g_0K$ is a horocycle in Ξ. Thus G acts on Ξ by left translation. Furthermore, this action is transitive: consider the horocycles $gNg^{-1} \cdot hK$ and $N \cdot K$. Next consider the Iwasawa decomposition $h^{-1}g = kan$ and note that

$$\begin{aligned} hkN \cdot K &= hkN(hk)^{-1}hk \cdot K \\ &= g(an)^{-1}N(an)g^{-1}hk \cdot K \\ &= gNg^{-1}h \cdot K. \end{aligned}$$

One then has the following theorem of Helgason.

Theorem 3.3 (Theorem II.1.1 in [23])**.** *The group G acts transitively on Ξ, and the isotropic subgroup of G which fixes the horocycle $N \cdot K$ is MN.*

In other words, we can make the identification

$$\Xi \cong G/MN$$

In certain ways, Ξ behaves more simply than G/K, as exemplified by the following important decomposition theorem.

Theorem 3.4. *(Proposition II.1.4 in [23]).*

1. *The map*

$$\begin{aligned} K/M \times A &\to G/K \\ (kM, a) &\mapsto kaK \end{aligned}$$

is a surjection, and its restriction to $K/M \times A^+$ is a diffeomorphism onto its dense image in G/K.

2. *The map*

$$\begin{aligned} K/M \times A &\to G/MN \\ (kM, a) &\mapsto kaMN \end{aligned}$$

is a diffeomorphism.

Proof. Note that (1) is well-defined because $kmaK = k(mam^{-1})mK = kaK$. The surjection of the map in (i) then follows immediately from the decomposition $G = KAK$.

The map in (2) is well-defined by a similar argument. To see that it is surjective, we note that G acts transitively on Ξ and observe that

$$kanMN = kaMN$$

for each $k \in K$, $a \in A$ and $n \in N$. Surjectivity then follows from the Iwasawa decomposition.

Next we show that (2) is injective. Suppose that $k_1a_1MN = k_2a_2MN$. If follows that $a_2^{-1}k_2^{-1}k_1a_1 = mn$ for some $m \in M$ and $n \in N$. Then

$$\begin{aligned} k_2^{-1}k_1 &= a_2mna_1^{-1} \\ &= ma_2a_1^{-1}(a_1na_1^{-1}). \end{aligned}$$

Since $a_1na_1^{-1} \in N$, $a_2a_1^{-1} \in A$ and $k_2^{-1}k_1 \in K$, it follows from the uniqueness of the Iwasawa decomposition that $a_2a_1^{-1} = e$ and $k_2^{-1}k_1 = m \in M$. Thus $a_1 = a_2$ and $k_1M = k_2M$. Thus (2) is injective.

The proof that (2) is a diffeomorphism may be found in [23]. □

For each $w \in W$, define a set of horocycles by $\Xi_w = NAm_w \cdot MN \subseteq G/MN$, where m_w is a representative in M' of the Weyl group element $w \in W = M'/M$. Using the fact that A normalizes N and that M' normalizes A, we obtain the following identity for each Bruhat cell:

$$MANm_wMAN = MNAm_wMN = NAm_wMN.$$

The Bruhat decomposition then implies (see [23, p. 63]) that Ξ decomposes disjointly as

$$\Xi = \dot{\bigcup_{w \in W}} \Xi_w,$$

Furthermore, from the denseness of the embedding in (3.7), we see that Ξ_{w^*} is an open, dense subset of Ξ. In fact, we can write $\xi \in \Xi_{w^*}$ as

$$\xi = na(\xi) \cdot \xi^*,$$

where $n \in N$ and $a(\xi) \in A$. The next theorem shows that $a(\xi)$ is uniquely determined by ξ.

Theorem 3.5 (Proposition II.1.5 in [23]). *Each element $gMN \in G/MN$ may be written in the form*

$$gMN = nam_wMN,$$

where $n \in N$, $a \in A$, and $m_w \in M'$ is a representative of $w \in W$. Furthermore a and w are uniquely determined.

As a corollary of this result, we may make an identification

$$MN\backslash G/MN \cong A \times W,$$

which should be viewed in analogy with (3.10).

Finally, we take a moment to identify which horocycles pass through a given point in G/K.

Theorem 3.6. *A horocycle $hN \cdot K \in \Xi$ contains $gK \in G/K$ if and only if there is $k \in K$ such that $hMN = gkMN$.*

Proof. First, we note that it is clear that $gK \in gkN \cdot K$ for all $k \in K$.

Next, we claim that $gK \in gaN \cdot K$ (where $a \in A$) if and only if $a = e$. In fact, suppose that $gK \in ga'N \cdot K$ for some $a' \in A$. It immediately follows that $K = aN \cdot K$. In other fords, there is $n \in N$ such that $K = anK$. The uniqueness of the Iwasawa decomposition then shows that $n = e$ and $a = e$, as we wanted to show.

Now fix $hMN \in G/MN$ such that $gK \in hN \cdot K$. By Theorem 3.4, we may write $hMN = gkaMN$ for some $k \in K$ and $a \in A$. By the previous paragraph, we see that $gK \in gkN \cdot K$ and that $gK = gkK \in gkaN \cdot K$ if and only if $a = e$. Thus we are done. □

3.3.3 Tangent Spaces of Horocycles and Geometry on the Symmetric Space

In this section, we prove some results about the relationship between horocycles and the geometry on the symmetric space. The tangent spaces of horocycles play an important role in these results.

It is a result of Cartan that the maximal flats of G/K have the form $gA \cdot K \subseteq G/K$, where $g \in G$ (see Section V.6 in [21]). We begin by showing that horocycles and maximal flats are, indeed, embedded submanifolds of G/K.

Lemma 3.7. *Consider the canonical projection $\pi : G \to G/K$. The restrictions $\pi|_N : N \to N \cdot K \subseteq G/K$ and $\pi|_A : A \to A \cdot K \subseteq G/K$ are smooth embeddings.*

Proof. Note that the tangent space of $N \subseteq G$ at the identity may be identified with $\mathfrak{n} \subseteq \mathfrak{g}$ and that the tangent space of $A \subseteq G$ at the identity may be identified with $\mathfrak{a}$. Furthermore, there is a natural identification of $T_{eK}G/K$ with $\mathfrak{p}$ such that the canonical projection $p : G \to G/K$ induces a differential $d\pi_e : \mathfrak{g} \to \mathfrak{p}$ which is equal to the orthogonal projection $p_{\mathfrak{p}}$ of $\mathfrak{g}$ onto $\mathfrak{p}$.

Our next claim is that $\pi|_N : N \to N \cdot K \subseteq G/K$ and $\pi|_A : A \to A \cdot K \subseteq G/K$ are smooth embeddings. First we recall from the discussion surrounding 3.12 that the differential $d\pi_e = p_{\mathfrak{p}}$ restricts so that $d\pi|_{\mathfrak{n}} = p_{\mathfrak{p}}|_{\mathfrak{n}}$ and $d\pi|_{\mathfrak{a}} = p_{\mathfrak{p}}|_{\mathfrak{a}}$ are linear isomorphisms onto their images. Because $\pi : G \to G/K$ is equivariant with respect to the left-actions of G on G and G/K, it follows that $d(\pi|_N)_n : T|_n(N) \to T|_{nK}(G/K)$ and $d(\pi|_A)_a : T|_a(A) \to T|_{aK}(G/K)$ are linear isomorphisms for each

$n \in N$ and $a \in A$. The injectivity of $\pi|_N$ and $\pi|_A$ follows immediately from the Iwasawa decomposition on G.

We still need to show that $\pi|_N$ and $\pi|_A$ are homeomorphisms onto their images in G/K. Since they are already known to be continuous and injective, we need only show that they are closed maps. Suppose that $F \subseteq N$ is closed in N. Since N is a closed subgroup of G we see that F is also closed in G. Then $\pi(F) \subseteq G/K$ is closed if and only if $\pi^{-1}(\pi(F)) \subset G$ is closed. But $\pi^{-1}(\pi(F)) = FK \subseteq G$. It is well known that the product of a closed set and a compact set is closed for any topological group (see [8, p. 28], for instance). Thus $FK \subseteq G$ is closed and it follows that $p(F)$ is closed. The same argument shows that $\pi|_A$ is a homeomorphism onto its image in G/K. □

Corollary 3.8. *Each horocycle $gN \cdot K$ and maximal flat $gA \cdot K$ is an embedded submanifold of G/K.*

Proof. This corollary follows immediately from the fact that $\tau_g : x \to g \cdot x$ is a diffeomorphism of G/K. □

Recall that the geodesics in G/K which pass through the basepoint eK are precisely the curves of the form

$$\gamma_X(t) = \exp(tX)K,$$

where $X \in \mathfrak{p}$. More generally, the geodesics in G/K which pass through a point gK are precisely curves of the form

$$\gamma_{gK,X}(t) = g\exp(tX)K,$$

The next theorem shows that each horocycle is indeed orthogonal to a family of geodesics and, in fact, orthogonal to a maximal flat:

Theorem 3.9 (See Exercise VI.B.2 in [21])**.** *The horocycle $gN \cdot K$ is orthogonal to the maximal flat $gA \cdot K$ at gK. In particular, $gN \cdot K$ is orthogonal to the geodesic $\gamma_{gK,H}$ for each $H \in \mathfrak{a}$.*

Proof. First we will show that $N \cdot K \subseteq G/K$ and $A \cdot K \subseteq G/K$ are orthogonal at eK. It will then follow that $gN \cdot K$ is orthogonal to $gA \cdot K$ at gK because the metric on G/K is left-invariant under G.

By Lemma 3.7, we see that $N \cdot K$ and $A \cdot K$ are embedded submanifolds of G/K with tangent spaces $T_{eK}(N \cdot K) = d\pi_e(\mathfrak{n}) = p_{\mathfrak{p}}(\mathfrak{n})$ and $T_{eK}(A \cdot K) = d\pi_e(\mathfrak{a}) = p_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{a}$. We recall from the discussion following 3.12 that $p_{\mathfrak{p}}(\mathfrak{n}) \perp \mathfrak{a}$ and so we are done. □

Corollary 3.10. *The horocycle $kaN \cdot K$ is orthogonal to the maximal flat $kA \cdot K = kAk^{-1} \cdot K$ at kaK. In other words, $kaN \cdot K$ is orthogonal to the geodesic $\gamma_{\mathrm{ad}(k)H}$ for all $H \in \mathfrak{a}$.*

An Iwasawa decomposition argument shows that $kA \cdot K \cap kaN \cdot K = \{kaK\} \subseteq G/K$. Thus, the previous theorem shows that each horocycle is orthogonal to a unique maximal flat. Since kM uniquely identifies which maximal flat the horocycle $kaMN$ is orthogonal to, Helgason refers to $kM \in K/M$ as the **normal** for $kaMN$.

Theorem 3.11. *([23, Proposition 1.7(ii)]). Each $gK \in G/K$ is contained in a unique horocycle $kaMN$ with a given normal $kM \in K/M$.*

Proof. To show existence, suppose that $gK \in G/K$. Consider the Iwasawa decomposition $g^{-1}k = k_1a_1n_1$. Then $g = kn_1^{-1}a_1^{-1}k_1^{-1}$. In particular $gK = ka_1^{-1}n_2K$ for some $n_2 \in N$ because A normalizes N. Thus, gK is contained in the horocycle $ka_1^{-1}N \cdot K$, which has normal kM.

Conversely, if $gK \in kaN \cdot K$ for some $a \in A$, then by Theorem 3.6, we see that $kaMN = gk_1MN$ for some $k_1 \in K$. Thus there are $m_1, m_2 \in M$ and $n_1, n_2 \in N$ such that $kam_1n_1 = gk_1m_2n_2$. Then $g^{-1}k = k_1m_2(n_2n_1^{-1})m_1^{-1}a^{-1}$. Since M commutes with A and both M and A normalize N, we see that we can write $g^{-1}k = k_3a^{-1}n_3$. Hence a^{-1} is the Iwasawa A-component of $g^{-1}k$ and is thus uniquely identified by $g \in G$ and $kM \in K/M$. □

In the previous section, we classified which horocycles contain a given point in G/K. In this section, we consider the question of determining which horocycles pass through a given point and have the same tangent space at that point.

Theorem 3.12. *([23, Proposition 1.7(ii)] The horocycles $kN \cdot K$ and $k'N \cdot K$ have the same tangent space at eK if and only if $k^{-1}k' \in M'$. Thus there are $\#W$ distinct horocycles which pass through a given point in G/K and possess the same given tangent space.*

Proof. First we note that $kN \cdot K$ and $k'N \cdot K$ have the same tangent space if and only if $k^{-1}k'$ and $N \cdot K$ have the same tangent space. Thus we may assume that $k' = e$ without loss of generality.

We use the notation $\mathfrak{q} = p_{\mathfrak{p}}(\mathfrak{n})$ to denote the tangent space of $N \cdot K$ at eK. Note that $kN \cdot K = kNk^{-1} \cdot K$. Thus, the tangent space of $kN \cdot K$ at eK is

$$\begin{aligned} T_eK(kN \cdot K) &= \mathrm{Ad}(k)T_{eK}(N \cdot K) \\ &= \mathrm{Ad}(k)p_{\mathfrak{p}}(\mathfrak{n}) \\ &= p_{\mathfrak{p}}(\mathrm{Ad}(k)\mathfrak{n}), \end{aligned}$$

where the last equality comes from the fact that $\mathrm{Ad}(k)\mathfrak{p} \subseteq \mathfrak{p}$. Therefore, we need to show that $p_{\mathfrak{p}}(\mathfrak{n}) = p_{\mathfrak{p}}(\mathrm{Ad}(k)\mathfrak{n})$ if and only if $k \in M'$.

The key is to note that $\mathrm{Ad}(k)$ acts as an isometry on $\mathfrak{p}$ with respect to B for each $k \in K$ (since B is ad-invariant). Thus, the orthogonal decomposition

$$\mathfrak{p} = \mathfrak{a} \oplus p_{\mathfrak{p}}(\mathfrak{n})$$

(see 3.12) turns into the orthogonal decomposition

$$\mathfrak{p} = \mathrm{Ad}(k)\mathfrak{a} \oplus p_{\mathfrak{p}}(\mathrm{Ad}(k)\mathfrak{n}).$$

Hence $p_{\mathfrak{p}}(\mathfrak{n}) = p_{\mathfrak{p}}(\mathrm{Ad}(k)\mathfrak{n})$ if and only if $\mathfrak{a} = \mathrm{Ad}(k)\mathfrak{a}$, which is equivalent to $k \in M'$ by definition. Thus we are done.

We have already seen a horocycle $gN \cdot K$ passes through eK if and only if $g \in K$ (see Theorem 3.6). Since two horocycles $kN \cdot K$ and $k'N \cdot K$ (where $k, k' \in K$) are equal equal if and only if $k^{-1}k' \in M$, we can identify the set of horocycles passing through eK having a fixed tangent space with $W = M'/M$. Left translation by elements of G then shows that there are $\#W$ horocycles passing through a given point in G/K which have a given tangent space. □

3.3.4 Geometry on the Horocycle Space

In this section we discuss some aspects of the geometry on the horocycle space G/MN of a Riemannian symmetric space G/K of noncompact type. The first thing we note is that G/MN is not a symmetric space in general and, in fact, not even a reductive homogeneous space.

The fact that G/MN is not a symmetric space (with respect to G) follows from the fact that $\mathfrak{m} \oplus \mathfrak{n}$ is a degenerate subspace of $\mathfrak{g}$. In fact, from the results of Section 3.2.1, we see that $\mathfrak{n} \in (\mathfrak{m} \oplus \mathfrak{n}) \cap (\mathfrak{m} \oplus \mathfrak{n})^{\perp}$, so $\mathfrak{m} \oplus \mathfrak{n}$ is clearly degenerate. However, if G/MN were a symmetric space, then we would have there would be an involution τ such that $\mathfrak{g} = (\mathfrak{m} \oplus \mathfrak{n}) \oplus \mathfrak{q}$, where $\mathfrak{m} \oplus \mathfrak{n}$ is the $+1$-eigenspace and $\mathfrak{q}$ is the -1-eigenspace of τ. The same argument used to prove that $\mathfrak{k} \perp \mathfrak{p}$ would show that $(\mathfrak{m} \oplus \mathfrak{n}) \perp \mathfrak{q}$. But since $\mathfrak{m} \oplus \mathfrak{n}$ is degenerate, it is impossible to write an orthogonal (with respect to B) decomposition $\mathfrak{g} = (\mathfrak{m} \oplus \mathfrak{n}) \oplus \mathfrak{q}$ for any subspace $\mathfrak{q} \subseteq \mathfrak{g}$. Thus G/MN is not a symmetric space.

We say that a homogeneous space G/H, where H is a closed subgroup of G, is **reductive** if there is an $\mathrm{ad}(\mathfrak{h})$-invariant subspace $\mathfrak{q} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}$. In particular, it is clear that symmetric spaces are always reductive. The important property of reductive homogeneous spaces is that if G is semisimple, then the Killing form on G can be used to produce a G-invariant pseudo-Riemannian metric on G/H, which makes it possible to define geodesics, parallel transport, etc. on G/H.

Unfortunately, G/MN is not, in general, reductive (see [23, p. 65]). What can be said, then about the geometry on G/MN? Helgason defines curves which resemble geodesics in the following way. Let γ be a **regular** geodesic–that is, a geodesic such that the stabilizer $G_{\gamma} = \{g \in G | g \cdot \gamma = \gamma\}$ has minimal dimension. Next, we pick a point $x = gK \in \gamma \subseteq G/X$. Then isotropy subgroup at $x \in G/K$ is $K_x = gKg^{-1}$. Consider the Lie algebra $\mathfrak{k}_x = \mathrm{Ad}(g)\mathfrak{k}$ of K_x and its orthogonal complement $\mathfrak{p}_x = \mathrm{ad}(g)\mathfrak{p}$. Then the quotient map $\pi_x : g \mapsto g \cdot x$ differentiates to an isomorphism $(d\pi_x)_e \mathfrak{p}_x \to T_x(G/K)$.

Finally, we choose $d_x \in \mathfrak{p}_x$ such that $(d\pi_x)_e(d_x)$ is the tangent vector of γ at $x \in G/K$. Because γ is a regular curve, it follows that the centralizer of d_x in $\mathfrak{p}_x$ has minimal dimension. Thus, $\mathfrak{a}_x = Z_{\mathfrak{p}_x}(g_x)$ is a maximal abelian subspace of $\mathfrak{p}_x$ and d_x lies in a Weyl chamber $\mathfrak{a}_x^+$. Denote by N_x the nilpotent group which is constructed with the weyl chamber $\mathfrak{a}_x^+$. It can be shown (see [23, p. 66]) that

$N_x = N_y$ for all $x, y \in \gamma$. Thus it makes sense to write $N_\gamma = N_x$ for any x along γ. Then one defines

$$\Gamma_\gamma(t) = N_\gamma \cdot \gamma(t) \in \Xi$$

for $t \in \mathbb{R}$. We say that Γ_γ is a **geodesic** in $\Xi = G/MN$.

Lemma 3.13 (Lemma II.1.9 in [23]). *For any $g \in G$, we have that $g \cdot \Gamma_\gamma(t) = \Gamma_{g\cdot\gamma}(t)$.*

A **transvection** along a geodesic $\gamma_{gK,X}$ in G/K is defined to be a map of the form $x \mapsto g\exp(tX)g^{-1} \cdot x$, where $g \in G$ and $X \in \mathfrak{p}$. It is an isometry of G/K that, when restricted to $\gamma_{gk,X}$, corresponds to a shift along the geodesic.

Lemma 3.14 (Corollary II.1.10 in [23]). *Any two horocycles on a given geodesic Γ_γ correspond in G/K under a transvection along γ.*

Given an element $m \in M'$, we say that the m-**reflection** about a point $naK \in G/K$ is the map $x \mapsto (na)m(na)^{-1} \cdot x$. Note that the reflections about a point $x \in G/K$ preserve the tangent spaces of horocycles passing through x.

Theorem 3.15 (Proposition II.1.12 in [23]). *Given any two horocycles ξ_1 and ξ_2, it is possible to write $\xi_1 = s\tau\xi_2$, where τ is a transvection along a geodesic γ orthogonal to both ξ_1 and ξ_2 and s is a reflection about $\xi_1 \cap \gamma$.*

3.4 The Radon Transform for Riemannian Symmetric Spaces

Recall that functions on G/K may be thought of as right-K-invariant functions on G. Similarly, functions on G/MN are identified with right-MN-invariant functions on G. We can, for instance, consider the natural projections $P_K : C^\infty(G) \to C^\infty(G/K)$ and $P_{MN} : \mathcal{D}(G) \to \mathcal{D}(G/MN)$ by

$$P_K f(gK) = \int_K f(gk)dk$$
$$P_{MN} f(gMN) = \int_{MN} f(gmn)dm\,dn.$$

In fact, these projections generalize to projections $P_K : \mathcal{D}'(G) \to \mathcal{D}'(G/K)$ and $P_{MN} : \mathcal{D}'(G) \to \mathcal{D}'(G/MN)$. Note that these projections are both intertwining operators for the left-regular action of G.

Note that compactly-supported functions on G/K correspond to campactly-supported right-K-invariant functions on G (because K is compact). In other words, $\mathcal{D}(G/K) \subseteq \mathcal{D}(G)$. Thus, it is possible to construct a G-intertwining operator $\mathcal{R} : \mathcal{D}(G/K) \to \mathcal{D}(G/MN)$ by simply restricting $P_M N$. That is, we define

$\mathcal{R} = P_{MN}|_{\mathcal{D}(G/K)}$. In particular,

$$\begin{aligned}\mathcal{R}f(gMN) &= \int_{MN} f(gmnK)dm\,dn \\ &= \int_N f(gnK)dn.\end{aligned}$$

for each $f \in \mathcal{D}(G/K)$.

In fact, if we recall that $gMN \in G/MN$ corresponds to the horocycle $gN \cdot K \subseteq G/K$, then we see that

$$\mathcal{R}f(\xi) = \int_\xi f(x)dx$$

for each horocycle $\xi \in \Xi$. It is for this reason that we refer to $\mathcal{R}$ as the **Radon transform** for G/K.

Similarly, the **dual Radon transform** $\mathcal{R}^\vee : \mathcal{D}(G/MN) \to \mathcal{D}(G/K)$ is defined by

$$\begin{aligned}\mathcal{R}^\vee f(gK) &= \int_K f(gkMN)dk \\ &= \int_{K/M} f(gkMN)dk.\end{aligned}$$

for each $f \in \mathcal{D}(G/MN)$.

Because the horocycles passing through a point $gK \in G/K$ are precisly those of the form $gkMN$, where $k \in K/M$ (see Theorem 3.6, we see that

$$\mathcal{R}^\vee f(x) = \int_{\xi \ni x} f(\xi)d\xi,$$

and thus the dual Radon transform generalizes the dual Radon transform for Euclidean space.

3.5 Spherical Representations and Gelfand Pairs

In this section, we review the theory of harmonic analysis on Gelfand Pairs, which generalize the theory of harmonic analysis on locally compact abelian groups. Most of the theorems and their proofs (with the exception of the direct integral theory) appear in Chapter 6 of [8], and these notes closely follow the exposition there. Helgason uses a more geometric approach in Chapter IV of [22] to study the case of Riemannian symmetric spaces, where the theory of invariant differential operators is emphasized.

Suppose now that G is any locally compact topological group and that K is a compact subgroup. Consider the convolution algebra $L^1(G)$, and note that $L^1(G)$ is in fact a Banach $*$-algebra with the involution $*$ given by $f^*(x) = \overline{f(x^{-1})}$. One can show that the space $L^1(G)^\# = L^1(K\backslash G/K)$ of bi-K-invariant functions in $L^1(G)$ is a closed subspace of $L^1(G)$ and closed under convolutions. Furthermore, there is a projection $\# : L^1(G) \to L^1(G)^\#$ given by

$$f^\#(g) = \int_K \int_K f(k_1 g k_2) dk_1 dk_2.$$

Definition 3.16. *If G is a locally compact group and K is a compact subgroup, then we say that G/K is a **Gelfand pair** if $L^1(G)^\#$ is a commutative Banach $*$-algebra.*

Notice that if G is abelian and $K = \{e\}$ is the trivial subgroup, then G/K is clearly a Gelfand pair. In fact, the concept of a Gelfand pair is intended to be a tool for generalizing the techniques of harmonic analysis on abelian groups. Another important class of examples is given by Riemannian symmetric spaces, as demonstrated by the following lemma.

Theorem 3.17. *Suppose that K is a compact closed subgroup of a locally compact topological group G. If there is an involution θ such that $\theta(g) \in Kg^{-1}K$ for each $g \in G$, then G/K is a Gelfand pair.*

Corollary 3.18. *Every Riemmanian symmetric space (of either compact or non-compact type) is a Gelfand pair.*

Proof. This follows immediately from the KAK decomposition (we note that the KAK decomposition holds for both compact- and non-compact-type Riemannian symmetric spaces (see [21, Theorem V.6.7])). □

The next lemma is important for some integration arguments on G:

Theorem 3.19. *If G/K is a Gelfand pair, then G is a unimodular group (that is, the Haar measure on G is both left- and right-G invariant and invariant under the group inverstion).*

For harmonic analysis on an abelian locally compact group G, one studies the space $\widehat{G}$ of irreducible representations of G. By Schur's Lemma, it is easy to show that each irreducible representation of G is one-dimensional. Thus, G is the character group of G and consists of all of the continuous homomorphisms $G \to S^1$, where S^1 is the group of all complex numbers of modulus one. Finally, the characters ϕ in $\widehat{G}$ are precisely the bounded, continuous positive-definite functions on G such that

$$f \mapsto \int_G f(x)\overline{\phi(x)} dx$$

is a character of the Banach $*$-algebra $L^1(G)$.

For Gelfand pairs we consider the following generalization of the notion of a character for an abelian group.

Definition 3.20. *A continuous, bounded, bi-K-invariant function $\phi \in C_b(G)^\#$ is said to be* ***spherical*** *if the mapping*

$$\chi_\phi : f \mapsto \int_G f(x)\phi(x^{-1})dx$$

is a character of the Banach $*$-algebra $L^1(G)^\#$.

The appearance of $\phi(x^{-1})$ in the above definition in place of $\overline{\phi(x)}$ may be confusing at first, but we will ultimately be concerned with positive-definite spherical functions, which, like all positive-definite functions, have the property that $\phi(x^{-1}) = \overline{\phi(x)}$.

Theorem 3.21. *Every character of $L^1(G)^\#$ is given by a spherical function.*

The analogies with characters for abelian groups continue in the following two theorems.

Theorem 3.22. *A continuous, bounded bi-K-invariant function $\phi \in C_b(G)^\#$ is spherical if and only if*

1. $\phi(e) = 1$
2. *For each $f \in L^\infty(G)^\#$, there is $\chi_\phi(f) \in \mathbb{C}$ such that one has $f * \phi = \chi_\phi(f)\phi$, where $\chi_\phi(f) = \int_G f(x)\phi(x^{-1})dx$.*

As a consequence of the following theorem, it is easy to see that any spherical function on G will in fact be bi-K-invariant. Thus, one may think of it as a function on G, as a function on G/K, or as a function on $K\backslash G/K$.

Theorem 3.23 (Proposition IV.2.2 in [22])**.** *A continuous bounded function $\phi : G \to \mathbb{C}$ is a spherical function if and only if ϕ is not identically zero and*

$$\int_K \phi(xky)dk = \phi(x)\phi(y) \tag{3.13}$$

for all $x, y \in G$.

For abelian groups, a character is a one-dimensional unitary group representation. Similarly, spherical functions on a Gelfand pair G/K are closely related to a particular type of representation of G, called a spherical representation.

Definition 3.24. *Let G/K be a Gelfand pair. We say that a Hilbert space representation $(\pi, \mathcal{H})$ of G is* ***spherical*** *if there is a nonzero cyclic vector $v \in \mathcal{H}$ such that $\pi(k)v = v$ for all $k \in K$.*

Theorem 3.25. *Suppose that G is a Lie group with a closed compact subgroup K.*

1. *If $(\pi, \mathcal{H})$ is a of G such that there is a nonzero cyclic vector $v \in \dim \mathcal{H}^K$ and $\dim \mathcal{H}^K = 1$, then π is irreducible.*
2. *G/K is a Gelfand pair if and only if $\dim \mathcal{H}^K \leq 1$ for every irreducible unitary representation $(\pi, \mathcal{H})$ of G.*

The next theorem shows that the positive-definite spherical functions are precisely those given by certain matrix coefficients of certain irreducible untiary spherical representations:

Lemma 3.26. *(Theorem IV.3.7 in [22]) Suppose that $(\pi, \mathcal{H})$ is an irreducible unitary spherical representation of G with a spherical vector e. Then the function ϕ_π on G given by*

$$\phi_\pi(x) = \langle e, \pi(g)e \rangle$$

is a positive-definite spherical function. Furthermore, every positive-definite spherical function takes the form ϕ_π for an irreducible unitary spherical representation π that is unique up to unitary equivalence.

Proof. Suppose that $(\pi, \mathcal{H})$ is an irreducible unitary spherical representation of G with a spherical vector e. We will show that ϕ_π is spherical by demonstrating that it satisfies the condition of Lemma 3.23. Note that the orthogonal projection P from $\mathcal{H}$ to $\mathcal{H}^K$ is given by:

$$P(v) = \int_K \pi(k)v \, dk.$$

Since $P(\pi(y)e) \in \mathcal{H}^K$ and $\dim \mathcal{H}^K = 1$, it follows that $P(\pi(y)e) = ce$ for some nonzero $c \in \mathbb{C}$. But then

$$\begin{aligned} c &= \langle P(\pi(y)e), e \rangle \\ &= \int_K \langle \pi(ky)e, e \rangle \\ &= \langle \pi(y)e, e \rangle. \end{aligned}$$

Hence

$$\begin{aligned} \int_K \phi_\pi(xky)dk &= \int_K \langle e, \pi(xky)e \rangle \\ &= \left\langle \pi(x^{-1})e, \int \pi(k)\pi(y)e \, dk \right\rangle \\ &= \left\langle \pi(x^{-1})e, P(\pi(y)e) \right\rangle \\ &= \left\langle \pi(x^{-1})e, \langle \pi(y)e, e \rangle e \right\rangle \\ &= \langle e, \pi(x)e \rangle \langle e, \pi(y)e \rangle \\ &= \phi_\pi(x)\phi_\pi(y) \end{aligned}$$

On the other hand, suppose that ϕ is a positive-definite spherical function on G. Because ϕ is positive definite, we recall from Section 2.4 that there is a representation $(\pi, \mathcal{H})$ of G with a nonzero cyclic vector $v \in \mathcal{H}$ such that

$$\phi(g) = \langle v, \pi(g)v \rangle$$

It follows immediately from the bi-K-invariance of ϕ that $v \in \mathcal{H}^K$.

It remains to be shown that π is irreducible. To that end, we will show that $\dim \mathcal{H}^K = 1$. Fix $y \in G$. From Lemma 3.23, we see that

$$\begin{aligned}\langle \pi(x^{-1})v, P(\pi(y)v) \rangle &= \int_K \phi(xky)dk \\ &= \phi(x)\phi(y) \\ &= \langle \pi(x^{-1})v, \phi(y)v \rangle\end{aligned}$$

for all $x \in G$. Recall that v is cyclic; that is, $\langle \pi(G)v \rangle$ is dense in $\mathcal{H}$, It follows that $P(\pi(y)v) = \phi(y)v$. Using again the fact that v is cyclic, we see that

$$\dim(\text{range } P) = 1.$$

In other words, $\dim \mathcal{H}^K = 1$, and thus $\mathcal{H}$ is irreducible. □

One can show that in fact, the positive-definite spherical functions (together with the zero function) are precisely the extremal points of the compact convex space $\mathcal{P}(G)^{\#}$ of positive-definite functions $\phi \in L^\infty(G)$ such that $||\phi||_\infty = \phi(e) \leq 1$. The Krein-Milman Theorem then suggests that all other positive-definite functions should be constructed as limits of convex combinations of positive-definite spherical functions.

We will see that understanding the commutative Banach $*$-algebra $L^1(G)^{\#}$ is the key to determining the decomposition of the regular representation of G on $L^2(G/K)$. To that end, we begin by considering the Gelfand transform on $L^1(G)^{\#}$, which in this context we refer to as the **spherical Fourier transform**. We have seen that the character space $\widehat{L^1(G)^{\#}}$ for $L^1(G)^{\#}$ may be identified with the set of spherical functions on G and with the space of spherical representations of G. For any $f \in L^1(G)^{\#}$, one sees that the spherical Fourier transform $\hat{} : L^1(G)^{\#} \to C(\widehat{L^1(G)^{\#}})$ is given by

$$\widehat{f}(\pi) = \chi_{\phi_\pi}(f) = \int_G f(x)\phi_\pi(x^{-1})dx.$$

This Fourier transform may readily be seen to have the following basic properties:

1. $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ for $f, g \in L^1(G)^{\#}$
2. $\widehat{f^*} = \overline{\widehat{f}}$
3. $\widehat{f} \in C_0(X)$ and $||\widehat{f}||_\infty \leq ||f||_1$

These may be proved directly, but they also follow immediately from the fact that $\hat{}$ is a Gelfand transform of a commutative Banach $*$-algebra.

It can be shown that if $f \in \mathcal{P}(G)^{\#}$, then the support of $\widehat{f}$ on the space of positive-definite spherical functions is sufficient to recover f. This is essentially true by the Krein-Milman theorem, as mentioned earlier. We denote by $\widehat{G/K}$ the space of equivalence classes of irreducible unitary spherical representations of G. This space may be identified, as seen by Theorem 3.26, with the space of positive-definite spherical functions G. In fact, one has the following result:

Theorem 3.27. *Let $\mathcal{P}^1(G)^\#$ denote the space of all finite linear combinations of positive-definite functions in $\mathcal{P}(G)^\#$. For each irreducible unitary spherical representation π, let ϕ_π denote the corresponding spherical function. Then there is a unique positive Borel measure μ on $\widehat{G/K}$ such that*

1. $\widehat{f} \in L^1(\widehat{G/K})$ *for all* $f \in \mathcal{P}^1(G)^\#$
2. $f(g) = \int_{\widehat{G/K}} \widehat{f}(\pi)\phi(g)d\mu(\pi)$ *for all* $f \in \mathcal{P}^1(G)^\#$ *(in particular,* $f(e) = ||\widehat{f}||_1$*)*
3. $\int_G |f(g)|^2 dg = \int_{\widehat{G/K}} |\widehat{f}(\pi)|^2 d\mu(\pi)$ *for all* $f \in C_c(G)^\#$*, so that the Fourier transform extends to a unitary operator* $\hat{}: L^2(G)^\# \to L^2(\widehat{G/K})$.

We refer to μ as the ***Plancherel measure*** *for G/K.*

The above theorem provides all of the essential details about harmonic analysis on spaces of bi-K-invariant functions on G. We would like to understand harmonic analysis on spaces of functions on G/K, which may be thought of as right-invariant functions on G.

To begin, we recall that there is a unique G-invariant measure on G/K. In fact, we may specify this G-invariant measure as follows. Consider the canonical projection $p : G \to G/K$. We recall that $f \mapsto f \circ p$ defines a continuous embedding $C_c(G/K) \to C_c(G)$. The G-invariant measure on G/K is then defined by

$$\int_{G/K} f(gK)d(gK) = \int_G f(p(g))dg.$$

for all $f \in C_c(G/K)$. This definition gives rise to a natural unitary identification $L^2(G)^\# \cong L^2(G/K)^K$, where $L^2(G/K)^K$ denotes the left-K-invariant square-integrable functions on G/K. We will use the Fourier theory of $L^2(G/K)^\#$ in Theorem 3.27 to generate harmonic analysis on $L^2(G/K)$.

Next we recall the **operator-valued Fourier transform on** G. For any $f \in L^1(G)$ and any $(\pi, \mathcal{H}) \in \widehat{G}$, we define an operator $\widehat{f}(\pi) \in B(\mathcal{H})$ by

$$\widehat{f}(\pi) = \int_G f(g)\pi(g^{-1})dg.$$

Note that $\widehat{f}(\pi)$ is very closely related to the integrated representation of π, which is given by

$$\pi(f) = \int_G f(g)\pi(g)dg.$$

In fact, it follows that $\widehat{f}(\pi) = \pi(f^\vee)$, where $f^\vee(g) = f(g^{-1})$ (here it is necessary to use that G is unimodular). Another quick computation shows that $\widehat{f}(\pi)^* = \pi(f)$. The Fourier transform and the integrated representations will both be useful in different contexts.

This transform has all the expected properties of a Fourier transform:

1. $\widehat{f * g}(\pi) = \widehat{f}(\pi)\widehat{g}(\pi)$ for $f, g \in L^1(G)^{\#}$
2. $\widehat{f^*} = \widehat{f}(\pi)^*$
3. $||\widehat{f}(\pi)|| \leq ||f||_1$

for $f, g \in L^1(G)$ and $\pi \in \widehat{G}$. Furthermore, if $g \in G$ and $f \in L^1(G)$, then

$$\begin{aligned}\widehat{L_g f}(\pi) &= \int_G f(g^{-1}x)\pi(x^{-1})dx \\ &= \int_G f(x)\pi((gx)^{-1})dx = \widehat{f}(\pi)\pi(g^{-1}).\end{aligned}$$

Similarly, on the level of integrated representations, we have that

$$\begin{aligned}\pi(L_g f) &= \int_G f(g^{-1}x)\pi(x)dx \\ &= \int_G f(x)\pi(gx)dx = \pi(g)\pi(f).\end{aligned}$$

Recall that functions on G/K are naturally identified with right-K-invariant functions on G. Furthermore, for such functions the projection $\#$ defined at the beginning of the section takes on the simplified form

$$f^{\#}(g) = \int_K f(kg)dk$$

for each $f \in L^1(G/K) \subseteq L^1(G)$.

Let $\mathcal{P}^1(G/K)$ denote the space of all finite linear combinations of positive-definite functions in $\in L^1(G/K)$. Fix $f \in \mathcal{P}^1(G/K)$. Then $f^{\#} \in \mathcal{P}^1(G)^{\#}$. It is not difficult to use the right-invariance of f to see that $f(e) = f^{\#}(e)$.

Here we hit a small notational annoyance. Note that $f^{\#}$ may considered an element of either $L^1(G)$ or $L^1(G)^{\#}$, and we have a different Fourier transform for each space. For now, we let $\widetilde{f^{\#}}$ denote the Fourier transform of $f^{\#}$ as an element of $L^1(G)^{\#}$ and let $\widehat{f^{\#}}$ denote the Fourier transform of f as an element of $L^1(G)$. There is no real difficulty here, however, because the two transforms are closely related as follows. For each $(\pi, \mathcal{H}_\pi) \in \widehat{G/K}$, let e_π denote a unit vector in $\mathcal{H}_\pi^K$. Then

$$\begin{aligned}\langle \widehat{f^{\#}}(\pi)e_\pi, e_\pi\rangle &= \int_G \langle f^{\#}(g)\pi(g^{-1})e_\pi, e_\pi\rangle dg \\ &= \int_G f^{\#}(g)\phi_\pi(g^{-1}) = \widetilde{f^{\#}}(\pi).\end{aligned}$$

Next we examine the connection between $\widehat{f^{\#}}$ and $\widehat{f}$. For each $(\pi, \mathcal{H}_\pi) \in \widehat{G/K}$, we note that the orthogonal projection $P : \mathcal{H}_\pi \to \mathcal{H}_\pi^K$ is given by

$$Pv = \int_K \pi(k)v dk.$$

We then see that

$$\begin{aligned}
\widehat{f^{\#}}(\pi) &= \int_G \int_K f(kg)dk\pi(g^{-1})dg \\
&= \int_K \int_G f(g)\pi((k^{-1}g)^{-1}dgdk \\
&= \int_K \widehat{f}(\pi)\pi(k)dk = \widehat{f}(\pi)P_\pi.
\end{aligned}$$

In particular, since $P_\pi e_\pi = e_\pi$, we see that $\langle \widehat{f}(\pi)e_\pi, e_\pi\rangle = \langle \widehat{f^{\#}}(\pi)e_\pi, e_\pi\rangle$.

We may now procede to the inversion formula for f. We see that

$$\begin{aligned}
f(e) = f^{\#}(e) &= \int_{\widehat{G/K}} \widetilde{f^{\#}}(\pi)d\mu(\pi) \\
&= \int_{\widehat{G/K}} \langle \widehat{f}(\pi)e_\pi, e_\pi\rangle d\mu(\pi).
\end{aligned}$$

This formula merely recovers the value of f at the identity, which does not sound impressive until we perform the following trick:

$$\begin{aligned}
f(g) = L_{g^{-1}}f(e) &= \int_{\widehat{G/K}} \langle \widehat{L_{g^{-1}}f}(\pi)e_\pi, e_\pi\rangle d\mu(\pi) \\
&= \int_{\widehat{G/K}} \langle \widehat{f}(\pi)\pi(g)e_\pi, e_\pi\rangle d\mu(\pi).
\end{aligned}$$

Proceeding to the L^2-theory, we claim that the regular representation $(L, L^2(G/K))$ decomposes into a direct integral over all spherical representations with respect to the Plancherel measure:

$$L^2(G/K) \cong_G \int_{\widehat{G/K}}^{\oplus} \mathcal{H}_\pi d\mu(\pi). \tag{3.14}$$

In order to correctly define the space $\int_{\widehat{G/K}}^{\oplus} \mathcal{H}_\pi d\mu(\pi)$, we need to construct measureable frames. In fact, we begin by considering the section

$$\begin{aligned}
e : \widehat{G/K} &\to \dot{\bigcup_{\pi\in\widehat{G/K}}} \mathcal{H}_\pi \\
\pi &\mapsto e_\pi.
\end{aligned}$$

Then choose a countable dense subset $\{g_n\}_{n\in\mathbb{N}}$ of G (since G is a Lie group, it is separable). Our frame of measurable sections will be the collection of sections $g_n \cdot e$ defined by $\pi \mapsto \pi(g_n)e$. Since e_π is a conical vector in $\mathcal{H}_\pi$ for each $\pi \in \widehat{G/K}$, it follows that $\langle \{g_n \cdot e(\pi)\}_{n\in\mathbb{N}}\rangle = \langle\{\pi(g_n)e_\pi\}_{n\in\mathbb{N}}\rangle$ is dense in $\mathcal{H}_\pi$ for each $\pi \in \widehat{G/K}$.

We thus have a measurable bundle of Hilbert spaces which produces the direct integral in 3.14, and G acts unitarily on $\int_{\widehat{G/K}}^{\oplus} \mathcal{H}_\pi d\mu(\pi)$ by

$$g \cdot \int_{\widehat{G/K}}^{\oplus} v_\pi d\mu(\pi) = \int_{\widehat{G/K}}^{\oplus} \pi(g) v_\pi d\mu(\pi),$$

where $\int_{\widehat{G/K}}^{\oplus} v_\pi d\mu(\pi)$ denotes a square-integrable section $\pi \mapsto v_\pi$ in $\int_{\widehat{G/K}}^{\oplus} \mathcal{H}_\pi d\mu(\pi)$.

Finally, we define the operator

$$\begin{aligned} T : C_c(G/K) &\to \int_{\widehat{G/K}}^{\oplus} \mathcal{H}_\pi d\mu(\pi) \\ f &\mapsto \int_{\widehat{G/K}}^{\oplus} \pi(f) e_\pi d\mu(\pi). \end{aligned}$$

Note that we have used integrated representations here instead of the Fourier transform; this is done in order that T be an intertwining operator, which follows from the fact that $\pi(L_g f) = \pi(g)\pi(f)$ for all $g \in G$.

It remains only to be shown that T extends continuously to a unitary intertwining operator between $L^2(G/K)$ and $\int_{\widehat{G/K}}^{\oplus} \mathcal{H}_\pi d\mu(\pi)$. To prove this claim, we recall that if $f \in C_c(G/K)$, then $f * f^*$ is a positive-definite function on G. That is, $f * f^* \in \mathcal{P}^1(G/K)$. A quick computation shows that

$$f * f^*(e) = \int_G f(g)\overline{f((g^{-1})^{-1})} dg = ||f||_2^2.$$

But the inversion formula given above for elements of $\mathcal{P}^1(G/K)$ shows us that

$$\begin{aligned} ||f||_2^2 = f * f^*(e) &= \int_{\widehat{G/K}} \langle \widehat{f * f^*}(\pi) e_\pi, e_\pi \rangle d\mu(\pi) \\ &= \int_{\widehat{G/K}} \langle \widehat{f}(\pi)\widehat{f}(\pi)^* e_\pi, e_\pi \rangle d\mu(\pi) \\ &= \int_{\widehat{G/K}} \langle \pi(f)^*\pi(f) e_\pi, e_\pi \rangle d\mu(\pi) \\ &= \int_{\widehat{G/K}} ||\pi(f) e_\pi||^2 d\mu(\pi) = ||Tf||^2. \end{aligned}$$

One can thus show that T extends to a unitary intertwining operator.

We have so far come at the theory of spherical functions from an "integral calculus" point of view. In the case of a symmetric space, it is possible to view the theory from a "differential calculus" point of view. In particular, this approach is taken by Helgason in Chapter IV of [22].

Suppose that G/K is a Riemmanian symmetric space with an involution θ. In this particular case, one also has that the algebra $\mathbb{D}(G/K)$ of left G-invariant differential operators on G/K is abelian (see Corollary II.5.4 in [22]). It is thus natural to look for functions which are joint eigenvectors for the operators in $\mathbb{D}(G/K)$. In fact, these are precisely the spherical functions:

Theorem 3.28. *A function $\phi \in C^\infty(G/K)$ is a spherical function if*

1. *ϕ is left-invariant under translations by elements of K.*
2. *ϕ is an eigenfunction for every differential operator in $\mathbb{D}(G/K)$.*
3. *$\phi(eK) = 1$.*

In particular, spherical functions are eigenfunctions of the Laplace operator. An application of the elliptic regularity theorem shows that any distribution in $\mathcal{D}'(G)$ which satisfies the conditions of Definition 3.28 is automatically an analytic function (see [23, p. 105]), which is why we speak of spherical *functions* rather than spherical *distributions*. It is this definition, in fact, which we will generalize to the context of the horocycle space.

3.6 Conical Representations

In this section we assume that G/K is a noncompact-type Riemannian symmetric space. Just as was the case for a symmetric space G/K, it can be shown that the algebra $\mathbb{D}(G/MN)$ of left-G-invariant differential operators on a horocycle space G/MN is commutative (see Theorem II.2.2 in [23]), and it is natural to look for joint eigendistributions.

Definition 3.29. *A distribution $\phi \in \mathcal{D}'(G/MN)$ is called a* ***conical distribution*** *if it is an eigendistribution for every differential operator in $\mathbb{D}(G/MN)$. If ϕ is in fact a smooth function on G, then we say that it is a* ***conical function****.*

As before, we notice that a conical distribution on G/MN may be considered to be a bi-MN-invariant distribution on G. In contrast to the situation for spherical functions, a conical distribution need not be analytic and need not be a function at all.

The analogue for a horocycle space of a spherical representation is called a *conical representation.*

Definition 3.30. *A Hilbert space representation $(\pi, \mathcal{H})$ of G is said to be* ***conical*** *if there is a nonzero cyclic distribution vector v in $\mathcal{H}^{-\infty}$ such that $\pi(MN)v = v$. In this case, v is said to be a* ***conical distribution vector*** *for π.*

Suppose that $(\pi, \mathcal{H})$ is a conical representation of G with a conical unit vector $v \in \mathcal{H}^{MN}$. In this case, one obtains a conical *function* $\psi_{\pi,v}$ by

$$\psi_{\pi,v}(g) = \langle v, \pi(g)v \rangle. \tag{3.15}$$

Note the similarity with the way in which spherical representations give rise to spherical functions.

In general, a conical representation $(\pi, \mathcal{H})$ of G might not have a conical vector but rather may have merely a conical distribution vector. In this case, each $v \in (\mathcal{H}^{-\infty})^{MN}$ gives rise to a conical distribution $\psi_{\pi,v}$ on G in the following way:

Suppose that π is a conical representation of G with conical vector $v \in (\mathcal{H}^{-\infty})^{MN}$. For each $v \in \mathcal{H}^{-\infty}$ and $f \in \mathcal{D}(G)$, consider as in Section 2.5 the vector

$$\pi(f)v = \int_G f(g)\pi(g)v \, dg \in \mathcal{H}^\infty$$

We then define a conical distribution $\psi_{\pi,v}$ on G by

$$\langle \psi_{\pi,v}, f\rangle = \langle v, \pi(f)v\rangle$$

Another contrast with spherical representations is that an irreducible conical representation $(\pi, \mathcal{H})$ may have the property that $\dim(\mathcal{H}^{-\infty})^{MN} > 1$, as we shall see later.

3.7 Finite-Dimensional Representations and Weyl's Unitary Trick

The easiest representations to construct and classify are those which are finite-dimensional. For that reason, we will later be interested in studying, for infinite-dimensional Riemannian symmetric spaces, the analogues of finite-dimensional conical representations. Those representations will no longer be finite-dimensional, but they will inherit many of the features of finite-dimensional conical representations. To that end, we review the relevant material on finite-dimensional representations.

The material in this section is almost entirely classical and very well known. For a treatment of Weyl's Unitary Trick, see Section VII.1 of [27]. The highest-weight theorem may be found in any standard reference on Lie groups, including Section V.2 of [27]. For a more algebraic treatment, see Chapter 3 of [16]. Results about finite-dimensional spherical and conical representations may be found in Section V.4 of [22] and Section II.4 of [23], respectively.

As before, we suppose that G/K is a noncompact-type Riemannian symmetric space with involution θ. For this section, we will further assume that G/K is the c-dual of a simply-connected compact-type Riemannian symmetric space U/K. In other words, we have that $\mathfrak{g}_\mathbb{C} = \mathfrak{u}_\mathbb{C}$ and also have the decompositions

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} \oplus \mathfrak{p} \\ \mathfrak{u} &= \mathfrak{k} \oplus i\mathfrak{p}, \end{aligned}$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ on $\mathfrak{g}$. Furthermore, G and U share the same complexified group $G_\mathbb{C} = U_\mathbb{C}$.

Theorem 3.31 (Weyl's Unitary Trick). *([27, Proposition 7.15]) There is are one-to-one correspondences between the following categories of representations on a finite-dimensional vector space V, under which corresponding representations have the same algebra of intertwining operators:*

1. *representations of G on V*

2. *representations of* $\mathfrak{g}$ *on* V

3. *complex-linear representations of* $\mathfrak{g}_{\mathbb{C}}$ *on* V

4. *holomorphic representations of* $G_{\mathbb{C}}$ *on* V

5. *representations of* U *on* V

6. *representations of* $\mathfrak{u}$ *on* V

Proof. We briefly sketch an outline of the proof. Begin with a representation π of G on V. We can differentiate π to yield a representation of $\mathfrak{g}$ on V. Note that any two representations of G with the same derived representation are equivalent, so passing from (1) to (2) is injective. We can extend the real-linear representation of $\mathfrak{g}$ on V to a complex-linear representation $\mathfrak{g}_{\mathbb{C}}$, and this process is bijective. We can similarly extend representations of $\mathfrak{u}$ to $\mathfrak{u}_{\mathbb{C}}$. This gives the correspondences (2) $\leftrightarrow$ (3) and (6) $\leftrightarrow$ (3). Since $G_{\mathbb{C}}$ is simply-connected, there is a correspondence between holomorphic representations of $G_{\mathbb{C}}$ and complex-linear representations of $\mathfrak{g}_{\mathbb{C}}$ given by differentiation. This gives the bijective correspondence (3) $\leftrightarrow$ (4). We can restrict the holomorphic representation of $G_{\mathbb{C}}$ to the closed subgroups U and G, giving (4) $\to$ (5) and (4) $\to$ (1). Finally, differentiating a representation of U gives a representation of $\mathfrak{u}$ and this correspondence is bijective because U is simply connected, yielding the correspondences (5) $\leftrightarrow$ (6). $\square$

At this point we recall Theorem 2.6, from which it follows that every finite-dimensional representation of U is equivalent to a unitary representation and thus decomposes into a direct sum of irreducible representations of U.[3] At any rate, classifying finite-dimensional representations of G can be reduced to classifying irreducible unitary representations of U. For that reason, our next step is to briefly review the highest-weight classification of irreducible representations of the compact group U.

We use the notation of Section 3.2. In particular, we have a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$.[4] Now let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. It can be shown that $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{u}$ and that $\widetilde{\mathfrak{h}} = \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. In other words, $\mathfrak{h}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}$ is a maximal abelian subalgebra of $\mathfrak{g}_{\mathbb{C}}$. For each $\alpha \in \mathfrak{a}_{\mathbb{C}}^*$, we define the space

$$\mathfrak{g}_{\mathbb{C},\alpha} = \{Y \in \mathfrak{g}_{\mathbb{C}} \mid [H, Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{h}_{\mathbb{C}}\}.$$

If $\mathfrak{g}_{\mathbb{C},\alpha} \neq 0$, then we say that α is a **root** for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and denote the set of all such roots by $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Note the distinction between the *restricted roots* in $\Sigma(\mathfrak{g}, \mathfrak{a})$ and the roots in $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. As with the restricted roots, we choose a positive root subsystem $\Delta^+ \subseteq \Delta$. Then $\Delta = (\Delta^+)\dot{\cup}(-\Delta^+)$.

[3] In contrast, finite-dimensional irreducible representations of the noncompact semisimple group G are typically not unitary.

[4] In the literature it is standard to write $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$ rather than $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$. Thus all instances of $\mathfrak{a}$ or $\mathfrak{a}^*$ will be off by a factor of i from the literature on compact-type symmetric spaces.

Now consider an irreducible unitary representation (π, V) of U. The derived representation of $\mathfrak{u}$ then acts on V by skew-adjoint operators, whose eigenvalues are purely imaginary. For each $\lambda \in i\mathfrak{h}^*$, we define the **weight space**

$$V_\lambda = \{v \in V | \mathrm{d}\pi(H)v = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}.$$

If $V_\lambda \neq \{0\}$, then we say that λ is a weight for π and denote the set of all weights for π by $\Delta(\pi)$. Because $\mathrm{d}\pi(\mathfrak{h})$ is an abelian Lie algebra of skew-adjoint operators on V, it follows that V decomposes into joint eigenspaces. In other words,

$$V = \bigoplus_{\lambda \in \Delta(\pi)} V_\lambda.$$

We say that a weight $\lambda \in i\mathfrak{h}^*$ is **dominant** if $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Delta^+$. We can now review the famous Highest-Weight Theorem, which classifies irreducible representations of compact groups. We say that a weight $\lambda \in i\mathfrak{h}^*$ is **integral** if $\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha, \rangle} \in \mathbb{Z}$ for each $\alpha \in \Delta^+$. We denote the set of all dominant, integral weights by $\Lambda^+(\mathfrak{u}, \mathfrak{h})$.

Theorem 3.32. *(The Highest-Weight Theorem; see Theorem 5.110 in [27])*
Let U be a simply-connected compact group.

1. *If (π, V) is an irreducible representation of U, then there is a unique dominant integral weight $\lambda \in \Lambda^+(\mathfrak{u}, \mathfrak{h})$ such that $\lambda \in \Delta(\pi)$ and*

$$\mathrm{d}\pi(X)v = 0$$

   *for all $v \in V_\lambda$ and $X \in \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\mathbb{C},\alpha}$. One says that λ is the **highest weight** of π and that elements of V_λ are **highest-weight vectors**. Furthermore,* $\dim V_\lambda = 1$.

2. *If (π, V) is an irreducible representation of U, then $\dim V_\lambda = \dim V_{w\lambda}$ for any w in the Weyl group $W = N_U(\mathfrak{h})/Z_U(\mathfrak{h})$ and any $\lambda \in i\mathfrak{h}^*$.*

3. *Two representations of U are equivalent if and only if they possess the same highest weight.*

4. *Each dominant integral weight $\lambda \in \Lambda^+(\mathfrak{u}, \mathfrak{h})$ is the highest weight of some irreducible unitary representation of U. We denote such a representation by $(\pi_\mu, \mathcal{H}_\mu)$.*

Together with Weyl's Unitary Trick, The Highest-Weight Theorem provides a parameterization of all finite-dimensional irreducible representations of semisimple Lie groups.

3.8 Finite-Dimensional Conical and Spherical Representations

The problem of determining which finite-dimensional representations are spherical or conical is solved by some classical results of Helgason, which we state in this section.

Theorem 3.33 (The Cartan-Helgason Theorem). *([22, p. 535]) Suppose that U/K is a compact-type symmetric space with c-dual G/K and that (π, V) be an irreducible representation of U with highest-weight $\lambda \in i\mathfrak{h}^*$. We recall that $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{a}$, where $\mathfrak{t} \subseteq \mathfrak{m}$ is a maximal abelian subalgebra of $\mathfrak{m}$. Suppose further that U is simply-connected. Then the following are equivalent:*

1. *π is a spherical representation of U.*
2. *$\pi(M)v = v$ for each highest-weight vector $v \in V_\lambda$.*
3. *$\lambda(\mathfrak{t}) = 0$ and also*
$$\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N} \text{ for all } \alpha \in \Sigma^+$$

Proof. We prove that (1) $\iff$ (2). Suppose that π is an irreducible spherical representation of U. We use Weyl's trick to consider π as a representation of G. Then we recall that the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Furthermore, we have a root-system $\Delta = \Delta(\mathfrak{u}, \mathfrak{h}) \subseteq i\mathfrak{h}^*$ corresponding to the Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{u}$. Choose a positive subsytem $\Delta^+ \subseteq \Delta$ so that $\Delta^+|_\mathfrak{a} \subseteq \Sigma^+$. the triangular decomposition $\mathfrak{u}_\mathbb{C} = \mathfrak{g}_\mathbb{C} = \overline{\mathfrak{n}_\mathfrak{h}} \oplus \mathfrak{h}_\mathbb{C} \oplus \mathfrak{n}_\mathfrak{h}$, where

$$\mathfrak{n}_\mathfrak{h} = \bigoplus_{\alpha \in \Delta^+(\mathfrak{u}, \mathfrak{h})} \mathfrak{u}_{\alpha, \mathbb{C}} \subseteq \mathfrak{u}_\mathbb{C}$$

is the positive-root nilpotent algebra coming from the root system Δ. One shows that $\mathfrak{n} \subseteq \mathfrak{n}_\mathfrak{h}$.

If $v \in V_\lambda$ is a highest-weight vector for π, then we see that $\pi(\mathfrak{n} \oplus \mathfrak{a})v \subseteq \mathbb{C}v$. Because π is irreducible, we know that $\pi(\mathfrak{U}(\mathfrak{g}))v = V$, where $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Hence, by the Poincare-Birkhoff-Witt theorem it follows that $\pi(\mathfrak{U}(\mathfrak{k}))v = V$ and thus that $\langle \pi(K)v \rangle = V$.

Now recall that the orthogonal projection $P : V \to V^K$ is given by

$$Pw = \int_K \pi(k)w dk.$$

It follows that $Pv \in V^K \backslash \{0\}$ (in fact, if $Pv = 0$, then V^K and $\langle \pi(K)v \rangle$ are orthogonal, which contradicts the fact that $\langle \pi(K)v \rangle = V$). We write $Pv \equiv e$. Because P is a K-intertwining operator, $\mathfrak{t} \subseteq \mathfrak{k}$, and $\pi(\mathfrak{t})v \subseteq \mathbb{C}v$, we see that $\pi(X)v = 0$ for all $X \in \mathfrak{t}$ and thus that $\lambda(\mathfrak{t}) = 0$. A similar argument shows that $\pi(g)v = v$ for all $g \in K \cap \exp(i\mathfrak{a})$.

Next, one sees that

$$\mathfrak{m}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \bigoplus_{\beta \in \Delta^+ \text{ s.t. } \beta|_{\mathfrak{a}}=0} (\mathfrak{g}_{\mathbb{C},\beta} \oplus \mathfrak{g}_{\mathbb{C},-\beta}) .$$

One sees clearly that $\pi(X) = 0$ for each $X \in \mathfrak{g}_{\mathbb{C},\beta}$ such that $\beta \in \Delta^+$, because λ is a highest weight for π and thus $\lambda + \beta$ is not a weight of the representation. Now suppose that $\beta \in \Delta^+$ and $\beta|_{\mathfrak{a}} = 0$. Then $\langle \lambda, \beta \rangle = 0$ under the Killing form on $i\mathfrak{h}^*$ because $\lambda(\mathfrak{t}) = 0$ but $\beta(\mathfrak{a}) = 0$. Thus, the Weyl-group reflection

$$w_\beta : \gamma \mapsto \gamma - 2\frac{\langle \gamma, \beta \rangle}{\langle \beta, \beta \rangle}\beta$$

reflects $\lambda + \beta$ into $\lambda - \beta$. It follows that $\lambda - \beta$ is not a weight of the representation (since the set of weights of π is invariant under the Weyl group) and thus that $\pi(X)v = 0$ for all $X \in \mathfrak{g}_{\mathbb{C},-\beta}$.

Thus $\pi(X)v = 0$ for all $X \in \mathfrak{m}$. It follows that $\pi(m)v = v$ for all $v \in M_0 = \exp(\mathfrak{m})$. Since $M = M_0(\exp(i\mathfrak{a}) \cap K$, we see that $\pi(M)v = v$ and we are done showing that (1) $\implies$ (2).

Now suppose that π is an irreducible representation of U with highest-weight λ and highest-weight vector $v \in V_\lambda$ such that $\pi(M)v = v$. We must show that $Pv \in V^K$ is a nonzero vector. Because $\mathfrak{t} \subseteq \mathfrak{m}$, it follows immediately that $\lambda(\mathfrak{t}) = 0$.

Because π is a *unitary* representation of U, we have that $\pi(u)^* = \pi(u^{-1})$ for all $u \in U$. Since $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$, it follows that $\pi(X)^* = \pi(X)$ for $X \in \mathfrak{p}$ and $\pi(X)^* = -\pi(X)$ for $X \in \mathfrak{k}$, so that $\pi(g)^* = \pi(\theta(g))$ for all $g \in G$. In particular, for each $\overline{n} \in \overline{N}$, we have that

$$\langle \pi(\overline{n})v, v \rangle = \langle v, \pi(\theta(\overline{n}))v \rangle = \langle v, v \rangle,$$

since $\theta(\overline{n}) \in N$. Next, for each $\overline{n} \in \overline{N}$, we write $\overline{n} = k(\overline{n})a(\overline{n})n(\overline{n})$ for the Iwasawa decomposition. Next, we note that

$$\begin{aligned} e \equiv Pv &= \int_{K/M} \pi(k)v dk \\ &= \int_{\overline{N}} \pi(k(\overline{n}))v \cdot a(\overline{n})^{-2\rho} d\overline{n} \\ &= \int_{\overline{N}} \pi(\overline{n}n(\overline{n})^{-1}a(\overline{n})^{-1})v \cdot a(\overline{n})^{-2\rho} d\overline{n} \\ &= \int_{\overline{N}} \pi(\overline{n})v \cdot a(\overline{n})^{-\lambda-2\rho} d\overline{n}, \end{aligned}$$

where we have used the fact that $\pi(n)v = v$ and $\pi(a)v = a^\lambda v$ for all $n \in N$ and $a \in A$.

Thus, we have

$$\begin{aligned} \langle e, v \rangle &= \int_{\overline{N}} \pi(\overline{n})\langle v, v \rangle \cdot a(\overline{n})^{-\lambda-2\rho} d\overline{n} \\ &= \langle v, v \rangle \int_{\overline{N}} \pi(\overline{n})\langle v, v \rangle \cdot a(\overline{n})^{-\lambda-2\rho} d\overline{n} \end{aligned}$$

and thus $\langle e, v\rangle > 0$. Hence $e \in V^K\backslash\{0\}$. □

If π is spherical, then we say by abuse of notation that $\lambda|_{i\mathfrak{a}}$ is the **highest weight** of π. Note that there is a natural identification of purely imaginary weights on $i\mathfrak{a}$ with purely real weights on $\mathfrak{a}$. Thus, the highest restricted roots may be identified with elements of $\mathfrak{a}^*$. We write

$$\Lambda^+ \equiv \Lambda^+(\mathfrak{g}, \mathfrak{a}) \equiv \left\{ \mu \in \mathfrak{a}^* \,\middle|\, \frac{\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{N} \text{ for all } \alpha \in \Sigma^+ \right\}$$

and note that each element of Λ^+ corresponds to unique irreducible spherical representations of U and G.

Moreover, Λ^+ is a semilattice. In fact, define linear functionals $\xi_j \in \mathfrak{a}^*$ by

$$\frac{\langle \xi_i, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle} = \delta_{i,j} \text{ for } 1 \le j \le r \quad . \tag{3.16}$$

Then $\xi_1, \ldots, \xi_r \in \Lambda^+$ and

$$\Lambda^+ = \mathbb{Z}^+\xi_1 + \cdots + \mathbb{Z}^+\xi_r = \left\{ \sum_{j=1}^r n_j\xi_j \,\middle|\, n_j \in \mathbb{Z}^+ \right\} .$$

The weights ξ_j are called the *fundamental weights* for $(\mathfrak{g}, \mathfrak{a})$. Note that each element of Λ^+ corresponds to a unique irreducible spherical representation of U.

In fact, a corollary of the Cartan-Helgason Theorem is that the finite-dimensional conical and spherical representations of G are the same.

Theorem 3.34. *([23, p. 119]) Suppose that (π, V) is an irreducible finite-dimensional representation of G. Then π is spherical if and only if it is conical, in which case V^{MN} consists of the highest-weight vectors of π.*

Now that the irreducible finite-dimensional spherical and conical representations have been parameterized, one may ask more generally about finite-dimensional spherical and conical representations that may not be irreducible.

To that end, suppose that $(\pi_\mu, \mathcal{H}_\mu)$ is an irreducible K-spherical representation of G with highest weight μ and that $(\sigma, \mathcal{H})$ is a unitary primary representation of G consisting of representations of type μ. By [17, Lemma 1.5], all cyclic primary representations of a compact group are finite-dimensional, and hence σ extends uniquely to a holomorphic spherical representation of $G_{\mathbb{C}}$. Because it is a finite-dimensional spherical representation, σ is automatically a conical representation of $G_{\mathbb{C}}$. In fact, as the following result shows, the MN-invariant vectors of σ are precisely the highest-weight vectors of irreducible subrepresentations of σ.[5]

Lemma 3.35. *Suppose, as above, that $(\sigma, \mathcal{H})$ is a unitary primary representation of a compact group G consisting of representations with highest weight μ. If $v \in$*

[5]The lemma is likely known by specialists, but we were not able to find a citation in the literature.

$\mathcal{H}^{MN}\backslash\{0\}$, then v is a highest-weight vector that generates an irreducible spherical representation of G. Furthermore, if $v, w \in \mathcal{H}^{MN}\backslash\{0\}$ and $v \perp w$, then $\langle\pi(G)v\rangle \perp \langle\pi(G)w\rangle$.

Proof. Let $v \in \mathcal{H}^{MN}\backslash\{0\}$, and consider $W = \langle\sigma(G)v\rangle$. We can write $W = W_1 \bigoplus \cdots \bigoplus W_n$ where each W_i gives an irreducible representation of G that is equivalent to $\mathcal{H}_\mu$. It must be a finite direct sum because all cyclic primary representations of compact groups are finite-dimensional (see [17]). For each i, let v_i be the orthogonal projection of v onto W_i. Then $v = v_1 + \cdots + v_n$. Since each W_i is a G-invariant subspace, it follows that each vector v_i is also invariant under MN. Because W_i is irreducible, we see that v_i must be a (nonzero) highest-weight vector of weight μ (see [16, Theorem 12.3.13]). Hence v is a weight vector of weight μ.

Suppose that W is not irreducible (that is, $n > 1$). Because W is cyclic, there must be $g_1, \ldots, g_k \in G$ and $c_1, \ldots, c_k \in \mathbb{C}$ such that $\sum_{i=0}^k c_i\pi(g_i)v = v_1$ (it is sufficient to consider finite linear combinations because W is finite-dimensional). It follows from the invariance of each space W_k that $\sum_{i=0}^k c_i\pi(g_i)v_1 = v_1$ and $\sum_{i=0}^k c_i\pi(g_i)v_2 = 0$. Because W_1 and W_2 give equivalent representations of G and all highest-weight vectors of an irreducible representation are constant multiples of each other, this is a contradiction. Thus W is irreducible and $v = v_1$ is a highest-weight vector for W.

Now suppose that v and w are nonzero MN-invariant vectors in $\mathcal{H}$ such that $v \perp w$. Write $V = \langle\pi(G)v\rangle$ and $W = \langle\pi(G)w\rangle$. By the above, we know that V and W are irreducible representations of G with highest-weight vectors v and w, respectively. Hence, either $V \cap W = \{0\}$ or $V = W$. Because the space of highest-weight vectors of an irreducible representation of G is one dimensional and $v \perp w$, we cannot have $V = W$. Thus $V \cap W = \{0\}$.

Now consider the invariant subspace $Z = V + W$ and the corresponding orthogonal projection $p : Z \to W$, which is an intertwining operator for π because W is an invariant subspace of Z. Hence, $p(v) \in \mathcal{H}^{MN}$ and so $p(v) = cw$ for some $c \in \mathbb{C}$. Since $v \perp w$, we see that $c = 0$ and thus $v \in \ker p$. Moreover, it is clear that $\ker p$ is a U-invariant subspace of Z, so it follows that $V = \langle\pi(U)v\rangle \subseteq \ker p$. Hence $V \perp W$ as we wished to show. □

3.8.1 Applications to Harmonic Analysis

The importance of finite-dimensional spherical representations of a group G for harmonic analysis may be seen by the fact that each finite-dimensional irreducible representation (π, V) of G is contained in the regular representation of G on $C^\infty_{\text{fin}}(G)$. In fact, let (π, V) be a finite-dimensional irreducible representation of G. Fix an inner product on $\mathcal{H}$ such that the corresponding representation of U is unitary (this inner product is unique up to multiplication by a constant). However, π will not be a unitary representation of G.

What we can say, however, is that $\pi(k)$ is unitary for each $k \in K$ and $\pi(\exp X)$ is self-adjoint for each $X \in \mathfrak{p}$ (since $\mathrm{d}\pi$ acts by skew-adjoint operators on $i\mathfrak{p}$, it acts by self-adjoint operators on $\mathfrak{p}$). That is,

$$\langle \pi(g)v, w\rangle = \langle v, \pi(\theta(g^{-1}))\rangle.$$

In other words, the contragredient representation (π^*, V^*) is given by

$$\pi^*(g) = \overline{\pi(\theta(g))}$$

under the conjugate-linear identification $V \cong V^*$ given by the inner product on V. It can be shown (see Corollary II.4.13 in [23]) that if π_μ is the finite-dimensional spherical/conical representation with highest-weight $\mu \in \Lambda^+$, then $\pi_\mu^* = \pi_{-w^*\mu}$, where $w^* \in W$ is again the longest element of the Weyl group.

For each $u, v \in \mathcal{H}$ the **matrix coefficient function**

$$\pi_{u,v}(g) = \langle \pi(g^{-1})u, v\rangle$$

extends to a holomorphic function on $G^{\mathbb{C}}$ by Weyl's Unitary Trick. For each $v \in V$, the map

$$u \mapsto \pi_{u,v}$$

is a linear intertwining operator from (π, V) into $(L, C^\infty_{\text{fin}}(G))$. It is injective because π is irreducible.

Furthermore, if (π, V) is a finite-dimensional irreducible spherical representation of G such that $e \in V^K$ is a unit vector, then $\pi_{u,e}$ is a right-K-invariant smooth function on G for each $u \in V$: in fact,

$$\begin{aligned}\pi_{u,e}(gk) &= \langle \pi((gk)^{-1})u, e\rangle \\ &= \langle \pi(g^{-1})u, \pi(k)e\rangle \\ &= \langle \pi(g^{-1})u, e\rangle = \pi_{u,e}(g)\end{aligned}$$

for all $g \in G$ and $k \in K$. Here we have used the fact that $\pi|_K$ is unitary. Using the identification of right-K-invariant functions on G with functions on G/K, we see that

$$u \mapsto \pi_{u,e}$$

gives an intertwining operator from (π, V) into $(L, C^\infty_{\text{fin}}(G/K))$. In fact, it can be shown that (up to a normalizing factor) $\pi_{e,e}$ is a spherical function on G (see [23, p. 106]).

We can, in fact, obtain a more explicit formula for the spherical function $\pi_{e,e}$. Suppose π is a spherical representation with highest-weight $\mu \in \Lambda^+$ and highest-weight vector v. For each $g \in G$, write $g = k(g)a(g)n(g)$ be the Iwasawa decomposition. Furthermore, consider the logarithm $\log : A \to \mathfrak{a}$. Using the fact that $e = P(v) = \int_K \pi(k)v dk$ and that v is a μ-weight vector, we arrive at Harish-Chandra's famous integral formula (see also Theorem IV.4.3 in [22]):

$$
\begin{aligned}
\pi_{e,e}(g) &= \pi_{e,e}(g) \\
&= \int_K \langle \pi(g^{-1})\pi(k)v, \pi(k)e\rangle dk \\
&= \int_K \langle \pi(\theta(k(g^{-1}k)a(g^{-1}k)n(g^{-1}k))v, e\rangle dk \\
&= \int_K \langle \pi(a(g^{-1}k))v, e\rangle dk \\
&= \int_K \langle e^{\mu(\log(a(g^{-1}k)))}v, e\rangle dk \\
&= \int_K e^{\mu(\log(a(g^{-1}k)))}\langle v, e\rangle, \qquad (3.17)
\end{aligned}
$$

where we note that $\langle v, e\rangle$ is a constant.

Now suppose that (π, V) is a finite-dimensional irreducible conical representation of G such that $v \in V^{MN}$ is a unit vector. We choose a unit vector $v^* \in V$ as a highest-weight representation for the contragredient representation π^*. That is, v^* is invariant under $\theta(MN) = M\overline{N}$. In other words, v^* may be thought of as a *lowest-weight* vector for π.

Under these constructions, π_{u,v^*} is a right-MN-invariant smooth function on G for each $u \in V$: in fact,

$$
\begin{aligned}
\pi_{u,v^*}(gmn) &= \langle \pi((gmn)^{-1})u, v^*\rangle \\
&= \langle \pi(n^{-1})\pi(m^{-1})\pi(g^{-1})u, v\rangle \\
&= \langle \pi(g^{-1})u, \pi(m)\pi(\theta(n^{-1}))v^*\rangle \\
&= \langle \pi(g^{-1})u, v^*\rangle = \pi_{u,v*}(g)
\end{aligned}
$$

for all $g \in G$, $m \in M$, and $n \in N$. Here we have used the fact that $\pi(m)$ is unitary for $m \in M \subseteq K$ and that $v^* \in \pi^{M\overline{N}}$. Furthermore, it can be shown that (up to a normalizing constant) ψ_{v,v^*} is a conical function (see [23, p. 113]). Using the identification of right-MN-invariant functions on G with functions on G/MN, we see that

$$u \mapsto \pi_{u,v*}$$

gives an intertwining operator from (π, V) into $(L, C^\infty_{\text{fin}}(G/MN))$.

We can obtain a more explicit formula for the conical function π_{v,v^*} on a dense subset of G. Suppose that π is a conical representation with highest-weight $\mu \in \Lambda^+$, highest-weight vector v, and lowest-weight vector v^*. For each $g \in \overline{N}MAN$, write $g = n'\alpha(g)m_{w^*}mn$, where $m \in M$, n and n' are in N, and $\alpha(g) \in A$. Furthermore, consider the logarithm $\log : A \to \mathfrak{a}$. The following formula may also be found in

Theorem II.4.6 of [23]:

$$\begin{aligned}\pi_{v,v*}(g) &= \langle \pi((n'\alpha(g)m_{w^*}mn)^{-1})v, v^*\rangle \\ &= \langle \pi(m_{w^*}^{-1}\alpha(g)^{-1})v, v^*\rangle \\ &= \langle \pi(m_{w^*}^{-1})e^{-\mu(\log\alpha(g))}v, v^*\rangle \\ &= e^{-\mu(\log\alpha(g))}\pi_{v,v^*}(m_{w^*})\end{aligned}$$

for all $g \in \overline{N}MAN$, where we note that $\pi_{v,v^*}(m_{w^*})$ is a constant.

Next we consider how these representations allow us to decompose function spaces. Because $\dim V^K = 1$ and $\dim V^{MN} = 1$, it is possible to show that (π, V) appears in $(L, C^\infty_{\text{fin}}(G/K))$ and $(L, C^\infty_{\text{fin}}(G/MN))$, respectively, with multiplicity one. In fact, it follows from Lemma II.4.14 and Proposition II.4.15 in [23] that

$$C^\infty_{\text{fin}}(G/MN) \cong_G \sum_{\lambda\in\Lambda^+(\mathfrak{g},\mathfrak{a})}^{\oplus} \mathcal{H}_\lambda,$$

where $\sum^\oplus$ denotes an algebraic direct sum. From Corollary 12.3.15 in [16], we know that

$$C^\infty_{\text{fin}}(G/K) \cong_G \sum_{\lambda\in\Lambda^+(\mathfrak{g},\mathfrak{a})}^{\oplus} \mathcal{H}_\lambda.$$

Furthermore, one can show (see [22, Theorem V.4.3]) that

$$C^\infty_{\text{fin}}(U/K) \cong_U \sum_{\lambda^+\in\Lambda(\mathfrak{g},\mathfrak{a})}^{\oplus} \mathcal{H}_\lambda.$$

In other words, there are very natural identifications of smooth, G-finite smooth functions on U/K, G/K and G/MN. Consider the mapping defined by

$$\pi_{u,e} \mapsto \pi_{u,v}$$

for each spherical/conical representation $(\pi, \mathcal{H})$ and each $u \in \mathcal{H}$, where $e \in \mathcal{H}^K$ and $v \in \mathcal{H}^{MN}$ are unit vectors. This mapping may be extended by linearity to yield a G-intertwining operator from $C^\infty_{\text{fin}}(G/K)$ to $C^\infty_{\text{fin}}(G/MN)$. This intertwining operator is given by a form of the celebrated Radon transform and may be defined in terms of integral operators.

3.9 Unitary Spherical and Conical Representations

We are primarily concerned in this thesis with studying the analogue of finite-dimensional conical representations for infinite-dimensional symmetric spaces, none of which are unitary. However, we briefly review the construction of unitary conical representations in order to show the important role that they play in harmonic analysis on noncompact-type Riemannian symmetric spaces, which provides an important motivation for extending the theory of conical representations to the infinite-dimensional context.

The results in this section are primarily due to Harish-Chandra (for G/K) and Helgason (for G/MN). Helgason's exposition of Harmonic analysis on G/K may be found in Chapter IV of [22] and his exposition of analysis on G/MN may be found in Chapters II and VI of [23] and in the earlier paper [20]. A simpler proof of the Plancherel formula for G/MN was provided by Ronald Lipsman in [29]; it is simple enough that we shall provide a brief outline here. For a good overview of these topics from a representation-theory perspective, see [38].

The standard construction of unitary spherical and conical representations for noncompact-type Riemannian symmetric spaces uses a technique known as *parabolic induction*. As before, let G/K be a Riemannian symmetric space of noncompact type and use the notation in Section 3.2.

We begin by choosing a one-dimensional representation of A, which may be identified with an element λ of $\mathfrak{a}_{\mathbb{C}}^*$. Due to the fact that $MAN/MN \cong A$, there is a well-defined extension to a representation $1 \otimes \lambda \otimes 1$ of MAN such that

$$(1 \otimes \lambda \otimes 1)(man) = e^{\lambda(\log(a))} \equiv a^{\lambda}$$

for all $m \in M$, $a \in A$, and $n \in N$.

We then define the **spherical principal series representation** $(\sigma_\lambda, \mathcal{K}_\lambda)$ by setting[6]

$$\mathcal{K}_\lambda = \left\{ \psi : G \to \mathbb{C} \,\middle|\, \psi(gman) = a^{-\lambda-\rho}\psi(g) \text{ and } ||\psi||^2 \equiv \int_K |\psi(k)|^2 dk < \infty \right\}$$

and letting σ_λ act on $\mathcal{K}_\lambda$ by

$$\sigma_\lambda(g)\psi(h) = \psi(g^{-1}h).$$

In the terminology of induced representations, one writes $\sigma_\lambda = \mathrm{Ind}_{MAN}^G(1 \otimes \lambda \otimes 1)$ and says that σ_λ is the representation of G **induced by** the representation $1\otimes\lambda\otimes 1$ of the parabolic subgroup MAN. The factor of ρ comes from the fact that the Haar measure on $G = KAN$ is given by $dg = a^{2\rho}\, dk\, da\, dn$.

There are several ways to interpret the space $\mathcal{K}_\lambda$. It can be viewed as a space of square-integrable sections of a particular homogeneous line bundle over G/MAN. Furthermore, using the Iwasawa decomposition, one can show that there is a diffeomorphism between G/MAN and K/M which gives a natural identification of $\mathcal{K}_\lambda$ with $L^2(K/M)$, on which the action is given by:

$$\sigma_\lambda(g)f(hM) = a(g^{-1}h)^{-\lambda-\rho} f(k(g^{-1}h))$$

for all $f \in L^2(K/M)$ and $h \in K$, where

$$g^{-1}h = k(g^{-1}h)a(g^{-1}h)n(g^{-1}h)$$

[6]The literature typically denotes this representation by $(\pi_\lambda, \mathcal{H}_\lambda)$, but we need to reserve that notation for a later use.

is the Iwasawa decomposition of $g^{-1}h$ in G. This realization of $\mathcal{K}_\lambda$ is called the **compact picture**. Note that $\sigma_\lambda|_K$ is just the regular representation of K on $L^2(K/M)$.

One can further show that σ_λ is irreducible for almost all λ. Also, for each Weyl group element w, we have that $\sigma_\lambda \cong \sigma_{w\lambda}$ for almost all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Finally, σ_λ is a unitary representation if $\lambda \in i\mathfrak{a}^*$.

It is easy to see from the compact picture that σ_λ is a spherical representation. In particular, we note that the constant function $\mathbf{1}$ in $L^2(K/M)$ is K-invariant. The work of Harish-Chandra gives an explicit formula for the corresponding spherical functions. In particular, a generalization of the argument in (3.17), one can show that the positive-definite spherical function for $\mathcal{K}_\lambda$ has the form

$$\phi_\lambda(g) = \langle \mathbf{1}, \phi_\lambda(g)\mathbf{1} \rangle = \int_K a(gk)^{-\lambda-\rho} dk,$$

where $g = ka(g)n$ for some $k \in K$, $a(g) \in A$, and $n \in N$. This formula, when varied over all $\lambda \in i\mathfrak{a}^*$, provides all positive-definite spherical functions for G/K. In fact, when varied over all $\lambda \in i\mathfrak{a}_{\mathbb{C}}^*$, it provides all bounded spherical functions on G. Finally, one shows that $\phi_\lambda = \phi_\mu$ if and only if $\mu = w\lambda$ for some Weyl group element $w \in W$, which demonstrates the earlier claim that $\sigma_\lambda \cong \sigma_{w\lambda}$ for almost all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $w \in W$.

It is less obvious that σ_λ is a conical representation. Note that there is a continuous injection $\mathcal{H}_\lambda \hookrightarrow \mathcal{D}'(G/MN)$. For almost all $\lambda \in i\mathfrak{a}^*$, Helgason constructs $\#W$ distinct conical distributions on G/MN with eigenvalue $\lambda - \rho$ with respect to the action of $\mathfrak{a}$. It is not clear from the works of Helgason whether these conical distributions are continuous functionals on the space $(\mathcal{H}_\lambda)^\infty$ of smooth vectors for $\mathcal{H}_\lambda$. but this result may be seen in [29, p. 50]. In other words, one has that $\dim(\mathcal{H}_\lambda^{-\infty})^{MN} = \#W$ for almost all $\lambda \in i\mathfrak{a}^*$.

The question of whether all unitary irreducible conical representations are constructed by the unitary spherical principal series is a subtle one. In a certain moral sense, one expects the unitary spherical principal series to exhaust "almost all," if not all, unitary irreducible conical representations [23, p. 147]. To this end, Helgason was able to classify all conical distributions with the exception of certain singular eigenvalues [23, Theorem II.5.16]. For symmetric spaces G/K of rank one, the classification was completed by Hu (see [25] as well as Theorem II.6.18 and Theorem II.6.21 in [23]). However, for cases of rank higher than one it is not clear in the literature whether the answer is known.

3.9.1 Applications to Harmonic Analysis

In this section we briefly discuss the Plancherel formulas for noncompact-type Riemannian symmetric spaces and their associated horocycle spaces and note the role played by unitary spherical and conical representations.

We once again suppose that G/K is a noncompact-type Riemannian symmetric sapce and use the terminology of Section 3.2. Because M is a compact group and

N is a connected nilpotent group, we see that both groups are unimodular. We normalize the measure on N by

$$\int_N a(n)^{-2\rho} dn = 1.$$

One can show that MN is a unimodular group ([23, p. 82]) and that its Haar measure is (up to a constant) $d(mn) = dm\, dn$. Furthermore, G is a unimodular group because it is semisimple. It follows from Theorem 2.9 that G/MN possesses a G-invariant measure. Similarly, because G and K are both unimodular, the symmetric space G/K possesses a G-invariant measure.

We can now consider the unitary regular representations of G on $L^2(G/K)$ and $L^2(G/MN)$. The deep work of Harish-Chandra shows that the regular representation $(L_{G/K}, L^2(G/K))$ may be written as a direct integral of unitary spherical principal series representations (see, for instance, Sections 2.5 and 2.8 in [38]):

$$L^2(G/K) \cong_G \int_{i\mathfrak{a}^*/W}^{\oplus} \mathcal{K}_\lambda |\mathbf{c}(\lambda)|^{-2} d\lambda, \tag{3.18}$$

where the measure $|\mathbf{c}(\lambda)|^{-2} d\lambda$ is the Lebesgue measure on $i\mathfrak{a}^*/W$ weighted by $|\mathbf{c}(\lambda)|^{-2}$. Here $\mathbf{c}$ is the famous Harish-Chandra $\mathbf{c}$-function given by

$$\mathbf{c}(\lambda) = \int_{\overline{N}} a(\bar{n})^{-\lambda-\rho}\, d\bar{n}$$

for $\lambda \in i\mathfrak{a}^*$ with $\operatorname{Re}\langle \lambda, \alpha\rangle > 0$ for all $\alpha \in \Sigma^+$. It is W-invariant and may be extended meromorphically to all of $i\mathfrak{a}^*$, so that (3.18) is well-defined.

The direct-integral space $\int_{i\mathfrak{a}^*/W}^{\oplus} \mathcal{K}_\lambda |\mathbf{c}(\lambda)|^{-2} d\lambda$ may be realized geometrically as the space

$$L^2_W(\mathfrak{a}^*, L^2(K/M); (\#W)^{-1}|\mathbf{c}(\lambda)|^{-2} d\lambda) \cong L^2_W(\mathfrak{a}^* \times K/M, (\#W)^{-1}|\mathbf{c}(\lambda)|^{-2} d\lambda dk),$$

where the subscript W indicates that we consider W-invariant functions.

On the other hand, the decomposition of $(L_{G/MN}, L^2(G/MN))$ may be derived using more elementary methods, and we briefly sketch the argument here. By using induction in stages, one has that

$$L_{G/MN} \cong \operatorname{Ind}_{MN}^{G}(1) \cong \operatorname{Ind}_{MAN}^{G}\operatorname{Ind}_{MN}^{MAN}(1)$$

But we also have

$$\operatorname{Ind}_{MN}^{MAN}(1) \cong L^2(MAN/MN) \cong \int_{i\mathfrak{a}^*}^{\oplus} 1 \otimes \lambda \otimes 1\; d\lambda,$$

where the latter equality follows from the fact that $MAN/MN \cong A$. Here $d\lambda$ is Lebesgue measure on $i\mathfrak{a}^*$. Therefore, one has that

$$L_{G/MN} \cong \int_{i\mathfrak{a}^*}^{\oplus} \operatorname{Ind}_{MAN}^{G}(1 \otimes \lambda \otimes 1)$$

where we use the fact that induction commutes with taking direct integrals. Putting everything together, we have the following result (see Section 4.2 in [38]):

$$L^2(G/MN) \cong_G \int_{i\mathfrak{a}^*}^{\oplus} \mathcal{K}_\lambda d\lambda. \tag{3.19}$$

In fact, because $(\sigma_\lambda, \mathcal{H}_\lambda)$ is equivalent to $(\sigma_{w\lambda}, \mathcal{H}_{w\lambda})$ for almost all $\lambda \in i\mathfrak{a}^*$ and $w \in W$, one sees that

$$L^2(G/MN) \cong_G (\#W) \int_{i\mathfrak{a}^*/W}^{\oplus} \mathcal{K}_\lambda d\lambda \cong_G (\#W) L^2(G/K).$$

That is, $L^2(G/MN)$ is equivalent to a direct sum of $\#W$ copies of $L^2(G/K)$.

We noted earlier that the Radon transform is an itnertwining operator. It can also be shown that it is injective. Unfortunately, it does not extend to a unitary intertwining operator from $L^2(G/K)$ into $L^2(G/MN)$. However, it is possible to "twist" the Radon transform into a unitary intertwining operator, as shown in Section II.3.3 of [23], a variant of the Radon transform may be used to define an intertwining operator from $L^2(G/K)$ to the space $L^2_W(G/MN)$ of W-invariant functions on G/MN. A good exposition of the results may be found in Sections 4.2 and 4.3 of [38].

We briefly state the results here for completeness. Using the Laplace operators on K/M and A, it is possible to define a Schwartz space $\mathcal{S}(G/MN)$ of rapidly-decreasing functions on $G/MN \cong K/M \times A$. Then define a twisted Schwartz space

$$\mathcal{S}_\rho = \{f \in C^\infty(G/MN) : e^\rho f \in \mathcal{S}(G/MN)\},$$

where e^ρ is the function on G/MN defined by $e^\rho(kM, a) = e^{\rho(\log(a))}$. We also use the notation $a^\rho = e^{\rho(\log(a))}$ for each $a \in A$. Now define an operator $\Lambda : \mathcal{S}_\rho(G/MN) \to \mathcal{S}_\rho(G/MN)$ by

$$\Lambda f = e^{-\rho} \mathcal{F}_A^{-1}(\mathbf{c}^{-1} \mathcal{F}_A(e^\rho f)),$$

where $\mathcal{F}_A : \mathcal{S}(K/M \times A) \to \mathcal{S}(K/M \times \mathfrak{a}^*)$ is the classical Fourier transform in the A variable—that is,

$$\mathcal{F}_A f(kM, \lambda) = \int_A f(kM, a) e^{-i\lambda(\log(a))} da.$$

for each $\lambda \in \mathfrak{a}^*$. Then it can be shown that the composition $\Lambda\mathcal{R}$ extends to an intertwining operator that is a partial isometry from $L^2(G/K)$ into $L^2(G/MN)$.

Chapter 4

Direct Limits of Groups and Symmetric Spaces

Motivated in part by applications to physics, there has been an increasing amount of work done on infinite-dimensional Lie groups since the 1970s. These are topological groups which are locally modeled on locally convex topological vector spaces over $\mathbb{R}$ (in the same way that finite-dimensional Lie groups are modeled on finite-dimensional vector spaces over $\mathbb{R}$). The simplest infinite-dimensional Lie groups which may be considered are those which are formed by taking direct limits of finite-dimensional Lie groups. They occupy a sort of "middle ground" between finite-dimensional groups and other infinite-dimensional groups with finer topologies, in that they inherit many of the properties of the former but already exhibit some of the pathologies of the latter.

We refer the reader to [11] and [35] for a good overview of the basic properties of direct-limit groups. See [36] and [33] for some details about the construction of smooth manifold structures on direct-limit groups. See also [52] for an in-depth study of direct limits of abelian and nilpotent groups and for applications of direct-limit groups to physics.

4.1 Review of Direct Limits and Projective Limits

We begin in this section by very briefly reviewing several basic definitions and results about direct limits and projective limits. See, for instance, the appendices in [35] for more details.

Suppose that for each $n \in \mathbb{N}$ one has a topological space X_n and continuous embeddings $p_n^{n+1} : X_n \to X_{n+1}$, which we refer to as **inclusion maps**.[1] By repeated composition of these inclusion maps, we construct continuous maps $p_n^k : X_n \to X_k$ for any $n \leq k$. Note that $p_n^k \circ p_m^n = p_m^k$ for all $m \leq n \leq k$. We say that $\{X_n\}_{n\in\mathbb{N}}$ together with the inclusion maps forms a **direct system**.

Next, we define an equivalence relation $\sim$ on the disjoint union $\dot\sqcup_{n\in\mathbb{N}} X_n$ as follows: for $x \in X_n$ and $y \in X_m$, where $n \leq m$, we write $x \sim y$ if $p_n^m(x) = y$. We then define

$$X_\infty \equiv \varinjlim X_n \equiv \left(\dot{\bigsqcup_{n\in\mathbb{N}}} X_n \right) / \sim$$

and say that X_∞ is the **direct limit** of $\{X_n\}_{n\in\mathbb{N}}$. Note the the inclusion map from X_n to $\dot\sqcup_{n\in\mathbb{N}} X_n$ factors through the quotient to give an injective map $p_n : X_n \to X_\infty$. We then give X_∞ the weakest topology such that p_n is continuous for each $n \in \mathbb{N}$. The direct limit possesses two important properties:

[1] We warn the reader that it is not always assumed in the literature that the inclusion maps are injective.

Lemma 4.1. *Let Y be a topological space. Suppose that $\{X_n, p_n^{n+1}\}$ is a direct system of topological spaces and suppose that for each $n \in \mathbb{N}$ we are given a continuous map $f_n : X_n \to Y$ so that the diagram*

$$\begin{array}{ccc} X_k & \xrightarrow{f_k} & Y \\ {\scriptstyle p_n^k}\uparrow & \nearrow_{f_n} & \\ X_n & & \end{array}$$

commutes for each $n \leq k$. Then there is a unique continuous map $f_\infty : X_\infty \to Y$ such that

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y \\ {\scriptstyle p_n}\uparrow & \nearrow_{f_n} & \\ X_n & & \end{array}$$

commutes for each $n \in \mathbb{N}$.

Lemma 4.2. *Suppose that $\{X_n, p_n^{n+1}\}$ and $\{Y_n, q_n^{n+1}\}$ are direct systems of topological spaces and suppose that for each $n \in \mathbb{N}$ we are given a continuous map $f_n : X_n \to Y_n$ so that the diagram*

$$\begin{array}{ccc} X_k & \xrightarrow{f_k} & Y_k \\ {\scriptstyle p_n^k}\uparrow & & \uparrow{\scriptstyle q_n^k} \\ X_n & \xrightarrow[f_n]{} & Y_n \end{array}$$

commutes for each $n \leq k$. Then there is a unique continuous map $f_\infty : X_\infty \to Y_\infty$ such that

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ {\scriptstyle p_n}\uparrow & & \uparrow{\scriptstyle q_n} \\ X_n & \xrightarrow[f_n]{} & Y_n \end{array}$$

commutes for each $n \in \mathbb{N}$.

In fact, these can be taken to be a sort of universal property for direct limits. Following the construction of direct limits of topological spaces, it is possible to define direct limits for the categories of topological groups, vector spaces, and Lie algebras which satisfy the previous two lemmas.

The prototypical example of a direct system is that of a collection $\{X_k\}_{k\in\mathbb{N}}$ of topological spaces such that X_k is a closed subset of X_m whenever $k \leq m$. Then we can identify $\varinjlim X_n$ with the set $X_\infty = \bigcup_{m\in\mathbb{N}}$ given by the topology where a set $A \subset X_\infty$ is open if and only if $A \cap X_n$ is an open subset of X_n for each $n \in \mathbb{N}$.

If $\{G_k\}_{k\in\mathbb{N}}$ is a collection of topological groups such that G_k is a closed subgroup of G_m whenever $k \leq m$, then we form the direct limit $G_\infty = \cup_{n\in\mathbb{N}} G_n$ in the topological category. The group product is obvious: if $a, b \in G_n$, then their product in G_∞ is equal to the group product under G_n. One uses Lemma 4.2 to show that the group product and inverse on G_∞ are continuous.

For example, consider the groups $\mathrm{SU}(n)$ for each $n \in \mathbb{N}$. We see that $\mathrm{SU}(n) \leq \mathrm{SU}(n+1)$ under the identification

$$\begin{aligned} \mathrm{SU}(n) &\mapsto \mathrm{SU}(n+1) \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

We can form the direct-limit group $\mathrm{SU}(\infty) = \varinjlim \mathrm{SU}(n) = \bigcup_{n\in\mathbb{N}} \mathrm{SU}(n)$. One can think of $\mathrm{SU}(\infty)$ as consisting of all unitary operators on $\ell_2(\mathbb{C})$ which fix all but finitely many of the standard basis elements. Alternately, $\mathrm{SU}(\infty)$ may be thought of as consisting of infinite complex matrices which are equal to the identity matrix outside of a finite block in the upper-left corner.

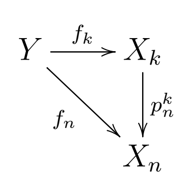
If $\{\mathcal{H}_k\}_k \in \mathbb{N}$ is a collection of Hilbert spaces such that such that $\mathcal{H}_k$ is a closed subgroup of $\mathcal{H}_m$ whenever $k \leq m$, then we form the direct limit $\mathcal{H}_\infty = \cup_{n\in\mathbb{N}}\mathcal{H}_n$ in the topological category. One uses Lemma 4.2 to show that the addition and constant multiplication on $\mathcal{H}_\infty$ are continuous. Furthermore, $\mathcal{H}_\infty = \cup_{n\in\mathbb{N}}\mathcal{H}_n$ carries a continuous inner product. However, $\mathcal{H}_\infty$ is not necessarily a Hilbert space and we must take the completion $\overline{\mathcal{H}_\infty}$ to obtain a Hilbert space.

Now suppose that for each $n \in \mathbb{N}$ one has a topological space X_n and continuous surjections $p_n^{n+1} : X_{n+1} \to X_n$, which we refer to as **projection maps**. By repeated composition of these inclusion maps, we construct continuous maps $p_n^k : X_k \to X_n$ for any $n \leq k$. Note that $p_m^n \circ p_n^k = p_m^k$ for all $m \leq n \leq k$. We say that $\{X_n\}_n$ together with the inclusion maps forms a **projective system**.

Next, we consider the Cartesian product $\prod_{n\in\mathbb{N}} X_n$ under the product topology. We denote by $\varprojlim X_n$ the set of all sequences $(x_n)_{n\in\mathbb{N}}$ such that $p_m^n(x_n) = x_m$. We give $\varprojlim X_n$ the topology it inherits as a subspace of the Cartesian product. Note that there are projection maps $p_n : X_\infty \to X_n$ defined by $p_n((x_m)_{m\in\mathbb{N}}) = x_n$. In fact, the topology on $\varprojlim X_n$ is the weakest topology such that the p_n is continuous for each $n \in \mathbb{N}$. In other words, we can form a basis for the topology on $\varprojlim X_n$ consisting of sets of the form $p_n^{-1}(A)$ where A is an open subset of X_n for some $n \in \mathbb{N}$. These sets are called **cylinder sets**.

Projective limits satisfy universal properties obtained by reversing the arrows for the corresponding properties of direct limits:

Lemma 4.3. *Let Y be a topological space. Suppose that $\{X_n, p_n^{n+1}\}$ is a projective system of topological spaces and suppose that for each $n \in \mathbb{N}$ we are given a continuous map $f_n : Y \to X_n$ so that the diagram*



commutes for each $n \leq k$*. Then there is a unique continuous map* $f_\infty : Y \to X_\infty$ *such that*

$$\begin{array}{ccc} Y & \xrightarrow{f_\infty} & X_\infty \\ & \searrow^{f_n} & \downarrow p_n \\ & & X_n \end{array}$$

commutes for each $n \in \mathbb{N}$*.*

Lemma 4.4. *Suppose that* $\{X_n, p_n^{n+1}\}$ *and* $\{Y_n, q_n^{n+1}\}$ *are projective systems of topological spaces and suppose that for each* $n \in \mathbb{N}$ *we are given a continuous map* $f_n : X_n \to Y_n$ *so that the diagram*

$$\begin{array}{ccc} X_k & \xrightarrow{f_k} & Y_k \\ \downarrow p_n^k & & \downarrow q_n^k \\ X_n & \xrightarrow[f_n]{} & Y_n \end{array}$$

commutes for each $n \leq k$*. Then there is a unique continuous map* $f_\infty : X_\infty \to Y_\infty$ *such that*

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ \downarrow p_n & & \downarrow q_n \\ X_n & \xrightarrow[f_n]{} & Y_n \end{array}$$

commutes for each $n \in \mathbb{N}$*.*

One may define projective limits in the category of topological groups by starting with the topological projective limit and defining the group product to be the restriction of the componentwise product of sequences in the Cartesian product. Projective limits of vector spaces and Lie algebras may be defined in similar ways.

Suppose that $(V_n, p_n^{n+1})_{n\in\mathbb{N}}$ is a direct system of topological vector spaces. Then we can define continuous projections $q_n^{n+1} : V_{n+1}^* \to V_n^*$ by $q_n^{n+1}(\lambda)v = \lambda(p_n^{n+1}v)$ for each $v \in V_n$. This allows us to form the projective limit $\varprojlim (V_n^*)$. In fact, one can show that

$$\left(\varinjlim V_n\right)^* \cong \varprojlim (V_n^*)\,.$$

4.2 Lie Algebras and Complexifications of Direct-Limit Groups

Suppose that $\{G_n\}_{n\in\mathbb{N}}$ is a direct system of Lie groups with inclusion maps $p_n^{n+1} : G_n \to G_{n+1}$. Then the differentiated map $\mathrm{d}p_n^{n+1} : \mathfrak{g}_n \to \mathfrak{g}_{n+1}$ is an injective Lie algebra homomorphism for each $n \in \mathbb{N}$ because each p_n^{n+1} is a smooth embedding. Thus $\{\mathfrak{g}_n\}_{n\in\mathbb{N}}$ is a direct system of Lie algebras with inclusion maps $\mathrm{d}p_n^{n+1} : \mathfrak{g}_n \to \mathfrak{g}_{n+1}$. Thus we have the direct-limit group $G_\infty = \varinjlim G_n$ and the direct-limit Lie algebra $\mathfrak{g}_\infty = \varinjlim \mathfrak{g}_n$.

It is natural to ask whether $\mathfrak{g}_\infty$ is the Lie algebra for G_∞ in some sense. To that end, consider the exponential maps $\exp_n : \mathfrak{g}_n \to G_n$ for each $n \in \mathbb{N}$. One notices that $p_n^{n+1} \circ \exp_n = \exp_{n+1} \circ \mathrm{d}p_n^{n+1}$ by the definition of the differentiated homomorphism $\mathrm{d}p_n^{n+1}$. In other words, the diagram

$$\begin{array}{ccc} \mathfrak{g}_{n+1} & \xrightarrow{\exp_{n+1}} & G_{n+1} \\ \Big\uparrow{\scriptstyle \mathrm{d}p_n^{n+1}} & & \Big\uparrow{\scriptstyle p_n^{n+1}} \\ \mathfrak{g}_n & \xrightarrow{\exp_n} & G_n \end{array}$$

commutes for each $n \in \mathbb{N}$. Thus we may consider the continuous map

$$\exp_\infty : \mathfrak{g}_\infty \to G_\infty$$

defined by $\exp_\infty(p_n(X)) = p_n(\exp_n(X))$ for all $X \in \mathfrak{g}_n$. Under certain technical conditions which include all of the classical direct-limit groups, it has been shown that $\exp_\infty$ is a local homeomorphism (see Proposition 7.1 in [33]).[2] However, we will not need to use this result for our purposes.

Now suppose that $\{U_n\}_{n\in\mathbb{N}}$ is a direct system of connected compact Lie groups with inclusion maps $p_n^{n+1} : U_n \to U_{n+1}$. Following the process described in Proposition 3.6 of [35], we construct complexifications of $\mathfrak{u}_\infty = \varinjlim \mathfrak{u}_n$ and $U_\infty = \varinjlim U_n$. As before, we consider the direct-limit group $U_\infty = \varinjlim U_n$ and its Lie algebra $\mathfrak{u}_\infty = \varinjlim \mathfrak{u}_n$. For each $n \in \mathbb{N}$, we consider the complexified Lie algebra $(\mathfrak{u}_n)_\mathbb{C} = \mathfrak{u}_n \otimes_\mathbb{R} \mathbb{C}$. Then the inclusion maps $\mathrm{d}p_n^{n+1} : \mathfrak{u}_n \to \mathfrak{u}_{n+1}$ may be complexified to yield complex-linear injective Lie algebra homomorphisms $(\mathrm{d}p_n^{n+1})_\mathbb{C} : (\mathfrak{u}_n)_\mathbb{C} \to (\mathfrak{u}_{n+1})_\mathbb{C}$. We may thus consider the complex Lie algebra

$$(\mathfrak{u}_\infty)_\mathbb{C} = \varinjlim\, (\mathfrak{u}_n)_\mathbb{C}.$$

Furthermore, because $(\mathrm{d}p_n^{n+1})_\mathbb{C}$ is the complexification of the linear map $\mathrm{d}p_n^{n+1}$, we see that the inclusions $i_n : \mathfrak{u}_n \to (\mathfrak{u}_n)_\mathbb{C}$ satisfy the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{u}_{n+1} & \xrightarrow{i_{n+1}} & (\mathfrak{u}_{n+1})_\mathbb{C} \\ \Big\uparrow{\scriptstyle \mathrm{d}p_n^{n+1}} & & \Big\uparrow{\scriptstyle (\mathrm{d}p_n^{n+1})_\mathbb{C}} \\ \mathfrak{u}_n & \xrightarrow{i_n} & (\mathfrak{u}_n)_\mathbb{C} \end{array}$$

We thus obtain an injective homomorphism $i_\infty : \mathfrak{u}_\infty \to (\mathfrak{u}_\infty)_\mathbb{C}$. One can show that $(\mathfrak{u}_\infty)_\mathbb{C}$ is the complexification of the Lie algebra $\mathfrak{u}_\infty$.

For each $n \in \mathbb{N}$, we consider the complexification $(U_n)_\mathbb{C}$ of the compact Lie group U_n. We recall that $(U_n)_\mathbb{C}$ has Lie algebra $(\mathfrak{u}_n)_\mathbb{C}$ and that U_n is the closed analytic subgroup of $(U_n)_\mathbb{C}$ corresponding to the Lie algebra $\mathfrak{u}_n$. By [27, Proposition 7.5], each homomorphism p_n^{n+1} induces a holomorphic homomorphism $(p_n^{n+1})_\mathbb{C}$:

[2] In fact, once the proper definitions for infinite-dimensional manifolds have been made, it can be shown under these technical conditions that $\exp_\infty$ is a local diffeomorphism (see Theorem 8.2).

$(U_n)_\mathbb{C} \to (U_{n+1})_\mathbb{C}$ whose differential is $(\mathrm{d}p_n^{n+1})_\mathbb{C}$. We may thus consider the direct-limit group

$$(U_\infty)_\mathbb{C} = \varinjlim \, (U_n)_\mathbb{C}.$$

Furthermore, by [27, Proposition 7.5] it follows that the inclusion maps $i_n : U_n \to (U_n)_\mathbb{C}$ satisfy the following commutative diagram:

$$\begin{array}{ccc} U_{n+1} & \xrightarrow{i_{n+1}} & (U_{n+1})_\mathbb{C} \\ {\scriptstyle p_n^{n+1}}\uparrow & & \uparrow{\scriptstyle (p_n^{n+1})_\mathbb{C}} \\ U_n & \xrightarrow{i_n} & (U_n)_\mathbb{C} \end{array}$$

We thus obtain a continuous injective homomorphism $i_\infty : U_\infty \to (U_\infty)_\mathbb{C}$. Because the image of U_n under i_n is closed in $(U_n)_\mathbb{C}$ for each $n \in \mathbb{N}$, we see that the image of U_∞ under i_∞ is a closed subgroup of $(U_\infty)_\mathbb{C}$. For these reasons, we say that $(U_\infty)_\mathbb{C}$ is the **complexification** of U_∞.

4.3 Direct Systems of Riemannian Symmetric Spaces

Suppose that $\{G_n\}_{n\in\mathbb{N}}$ is a direct system of semisimple Lie groups and that for each $n \in \mathbb{N}$ we have an involution $\theta_n : G_n \to G_n$ such that $G_n/(G_n)^\theta$ is a Riemannian symmetric space and the diagram

$$\begin{array}{ccc} G_{n+1} & \xrightarrow{\theta_{n+1}} & G_{n+1} \\ {\scriptstyle p_n^{n+1}}\uparrow & & \uparrow{\scriptstyle p_n^{n+1}} \\ G_n & \xrightarrow{\theta_n} & G_n \end{array} \tag{4.1}$$

commutes. We thus have a continuous involution $\theta_\infty : G_\infty \to G_\infty$. Write $K_n = (G_n)^\theta$ for each $n \in \mathbb{N}$. We see that $\{K_n\}_{n\in\mathbb{N}}$ forms a direct system with inclusion maps given by $p_n^{n+1}|_{K_n}$. Furthermore, (4.1) implies that $p_n^{n+1}(K_n) = p_n^{n+1}(G_n) \cap K_{n+1}$ for each $n \in \mathbb{N}$, so there are well-defined inclusion maps from the quotient space G_n/K_n to G_{n+1}/K_{n+1}. We thus obtain a direct system of homogeneous spaces $\{G_n/K_n\}_{n\in\mathbb{N}}$. Now construct the direct limits $G_\infty = \varinjlim G_n$, $K_\infty = \varinjlim K_n$, and $G_\infty/K_\infty = \varinjlim G_n/K_n$. Finally, one can show that $K_\infty = (G_\infty)^{\theta_\infty}$.

We say that G_∞/K_∞ is a **lim-Riemannian symmetric space**. If G_n/K_n is a compact-type symmetric space for each $n \in \mathbb{N}$, then G_∞/K_∞ is said to be a **lim-compact Riemmanian symmetric space**. Similarly, if G_n/K_n is a noncompact-type Riemannian symmetric space for all $n \in \mathbb{N}$, then G_∞/K_∞ is said to be a **lim-noncompact Riemannian symmetric space**.

For each $m \in \mathbb{N}$, denote the Killing form on $\mathfrak{g}_k$ by B_k. Note that for each $k \le m$, the Killing form $B_m : \mathfrak{g}_m \times \mathfrak{g}_m \to \mathbb{C}$ restricts to an $\mathrm{ad}(\mathfrak{g}_k)$-invariant bilinear form on $\mathfrak{g}_k$. If $\mathfrak{g}_k$ is a simple Lie algebra for all $k \in \mathbb{N}$, then all such $\mathrm{ad}(\mathfrak{g}_k)$-invariant bilinear forms are constant multiples of each other, and hence $B_m|_{\mathfrak{g}_k\times\mathfrak{g}_k} = cB_k$ for some constant $c \in \mathbb{C}$. In the interest of consistency, we replace each Killing form B_k in this case with a constant multiple in such a way that $B_m|_{\mathfrak{g}_k \times \mathfrak{g}_k} = B_k$ for all $k \leq m$. In other words, we will shall normalize the Killing forms of the $\mathfrak{g}_k$'s so that they are consistent with each other.

Similarly, if $\{\mathfrak{g}_k\}_{k \in \mathbb{N}}$ is a direct system of simple Lie algebras, then one constructs a direct system $\{\mathfrak{g}_k \times \mathfrak{g}_k\}_{k \in \mathbb{N}}$ of semisimple Lie groups. The same construction as before allows us to consistently normalize Killing forms on $\mathfrak{g}_k \times \mathfrak{g}_k$ for each $k \in \mathbb{N}$.

We now see how the notion of c-duals may be extended to lim-Riemannian symmetric spaces. Suppose that $\{U_n/K_n\}_{n \in \mathbb{N}}$ is a direct system of Riemannian symmetric spaces with involutions $\theta_n : U_n \to U_n$ and inclusion maps $p_n^{n+1} : U_n \to U_{n+1}$. We follow the constructions in Section 4.2 to produce a complexification $(U_\infty)_\mathbb{C} = \varinjlim (U_n)_\mathbb{C}$ for the lim-compact group $U_\infty = \varinjlim U_n$. To simplify notation we assume that $(U_n)_\mathbb{C} \subseteq (U_{n+1})_\mathbb{C}$ and therefore $U_n \subseteq U_{n+1}$ for each $n \in \mathbb{N}$.

We recall that the involutions $\theta_n : U_n \to U_n$ extend to holomorphic involutions $\theta_n : (U_n)_\mathbb{C} \to (U_n)_\mathbb{C}$. The fact that the diagram

$$\begin{array}{ccc} U_{n+1} & \xrightarrow{\theta_{n+1}} & U_{n+1} \\ \uparrow & & \uparrow \\ U_n & \xrightarrow{\theta_n} & U_n \end{array}$$

commutes implies that

$$\begin{array}{ccc} (U_{n+1})_\mathbb{C} & \xrightarrow{\theta_{n+1}} & (U_{n+1})_\mathbb{C} \\ \uparrow & & \uparrow \\ (U_n)_\mathbb{C} & \xrightarrow{\theta_n} & (U_n)_\mathbb{C} \end{array} \tag{4.2}$$

commutes by [27, Proposition 7.5].

As before, we write

$$\mathfrak{u}_n = \mathfrak{k}_n \oplus \widetilde{\mathfrak{p}}_n$$

for each $n \in \mathbb{N}$, where $\mathfrak{k}_n$ and $\widetilde{\mathfrak{p}}_n$ are the $+1$- and -1-eigenspaces of θ_n. From (4.1) it follows that

$$\mathfrak{k}_n = \mathfrak{k}_{n+1} \cap \mathfrak{u}_n \text{ and } \widetilde{\mathfrak{p}}_n = \widetilde{\mathfrak{p}}_{n+1} \cap \mathfrak{u}_n$$

and hence that $\mathfrak{k}_n \subseteq \mathfrak{k}_{n+1}$ and $\widetilde{\mathfrak{p}}_n \subseteq \widetilde{\mathfrak{p}}_{n+1}$. For each n, we construct the c-dual Lie algebra

$$\mathfrak{g}_n = \mathfrak{k}_n \oplus i\widetilde{\mathfrak{p}}_n \subseteq (\mathfrak{u}_n)_\mathbb{C}$$

and note that $\mathfrak{g}_n \subseteq \mathfrak{g}_{n+1}$. Finally, we construct the analytic subgroup G_n of $(U_n)_\mathbb{C}$ which corresponds to the Lie algebra $\mathfrak{g}_n$ and recall that G_n is closed in $(U_n)_\mathbb{C}$. Thus G_n is a closed subgroup of G_{n+1} for each n. It follows that the direct-limit group $G_\infty = \varinjlim G_n$ is a closed subgroup of $(U_n)_\mathbb{C}$ and possesses the direct-limit Lie algebra $\mathfrak{g}_\infty = \varinjlim \mathfrak{g}_n$.

Reviewing the construction of finite-dimensional c-dual spaces, we see that the complexified involution $\theta_n : (\mathfrak{u}_n)_\mathbb{C} \to (\mathfrak{u}_n)_\mathbb{C}$ restricts to an involution $\theta_n : \mathfrak{g}_n \to \mathfrak{g}_n$

and that $\mathfrak{k}_n$ and $i\widetilde{\mathfrak{p}}_n$ are the $+1$- and -1-eigenspaces of θ_n in $\mathfrak{g}_n$. Furthermore, because $\mathfrak{g}_n$ is θ_n-stable, the holomorphic involution $\theta_n : (U_n)_{\mathbb{C}} \to (U_n)_{\mathbb{C}}$ restricts to an involution $\theta_n : G_n \to G_n$ such that $(G_n)^{\theta_n} = K_n$. Finally, the restriction of (4.2) implies that the diagram

$$\begin{array}{ccc} G_{n+1} & \xrightarrow{\theta_{n+1}} & G_{n+1} \\ \uparrow & & \uparrow \\ G_n & \xrightarrow{\theta_n} & G_n \end{array}$$

commutes. Thus $\{G_n/K_n\}_{n\in\mathbb{N}}$ is a direct system of noncompact-type Riemannian symmetric spaces. We say that $G_\infty/K_\infty = \varinjlim G_n/K_n$ is the **c-dual** of U_∞/K_∞.

In order to align our notation with that of Chapter 3, we set $\mathfrak{p}_n = i\widetilde{\mathfrak{p}}_n$ for each n, so that

$$\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{p}_n$$
$$\mathfrak{u}_n = \mathfrak{k}_n \oplus i\mathfrak{p}_n.$$

Finally, we notice that

$$\mathfrak{g}_\infty = \mathfrak{k}_\infty \oplus \mathfrak{p}_\infty$$
$$\mathfrak{u}_\infty = \mathfrak{k}_\infty \oplus i\mathfrak{p}_\infty,$$

where $\mathfrak{k}_\infty = \varinjlim \mathfrak{k}_n$ and $\mathfrak{p}_\infty = \varinjlim \mathfrak{p}_n$ are the $+1$-and -1-eigenspaces of θ_∞ in $\mathfrak{g}_\infty$.

4.4 Propagated Direct Limits

As before, we assume that G_∞/K_∞ is a lim-noncompact Riemannian symmetric spaces which is the c-dual of a direct limit U_∞/K_∞ of simply-connected compact Riemannian symmetric space. We need to put some further technical conditions on G_∞/K_∞ in order to prove our results about conical representations. The first condition is that of *propagation*, which was introduced by Ólafsson and Wolf. See [42], [54], and [56] for more details on this construction.

We begin this section by examining the restricted root data of G_∞/K_∞, using the notation of Section 4.3. We recursively choose maximal commutative subspaces $\mathfrak{a}_k \subset \mathfrak{p}_k$ such that $\mathfrak{a}_n \subseteq \mathfrak{a}_k$ for $n \leq k$ and define $\mathfrak{a}_\infty = \varinjlim \mathfrak{a}_n$. We then obtain the restricted root system $\Sigma_n = \Sigma(\mathfrak{g}_n, \mathfrak{a}_n)$ for each $n \in \mathbb{N}$. Note that

$$\Sigma_n \subseteq \Sigma_k|_{\mathfrak{a}_n} \backslash \{0\}$$

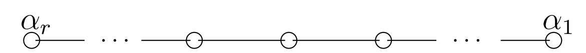
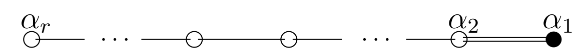
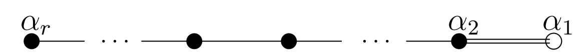
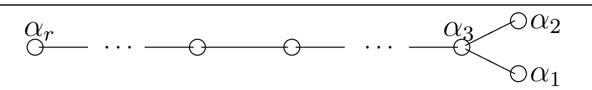
whenever $n \leq k$.

Next, we recursively choose positive subsystems $\Sigma_n^+ \subseteq \Sigma_n$ in such a way that

$$\Sigma_n^+ \subseteq \Sigma_k^+|_{\mathfrak{a}_n} \backslash \{0\}.$$

The projective limit $\Sigma_\infty^+ = \varprojlim \Sigma_n^+$ plays the role of the positive root subsystem for $(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$.

For each $n \in \mathbb{N}$, we let $(\Sigma_n)_0$ denote the set of nonmultipliable roots in Σ_n and set $(\Sigma_n)_0^+ = (\Sigma_n)_0 \cap \Sigma_n^+$. Denote the set of simple roots in $(\Sigma_n)_0^+$ by $\Psi_n = \{\alpha_1, \ldots, \alpha_{r_n}\}$, where $r_n = \dim \mathfrak{a}_n$. Since we will be dealing with direct limits we may assume that Σ, and hence Σ_0, is one of the classical root systems. We number the simple roots in the following way:

$\Psi = A_r$		$r \geqq 1$
$\Psi = B_r$		$r \geqq 2$
$\Psi = C_r$		$r \geqq 3$
$\Psi = D_r$		$r \geqq 4$

(4.3)

We are now ready to introduce the definition of propagated direct-limits of symmetric spaces.

Definition 4.5. *We say that a lim-noncompact symmetric space G_∞/K_∞ is **propagated** if*

1. *For each simple root $\alpha \in \Psi_k$ there is a unique simple root $\widetilde{\alpha} \in \Psi_n$ such that $\widetilde{\alpha}|_{\mathfrak{a}_k} = \alpha$, whenever $k \le n$.*

2. *There is a choice of ordering on the roots in Ψ_k for each $k \in \mathbb{N}$ such that either $\mathfrak{a}_n = \mathfrak{a}_k$ or else Ψ_k extends Ψ_n for $n \le k$ only by adding simple roots at the left end. (In particular, each Ψ_k has the same Dynkin diagram type.)*

We also introduce an analogous notion of propagation for lim-compact groups. Let $U_\infty = \varinjlim U_n$ be a direct limit of compact Lie groups. Choose a Cartan subalgebra $\mathfrak{h}_n \subseteq \mathfrak{g}_n$ for each n in such a way that $\mathfrak{h}_n \subseteq \mathfrak{h}_k$ whenever $n \le k$. One then obtains a root system $\Delta_n = \Delta(\mathfrak{g}_n, \mathfrak{h}_n)$ for each n. After recursively choosing positive subsystems $\Delta_n^+ \subseteq \Delta_n$ such that

$$\Delta_n^+ \subseteq \Delta_k^+|_{\mathfrak{h}_n} \setminus \{0\},$$

for $nleqk$, we arrive at a set Ξ_n of simple roots in Δ_n^+. We order these simple roots the same way as in Table 4.3.

Definition 4.6. *We say that the lim-compact group U_∞ is **propagated** if*

1. *For each simple root $\alpha \in \Xi_k$ there is a unique simple root $\widetilde{\alpha} \in \Xi_n$ such that $\widetilde{\alpha}|_{\mathfrak{h}_k} = \alpha$, whenever $k \le n$.*

2. *There is a choice of ordering on the roots in Ξ_k for each $k \in \mathbb{N}$ such that either $\mathfrak{h}_n = \mathfrak{h}_k$ or else Ξ_k extends Ξ_n for $n \le k$ only by adding simple roots at the left end.*

Suppose that U_∞ is a propagated direct limit of compact, simply-connected semisimple Lie groups. Then each U_k may be decomposed into a product of compact simple Lie groups, say $U_k = U_k^1 \times U_k^2 \times \cdots \times U_k^{d_k}$. We can recursively choose Cartan subalgebras $\mathfrak{h}_k = \mathfrak{h}_k^1 \oplus \mathfrak{h}_k^2 \oplus \cdots \oplus \mathfrak{h}_k^{d_k}$ where each $\mathfrak{h}_k^i$ is a Cartan subalgebra of $\mathfrak{u}_k^i$. The definition of propagation then implies that $d_n = d_m \equiv d$ for each $n, m \in \mathbb{N}$ and that the indices may be ordered in such a way that $\{U_k^i\}_{n\in\mathbb{N}}$ is a propagated direct system of compact simple Lie groups for each $1 \leq i \leq d$.

Following the exposition in [7], we make note of the details of each root system for later use. We identify $\mathfrak{a}$ with $\mathbb{R}^r$ so that, as usual, $\mathfrak{a} = \{(x_{r+1}, \ldots, x_1) \mid x_1 + \ldots + x_{r+1} = 0\}$ if $\Psi = A_r$ and otherwise $\mathfrak{a} = \mathbb{R}^r$. Set $e_1 = (0, \ldots, 0, 1)$, $e_2 = (0, \ldots, 0, 1, 0)$, $\ldots$, $e_n = (1, 0, \ldots, 0)$ where $n = r+1$ for A_r and otherwise $n = r$. We view the vectors e_j also as elements in $\mathfrak{a}^*$ via the standard inner product in $\mathbb{R}^{r+1}$ in the case $\Psi = A_r$ and otherwise $\mathbb{R}^r$. Note that in the case $\Psi = A_r$ this gives a map $\mathbb{R}^{r+1} \to \mathfrak{a}^*$ which is not injective.

For $\Psi = A_r$, we have $\Sigma_0^+ = \{e_j - e_i \mid 1 \leqq i < j \leqq n\}$ and $\alpha_j = e_{j+1} - e_j$, $j = 1, \ldots, r$. The Weyl group W consists of linear maps given by

$$w_\sigma(e_i) = e_\sigma(i)$$

for permutations in the symmetric group S_{r+1}. One can show that the fundamental weights are

$$\xi_j = 2 \sum_{i=j+1}^{r+1} e_i \,.$$

If Ψ is of type B_r then we have $\Sigma_0^+ = \{e_j \mid j = 1, \ldots, r\} \cup \{e_j \pm f_i \mid 1 \leqq i < j \leqq r\}$ and $\Psi = \{\alpha_1 = f_1\} \cup \{\alpha_i = e_i - e_{i-1} \mid i = 2, \ldots, r\}$. The Weyl group consists of linear maps generated by the involutions

$$w_i(e_i) = -e_i \text{ and } w_i(e_j) = e_j \text{ for } j \neq i$$

for $i \leq r$ and the maps

$$w_\sigma(e_i) = e_{\sigma(i)}$$

for permutations in the symmetric group S_r. Furthermore, one shows that the fundamental weights are

$$\xi_1 = \sum_{j=1}^{r} e_j \text{ and } \xi_j = 2 \sum_{i=j}^{r} e_i \,, \; j > 1 \,.$$

If Ψ is of type C_r then we have $\Sigma_0^+ = \{2e_j \mid j = 1, \ldots, r\} \cup \{e_j \pm e_i \mid 1 \leqq i < j \leqq r\}$ and $\Psi = \{\alpha_1 = 2e_1\} \cup \{\alpha_j = e_j - e_{j-1} \mid j = 2, \ldots, r\}$. The Weyl group consists of linear maps generated by the involutions

$$w_i(e_i) = -e_i \text{ and } w_i(e_j) = e_j \text{ for } j \neq i$$

for $i \leq r$ and the maps

$$w_\sigma(e_i) = e_{\sigma(i)}$$

for permutations in the symmetric group S_r. Furthermore, one shows that the fundamental weights are

$$\xi_j = 2\sum_{i=j}^{r} f_i$$

.

If Ψ is of type D_r then $\alpha_1 = e_1 + e_2$ and $\alpha_j = e_j - e_{j-1}$ for $j \geqq 2$. The Weyl group consists of linear maps generated by the involutions

$$w_i(e_i) = -e_i \text{ and } w_i(e_j) = e_j \text{ for } j \neq i$$

for $2 \leq i \leq r$ and the maps

$$w_\sigma(e_i) = e_{\sigma(i)}$$

for permutations in the symmetric group S_r. One shows that the fundamental weights are

$$\xi_1 = \sum_{i=1}^{r} e_i\,,\ \xi_2 = -e_1 + \sum_{j=2}^{r} e_j\,,\ \text{and } \xi_j = 2\sum_{i=j}^{r} e_i \ \text{ for } j \geqq 3\,.$$

Thus if we take a propagated symmetric space G_∞/K_∞ or a propagated direct-limit group U_∞, then one uses the above formulations to construct countable bases $\{e_1, e_2, \ldots\}$ for $\mathfrak{a}_\infty$ and $\mathfrak{h}_\infty$, respectively.

4.5 Admissible Direct Limits

We continue to examine the root data for lim-noncompact symmetric spaces G_∞/K_∞ by analogy with Section 3.2. For each $k \in \mathbb{N}$ and each restricted root $\alpha \in \Sigma_k$, we define as before the root space

$$\mathfrak{g}_{k,\alpha} = \{Y \in \mathfrak{g}_k \mid [H, Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{a}_k\}\,.$$

Next we define the subalgebras

$$\mathfrak{n}_k = \bigoplus_{\alpha\in\Sigma_k^+} \mathfrak{g}_{k,\alpha}$$

and

$$\mathfrak{m}_k = Z_{\mathfrak{k}_k}(\mathfrak{a}_k)$$

of $\mathfrak{g}_k$. Similarly, we define the subgroups $N_k = \exp(\mathfrak{n}_k)$ and $M_k = Z_{K_k}(\mathfrak{a}_k)$ of G_k.

For each $k \in K$, the conical representations of G_k are the representations which possess a nonzero vector (or, more generally, distribution vector) which is invariant under the action of the group M_kN_k. Hence, in order to define conical representations of G_∞, one would like to define a subgroup $M_\infty N_\infty = \varinjlim M_nN_n$. In order for such a group to be well-defined, we need to make a technical assumption that was first introduced in [24].

Definition 4.7. *A lim-noncompact symmetric space G_∞/K_∞ is said to be* ***admissible*** *if $M_kN_k \leq M_mN_m$ whenever $k \leq m$.*

As a consequence of the following lemmas, it is sufficient to assume that $\mathfrak{m}_k \subseteq \mathfrak{m}_m$ for $k \leq m$:

Lemma 4.8. *If G_∞/K_∞ is a lim-noncompact symmetric space, then $N_k \leq N_m$ for $k \leq m$.*

Proof. We will show that $\mathfrak{n}_k \subseteq \mathfrak{n}_m$. The result will then follow from the fact that $N_k = \exp \mathfrak{n}_k$ and $N_m = \exp \mathfrak{n}_m$.

In fact, it suffices to show that $\mathfrak{g}_{k,\alpha} \subseteq \mathfrak{n}_m$ for all $\alpha \in \Sigma_k^+$. Suppose that $X \in \mathfrak{g}_{k,\alpha}$. Consider the decomposition of X into $\mathfrak{a}_m$-root vectors:

$$X = \sum_{\beta \in \Sigma_m} X_\beta,$$

where $X_\beta \in \mathfrak{g}_{m,\beta}$ for each $(\mathfrak{g}_m, \mathfrak{a}_m)$-root β. Because this decomposition is unique and X is a root vector for $\mathfrak{a}_k \subseteq \mathfrak{a}_m$, it follows that $\beta|_{\mathfrak{a}_k} = \alpha$ for all $\beta \in \Sigma_m$ such that $X_\beta \neq 0$.

Now recall that we have made a consistent choice of positive root subsystems Σ_k^+ of Σ_k and Σ_m^+ of Σ_m. In other words, $\beta \in \Sigma_m$ is positive if $\beta|_{\mathfrak{a}_k}$ is positive. Since $\alpha \in \Sigma_k^+$, it follows that X is a sum of Σ_m^+-root vectors. Hence, $X \in N_m$. □

Due to the fact that M_k is typically a disconnected subgroup of G_n, it is not clear *a priori* that requiring $\mathfrak{m}_k \subseteq \mathfrak{m}_m$ for $k \leq m$ is sufficient to imply that $M_k \leq M_m$. However, the following lemma shows that this Lie algebra condition is, in fact, sufficient:

Lemma 4.9. *Suppose that G_∞/K_∞ is a propagated lim-noncompact symmetric space such that $\mathfrak{m}_k \subseteq \mathfrak{m}_m$ for all $k \leq m$. Then $M_k \leq M_m$ for $k \leq m$.*

Proof. By Theorem 7.53 in [27] we see that for each $k \in \mathbb{N}$ there is a finite discrete subgroup $F_k \subseteq \exp(i\mathfrak{a}_k) \cap K_k$ such that $M_k = F_k(M_k)_0$, where $(M_k)_0 = \exp \mathfrak{m}_k$ is the connected component of the identity in M_k. Because $\mathfrak{m}_k \subseteq \mathfrak{m}_m$ for all $k \leq m$, we see that $(M_k)_0 \leq (M_m)_0$. It is thus sufficient to show that $F_k \leq M_m$ for $k \leq m$. In fact, since $F_k \subseteq \exp(i\mathfrak{a}_k) \subseteq \exp(i\mathfrak{a}_m)$, it is clear that F_k centralizes $\mathfrak{a}_k$. Since $F_k \subseteq K_k \subseteq K_m$, we see that $F_k \leq M_m$, and the result follows. □

At this point we do not know whether every propagated direct limit of noncompact-type Riemannian symmetric spaces is admissible, but in any case this assumption is not a restrictive one, as it is satisfied by each of the classical direct limits, as we demonstrate in the next section.

4.6 Admissiblility of Classical Direct Limits

The classical propagated direct systems of Riemannian symmetric spaces may be found in Table 4.4, where each row gives a noncompact-type symmetric space

G_n/K_n and its simply-connected compact dual space U_n/K_n, and where the restricted roots exhibit the Dynkin diagram Ψ_n. For each row, the limit $G_\infty/K_\infty = \varinjlim G_n/K_n$ is propagated and also that it is possible to choose Cartan subalgebras of U_n for each $n \in \mathbb{N}$ so that $U_\infty = \varinjlim U_n$ is a propagated direct-limit group (see, for instance, [39, Section 2] or [54, Section 3]).

Note that in each row of Table 4.4, the symmetric space U_n/K_n is simply-connected. However, in certain rows the group U_n is not simply-connected. We may remove this obstruction simply by passing to the universal cover $\widetilde{U_n}$ of U_n. In fact, that the involution θ_n on $\mathfrak{u}_n$ integrates to an involution $\widetilde{\theta_n}$ on $\widetilde{U_n}$. Denote the fixed-point subgroup for $\widetilde{\theta_n}$ in $\widetilde{U_n}$ by $\widetilde{K_n}$. By simply-connectedness all of the inclusions on the Lie algebra level integrate to inclusions on the group level, so that $\widetilde{U_n}/\widetilde{K_n}$ forms a propagated direct system of compact-type symmetric spaces. Furthermore, one sees that if $p : \widetilde{U_n} \to U_n$ is the covering map, then $p\left(\widetilde{K_n}\right) \subseteq K_n$. Hence p factors to a covering map from $\widetilde{U_n}/\widetilde{K_n}$ to U_n/K_n (see [21, p. 213]). Since U_n/K_n is already simply-connected, we see that $\widetilde{U_n}/\widetilde{K_n}$ is diffeomorphic to U_n/K_n.

While we do not know whether it is possible to show that all propagated direct systems of Riemannian symmetric spaces are admissible in the sense of 4.7, the aim of this section is to show that each classical example is admissible. For the explicit matrix realizations of the compact-type Riemannian symmetric spaces, see [21, p. 446, 451–455].

Classical direct systems of irreducible Riemannian symmetric spaces (4.4)

	G_n	U_n	K_n	Ψ_n
1	$\mathrm{SL}(n,\mathbb{C})$	$\mathrm{SU}(n) \times \mathrm{SU}(n)$	$\mathrm{diag}\,\mathrm{SU}(n)$	A_{n-1}
2	$\mathrm{Spin}(2n+1,\mathbb{C})$	$\mathrm{Spin}(2n+1)\times$ $\mathrm{Spin}(2n+1)$	$\mathrm{diag}\,\mathrm{Spin}(2n+1)$	B_n
3	$\mathrm{Spin}(2n,\mathbb{C})$	$\mathrm{Spin}(2n)\times$ $\mathrm{Spin}(2n)$	$\mathrm{diag}\,\mathrm{Spin}(2n)$	D_n
4	$\mathrm{Sp}(n,\mathbb{C})$	$\mathrm{Sp}(n) \times \mathrm{Sp}(n)$	$\mathrm{diag}\,\mathrm{Sp}(n)$	C_n
5_1	$\mathrm{SU}(p, n-p)$	$\mathrm{SU}(n)$	$\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(n-p))$	C_p
5_2	$\mathrm{SU}(n,n)$	$\mathrm{SU}(2n)$	$\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n))$	C_n
6_1	$\mathrm{SO}_0(p, n-p)$	$\mathrm{SO}(n)$	$\mathrm{SO}(p) \times \mathrm{SO}(n-p)$	B_p
6_2	$\mathrm{SO}_0(n,n)$	$\mathrm{SO}(2n)$	$\mathrm{SO}(n) \times \mathrm{SO}(n)$	B_n
7_1	$\mathrm{Sp}(p, n-p)$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(p) \times \mathrm{Sp}(n-p)$	C_p
7_2	$\mathrm{Sp}(n,n)$	$\mathrm{Sp}(2n)$	$\mathrm{Sp}(n) \times \mathrm{Sp}(n)$	C_n
8	$\mathrm{SL}(n,\mathbb{R})$	$\mathrm{SU}(n)$	$\mathrm{SO}(n)$	A_{n-1}
9	$\mathrm{SL}(n,\mathbb{H})$	$\mathrm{SU}(2n)$	$\mathrm{Sp}(n)$	A_{n-1}
10_1	$\mathrm{SO}^*(4n)$	$\mathrm{SO}(4n)$	$\mathrm{U}(2n)$	C_n
10_2	$\mathrm{SO}^*(2(2n+1))$	$\mathrm{SO}(2(2n+1))$	$\mathrm{U}(2n+1)$	C_n
11	$\mathrm{Sp}(n,\mathbb{R})$	$\mathrm{Sp}(n)$	$\mathrm{U}(n)$	C_n

4.6.1 A General Strategy for Proving Admissibility

The embedding $G_n \hookrightarrow G_{n+1}$ takes the form

$$A \mapsto \begin{pmatrix} I & & \\ & A & \\ & & I \end{pmatrix} \tag{4.5}$$

for the systems in rows 5_2, 6_2, and 7_2. In all other cases in Table 4.4, the embedding $G_n \hookrightarrow G_{n+1}$ takes the form

$$A \mapsto \begin{pmatrix} A & \\ & I \end{pmatrix}, \tag{4.6}$$

where I is a 1×1, 2×2, or 4×4 identity matrix.

Suppose we can choose $\mathfrak{a}_n$ for each n in such a way that

$$\mathfrak{a}_{n+1} \subseteq \begin{pmatrix} * & 0 & * \\ 0 & \mathfrak{a}_n & 0 \\ * & 0 & * \end{pmatrix} \tag{4.7}$$

or

$$\mathfrak{a}_{n+1} \subseteq \begin{pmatrix} \mathfrak{a}_n & 0 \\ 0 & * \end{pmatrix} \tag{4.8}$$

(depending on the type of embedding $G_n \hookrightarrow G_{n+1}$). In this case, since $\mathfrak{a}_n$ commutes with $M_n = Z_{K_n}(\mathfrak{a}_n)$ by definition, it follows from (4.7) and (4.8) that $\mathfrak{a}_{n+1}$ commutes with

$$M_n \cong \begin{pmatrix} I & 0 & 0 \\ 0 & M_n & 0 \\ 0 & 0 & I \end{pmatrix}$$

or

$$M_n \cong \begin{pmatrix} M_n & 0 \\ 0 & I \end{pmatrix},$$

respectively, depending on the type of embedding $G_n \hookrightarrow G_{n+1}$. In other words, $M_n \leq Z_{K_{n+1}}(\mathfrak{a}_{n+1}) = M_{n+1}$

Hence, in order to prove that a propagated direct limit is admissible, it is sufficient to show that either (4.7) or (4.8) holds. In most cases, our proof of admissibility will take this form.

4.6.2 $U_n = L_n \times L_n$ and $K_n = \text{diag } L_n$

This case corresponds to the first four rows in Table 4.4. In this case, one sees that

$$\begin{aligned} \mathfrak{u}_n &= \mathfrak{l}_n \times \mathfrak{l}_n \\ \mathfrak{k}_n &= \{(X, X) \in \mathfrak{u}_n | X \in \mathfrak{l}_n\} \\ i\mathfrak{p}_n &= \{(X, -X) \in \mathfrak{u}_n | X \in \mathfrak{l}_n\}. \end{aligned}$$

Furthermore, if we fix a Cartan subalgebra $\mathfrak{h}_n \subseteq \mathfrak{l}_n$ for each n, then we can choose

$$i\mathfrak{a}_n = \{(X, -X) \in \mathfrak{u}_n | X \in \mathfrak{h}_n\}.$$

Now suppose that $g \in L_n$ and that $(g, g) \in M_n = Z_{K_n}(\mathfrak{a}_n)$. Then $g \in Z_{L_n}(\mathfrak{h}_n)$; that is, g centralizes the Cartan subalgebra $\mathfrak{h}_n$ of $\mathfrak{l}_n$. Since K_n is connected, it follows that $g \in H_n \equiv \exp(\mathfrak{h}_n)$. Thus $M_n = \text{diag } H_n$ for each n. It follows that $M_k \leq M_n$ for $k \leq n$.

4.6.3 $\operatorname{Rank}(G_\infty/K_\infty) \equiv \dim \mathfrak{a}_\infty < \infty$

This case corresponds to rows 5_1, 6_1, and 7_1 in Table 4.4. If $\dim \mathfrak{a}_\infty < \infty$, then for k large enough, one has $\mathfrak{a}_k = \mathfrak{a}_\infty$. Suppose $k \leq n$ and $g \in M_k$. That is, $g \in K_k$ and g centralizes $\mathfrak{a}_k$. But $\mathfrak{a}_k = \mathfrak{a}_n = \mathfrak{a}_\infty$ and $K_k \leq K_n$. Thus $g \in M_n$.

4.6.4 $\operatorname{Rank}(G_n/K_n) = \operatorname{Rank}(G_n)$ for all $n \in \mathbb{N}$

This case corresponds to rows 8 and 11 in Table 4.4. One has that $\mathfrak{a}_n$ is a Cartan subalgebra for $\mathfrak{g}_n$. In particular, $Z_{\mathfrak{g}_n}(\mathfrak{a}_n) = \mathfrak{a}_n$. Since $\mathfrak{a}_n \cap \mathfrak{k}_n = \{0\}$, one has that $\mathfrak{m}_n \equiv Z_{\mathfrak{k}_n}(\mathfrak{a}_n) = \{0\}$ for all $n \in \mathbb{N}$.

For example, if we let $G_n = \mathrm{SL}(n, \mathbb{R})$ and $K_n = \mathrm{SO}(n)$ and make the standard choice of $\mathfrak{a}_n = \{\operatorname{diag}(a_1, \ldots, a_n) | a_i \in \mathbb{R}\}$, then one has $M_n = \{\operatorname{diag}(\pm 1, \ldots, \pm 1)\}$. Thus $M_k \leq M_n$ for $k \leq n$.

4.6.5 $U_n/K_n = \mathrm{SU}(2n)/S(\mathrm{SU}(n) \times \mathrm{SU}(n))$

This case corresponds to row 5_2 in Table 4.4. One has $\mathfrak{g}_n = \mathfrak{su}(n,n)$, $\mathfrak{u}_n = \mathfrak{su}(2n)$, and $\mathfrak{k}_n = \mathfrak{s}(\mathfrak{su}(n) \oplus \mathfrak{su}(n))$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where

$$J_n = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}.$$

More explicitly, one has

$$\begin{aligned}
\mathfrak{u}_n &= \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} \in \mathrm{M}(2n, \mathbb{C}) \middle| \begin{array}{l} A^* = -A,\ D^* = -D, \\ \text{and } \operatorname{Tr}(A) + \operatorname{Tr}(D) = 0 \end{array} \right\} \\
\mathfrak{k}_n &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathrm{M}(2n, \mathbb{C}) \middle| \begin{array}{l} A^* = -A,\ D^* = -D, \\ \text{and } \operatorname{Tr}(A) + \operatorname{Tr}(D) = 0 \end{array} \right\} \\
i\mathfrak{p}_n &= \left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \in \mathrm{M}(2n, \mathbb{C}) \right\}.
\end{aligned}$$

We choose

$$i\mathfrak{a}_n = \left\{ \left(\begin{array}{ccc|ccc} & & & & & a_n \\ & & & & \iddots & \\ & & & a_1 & & \\ \hline & & -a_1 & & & \\ & \iddots & & & & \\ -a_n & & & & & \end{array} \right) \middle| \, a_i \in \mathbb{R} \right\}$$

Thus condition (4.7) is satisfied and so G_∞/K_∞ is admissible.

4.6.6 $U_n/K_n = \mathrm{SO}(2n)/(\mathrm{SO}(n) \times \mathrm{SO}(n))$

This case corresponds to row 6_2 in Table 4.4. One has $\mathfrak{g}_n = \mathfrak{so}(n,n)$, $\mathfrak{u}_n = \mathfrak{so}(2n)$, and $\mathfrak{k}_n = \mathfrak{so}(n) \oplus \mathfrak{so}(n)$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where

$$J_n = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}.$$

More explicitly, one has

$$\mathfrak{u}_n = \left\{ \begin{pmatrix} A & B \\ -B^{\mathrm{T}} & D \end{pmatrix} \in \mathrm{M}(2n, \mathbb{R}) \middle| A^{\mathrm{T}} = -A \text{ and } D^{\mathrm{T}} = -D \right\}$$
$$\mathfrak{k}_n = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathrm{M}(2n, \mathbb{R}) \middle| A^{\mathrm{T}} = -A \text{ and } D^{\mathrm{T}} = -D \right\}$$
$$i\mathfrak{p}_n = \left\{ \begin{pmatrix} 0 & B \\ -B^{\mathrm{T}} & 0 \end{pmatrix} \in \mathrm{M}(2n, \mathbb{R}) \right\}.$$

We choose

$$i\mathfrak{a}_n = \left\{ \left(\begin{array}{ccc|ccc} & & & & & a_n \\ & & & & \iddots & \\ & & & a_1 & & \\ \hline & & -a_1 & & & \\ & \iddots & & & & \\ -a_n & & & & & \end{array} \right) \middle| a_i \in \mathbb{R} \right\}.$$

Thus condition (4.8) is satisfied and so G_∞/K_∞ is admissible.

4.6.7 $U_n/K_n = \mathrm{Sp}(2n)/(\mathrm{Sp}(n) \times \mathrm{Sp}(n))$

This case corresponds to row 7_2 in Table 4.4. One has $\mathfrak{g}_n = \mathfrak{sp}(n, n)$, $\mathfrak{u}_n = \mathfrak{sp}(2n)$, and $\mathfrak{k}_n = \mathfrak{sp}(n) \oplus \mathfrak{sp}(n)$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where

$$J_n = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}.$$

More explicitly, one has

$$\mathfrak{u}_n = \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} \in \mathrm{M}(2n, \mathbb{H}) \middle| A^* = -A \text{ and } D^* = -D \right\}$$
$$\mathfrak{k}_n = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathrm{M}(2n, \mathbb{H}) \middle| A^* = -A \text{ and } D^* = -D \right\}$$
$$i\mathfrak{p}_n = \left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \in \mathrm{M}(2n, \mathbb{H}) \right\}.$$

We choose

$$i\mathfrak{a}_n = \left\{ \left(\begin{array}{ccc|ccc} & & & & & a_n \\ & & & & \iddots & \\ & & & a_1 & & \\ \hline & & -a_1 & & & \\ & \iddots & & & & \\ -a_n & & & & & \end{array} \right) \middle| a_i \in \mathbb{R} \right\}.$$

Thus condition (4.7) is satisfied and so G_∞/K_∞ is admissible.

4.6.8 $U_n/K_n = \mathrm{SU}(2n)/\mathrm{Sp}(n)$

This case corresponds to row 9 in Table 4.4. One has $\mathfrak{g}_n = \mathfrak{sl}(n,\mathbb{H})$, $\mathfrak{u}_n = \mathfrak{su}(2n)$ and $\mathfrak{k}_n = \mathfrak{sp}(n)$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where J_n is given by

$$J_n = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}. \tag{4.9}$$

One can also obtain the same symmetric space by using the involution $\widetilde{\theta}_n : A \mapsto \widetilde{J}_n A \widetilde{J}_n^{-1}$, where

$$\widetilde{J}_n = \left(\begin{array}{ccc|ccc} & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \\ \hline 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{array}\right). \tag{4.10}$$

The calculations will be easier if we use $\widetilde{\theta}_n$ instead of θ_n. However, we must use θ_n in order for the inclusions $U_n \to U_{n+1}$ to take the form of (4.6). We can move freely between these pictures, however, because $J_n = E_{\sigma_n} J_n E_{\sigma_n}^{-1}$, where $E_{\sigma_n} \in M(2n,\mathbb{C})$ is the permutation matrix corresponding to the permutation

$$\sigma = (1\ n)(2\ (n+1)) \cdots ((n-1)\ 2n) \in S_{2n}.$$

In other words, the rows and columns are interwoven, so that the first n basis elements of $\mathbb{C}^{2n}$ are mapped to odd-numbered basis elements and the final n basis elements of $\mathbb{C}^{2n}$ are sent to even-numbered basis elements.

We proceed by using $\widetilde{\theta}_n$. We have

$$\mathfrak{su}(2n) = \mathfrak{u}_n = \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} \in \mathrm{M}(2n,\mathbb{C}) \;\middle|\; \begin{array}{l} A^* = -A,\ D^* = -D, \text{ and} \\ \mathrm{Tr}(A) + \mathrm{Tr}(D) = 0 \end{array} \right\}$$

$$\mathfrak{sp}(n) \cong \mathfrak{k}_n = \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \in \mathrm{M}(2n,\mathbb{C}) \;\middle|\; \begin{array}{l} A^* = -A \\ \text{and } B^{\mathrm{T}} = B \end{array} \right\}$$

$$i\mathfrak{p}_n = \left\{ \begin{pmatrix} A & B \\ \overline{B} & -\overline{A} \end{pmatrix} \in \mathrm{M}(2n,\mathbb{C}) \;\middle|\; \begin{array}{l} A^* = -A, B^{\mathrm{T}} = -B, \\ \text{and } \mathrm{Tr}(A) = 0 \end{array} \right\}.$$

There is a $\widetilde{\theta}_n$-stable Cartan subalgebra

$$\widetilde{\mathfrak{h}}_n = \left\{ \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_{2n} \end{pmatrix} \;\middle|\; a_i \in \mathbb{R} \text{ and } \sum_{i=1}^{2n} a_i = 0 \right\}$$

for $\mathfrak{g}_n = \mathfrak{so}^*(4n)$, and we can choose

$$i\widetilde{\mathfrak{a}}_n = \left\{ \left(\begin{array}{ccc|ccc} ia_1 & & & & & \\ & \ddots & & & & \\ & & ia_n & & & \\ \hline & & & ia_1 & & \\ & & & & \ddots & \\ & & & & & ia_n \end{array} \right) \middle| \, a_i \in \mathbb{R} \text{ and } \sum_{i=1}^{n} a_i = 0 \right\}.$$

We now proceed to the θ_n picture. Conjugation of $\widetilde{\mathfrak{h}}_n$ by E_{σ_n} (followed by renumbering the indices) yields the θ_n-stable Cartan subalgebra

$$\mathfrak{h}_n = \widetilde{\mathfrak{h}}_n = \left\{ \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_{2n} \end{pmatrix} \middle| \, a_i \in \mathbb{R} \text{ and } \sum_{i=1}^{2n} a_i = 0 \right\}.$$

Finally, conjugation of $\widetilde{\mathfrak{a}}_n$ by E_{σ_n} yields

$$i\mathfrak{a}_n = \left\{ \begin{pmatrix} ia_1 & & & & & & \\ & ia_1 & & & & & \\ & & ia_2 & & & & \\ & & & ia_2 & & & \\ & & & & \ddots & & \\ & & & & & ia_n & \\ & & & & & & ia_n \end{pmatrix} \middle| \, a_i \in \mathbb{R} \text{ and } \sum_{i=1}^{n} a_i = 0 \right\}.$$

While condition (4.8) is not quite satisfied, we do have that

$$\mathfrak{a}_{n+1} \subseteq \begin{pmatrix} \mathfrak{a}_n + \mathbb{C}\mathrm{Id} & 0 \\ 0 & * \end{pmatrix}. \tag{4.11}$$

Since $\mathfrak{m}_n$ centralizes $\mathfrak{a}_n$, it follows that $\mathfrak{m}_n$ commutes with $\mathfrak{a}_n + \mathbb{C}\mathrm{Id}$. Thus by (4.11), it follows that $\mathfrak{m}_n$ commutes with $\mathfrak{a}_{n+1}$. Thus $\mathfrak{m}_m \subseteq \mathfrak{m}_n$ for $m \leq n$, and it follows that G_∞/K_∞ is admissible.

4.6.9 $U_n/K_n = \mathrm{SO}(4n)/\mathrm{U}(2n)$

This case corresponds to row 10_1 in Table 4.4. One has $\mathfrak{g}_n = \mathfrak{so}^*(4n)$, $\mathfrak{u}_n = so(4n)$ and $\mathfrak{k}_n = \mathfrak{u}(2n)$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where J_n is given by (4.9). As in the previous example, one can also obtain the same symmetric space by using the involution $\widetilde{\theta}_n : A \mapsto \widetilde{J}_n A \widetilde{J}_n^{-1}$, where $\widetilde{J}_n$ is given by (4.10).

We work first on the $\widetilde{\theta}_n$-side. We have

$$\mathfrak{so}(4n) = \mathfrak{u}_n = \left\{ \begin{pmatrix} A & B \\ -B^{\mathrm{T}} & D \end{pmatrix} \in \mathrm{M}(4n, \mathbb{R}) \middle| A^{\mathrm{T}} = -A \text{ and } D^{\mathrm{T}} = -D \right\}$$

$$\mathfrak{u}(2n) \cong \mathfrak{k}_n = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{M}(4n, \mathbb{R}) \middle| \begin{array}{l} A^{\mathrm{T}} = -A \\ \text{and } B^{\mathrm{T}} = B \end{array} \right\}$$

$$i\mathfrak{p}_n = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathrm{M}(4n, \mathbb{R}) \middle| \begin{array}{l} A^{\mathrm{T}} = -A \\ \text{and } B^{\mathrm{T}} = -B \end{array} \right\}.$$

There is a $\widetilde{\theta}_n$-stable Cartan subalgebra

$$\widetilde{\mathfrak{h}}_n = \left\{ \begin{pmatrix} 0 & a_1 & & & & & \\ -a_1 & 0 & & & & & \\ & & 0 & a_2 & & & \\ & & -a_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & a_{2n} \\ & & & & & -a_{2n} & 0 \end{pmatrix} \middle| a_i \in \mathbb{R} \right\}$$

and we can choose

$$i\widetilde{\mathfrak{a}}_n = \left\{ \left(\begin{array}{cccccc|cccccc} 0 & a_1 & & & & & & & & & & \\ -a_1 & 0 & & & & & & & & & & \\ & & \ddots & & & & & & & & & \\ & & & & 0 & a_n & & & & & & \\ & & & & -a_n & 0 & & & & & & \\ \hline & & & & & & 0 & -a_1 & & & & \\ & & & & & & a_1 & 0 & & & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & & & 0 & -a_n \\ & & & & & & & & & & a_n & 0 \end{array}\right) \middle| a_i \in \mathbb{R} \right\}.$$

Moving to the θ_n-picture, we conjugate everything by E_{σ_n} and renumber the indices to arrive at the θ_n-stable Cartan algebra

$$\mathfrak{h}_n = \left\{ \begin{pmatrix} 0 & 0 & a_1 & 0 & & & & & & & & & \\ 0 & 0 & 0 & a_2 & & & & & & & & & \\ -a_1 & 0 & 0 & 0 & & & & & & & & & \\ 0 & -a_2 & 0 & 0 & & & & & & & & & \\ & & & & 0 & 0 & a_3 & 0 & & & & & \\ & & & & 0 & 0 & 0 & a_4 & & & & & \\ & & & & -a_3 & 0 & 0 & 0 & & & & & \\ & & & & 0 & a_4 & 0 & 0 & & & & & \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & 0 & 0 & a_{2n-1} & 0 \\ & & & & & & & & & 0 & 0 & 0 & a_{2n} \\ & & & & & & & & & -a_{2n-1} & 0 & 0 & 0 \\ & & & & & & & & & 0 & -a_{2n} & 0 & 0 \end{pmatrix} \middle| a_i \in \mathbb{R} \right\}$$

and finally

$$i\mathfrak{a}_n = \left\{ \left(\begin{array}{cccccccccccccc} 0 & 0 & a_1 & 0 & & & & & & & & & & \\ 0 & 0 & 0 & -a_1 & & & & & & & & & & \\ -a_1 & 0 & 0 & 0 & & & & & & & & & & \\ 0 & a_1 & 0 & 0 & & & & & & & & & & \\ & & & & 0 & 0 & a_2 & 0 & & & & & & \\ & & & & 0 & 0 & 0 & -a_2 & & & & & & \\ & & & & -a_2 & 0 & 0 & 0 & & & & & & \\ & & & & 0 & a_2 & 0 & 0 & & & & & & \\ & & & & & & & & \ddots & & & & & \\ & & & & & & & & & 0 & 0 & a_n & 0 \\ & & & & & & & & & 0 & 0 & 0 & -a_n \\ & & & & & & & & & -a_n & 0 & 0 & 0 \\ & & & & & & & & & 0 & a_n & 0 & 0 \end{array} \right) \middle| \, a_i \in \mathbb{R} \right\}.$$

Hence $\mathfrak{a}_n$ is block-diagonal, and moving from $\mathfrak{a}_n$ to $\mathfrak{a}_{n+1}$ is simply a matter of adding another 4×4 block. Thus we see that condition (4.8) is satisfied and hence G_∞/K_∞ is admissible.

4.6.10 $U_n/K_n = \mathrm{SO}(2(2n+1))/\mathrm{U}(2n+1)$

This case corresponds to row 10_2 in Table 4.4. One has $\mathfrak{g}_n = \mathfrak{so}^*(2(2n+1))$, $\mathfrak{u}_n = so(4n)$ and $\mathfrak{k}_n = \mathfrak{u}(2n)$. As in the previous example, one can also obtain the same symmetric space by using the involution $\widetilde{\theta}_n : A \mapsto \widetilde{J}_n A \widetilde{J}_n^{-1}$, where $\widetilde{J}_n$ is given by (4.10).

We first work on the $\widetilde{\theta}_n$ side. We then have

$$\begin{aligned} \mathfrak{so}(2(2n+1)) = \mathfrak{u}_n &= \left\{ \begin{pmatrix} A & B \\ -B^{\mathrm{T}} & D \end{pmatrix} \in \mathrm{M}(2(2n+1), \mathbb{R}) \middle| \begin{array}{c} A^{\mathrm{T}} = -A \\ \text{and } D^{\mathrm{T}} = -D \end{array} \right\} \\ \mathfrak{u}(2n+1) \cong \mathfrak{k}_n &= \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{M}(2(2n+1), \mathbb{R}) \middle| \begin{array}{c} A^{\mathrm{T}} = -A \\ \text{and } B^{\mathrm{T}} = B \end{array} \right\} \\ i\mathfrak{p}_n &= \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathrm{M}(2(2n+1), \mathbb{R}) \middle| \begin{array}{c} A^{\mathrm{T}} = -A \\ \text{and } B^{\mathrm{T}} = -B \end{array} \right\}. \end{aligned}$$

There is a $\widetilde{\theta}_n$-stable Cartan subalgebra

$$\widetilde{\mathfrak{h}}_n = \left\{ \left(\begin{array}{ccccccc|ccccccc} 0 & & & & & & & a_1 & & & & & & \\ & 0 & a_2 & & & & & & & & & & & \\ & -a_2 & 0 & & & & & & & & & & & \\ & & & \ddots & & & & & & & & & & \\ & & & & & 0 & a_{n+1} & & & & & & & \\ & & & & & -a_{n+1} & 0 & & & & & & & \\ \hline -a_1 & & & & & & & 0 & & & & & & \\ & & & & & & & & 0 & a_{n+2} & & & & \\ & & & & & & & & -a_{n+2} & 0 & & & & \\ & & & & & & & & & & \ddots & & & \\ & & & & & & & & & & & 0 & a_{2n+1} \\ & & & & & & & & & & & -a_{2n+1} & 0 \end{array} \right) \middle| \, a_i \in \mathbb{R} \right\}.$$

and we can choose

$$i\widetilde{\mathfrak{a}}_n = \left\{ \left(\begin{array}{cccccc|cccccc}
0 & & & & & & & & & & & \\
& 0 & a_1 & & & & & & & & & \\
& -a_1 & 0 & & & & & & & & & \\
& & & \ddots & & & & & & & & \\
& & & & 0 & a_n & & & & & & \\
& & & & -a_n & 0 & & & & & & \\ \hline
& & & & & & 0 & & & & & \\
& & & & & & & 0 & -a_1 & & & \\
& & & & & & & a_1 & 0 & & & \\
& & & & & & & & & \ddots & & \\
& & & & & & & & & & 0 & -a_n \\
& & & & & & & & & & a_n & 0
\end{array} \right) \middle| \; a_i \in \mathbb{R} \right\}.$$

Moving to the θ_n-picture, we conjugate everything by E_{σ_n} and renumber the indices to arrive at the θ_n-stable Cartan algebra

$$\mathfrak{h}_n = \left\{ \left(\begin{array}{ccccccccccc}
0 & a_1 & & & & & & & & & \\
-a_1 & 0 & & & & & & & & & \\
& & 0 & 0 & a_2 & 0 & & & & & \\
& & 0 & 0 & 0 & a_3 & & & & & \\
& & -a_2 & 0 & 0 & 0 & & & & & \\
& & 0 & -a_3 & 0 & 0 & & & & & \\
& & & & & & \ddots & & & & \\
& & & & & & & 0 & 0 & a_{2n-1} & 0 \\
& & & & & & & 0 & 0 & 0 & a_{2n} \\
& & & & & & & -a_{2n-1} & 0 & 0 & 0 \\
& & & & & & & 0 & -a_{2n} & 0 & 0
\end{array} \right) \middle| \; a_i \in \mathbb{R} \right\}$$

and finally

$$i\mathfrak{a}_n = \left\{ \left(\begin{array}{ccccccccccc}
0 & 0 & & & & & & & & & \\
0 & 0 & & & & & & & & & \\
& & 0 & 0 & a_1 & 0 & & & & & \\
& & 0 & 0 & 0 & -a_1 & & & & & \\
& & -a_1 & 0 & 0 & 0 & & & & & \\
& & 0 & a_1 & 0 & 0 & & & & & \\
& & & & & & \ddots & & & & \\
& & & & & & & 0 & 0 & a_n & 0 \\
& & & & & & & 0 & 0 & 0 & -a_n \\
& & & & & & & -a_n & 0 & 0 & 0 \\
& & & & & & & 0 & a_n & 0 & 0
\end{array} \right) \middle| \; a_i \in \mathbb{R} \right\}.$$

Hence $\mathfrak{a}_n$ is block-diagonal, and moving from $\mathfrak{a}_n$ to $\mathfrak{a}_{n+1}$ is simply a matter of adding another 4×4 block. Thus we see that condition (4.8) is satisfied and hence G_∞ / K_∞ is admissible.

Chapter 5

Representations of Direct-Limit Groups

In this chapter we review some important results about representations for direct-limit groups and lim-Riemannian symmetric spaces. See [45] for a quite comprehensive overview of representation theory for classical direct limits of symmetric spaces. See also [11] and [35] for many basic results on representations of direct-limit groups.

We begin this chapter by reviewing how one can construct representations of a direct-limit group by forming a direct limit of representations of finite-dimensional Lie groups. This construction provides the simplest way to construct unitary or even irreducible unitary representations for direct-limit groups.

Next we begin to tackle the issue of smoothness for representations of direct-limit groups. We review several useful results from the literature (especially from [33] and [11]) which provide equivalent conditions for smoothness.

Next we discuss a generalization of Weyl's unitary trick which identifies smooth representations of a lim-compact symmetric space U_∞/K_∞ with smooth representations of its c-dual G_∞/K_∞. This brings up the question of unitarizability of representations of the lim-compact group U_∞, which is unfortunately rather subtle.

Making things more concrete, we follow earlier constructions in [39],[54], and [56] to define highest-weight representations for U_∞. We end the chapter by recalling the main result of [7] on spherical representations.

5.1 Direct Limits of Representations

Suppose that $G_\infty = \varinjlim G_n$ is a direct-limit group with inclusion maps $p_n^{n+1} : G_n \to G_{n+1}$ and that for each $n \in \mathbb{N}$ we are given a continuous Hilbert representation $(\pi_n, \mathcal{H}_n)$ and partial isometries $j_n^{n+1} : \mathcal{H}_n \to \mathcal{H}_{n+1}$ such that the diagram

$$\begin{array}{ccc} G_{n+1} \times \mathcal{H}_{n+1} & \xrightarrow{\pi_{n+1}} & \mathcal{H}_{n+1} \\ {\scriptstyle p_n^{n+1} \times j_n^{n+1}} \uparrow & & \uparrow {\scriptstyle j_n^{n+1}} \\ G_n \times \mathcal{H}_n & \xrightarrow{\pi_n} & \mathcal{H}_n \end{array}$$

commutes (see Section 2 in [33]). A continuous map $\pi_\infty : G_\infty \times \mathcal{H}_\infty \to \mathcal{H}_\infty$ is induced, where $\mathcal{H}_\infty = \varinjlim \mathcal{H}_n$. It may be readily shown that π_∞ is in fact a continuous representation of G_∞ on $\mathcal{H}_\infty$.

Suppose further that for all $g \in G_\infty$, $\pi(g)$ is a bounded operator on $\mathcal{H}_\infty$ under the natural pre-Hilbert space structure on $\mathcal{H}_\infty$. Then π_∞ extends by continuity to a continuous representation on the Hilbert space completion $\overline{\mathcal{H}_\infty}$ (see, for instance, Proposition B.10 in [35]). One can also show that if π_n is unitary for each $n \in \mathbb{N}$, then π_∞ is a unitary representation of G_∞ on $\overline{\mathcal{H}_\infty}$.

For a more intuitive perspective on this situation, suppose that $\{G_n\}_{n\in\mathbb{N}}$ is an increasing sequence of Lie groups (i.e., G_n is a closed subgroup of G_m for $n \leq m$) and that for each n we are provided with a continuous Hilbert representation $(\pi_n, \mathcal{H}_n)$ such that $(\pi_n, \mathcal{H})$ is equivalent (by a unitary intertwining operator) to a subrepresentation of $(\pi_{n+1}|_{G_n}, \mathcal{H}_{n+1})$. Then one has a direct system of representations and may form a direct-limit representation $(\pi_\infty, \overline{\mathcal{H}_\infty})$ of G_∞.

One of the key tools in representation theory is the study of intertwining operators for representations. It is clear that an operator $T \in \mathcal{B}(\mathcal{H})$ is an intertwining operator for a Hibert representation $(\pi, \mathcal{H})$ of a direct-limit group $G_\infty = \varinjlim G_n$ if and only if it is an intertwining operator for $\pi|_{G_n}$ for each n. If π is a direct-limit representation, then we can say more:

Lemma 5.1. *([28]) If $(\pi, \mathcal{H}) = (\varinjlim \pi_n, \overline{\varinjlim \mathcal{H}_n})$ is a direct limit of Hilbert representations, then a bounded operator $T \in \mathcal{B}(\mathcal{H})$ is an intertwining operator for π if and only if $T|_{\mathcal{H}_n}$ is an intertwining operator for $\pi|_{G_n}$ for each $n \in \mathbb{N}$.*

Proof. One direction is obvious. To prove the other direction, we suppose that $T|_{\mathcal{H}_n}$ is an intertwining operator for $\pi|_{G_n}$ for each $n \in \mathbb{N}$. It is thus clear that $T\pi(g)v = \pi(g)Tv$ for any $g \in G_\infty$ and any v in the algebraic direct limit space $\mathcal{H}_\infty = \varinjlim \mathcal{H}_n$. The lemma follows since $\mathcal{H}_\infty$ is a dense subspace of $\mathcal{H}$ and since $\pi(g)$ is continuous for each $g \in G_\infty$. □

Direct-limit representations are the easiest representations to construct for G_∞. The following theorem shows that they may be in fact be used to construct a large class of irreducible unitary representations:

Theorem 5.2. *([28]) Suppose that $\{G_n\}_{n\in\mathbb{N}}$ is a direct system of locally compact groups and that $\{(\pi_n, \mathcal{H}_n)\}_{n\in\mathbb{N}}$ is a compatible direct system of irreducible unitary representations of G_n for each $n \in \mathbb{N}$. Then $(\pi, \mathcal{H}) \equiv (\varinjlim \pi_n, \overline{\varinjlim \mathcal{H}_n})$ is an irreducible unitary representation of G_∞.*

Proof. Suppose that $T \in \mathcal{B}(\mathcal{H})$ is an intertwining operator for π. Then $T|_{\mathcal{H}_n}$ is a G_n-intertwining operator for π_n. Since π_n is irreducible, it follows from Schur's Lemma that $T|_{\mathcal{H}_n} = c\,\mathrm{Id}$ for some constant $c \in \mathbb{C}$. Because $\mathcal{H}_n \subseteq \mathcal{H}_k$ for $n \leq k$, we see that the constant is independent of n. Thus, $T|_{\mathcal{H}_\infty} = c\,\mathrm{Id}$, where $\mathcal{H}_\infty = \varinjlim \mathcal{H}_n$ is the algebraic direct limit space. By continuity we then have that $T = c\,\mathrm{Id}$ since $\mathcal{H}_\infty$ is a dense subspace of $\mathcal{H}$. Because the intertwining operator $T \in \mathcal{B}(\mathcal{H})$ was arbitrary, it follows immediately that $\mathcal{H}$ is an irreducible representation. □

We caution the reader that there are many examples of irreducible representations of direct-limit groups which are not given by direct limits of irreducible representations (see [11, p. 971]).

5.2 Smoothness and Local Finiteness

Just as for finite-dimensional Lie groups, it is natural to try to gather information about a representation of a direct-limit group by differentiating it to obtain a representation of its Lie algebra. We begin by defining a notion of smoothness for representations of direct-limit groups[1].

Definition 5.3. *Suppose that $(\pi, \mathcal{H})$ is a continuous Hilbert representation of a direct-limit group $G_\infty = \varinjlim G_n$ and that $v \in \mathcal{H}$. We say that v is a* ***smooth vector*** *for π if it is a smooth vector for the restricted representation $(\pi|_{G_n}, \mathcal{H})$ of G_n for each $n \in \mathbb{N}$. We denote by $\mathcal{H}^\infty$ the space of all smooth vectors for π.*

Similarly, we say that v is a ***locally finite vector*** *for π if it is a G_n-finite vector for the restricted representation $(\pi|_{G_n}, \mathcal{H})$ of G_n for each $n \in \mathbb{N}$. We denote by $\mathcal{H}^{fin}$ the space of locally finite vectors for π. Note that $\mathcal{H}^{fin} \subseteq \mathcal{H}^\infty$.*

Given a Hilbert representation $(\pi, \mathcal{H})$ of G_∞, we may construct a representation of $\mathfrak{g}_\infty$ on $\mathcal{H}^\infty$ as follows. For each $n \in \mathbb{N}$, we have the differentiated representation $\mathrm{d}(\pi|_{G_n})$ of $\mathfrak{g}_n$ on $\mathcal{H}_n$ with

$$\mathrm{d}(\pi|_{G_n}) = \lim_{t \to 0} \frac{\pi|_{G_n}(\exp tX)v - v}{t}.$$

for each $X \in \mathfrak{g}_n$ and $v \in \mathcal{H}^\infty$. We see that

$$\begin{aligned} \mathrm{d}(\pi|_{G_{n+1}})(X)v &= \lim_{t \to 0} \frac{\pi|_{G_{n+1}}(\exp(tX))v - v}{t} \\ &= \lim_{t \to 0} \frac{\pi|_{G_n}(\exp tX)v - v}{t} \\ &= \mathrm{d}(\pi|_{G_n})(X)v, \end{aligned}$$

and thus there is a well-defined map $\mathrm{d}\pi(X) : \mathcal{H}^\infty \to \mathcal{H}^\infty$ for each $X \in \mathfrak{g}_\infty = \varinjlim \mathfrak{g}_n$, given by $\mathrm{d}\pi(X)v = \mathrm{d}(\pi|_{G_n})v$ for each $X \in \mathfrak{g}_n$. It is a straightforward argument to show that

$$\mathrm{d}\pi(X + Y)v = \pi(X)v + \pi(Y)v$$

and

$$\mathrm{d}\pi([X, Y])v = \pi(X)\pi(Y)v - \pi(X)\pi(Y)v$$

for all $v \in \mathcal{H}^\infty$ and $X, Y \in \mathfrak{g}_\infty$.

It is not at all clear from the definitions that a representation of G_∞ is guaranteed to possess any smooth vectors or locally-finite vectors. In fact, the existence of smooth vectors is far more subtle for representations of infinite-dimensional Lie groups than for finite-dimensional Lie groups, where every continuous representation on a Frechet space admits a dense subspace of smooth vectors. There are examples of unitary representations of Banach-Lie groups which do not possess any C^1 vectors, much less any smooth vectors (see [3]). For direct-limit groups, however, a beautiful theorem of Danilenko shows that unitary representations always admit smooth vectors.

[1] The question of how to put a smooth structure on direct limit groups such as G_∞ has been explored extensively in [15] and [33], where it is shown that under certain technical growth conditions on the G_n's, it is possible to put a smooth structure on G_∞ that is consistent with Definition 5.3:

Theorem 5.4. *([6]; see also [36, Theorem 11.3]) Suppose that $(\pi, \mathcal{H})$ is a unitary representation of a countable direct limit of locally compact topological groups. Then $\mathcal{H}^\infty$ is a dense subspace of $\mathcal{H}$.*

We may thus consider the space $\mathcal{H}^{-\infty} = (\mathcal{H}^\infty)'$ of **distribution vectors** for a unitary representation $(\pi, \mathcal{H})$ of G_∞ and obtain dense embeddings

$$\mathcal{H}^\infty \hookrightarrow \mathcal{H} \hookrightarrow, \mathcal{H}^{-\infty}$$

as we saw for representations of finite-dimensional Lie groups.

Some representations may consist entirely of smooth vectors:

Definition 5.5. *Suppose that G_∞ is a direct-limit Lie group. We say that a continuous Hilbert representation $(\pi, \mathcal{H})$ of G_∞ is **smooth** if $\mathcal{H}^\infty = \mathcal{H}$.*

*If G_∞ is a direct limit of complex Lie groups, then a continuous Hilbert representation $(\pi, \mathcal{H})$ of G_∞ is **holmorphic** if $\pi|_{G_n}$ is holomorphic for each $n \in \mathbb{N}$.*

In fact, we will be primarily concerned with smooth representations in this thesis. They play a role for direct-limit groups that is similar to the role played by finite-dimensional representations for finite-dimensional Lie groups. There are several conditions which are equivalent to smoothness:

Theorem 5.6. *Let $(\pi, \mathcal{H})$ be a continuous Hilbert representation of a Lie group G. Then the following are equivalent:*

1. *π is smooth*
2. *There is a Lie algebra representation $\mathrm{d}\pi : \mathfrak{g} \to \mathcal{B}(\mathcal{H})$ (for which $\mathfrak{g}$ acts by bounded operators) such that*

   $$\pi(\exp X) = \exp(\mathrm{d}\pi(X)) \tag{5.1}$$

   for each $X \in \mathfrak{g}$ (i.e., π is analytic).
3. *π is norm-continuous.*

Proof. First we prove (1) $\to$ (2). Suppose that π is smooth. Then $\mathcal{H}^\infty = \mathcal{H}$ and it follows that for each $X \in \mathfrak{g}$, we have a strongly-continuous one-parameter group $\{Q(t)\}_{t\in\mathbb{R}}$ of bounded operators on $\mathcal{H}$ given by

$$Q(t) = \pi(\exp tX).$$

Since $\mathcal{H}^\infty = \mathcal{H}$, we see that the limit

$$\mathrm{d}\pi(X)v = \lim_{t\to 0} \frac{\pi(\exp tX)v - v}{t}$$

exists in $\mathcal{H}$ for all $v \in \mathcal{H}$. Following the terminology of [51, p. 375], we have that the domain of $\mathrm{d}\pi(X)$ is all of $\mathcal{H}$ (i.e., $\mathcal{D}(\mathrm{d}\pi(X)) = \mathcal{H}$). By [51, Theorem 13.36], this implies that $\mathrm{d}\pi(X) \in B(\mathcal{H})$ and that

$$\pi(\exp tX) = \exp(t\mathrm{d}\pi(X))$$

for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. This establishes (5.1).

Next we demonstrate that (2) $\to$ (3). Suppose that (5.1) holds. Then

$$\begin{aligned}||\pi(\exp(X)) - \mathrm{Id}|| &= ||\exp(\mathrm{d}\pi(X)) - \mathrm{Id}|| \\ &= \left|\left|\sum_{n=1}^{\infty} \frac{\mathrm{d}\pi(X)^n}{n!}\right|\right| \\ &\leq \sum_{n=1}^{\infty} \frac{||\mathrm{d}\pi(X)||^n}{n!} \\ &= \exp(||\mathrm{d}\pi(X)||) - 1\end{aligned}$$

for all $X \in \mathfrak{g}$.

Let $X_1, \ldots, X_d$ be a basis for $\mathfrak{g}$, where $d = \dim \mathfrak{g}$, and set

$$M = \max_{1 \leq i \leq d} ||\mathrm{d}\pi(X_i)||.$$

It follows that

$$\left|\left|\mathrm{d}\pi\left(\sum_{i=1}^{d} c_i X_i\right)\right|\right| \leq \left(\sum_{i=1}^{d} c_i\right) M$$

whenever $c_i \in \mathbb{R}$ for $1 \leq i \leq d$. Hence, it follows that if $X = \sum_{i=1}^d c_i X_i$ with $\sum_{i=1}^d c_i < \epsilon$, then $||\pi(\exp(X)) - Id|| \leq \exp(\epsilon M) - 1$. Thus, we see that $X \mapsto \pi(\exp X)$ is norm-continuous. The result then follows that π is norm-continuous from the fact that $\exp : \mathfrak{g} \to G$ is a local diffeomorphism.

Finally, (3) $\to$ (1) is a straightforward application of [51, Theorem-13.36], which says that if $\lim_{t\to 0} ||\pi(\exp(tX)) - \mathrm{Id}|| = 0$ for all $X \in \mathfrak{g}$, then the infinitesimal generator is a bounded operator (that is, the differential exists everywhere). □

It is certainly possible to construct continuous unitary representations of direct-limit groups which possess no locally finite vectors. This behavior is already present for finite-dimensional Lie groups, however: an irreducible infinite-dimensional representation of a noncompact Lie group G does not possess any G-finite vectors. More surprisingly, it is possible to construct an irreducible unitary representation of a lim-compact group which has no locally finite vectors ([37]). However, Corollary **??** will show that smooth representations of connected lim-compact groups always consist entirely of locally finite vectors.

It is well known that every continuous, finite-dimensional representation of a Lie group is smooth. However, it is also possible to construct infinite-dimensional Hilbert representations which are smooth. Suppose that U is a compact Lie group and that (π, V) is a finite-dimensional representation of U. Without loss of generality, we may assume that π is unitary. Now consider the representation

$$(\infty \cdot \pi, \infty \cdot V) \equiv \left(\bigoplus_{n\in\mathbb{N}} \pi, \bigoplus_{n\in\mathbb{N}} V\right)$$

constructed by taking a Hilbert space direct sum of countably many copies of (π, V). For each $v \in \infty \cdot V$, we consider the closed invariant subspace

$$W = \overline{\langle(\infty \cdot \pi)(U)v\rangle}$$

generated by v. Then W gives a cyclic primary representation of U and decomposes into a direct sum of representations equivalent to (π, V). From [17] we see that every cyclic primary representation of the compact group U is finite-dimensional. Thus $\dim W < \infty$ and so v is a U-finite vector.

In fact, the next theorem shows that in a certain sense, primary representations (or more precisely, finite direct sums of them) provide the only way to obtain infinite-dimensional smooth representations of U:

Theorem 5.7. *Let $(\pi, \mathcal{H})$ be a unitary representation of a compact Lie group U. Then the following are equivalent.*

1. *π is smooth.*

2. *π decomposes into a finite direct sum of primary representations of U.*

3. *π is locally finite.*

Before we prove this theorem, we need to introduce the following useful lemma, which we will also make use of several times in the next chapter:

Lemma 5.8. *Let G be a topological group and let $(\pi, \mathcal{H})$ be a unitary representation of G. Let $\mathcal{A}$ be a finite or countably infinite index set, and suppose that*

$$v = \sum_{i \in \mathcal{A}} v_i,$$

where $v_i \in \mathcal{H}$ for each $i \in \mathcal{A}$ and where $\langle\pi(G)v_i\rangle$ and $\langle\pi(G)v_j\rangle$ give mutually distinct irreducible representations of G for $i \neq j$. Then

$$\langle\pi(G)v\rangle = \bigoplus_{i \in \mathcal{A}} \langle\pi(G)v_i\rangle.$$

Proof (of Lemma 5.8). Write $V = \langle\pi(G)v\rangle$. The fact that $V_i = \langle\pi(G)v_i\rangle$ and $V_j = \langle\pi(G)v_j\rangle$ give disjoint representations of G for $i \neq j$ implies that $V_i \perp V_j$. It is obvious that

$$\langle\pi(G)v\rangle \subseteq \bigoplus_{i \in \mathcal{A}} \langle\pi(G)v_i\rangle,$$

so we prove the opposite containment. It suffices to show that $v_i \in V$ for all $i \in \mathcal{A}$.

Suppose that $v_i \notin V$ for some $i \in \mathcal{A}$. Define

$$w = \sum_{j \neq i} v_j \text{ and } W = \langle\pi(G)w\rangle \subseteq \bigoplus_{j \neq i} V_j.$$

Then $V_i \perp W$ and $v = v_i + w$. Furthermore, V_i and W give disjoint representations of G.

Now let $c_1, \ldots c_k \in \mathbb{C}$ and $g_1, \ldots g_k \in G$. Then

$$\sum_{j=1}^{k} c_j \pi(g_j) v = \left(\sum_{j=1}^{k} c_j \pi(g_j) v_i \right) + \left(\sum_{j=1}^{k} c_j \pi(g_j) w \right).$$

Because $v_i \notin V$ and V_i is irreducible, we see that $V \cap V_i = \emptyset$. It follows that

$$\sum_{j=1}^{k} c_j \pi(g_j) v_i = 0 \text{ if and only if } \sum_{j=1}^{k} c_j \pi(g_j) w = 0.$$

Hence there is a well-defined, nonzero intertwining operator $L : V_i \to W$ such that $L(v_i) = w$, which contradicts the fact that V_i and W give disjoint representations of G. □

Proof (of Theorem 5.7). Let $(\pi, \mathcal{H})$ be a unitary representation of U. Then we can write

$$\mathcal{H} \cong_G \bigoplus_{\delta \in \widehat{G}} \mathcal{H}_\delta,$$

where $\mathcal{H}_\delta$ is the space of δ-isotypic vectors for each $\delta \in \widehat{G}$ (that is, vectors in $\mathcal{H}_\delta$ generate primary representations that are direct sums of copies of δ). Then π is a finite direct sum of primary representations if and only if $\mathcal{H}_\delta = \{0\}$ for all but finitely many $\delta \in \widehat{G}$.

We begin by showing that (2) $\Longrightarrow$ (3). Suppose that

$$\mathcal{H} \cong_U \bigoplus_{i=1}^{n} \mathcal{H}_{\delta_i},$$

where $\delta_i \in \widehat{U}$ for each i. We will show that π is smooth. For each $v \in \mathcal{H}$, we can write $v = v_1 + \cdots + v_n$, where $v_i \in \mathcal{H}_{\delta_i}$. Then

$$\langle \pi(U) v \rangle \subseteq \bigoplus_{i=1}^{n} \langle \pi(U) v_i \rangle.$$

However, because each space $\langle \pi(U) v_i \rangle$ gives a cyclic primary representation of U, we see that it is finite-dimensional (see [17]). Thus v is U-finite. Because $v \in \mathcal{H}$ was arbitrary, it follows that π is locally finite.

Next we prove (3) $\Longleftrightarrow$ (1); that is, π is smooth if and only if it is locally finite. If $\mathcal{H}^{\text{fin}} = \mathcal{H}$ then it is obvious that $\mathcal{H}^\infty = \mathcal{H}$. For the other direction, recall from the highest-weight theorem that irreducible representations of compact connected Lie groups are parametrized by a discrete semilattice $\Lambda^+(\mathfrak{g}, \mathfrak{h}) \subseteq i\mathfrak{h}^*$ of dominant integral weights, where $\mathfrak{h}$ denotes a Cartan subalgebra of $\mathfrak{g}$. Let $\mathcal{S}$ denote the set of all weights $\lambda \in \Lambda^+(\mathfrak{g}, \mathfrak{h})$ such that λ appears as the highest weight of a subrepresentation of π.

If $\lambda \in \mathcal{S}$, then $\lambda(X)$ is an eigenvalue of $d\pi(X)$ for each $X \in \mathfrak{h}$. But Theorem 5.6 implies that $||d\pi(X)|| < \infty$ for each $X \in \mathfrak{h}$. Thus, since $\Lambda^+(\mathfrak{g}, \mathfrak{h})$ is a semilattice, it follows that $\{\lambda(X) : \lambda \in \mathcal{S}\}$ is finite for each X. Hence $\mathcal{S}$ is finite because $\mathfrak{h}$ is finite-dimensional. Because π is a direct sum of finitely many primary representations, we see that π is locally finite by the above argument.

Finally, we show that (1) $\Longrightarrow$ (2); that is, if π is smooth, then it decomposes into a finite direct sum of primary representations. Suppose

$$\mathcal{H} \cong_U \bigoplus_{i=1}^{\infty} \mathcal{H}_{\delta_i},$$

where $\delta_i \in \widehat{U}$ and $\mathcal{H}_{\delta_i} \neq \{0\}$ for each i. We will show that π is not smooth.

For each $i \in \mathbb{N}$, choose a nonzero unit vector $v_i \in \mathcal{H}_{\delta_i}$ such that $\langle \pi(U)v_i \rangle$ is irreducible. Note that $v_i \perp v_j$ for $i \neq j$. Furthermore, $\langle \pi(U)v_j \rangle$ give primary representations of type δ_i and δ_j, respectively, and are therefore disjoint. Consider the vector

$$v \equiv \sum_{i=1}^{\infty} \frac{1}{2^i} v_i \in \bigoplus_{i=1}^{\infty} \mathcal{H}_{\delta_i}.$$

For each $i \in \mathbb{N}$, we define

$$w_j \equiv \sum_{i \neq j} \frac{1}{2^i} v_i \in \bigoplus_{i \neq j} \mathcal{H}_{\delta_i}.$$

It is clear that the representation of U on $\langle \pi(U)v_i \rangle$ is disjoint from the representation on $\langle \pi(U)w_i \rangle$. Since $v = v_i + w_i$, Lemma 5.8 implies that $v_i \in \langle \pi(U)v \rangle$. Because this is true for each $i \in \mathbb{N}$ and $v_i \perp v_j$ for $i \neq j$, it follows that $\langle \pi(U)v \rangle$ is infinite-dimensional. Therefore, π is not smooth, since we have already shown that a representation is smooth if and only if it is locally finite. □

Corollary 5.9. *Suppose that $(\pi, \mathcal{H})$ is a continuous Hilbert representation of a connected lim-compact group U_∞. Then $\mathcal{H}^\infty = \mathcal{H}$ if and only if $\mathcal{H}^{fin} = \mathcal{H}$.*

The following corollaries restate the conclusion of the previous theorem in terms of weights. A slightly different proof may be found in Lemma 3.5 and Proposition 3.6 of [35].

Corollary 5.10. *Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{u}$, and suppose that $(\pi, \mathcal{H})$ is a unitary representation of U. Then π is smooth if and only if $\#\Delta(\pi) < \infty$ (that is, π has only finitely many weights).*

Proof. Let $H \leq U$ be the maximal torus corresponding to $\mathfrak{h}$. If π is smooth, then in particular $\pi|_H$ is smooth and thus there are only finitely many equivalence classes of irreducible (i.e., one-dimensional) representations of H_n which appear in

$(\pi|_H, \mathcal{H})$. Thus $\mathcal{H}$ decomposes under $\mathrm{d}\pi|_{\mathfrak{h}}$ into finitely many weight spaces and we are done.

Now suppose that π is not smooth. By Theorem 5.7, there are infinitely many inequivalent equivalence classes of irreducible representations of G_n which appear in $(\pi, \mathcal{H})$. Because they are mutually inequivalent, these irreducible representations have mutually distinct highest weights and hence $\Delta(\pi)$ is an infinite set. □

Corollary 5.11. *Suppose that U_∞ is a lim-compact group. As before, we fix a subalgebra $\mathfrak{h}_\infty = \varinjlim \mathfrak{h}_n$ in $\mathfrak{u}_\infty$, where each $\mathfrak{h}_n$ is a Cartan subalgebra of $\mathfrak{u}_n$. Suppose that $(\pi, \mathcal{H})$ is a unitary representation of U. Then π is smooth if and only if $\#\Delta(\pi|_{U_n}) < \infty$ (that is, π has only finitely many weights) for each $n \in \mathbb{N}$.*

Proof. This result follows immediately from Corollary 5.10 and the definition of smoothness for direct-limit groups. □

Suppose now that U_∞ is a propagated lim-compact group. We recursively choose a countable orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ for $\mathfrak{h}_\infty$ as in Section 4.4. Consider the supremum norm of a weight $\lambda \in i\mathfrak{h}_n^*$, given by

$$||\lambda||_\infty = \max_{1\le i\le r_n} |\lambda(e_i)|$$

We then obtain the following useful theorem, which is a modification of Proposition 3.14 in [35].

Theorem 5.12. *A unitary representation $(\pi, \mathcal{H})$ of a propagated direct limit U_∞ of simply-connected compact semisimple Lie groups is smooth if and only if there is $M > 0$ such that for all n one has $||\lambda||_\infty < M$ for each weight $\lambda \in i\mathfrak{h}_n^*$ that appears as the highest weight for an irreducible subrepresentation of $\pi|_{U_n}$.*

Proof. First we prove the theorem in the case that U_∞ is a direct limit of compact simple Lie groups.

Let $(\pi, \mathcal{H})$ be a unitary representation of U_∞. Suppose there is $M \in \mathbb{N}$ such that for all n one has $||\lambda||_\infty < M$ for each weight $\lambda \in i\mathfrak{h}_n^*$ that appears as the highest weight for an irreducible subrepresentation of $\pi|_{U_n}$. If $\lambda \in i\mathfrak{h}^*$ is a highest weight which appears in $\pi|_{U_n}$, then it has the form

$$\lambda = \sum_{i=1}^{r_n} a_i e_i, \text{ where } a_i \in \mathbb{Z} \text{ and } -M \le a_i \le M.$$

Thus, there are only $(2M)^{r_n}$ possible values for λ. In other words, $\pi|_{U_n}$ may be written as a direct sum of finitely many primary representations and is thus smooth by Theorem 5.7. Because $n \in \mathbb{N}$ was arbitrary, we have that π is smooth.

To prove the other direction, suppose that for each $M > 0$ there is $n \in \mathbb{N}$ and a highest weight $\lambda \in i\mathfrak{h}_n^*$ of an irreducible subrepresentation of $\pi|_{U_n}$ such that $||\lambda||_\infty > M$. Fix $M > 0$ and pick $n \in \mathbb{N}$ and $\lambda \in i\mathfrak{h}_n^*$ satisfying those conditions. Then $\lambda = \sum_{i=1}^{r_n} c_i e_i$, where $c_i \in \mathbb{Z}$ for each i. Because $||\lambda||_\infty > M$, we see that there is some index j such that $|c_j| > M$.

From the details in Section 4.4, there is a Weyl group element $w \in W(\mathfrak{g}_n, \mathfrak{a}_n)$ such that $w(e_1) = e_i$ and $w(e_i) = e_1$. Then $|w\lambda(e_1)| = |c_j| > M$. By the Highest-Weight Theorem, we see that $w\lambda \in \Delta(\pi|_{U_n})$; that is, $w\lambda$ is a $\mathfrak{h}_n$-weight for $\pi|_{U_n}$. It is then clear that $(w\lambda)|_{\mathfrak{h}_k}$ is an $\mathfrak{h}_k$-weight for $\pi|_{U_k}$ whenever $k \leq n$ (since every $w\lambda$-weight vector in $\mathcal{H}$ is automatically a $(w\lambda)_{\mathfrak{h}_k}$-weight vector). Furthermore, since $|(w\lambda|_{\mathfrak{t}_n})(e_1)| = |c_j| > M$, we see that $||w\lambda|_{\mathfrak{t}_n}||_\infty > M$.

Thus, if $k \in \mathbb{N}$ is fixed, then for each $M \in \mathbb{N}$ there is a weight $\lambda \in \Delta(\pi|_{U_k})$ such that $||\lambda|| > M$. Hence $\Delta(\pi|_{U_k})$ is not a finite set and thus by Corollary 5.11 it follows that π is not smooth.

Suppose more generally that U_∞ is a propagated direct limit of semisimple Lie groups. Then we can write $U_k = U_k^1 \times U_k^2 \times \cdots \times U_k^d$ for all $k \in \mathbb{N}$ in such a way that $\{U_n^i\}_{n\in\mathbb{N}}$ is a propagated direct system of compact simple Lie groups for each $1 \leq i \leq d$. We can then recursively choose Cartan subalgebras $\mathfrak{h}_n = \mathfrak{h}_n^i \oplus \mathfrak{h}_n^2 \oplus \cdots \oplus \mathfrak{h}_n^d$, where $\mathfrak{h}_n^i$ is a Cartan subalgebra of $\mathfrak{u}_n^i$ for each i and n. A weight in $\lambda \in i\mathfrak{h}_n^*$ is dominant integral if and only if $\lambda|_{\mathfrak{h}_n^i}$ is dominant integral for each $1 \leq i \leq d$. Since U_∞^i is a propagated direct limit of compact simple Lie groups, it follows that there is $M_i > 0$ such that for all $n \in \mathbb{N}$ one has that $||\lambda||_\infty < M_i$ for each highest weight $\lambda \in \mathfrak{h}_n^*$ appearing in $\pi|_{U_n^i}$. Since $\max_{1\leq i\leq d} M_i < \infty$, we are done. □

We end the section with the following remarkable result, which implies that the smoothness of a representation of a direct limit of simple groups is controlled by the smoothness of the restriction to any nontrivial one-dimensional analytic subgroup.

Theorem 5.13. *Let U be a compact simple Lie group. Then a unitary representation $(\pi, \mathcal{H})$ of U is smooth if and only if there is $X \in \mathfrak{u}\backslash\{0\}$ such that $\mathrm{d}\pi(X)$ is a bounded operator on $\mathcal{H}$.*

Proof. One direction is obvious. To show the other direction, suppose that $(\pi, \mathcal{H})$ is a non-smooth unitary representation of U. We will show that $\mathrm{d}\pi(X)$ has an unbounded spectrum. Let $\mathfrak{h}$ be any Cartan subalgebra for U.

Because π is not smooth, it follows that there is for each $M > 0$ weight $\lambda \in \Delta(\pi)$ with $||\lambda||_\infty > M$. As in the proof of Theorem 5.12, we see that for each Weyl-group element $w \in W(\mathfrak{u}, \mathfrak{h})$, the weight $w\lambda$ is in $\Delta(\pi)$. If we write $\lambda = \sum_{i=1}^r a_i e_i$, then there is some j such that $|a_j| > M$. We can use the Weyl group to permute the basis elements so that a_j appears as the i^{th} coefficient of a weight in $\Delta(\pi|_U)$. Thus we have that the set

$$\{\langle \lambda, e_i\rangle | \lambda \in \Delta(\pi)\}$$

of i^{th} coefficients of weights of π is unbounded for all $i \leq r$.

In other words, one has for each $n \in \mathbb{N}$ that the set of weights in $\Delta(\pi)$ is unbounded in every direction on $\mathfrak{h}$. It follows that $\mathrm{d}\pi(X)$ has an unbounded spectrum for all $X \in \mathfrak{h}$. Because every element of $\mathfrak{u}$ is contained in some Cartan subalgebra, the result follows. □

Corollary 5.14. *Let U_∞ be a direct limit of compact simple Lie groups. Then a unitary representation $(\pi, \mathcal{H})$ of U_∞ is smooth if and only if there is $X \in \mathfrak{u}\backslash\{0\}$ such that $\mathrm{d}\pi(X)$ is a bounded operator on $\mathcal{H}$.*

Proof. This corollary follows immediately by applying Lemma 5.13 to U_n for each n in $\mathbb{N}$. $\square$

Note that this result is false for non-simple compact groups: suppose that J and T are compact Lie groups, that $(\pi, \mathcal{H})$ is a smooth unitary representation of J, and that $(\sigma, \mathcal{K})$ is a non-smooth unitary representation of T. Then the outer tensor product representation $(\pi \boxtimes \sigma, \mathcal{H} \otimes \mathcal{K})$ of $J \times T$ has the property that $\pi|_J$ is smooth but $\pi|_T$ is non-smooth.

5.3 Generalizing Weyl's Unitary Trick

Weyl's Unitary Trick plays a crucial role in understanding finite-dimensional representations of finite-dimensional Lie groups. There is a natural extension of Weyl's Unitary Trick to smooth representations of direct-limit groups. The first step is to extend Weyl's unitary trick to smooth representations of finite-dimensional groups. We begin with a well-known lemma on intertwining operators of smooth representations.

Lemma 5.15. *Suppose that $(\pi, \mathcal{H})$ is a smooth Hilbert representation of a Lie group G. Then the derived representation $\mathrm{d}\pi : \mathfrak{g} \to \mathcal{B}(\mathcal{H})$ possesses the same algebra of intertwining operators as π.*

Proof. Suppose that T is an intertwining operator for $\mathrm{d}\pi$. That is, $\mathrm{d}\pi(X)T = T\mathrm{d}\pi(X)$ for all $X \in \mathfrak{g}$. It immediately follows that $\pi(\exp(X))T = T\pi(\exp(X))$ for all $X \in \mathfrak{g}$ and thus T is an intertwining operator for π by Theorem 5.6. Next suppose that T is an intertwining operator for π. Then $T\pi(\exp tX) = \pi(\exp(tX))T$ for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. It follows by differentiation at $t = 0$ that $T\mathrm{d}\pi(X) = \mathrm{d}\pi(X)T$ for all $X \in \mathfrak{g}$. $\square$

Now we are ready to extend Weyl's Trick to smooth representations of finite-dimensional groups.

Theorem 5.16. *Suppose that U is a compact Lie group and that G is a (not necessarily compact) closed subgroup of $U_{\mathbb{C}}$ such that $U_{\mathbb{C}}$ is a complexification of G. There are one-to-one correspondences between the following categories of representations on $\mathcal{H}$ which preserve the algebras of intertwining operators:*

1. *Locally-finite representations of G on $\mathcal{H}$*
2. *Holomorphic representations of $U_{\mathbb{C}}$ on $\mathcal{H}$*
3. *Smooth representations of U on $\mathcal{H}$*

Proof. We will construct the correspondences (1) $\to$ (2) and (2) $\to$ (1). The proofs for (2) $\to$ (3) and (3) $\to$ (2) are identical.

One passes from (2) to (1) quite easily: if $(\pi, \mathcal{H})$ is a holomorphic representation of $U_{\mathbb{C}}$, then it is clear that $\pi|_G$ is a smooth representation of G.

To construct (1) $\to$ (2), we suppose that $(\pi, \mathcal{H})$ is a smooth representation of G. We wish to construct a holomorphic representation $\pi_{\mathbb{C}}$ of $U_{\mathbb{C}}$ on $\mathcal{H}$ such that $\pi_{\mathbb{C}}|_G =$

π. First we notice that each vector $v \in \mathcal{H}$ is contained in a finite-dimensional G-invariant subspace W. Write π^W for the subrepresentation of π corresponding to W. By the finite-dimensional Weyl Trick, we see that π^W uniquely extends to a holomorphic representation $\pi_{\mathbb{C}}^W$ of $U_{\mathbb{C}}$ on W. We define $\pi_{\mathbb{C}}(g)v = \pi_{\mathbb{C}}^W(g)v$ for each $v \in W$ and $g \in U_{\mathbb{C}}$. If V and W are finite-dimensional invariant subspaces of $\mathcal{H}$ and $v \in V \cap W$, then the uniqueness of the holomorphic extension shows that $\pi_{\mathbb{C}}^W(g)v = \pi_{\mathbb{C}}^V(g)v$ and thus $\pi_{\mathbb{C}}$ is well-defined.

It is clear that $\pi_{\mathbb{C}}$ is a vector space representation of $U_{\mathbb{C}}$ on $\mathcal{H}$, but we must still show that $\pi_{\mathbb{C}}$ acts by bounded operators and that it acts holomorphically. Since π is smooth, Theorem 5.6 implies the existence of a Lie algebra representation $\mathrm{d}\pi : \mathfrak{g} \to \mathcal{B}(\mathcal{H})$ such that

$$\pi(\exp X) = \exp(\mathrm{d}\pi(X))$$

for all $X \in \mathcal{B}(\mathcal{H})$. Notice that $\mathrm{d}\pi$ uniquely extends to a complex-linear Lie algebra representation $\mathrm{d}\pi_{\mathbb{C}} : \mathfrak{u}_{\mathbb{C}} \to \mathcal{B}(\mathcal{H})$ by setting

$$\mathrm{d}\pi_{\mathbb{C}}(X + iY) = \mathrm{d}\pi(X) + i\mathrm{d}\pi(Y)$$

for all $X, Y \in \mathfrak{g}$.

By restricting to finite-dimensional invariant subspaces of $\mathcal{H}$ and applying the finite-dimensional Unitary Trick, we verify that

$$\pi_{\mathbb{C}}(\exp X)v = \exp(\mathrm{d}\pi_{\mathbb{C}}(X))v \tag{5.2}$$

for all $X \in \mathfrak{u}_{\mathbb{C}}$ and $v \in \mathcal{H}$. In particular, we see that $\pi_{\mathbb{C}}(g) \in \mathcal{B}(\mathcal{H})$ for all $g \in U_{\mathbb{C}}$ and also that $\pi_{\mathbb{C}}$ is smooth.

Next, we note that π^C gives a holomorphic representation on W for every finite-dimensional U-invariant subspace of $\mathcal{H}$. Since every vector in $\mathcal{H}$ is contained in such a finite-dimensional invariant subspace, we see that the map

$$\begin{aligned} U_{\mathbb{C}} &\mapsto \mathcal{H} \\ g &\mapsto \pi_{\mathbb{C}}(g)v \end{aligned}$$

is holomorphic for each $v \in V$. Thus $\pi_{\mathbb{C}}$ is holomorphic.

It is clear that the real Lie algebra representation $\mathrm{d}\pi$ and the complex Lie algebra representation $\mathrm{d}\pi_{\mathbb{C}}$ possess the same algebra of intertwining operators. Thus π and $\pi_{\mathbb{C}}$ possess the same algebra of intertwining operators by Lemma 5.15. Furthermore, the uniqueness of the complexification $\pi_{\mathbb{C}}$ follows from its uniqueness on every finite-dimensional invariant subspace of $\mathcal{H}$.

□

Our infinite-dimensional version of Weyl's Trick is then an immediate corollary (see [35, Proposition 3.6] for a partial version of this result and a different proof):

Corollary 5.17. *Suppose that G_∞/K_∞ is a lim-noncompact Riemannian symmetric spaces which is the c-dual of a lim-compact symmetric space U_∞/K_∞ where U_n/K_n and U_n are simply-connected for each n. Finally, let $\mathcal{H}$ be a Hilbert space. There are one-to-one correspondences between the following categories of representations on $\mathcal{H}$ which preserve the algebras of intertwining operators:*

1. *Locally-finite representations of G_∞ on $\mathcal{H}$*
2. *Holomorphic representations of $(U_\infty)_\mathbb{C}$ on $\mathcal{H}$*
3. *Smooth representations of U_∞ on $\mathcal{H}$*

Proof. This corollary follows immediately by applying Theorem 5.16 to representations of G_n, $(U_n)_\mathbb{C}$, and U_n on $\mathcal{H}$ for each $n \in \mathbb{N}$. □

There is one crucial aspect of the finite-dimensional version of Weyl's Unitary Trick which we have as yet failed to mention: every smooth (i.e., norm-continuous) Hilbert representation of a compact Lie group is unitarizable. This key property is what gives Weyl's Trick much of its power, since it allows us to treat finite-dimensional representations of noncompact semisimple Lie groups as if they were unitary. We take a moment, therefore, to explore what can be said about unitarizability of representations of U_∞.

The first thing we note is that the representation $(\pi|_{U_n}, \mathcal{H})$ may be unitarized for each $n \in \mathbb{N}$, because U_n is a compact group. Furthermore, a unitarization of $\pi|_{U_n}$ automatically unitarizes the restrictions $\pi|_{U_j}$ for $j \leq n$. However, it is not clear *a priori* whether or not it is possible to simultaneously unitarize $\pi|_{U_n}$ for all $n \in \mathbb{N}$, which is what would be required in order to unitarize π.

Recall that the trick we used to show that representations of compact groups are unitarizable was to integrate an inner product over the group using Haar measure. While U_∞ is not locally compact, and thus does not possess a Haar measure, one can show that it possesses the next-best thing:

Theorem 5.18. *([49, Proposition 13.6]). Let $\mathrm{UCB}(U_\infty)$ denote the Banach space of uniformly-continuous, bounded functions on G, then there is a continuous functional $\mu \in \mathrm{UCB}(U_\infty)^*$ such that*

1. *$\mu(\mathbf{1}) = \mathbf{1}$, where **1** is the constant-one function*
2. *$\mu(f) \geq 0$ whenever $f \geq 0$*
3. *$|\mu(f)| \leq ||f||_\infty$ for all $f \in \mathrm{UCB}(U_\infty)$*
4. *$\mu(R_g f) = \mu(L_g f) = \mu(f)$ for all $g \in U_\infty$ and $f \in \mathrm{UCB}(U_\infty)$*

*We say that μ is an **invariant mean** for U_∞.*

Proof. For each $\in \mathbb{N}$, we define a functional $\mu_n \in \mathrm{UCB}(U_\infty)^*$ by

$$\mu_n(f) = \int_{U_n} f|_{U_n}(g)dg$$

for each $f \in \mathrm{UCB}(\mathrm{U}_\infty)$. It is clear that each μ_n satisfies the first three conditions of an invariant mean. Furthermore, we see that $\mu_n(R_g f) = \mu_n(L_g f) = f$ whenever $g \in U_n \leq U_\infty$. Thus, any weak-$*$ cluster point of the set $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathrm{UCB}(U_\infty)^*$ will possess property (4). But by the Banach Alaoglu theorem, the unit ball in $\mathrm{UCB}(U_\infty)^*$ is weak-$*$ compact and thus our sequence must possess a cluster point (property (3) shows that the sequence is contained in the unit ball). □

Because $\mathrm{UCB}(U_\infty)$ is not separable, the unit ball in $\mathrm{UCB}(U_\infty)^*$ is not guaranteed to be weak-$*$ sequentially compact. Thus there is no reason to expect that $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathrm{UCB}(U_\infty)^*$ will possess a convergent sequence. In fact, an application of the Axiom of Choice is required to construct an invariant mean on U_∞. There are also an uncountable number of distinct invariant means on U_∞, so we are far from the uniqueness properties of Haar measures.

Invariant means in some ways behave as finitely-additive invariant integrals on U_∞. For that reason, we often us the notation

$$\mu(f) = \int_{U_\infty} f(g)d\mu(g),$$

although we must be careful to note that μ is not in any sense a countably-additive measure on U_∞.

Nevertheless, once a group G possesses an invariant mean, it is possible to use the "integration" trick to show that all uniformly bounded representations of G are unitarizable:

Theorem 5.19. *([49, Proposition 17.5]). Suppose that G is an amenable group and that π is a uniformly bounded continuous representation of G on a separable Hilbert space $\mathcal{H}$ (that is, $\sup_{g\in U_\infty} ||\pi(g)|| < \infty$). Then π is equivalent to a unitary representation.*

Proof. Let $M = \sup_{g\in U_\infty} ||\pi(g)||$. Clearly, $M = \sup_{g\in \mathrm{U}_\infty} ||\pi(g)^{-1}||$; it follows that

$$M^{-1}||u|| \leq ||\pi(g)u|| \leq M||u||$$

for all $g \in U_\infty$.

Now let μ be a bi-invariant mean on G. We denote the inner product on $\mathcal{H}$ by $\langle , \rangle_{\mathcal{H}}$ and define a new inner product $\langle , \rangle_\mu$ on $\mathcal{H}$ by

$$\langle u, v\rangle_\mu = \int_G \langle \pi(g)u, \pi(g)v\rangle_{\mathcal{H}} d\mu(g)$$

for all $u, v \in \mathcal{H}$. We use the fact that $g \mapsto \langle \pi(g)u, \pi(g)v\rangle_{\mathcal{H}}$ is a uniformly continuous, bounded function on G (since π is continuous and uniformly bounded). It is clear that $\langle , \rangle_\mu$ provides a positive semi-definite Hermitian form on $\mathcal{H}$.

Note that for $u \in \mathcal{H}$ one has that

$$0 < M^{-2}||u||^2_{\mathcal{H}} \leq ||u||^2_{\mu} = \int_G ||\pi(g)u||^2_{\mathcal{H}} d\mu(g) < M^2||u||^2_{\mathcal{H}}.$$

Thus $\langle , \rangle_\mu$ is strictly positive-definite and continuous with respect to $\langle , \rangle_{\mathcal{H}}$. □

Taking stock again of our situation, we see that all uniformly-bounded Hilbert representations of U_∞ are unitarizable. Furthermore, if a continuous Hilbert representation $(\pi, \mathcal{H})$ is unitarizable, then π is uniformly bounded. In fact, if an invertible bounded intertwining operator $T \in \mathrm{GL}(\mathcal{H})$ unitarizes π, then we see that $T\pi(g)T^{-1}$ is unitary and thus $||\pi(g)|| < ||T||||T^{-1}||$ for all $g \in U_\infty$.

Unfortunately, it is not possible to say much more, because it is possible to construct a smooth Hilbert representation of U_∞ which is not unitarizable, as we now show.

Consider the group $U_\infty = \mathrm{SU}(\infty) = \varinjlim \mathrm{SU}(2n)$. For each $n \in \mathbb{N}$, consider the standard representation π_n of $\mathrm{SU}(2n)$ on $\mathcal{H}_n = \mathbb{C}^{2n}$ (that is, $\pi_n(g)v = g \cdot v$ for all $g \in \mathrm{SU}(2n)$). By taking the direct limit, we may form a unitary representation $\pi = \varinjlim \pi_n$ of $\mathrm{SU}(\infty)$ on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{C}) = \overline{\varinjlim \mathbb{C}^{2n}}$ of square-summable sequences of complex numbers. Note that $\mathrm{SU}(2n)$ acts trivially on the orthogonal complement of $\mathcal{H}_n$. It follows that $\pi|_{\mathrm{SU}(2n)}$ decomposes into a direct sum of the standard representation π_n and infinitely many copies of the trivial irreducible representation. That is,

$$\pi|_{\mathrm{SU}(2n)} = \pi_n \oplus \infty \cdot \mathrm{Id}_{\mathrm{SU}(2n)},$$

where $\mathrm{Id}_{\mathrm{SU}(2n)}$ denotes the trivial irreducible representation of $\mathrm{SU}(2n)$ on $\mathbb{C}$. Thus, by Theorem 5.7, it follows that $\pi|_{\mathrm{SU}(2n)}$ is smooth for each $n \in \mathbb{N}$ and hence that π is smooth.

Now let $V_1 = \mathcal{H}_1$ and define $V_n = \mathcal{H}_n \ominus \mathcal{H}_{n-1}$ for each $n > 1$. Note that $\dim V_n = 2$ for each $n \in \mathbb{N}$. We now completely discard unitarity and choose some new inner product $\langle , \rangle_{V_n}$ on V_n under which $||\pi(g)|_{V_n}|| \geq n$ for some $g \in \mathrm{SU}(2n)$. For instance, if $\pi(g)v = w$, where $v, w \in V_n$ are linearly independent, then we can choose any inner product $\langle , \rangle_{V_n}$ on V_n such that $||v||_{V_n} = 1$ and $||w||_{V_n} = n$.

Next we define for each $n \in \mathbb{N}$ the finite-dimensional Hilbert space

$$\mathcal{K}_n = \bigoplus_{i=1}^{n} V_i,$$

where each V_i is given the new inner product we just defined. As vector spaces, $\mathcal{K}_n = \mathcal{H}_n$, but they possess different inner products. Now $\{(\pi_n, \mathcal{K}_n)\}_{n \in \mathbb{N}}$ forms a direct system of continuous Hilbert representations. We consider the representation $(\widetilde{\pi}_\infty, \mathcal{K}_\infty) = (\varinjlim \pi_n, \overline{\varinjlim \mathcal{K}_n})$. Note that $\pi|_{\mathrm{SU}(2n)}$ and $\widetilde{\pi}|_{\mathrm{SU}(2n)}$ possess the same irreducible subrepresentations for each $n \in \mathbb{N}$. In particular, $\widetilde{\pi}$ is smooth. Finally, it is clear that $\widetilde{\pi}$ is not uniformly bounded (since $\sup_{g \in \mathrm{SU}(2n)} ||\pi(g)|| \geq n$ for each $n \in \mathbb{N}$), and is therefore not unitarizable.

Heuristically, it seems that the smooth Hilbert representations of U_∞ which are not unitarizable have in some sense been given an unnatural or "incorrect" topology. For that reason, we will for the rest of the thesis work only with unitary representations of U_∞ and with smooth representations of G_∞ which correspond to smooth unitary representations of U_∞ under Weyl's Trick.

5.4 Highest-Weight Representations

Now suppose that G_∞/K_∞ is an admissible lim-noncompact symmetric space which is the c-dual of a lim-compact symmetric space U_∞/K_∞. We wish to construct irreducible spherical and conical representations for G_∞/K_∞ and U_∞/K_∞. The most natural way to do this would be to construct a direct limit of spherical/conical representations. The following lemma provides the foundation for this construction and is a generalization of a result proved by Ólafsson and Wolf in Lemma 5.8 of [42].

Theorem 5.20. *Let U_∞/K_∞ be a propagated lim-compact symmetric space such that U_n/K_n is simply connected for each $n \in \mathbb{N}$. Fix indices $n < m$ and dominant weights $\lambda \in \Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$ and $\mu \in \Lambda^+(\mathfrak{g}_m, \mathfrak{a}_m)$ such that $\mu|_{\mathfrak{a}_n} = \lambda$. Consider the irreducible spherical representations $(\pi_\mu, \mathcal{H}_\mu)$ and $(\pi_\lambda, \mathcal{H}_\lambda)$ of U_m and G_n, respectively, with respective highest weights μ and λ. Let w be a highest-weight vector for π_μ.*

Then the representation of U_n on $W = \langle \pi_\mu(U_n)w \rangle$ is equivalent to π_λ.

Proof. For each dominant weight ν in $\Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$, let w_ν be the orthogonal projection of w onto the space of π_ν-isotypic vectors in W. Then $w = \sum_\nu w_\nu$ (note that $w_\nu = 0$ for all but finitely many choices of ν).

Write $W_\nu = \langle \pi_\mu(U_n)w_\nu \rangle$ for each ν. Because W_ν consists of U_n-isotypic vectors of type ν, we see that the action of U_n on W_ν is U_n-isomorphic to a direct sum of copies of the irreducible representation $(\pi_\nu, \mathcal{H}_\nu)$ with highest-weight ν.

Since w is a U_m-highest-weight vector for π_μ, $\pi(M_mN_m)w = w$. In particular, $\pi(M_nN_n)w = w$. Since the space of isotypic vectors in W of type π_ν is invariant under G_n, it follows that w_ν is fixed under M_nN_n for each $\nu \in \Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$. Thus Lemma 3.35 shows that if $w_\nu \neq 0$, then W_ν is a U_n-irreducible subspace of W that is U_n-isomorphic to $\mathcal{H}_\nu$ and that w_ν is a highest-weight vector for W_ν. In particular, w_ν is a weight vector of weight ν.

On the other hand, since w is a U_m-weight vector of weight μ, it follows that it is a U_n-weight vector of weight $\lambda = \mu|_{\mathfrak{a}_n}$. But we also have that $w = \sum_\nu w_\nu$, where each w_ν is a weight vector of weight ν. Hence $w = w_\lambda$ and $W = W_\lambda$, and so we are done. □

We follow the construction in [54, p. 464–466], and more details may be found at that source. For each n, we denote the set of fundamental weights by $\xi_{n,1}, \dots \xi_{n,r_n}$, where $r_n = \dim \mathfrak{a}_n$ and where we have numbered the fundamental weights according to the roots as in Section 4.4. Suppose $k \leq n$. One can show that

$$\xi_{n,i}|_{\mathfrak{a}_k} = \xi_{k,i} \tag{5.3}$$

for all $n \in \mathbb{N}$ and $i \leq r_k$. Furthermore, one can check that $\xi_{n,i}|_{\mathfrak{a}_k} = 0$ for $r_k < i \leq r_n$. Thus

$$\Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n) = \mathbb{N}\xi_{n,1} + \cdots + \mathbb{N}\xi_{n,r_n} = \left\{ \sum_{j=1}^{r_n} c_j \xi_{n,j} \;\middle|\; c_j \in \mathbb{N} \right\} \tag{5.4}$$

and

$$\left(\sum_{j=1}^{r_n} c_j \xi_{n,j} \right) \Bigg|_{\mathfrak{a}_k} = \left(\sum_{j=1}^{r_k} c_j \xi_{k,j} \right) \in \Lambda^+(\mathfrak{g}_k, \mathfrak{a}_k) \tag{5.5}$$

whenever $k \leq n$.

We can thus form a projective limit

$$\Lambda^+ \equiv \Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty) = \varprojlim \Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n).$$

We say that $\Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ is the set of **dominant integral weights** for the restricted root system $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$. That is, Λ^+ consists of the elements λ of $\mathfrak{a}_\infty^* = \varprojlim \mathfrak{a}_n^*$ such that $\lambda|_{\mathfrak{a}_n}$ is dominant and integral for every n. Notice that (5.3) implies that for each $i \in \mathbb{N}$ there is a weight $\xi_i \in \mathfrak{a}_\infty^*$ such that $\xi_i|_{\mathfrak{a}_n} = \xi_{n,i}$ for each $n \in \mathbb{N}$.

If $\dim \mathfrak{a}_\infty = \infty$, then (5.4) and (5.4) imply that $\Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ is equal to the set of formal sums $\sum_{i \in \mathbb{N}} c_i \xi_i$ where $(c_i) \in \mathbb{N}$ is any sequence in $\mathbb{N}$. On the other hand, if $\mathfrak{a}_\infty$ is finite-dimensional, say with dimension r, then $\Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ is equal to the set of sums $\sum_{i=1}^{r} c_i \xi_i$ where $c_1, \ldots, c_r \in \mathbb{N}$.

Just as in the finite-dimensional case, weights in Λ^+ can be used to create highest-weight representations of U_∞. To see this, fix $\mu \in \Lambda^+$. For n in $\mathbb{N}$, let $(\pi_{\mu_n}, \mathcal{H}_{\mu_n})$ be the irreducible representation of U_n with highest weight $\mu_n \equiv \mu|_{\mathfrak{a}_n}$, and let $v_n \in \mathcal{H}_{\mu_n}$ be a nonzero highest-weight vector. By Theorem 5.20, we see that π_{μ_n} may be embedded unitarily into $\pi_{\mu_{n+1}}$ by identifying the respective highest-weight vectors v_n with v_{n+1}. The corresponding unitary representation of U_∞ constructed by the direct limit of π_{μ_n}, $n \in \mathbb{N}$ is denoted by

$$(\pi_\mu, \mathcal{H}_\mu) = \left(\varinjlim \pi_{\mu_n}, \overline{\varinjlim \mathcal{H}_{\mu_n}} \right),$$

where $\mathcal{H}_\mu = \overline{\varinjlim \mathcal{H}_{\mu_n}}$ is the Hilbert completion of the algebraic direct limit $\varinjlim \mathcal{H}_{\mu_n}$ of Hilbert spaces. We refer to π_μ as the **highest-weight representation with highest weight** μ. Note that a direct limit of irreducible representations of U_n is an irreducible representation of U_∞ by 5.2.

If $\dim \mathfrak{a}_\infty = \infty$, then we can write elements of $\mathfrak{a}^*$ as sequences $(a_i) \in \mathbb{Z}$ of integers, so that a sequence $(a_i) \in \mathbb{Z}$ corresponds to the formal sum $\sum_{i \in \mathbb{N}} a_i e_i \in \mathfrak{a}_\infty^*$. We now use this notation to write down the fundamental weights for $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ for some infinite Dynkin-diagram types.

If $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ has type A_∞, then

$$\xi_i = (0, \ldots, 0, 2, 2, 2, \ldots)$$

where the first i entries in ξ_i are zeros.

If $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ has type B_∞, then

$$\xi_1 = (1, 1, 1, \ldots) \text{ and } \xi_i = (0, \ldots, 0, 2, 2, 2, \ldots) \text{ for } i > 1,$$

where the first $i - 1$ entries in ξ_i are zero for $i > 1$.

If $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ has type C_∞, then

$$\xi_i = (0, \ldots, 0, 2, 2, 2, \ldots),$$

where the first $i - 1$ entries in ξ_i are zero.

If $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ has type D_∞, then

$$\xi_1 = (1, 1, 1, \ldots), \xi_2 = (-1, 1, 1, \ldots) \text{ and } \xi_i = (0, \ldots, 0, 2, 2, 2, \ldots) \text{ for } i \geq 3,$$

where the first $i - 1$ entries in ξ_i are zero for $i \geq 3$.

By examining the fundamental weights in each case and extending them to weights on $\mathfrak{h}_\infty$, it follows from the boundedness condition in Theorem 5.12 that a highest-weight representation $(\pi_\mu, \mathcal{H}_\mu)$ for $\lambda \in \Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ will be smooth if and only if we can write λ as a finite linear combination

$$\lambda = \sum_{i=1}^{n} c_i \xi_i,$$

where $c_i \in \mathbb{N}$ for each n. In particular, if $\dim \mathfrak{a}_\infty < \infty$, then every highest-weight representation $(\pi_\mu, \mathcal{H}_\mu)$ for $\lambda \in \Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ is smooth.

5.5 Spherical Representations for Lim-Compact Symmetric Spaces

In preparation for our study of conical representations, we end this chapter by reviewing the main result of our earlier paper [7], which concerned spherical representations for propagated lim-compact symmetric spaces.

Suppose that U_∞/K_∞ is a lim-compact symmetric space (as usual, we assume that U_n/K_n is simply-connected for each $n \in \mathbb{N}$ for the sake of clarity). The definitions of spherical representations and spherical functions are entirely analogous to the definitions for finite-dimensional symmetric spaces.

Definition 5.21. *A continuous unitary representation $(\pi, \mathcal{H})$ of U_∞ is said to be (K_∞-)**spherical** if there is a nonzero cyclic vector $v \in \mathcal{H}$ such that $\pi(K_n)v = v$ for each $n \in \mathbb{N}$.*

Definition 5.22. *(See [13]) A continuous, bi-K_∞-invariant function $\phi : U_\infty \to \mathbb{C}$ is said to be a **spherical function** if*

$$\phi(x)\phi(y) = \lim_{n\to\infty} \int_{K_n} \phi(xky)dk$$

for all $x, y \in U_\infty$.

It is natural to ask whether one may form an irreducible K_∞-spherical representation of U_∞ merely by taking a direct limit of irreducible unitary spherical representations of the K_n's. The most appealing candidates would be the unitary highest-weight representations constructed in the previous section. In [7] we showed that this scheme only works for certain symmetric spaces:

Theorem 5.23. *([7, Theorem 4.5]) Let $\mu \in \Lambda^+(\mathfrak{u}_\infty, \mathfrak{k}_\infty)$ and consider the corresponding unitary highest-weight representation $(\pi_\mu, \mathcal{H}_\mu)$ of U_∞. (Recall that π_μ was constructed as a direct limit of spherical representations.) Then π_μ is a spherical representation if and only if*

$$\text{Rank } U_\infty/K_\infty = \dim \mathfrak{a}_\infty < \infty,$$

that is, if U_∞/K_∞ is a symmetric space with a finite rank.

In the case that U_∞/K_∞ has finite rank, the function $\phi_\mu : U_\infty \to \mathbb{C}$ defined by

$$\phi_\mu(g) = \langle e, \pi(g)e \rangle,$$

where $e \in \mathcal{H}_\mu^K$ is a unit vector, is a positive-definite spherical function.

As a side note, the only classical finite-rank lim-compact symmetric spaces are the finite-rank Grassmannian spaces $\mathrm{SO}(p+\infty)/\mathrm{SO}(p) \times \mathrm{SO}(\infty)$, $\mathrm{SU}(p+\infty)/S(\mathrm{SU}(p) \times \mathrm{SU}(\infty))$, and $\mathrm{Sp}(p+\infty)/\mathrm{Sp}(p) \times \mathrm{Sp}(\infty)$, which correspond to the space of p-dimensional subspaces of $\mathbb{R}^\infty$, $\mathbb{C}^\infty$, and $\mathbb{H}^\infty$, respectively. The other classical lim-compact symmetric spaces in Table 4.4 all have infinite rank.

Theorem 5.23 demonstrates that there is a striking difference in behavior between finite-rank lim-Riemannian symmetric spaces and infinite-rank lim-Riemannian symmetric spaces, and we shall note this divergence of behavior again in the next chapter.

Finally, we note that for the case of a finite-rank lim-compact symmetric space U_∞/K_∞, the classification of spherical functions in [50] implies that the highest-weight representations $(\pi_\mu, \mathcal{H}_\mu)$ with highest-weight $\mu \in \Lambda^+(\mathfrak{g}_\infty, \mathfrak{k}_\infty)$ exhaust all irreducible spherical representations of U_∞.

Chapter 6

Conical Representations for Admissible Direct Limits

This chapter contains the main results of the thesis. In the first section, we give a natural definition for conical representations of admissible lim-noncompact symmetric spaces G_∞/K_∞. As before, we assume that G_∞/K_∞ is the c-dual of a propagated lim-compact symmetric space U_∞/K_∞. By using the generalization of Weyl's Unitary Trick from the previous chapter, each smooth cyclic representation of U_∞ gives rise to a smooth cyclic representation of G_∞, and it is natural to say that a smooth cyclic representation of U_∞ is conical if the corresponding representation of G_∞ is conical.

In fact, we will see that in some cases it is possible to define nonsmooth unitary representations of U_∞ which are conical but do not correspond to continuous Hilbert representations of G_∞. This is a strange situation which does not occur in the finite-dimensional case.

With these definitions, we classify all of the irreducible cyclic unitary representations of U_∞ which are conical. Next we see that smooth conical unitary representations of U_∞ decompose into a discrete direct sum of highest-weight representations.

Combining our results with Theorem 5.23, we will show that, if Rank $U_\infty/K_\infty = \infty$, then there are no smooth unitary representations of U_∞ which are both spherical and conical. On the other hand, if Rank $U_\infty/K_\infty < \infty$, then we will see that a smooth irreducible unitary representation of U_∞ is spherical if and only if it is conical. This situation is also in stark contrast to the situation for finite-dimensional symmetric spaces, for which finite-dimensional representations are spherical if and only if they are conical.

In the final section, we show how to disintegrate (possibly nonsmooth) conical representations into direct integrals of irreducible representations by integrating over a set of paths in a tree of highest weights. We also show that cyclic conical representations are always multiplicity-free representations (and hence are Type I representations).

6.1 Definition of Conical Representations

We begin by presenting our definition of conical representations for lim-Riemannian symmetric spaces. Let G_∞/K_∞ be the c-dual of a propagated lim-compact symmetric space U_∞/K_∞ such that U_n/K_n and U_n are simply-connected for each n and assume that G_∞/K_∞ is admissible.

For finite-dimensional symmetric spaces, it is possible to consider a finite-dimensional conical representation to be a representation of either G or U (where G/K is the c-dual of the compact symmetric space U/K). On the one hand, many harmonic analysis applications of conical representations appear on the horocycle space G/MN, so in a certain sense it is most natural to speak of conical representations of G. On the other hand, these representations are only unitary if we move to the compact group U.

Similarly, because unitarity is crucially important in the arguments which follow, we will mainly consider unitary conical representations of U_∞. However, it is important to remember that holomorphic representations of U_∞ correspond to holomorphic representations of G_∞ under Theorem 5.16, and vice versa.

We are now ready to present the definition:

Definition 6.1. *A unitary representation $(\pi, \mathcal{H})$ of U_∞ is* **conical** *if there is a nonzero cyclic vector $v \in \mathcal{H}^{\text{fin}}$ such that $\pi(M_n N_n)v = v$ for all $n \in \mathbb{N}$. In that case, we say that v is a* **conical vector** *for π.*

Notice that we do not require that conical representations of U_∞ be smooth. This opens the door to the possibility of constructing conical representations of U_∞ which do not correspond to representations of G_∞ under the generalized unitary trick, and indeed we will construct many examples of such representations in Section 6.5.

6.2 Classification of Conical Representations

In this section we begin to classify the unitary conical representations of U_∞. We determine which representations are irreducible and show how conical representations decompose into subrepresentations.

Theorem 6.2. *Suppose that U_∞/K_∞ is a propagated lim-compact symmetric space with U_n and U_n/K_n simply-connected for each n and such that the c-dual G_∞/K_∞ is admissible. Suppose further that $(\pi, \mathcal{H})$ is a conical representation with a conical vector v. For each n, write $\Gamma_n(\pi, v)$ for the set of highest weights μ in $\Lambda^+(\mathfrak{u}_n, \mathfrak{a}_n)$ such that the projection $v_\mu = \operatorname{pr}_{\mathcal{H}_\mu} v$ of v onto the space of U_n-isotypic vectors of type μ is nonzero. Then*

1. *For each $n \in \mathbb{N}$ and $\mu \in \Gamma_n(\pi, v)$, the action of U_∞ on $\overline{\langle \pi(U_\infty)v_\mu \rangle}$ gives a conical representation of U_∞ with conical vector v_μ.*

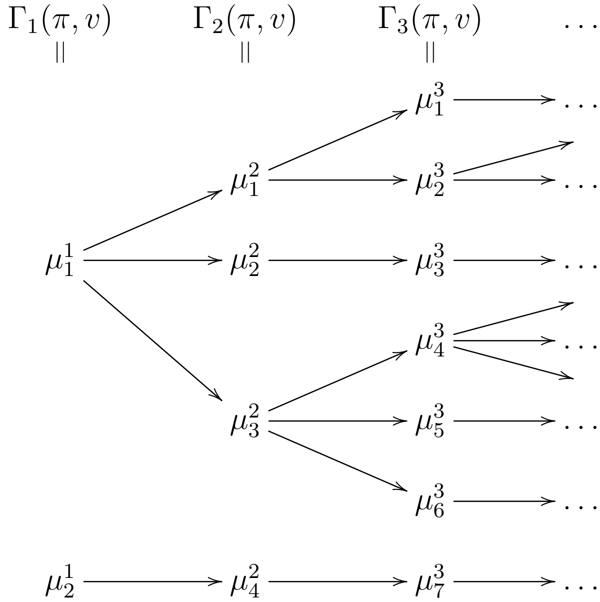
2. *π decomposes into an orthogonal direct sum of disjoint conical representations as follows:*
$$\mathcal{H} = \overline{\langle \pi(U_\infty)v \rangle} = \bigoplus_{\mu \in \Gamma_n(\pi, v)} \overline{\langle \pi(U_\infty)v_\mu \rangle}$$

3. *If π is irreducible, then π is equivalent to a highest-weight representation π_μ for some $\mu \in \Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$.*

4. *If π is irreducible, then $\dim \mathcal{H}^{M_\infty N_\infty} = 1$.*

Proof. For each $n \in \mathbb{N}$, the set $\Gamma_n(\pi, v)$ is finite because v is U_n-finite for all n. Then the decomposition of v into U_n-isotypic vectors may be written
$$v = \sum_{\mu \in \Gamma_n(\pi, v)} v_\mu,$$

FIGURE 6.1. Example of a highest-weight tree



where $v_\mu = \mathrm{pr}_{\mathcal{H}_\mu} v$. Since each isotypic subspace is U_n-invariant, it follows that $v_\mu \in \mathcal{H}^{M_n N_n}$ for each $\mu \in \Gamma_n(\pi, v)$. Note that $\langle \pi(U_n) v_\mu \rangle$ gives a primary representation of U_n of type μ. Hence, by Lemma 3.35, it is an irreducible representation with highest-weight vector v_μ.

We repeat the same process for U_{n+1}, writing the decomposition of v into U_{n+1}-isotypic vectors as

$$v = \sum_{\lambda \in \Gamma_{n+1}(\pi, v)} v_\lambda \tag{6.1}$$

By Theorem 5.20 it follows for each $\lambda \in \Gamma_{n+1}(\pi, v)$ that $\langle \pi(U_n) v_\lambda \rangle$ is a U_n-irreducible subspace for which v_λ is a highest-weight vector of weight $\lambda|_{\mathfrak{h}_n}$. In other words, v_λ is also a U_n-isotypic vector, so $\lambda|_{\mathfrak{h}_n} \in \Gamma_n(\pi, v)$. Furthermore, since (6.1) is a decomposition of v into U_n- and U_{n+1}-isotypic vectors, we see that for each $\mu \in \Gamma_n(\pi, v)$ there is $\lambda \in \Gamma_{n+1}(\pi, v)$ such that $\lambda|_{\mathfrak{h}_n} = \mu$.

In other words, if we consider all the highest weights of irreducible subrepresentations $\pi(U_n)$ and allow $n \in \mathbb{N}$ to vary, then the highest weights may be naturally arranged into a tree, as in Figure 6.1.

Next we prove (1). First note that $V_\lambda = \langle \pi(U_\infty) v_\lambda \rangle$ is a U_∞-invariant subspace of $\mathcal{H}$ for each $\lambda \in \Gamma_n(\pi, v)$. Suppose $m > n$, and write

$$u_\lambda = \sum_{\nu \in \Gamma_m(\pi, v) \text{ s.t. } \nu|_{\mathfrak{a}_n} = \lambda} v_\nu$$

for each $\lambda \in \Gamma_n(\pi, v)$. Then u_λ is a U_n-isotypic vector of type λ. Because $v = \sum_{\nu \in \Gamma_m(\pi, v)} v_\nu$, we see that $v = \sum_{\lambda \in \Gamma_n(\pi, v)} u_\lambda$ since every U_m-highest-weight vector v_ν appears as a summand in exactly one u_λ. Since $v = \sum_{\lambda \in \Gamma_n(\pi, v)} v_\lambda$ is also a decomposition of v into U_n-isotypic vectors, it follows that $v_\lambda = u_\lambda$ for each $\lambda \in$

$\underline{\Gamma_n(\pi, v)}$. In particular, v_λ is $M_m N_m$-invariant for all $m \geq n$. It follows that $V_\lambda = \underline{\langle \pi(U_\infty) v_\lambda \rangle}$ gives a conical representation of U_∞, proving (1).

To prove (2), we need to show that $V_{\mu_1} \perp V_{\mu_2}$ for all $\mu_1 \neq \mu_2$ in $\Gamma_n(\pi, v)$. It is sufficient to show that $V^m_{\mu_1} = \langle \pi(U_m) v_{\mu_1} \rangle$ and $V^m_{\mu_2} = \langle \pi(U_m) v_{\mu_2} \rangle$ are orthogonal for all m. We apply Lemma 5.8 to see that

$$\langle \pi(U_m) v_\lambda \rangle = \bigoplus_{\nu \in \Gamma_m(\pi,v) \text{ s.t } \nu|_{\mathfrak{a}_n} = \lambda} \langle \pi(U_m) v_\nu \rangle.$$

It follows that $\langle \pi(U_m) v_{\mu_1} \rangle$ and $\langle \pi(U_m) v_{\mu_2} \rangle$ are orthogonal for all m and hence that $V = \bigcup_m \langle \pi(U_m) v_{\mu_1} \rangle$ and $W = \bigcup_m \langle \pi(U_m) v_{\mu_2} \rangle$ are orthogonal G-invariant subspaces of $\mathcal{H}$, proving (2). Figure 6.2 demonstrates how the decomposition of U_m-representations matches the tree structure of the highest weights that was exhibited in Figure 6.1.

To prove (3), we assume that π is irreducible. Suppose that there is n such that $\#\Gamma_n(\pi, v) > 1$ (that is, there is more than one U_m-highest weight in $\pi|_{U_m}$). Then (2) produces orthogonal, nonzero invariant subspaces of $\mathcal{H}$, which contradicts the assumption that π is irreducible. Hence $\#\Gamma_n(\pi, v) = 1$ for all m.

For each n, let μ_n refer to the single element of $\Gamma_n(\pi, v)$. From this it follows that v is a U_m-highest-weight vector of weight μ_m for each m with the property that $\mu_m|_{\mathfrak{a}_n} = \mu_n$ for $m \geq n$. Furthermore, $V_n = \langle \pi(U_n) v \rangle$ is a U_n-irreducible subspace of $\mathcal{H}$ for each n, and we can write $\pi = \varinjlim \pi_n$, where π_n is the representation of U_n on V_n induced by π. Thus π is a highest-weight representation and (3) is proved.

To prove that $\dim \mathcal{H}^{M_\infty N_\infty} = 1$, suppose that v and w are nonzero conical vectors for π such that $v \perp w$. Write $V_n = \langle \pi(U_n) v \rangle$ and $W_n = \langle \pi(U_n) w \rangle$ for each n. We see that V_n and W_n are both equivalent to π_{μ_n} and have v and w as respective highest-weight vectors. By Lemma 3.35, it follows that $V_n \perp W_n$ for each n. Hence v and w generate nonzero, orthogonal invariant subspaces of $\mathcal{H}$, contradicting the irreducibility of π.

□

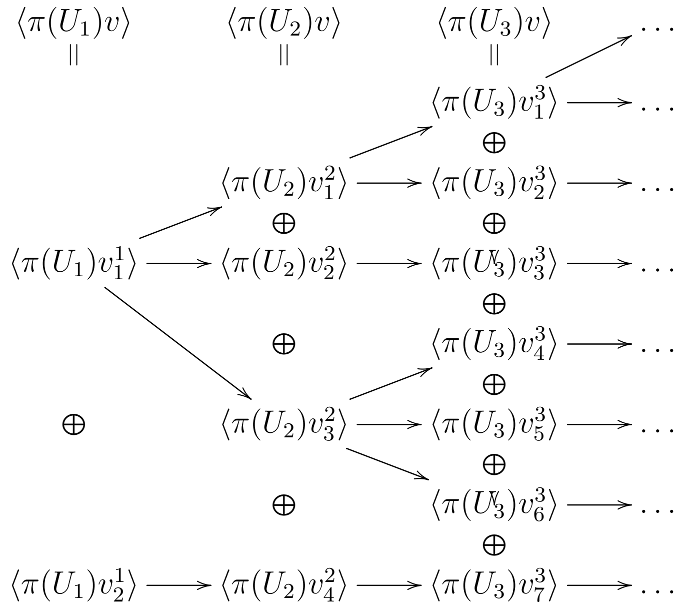
Notice that the maps $p_n^{n+1} : \Gamma_{n+1}(\pi, v) \to \Gamma_n(\pi, v)$ defined by $p_n(\lambda) = \lambda|_{\mathfrak{a}_n}$ define a projective system. We refer to the set $\Gamma(\pi, v) = \varprojlim \Gamma_n(\pi, v) \subseteq \Lambda^+(\mathfrak{u}_\infty, \mathfrak{a}_\infty)$ as the **highest-weight support** of π. If we arrange the highest weights in a tree as in Figure 6.1, then we see that elements of $\Gamma(\pi, v)$ correspond to infinite paths.

We now examine the connection between conical and spherical representations of G. Recall that for a finite-dimensional Riemannian symmetric space the irreducible finite-dimensional conical and spherical representations are identical. The situation is much different for infinite-dimensional symmetric spaces, as the following corollary shows.

Corollary 6.3. *If* $\mathrm{Rank}(U_\infty/K_\infty) < \infty$*, then a unitary irreducible representation is spherical if and only if it is conical. If* $\mathrm{Rank}(U_\infty/K_\infty) = \infty$*, then no unitary irreducible representation is both spherical and conical.*

Proof. By part (3) of Theorem 6.2, we see that the irreducible conical representations are precisely the highest-weight representations of U_∞ with highest weight

FIGURE 6.2. Example of a decomposition of $\langle \pi(U_n)v \rangle$ into U_n-isotypic subspaces (direct sums are taken vertically)



$\mu \in \Lambda^+(U_\infty, K_\infty)$. By Theorem 5.23, it follows that these highest-weight representations of U_∞ are spherical if and only if $\text{Rank}(U_\infty/K_\infty) < \infty$. Furthermore, if $\text{Rank}(U_\infty/K_\infty) < \infty$, then the spherical representations of U_∞ are exhausted by the irreducible highest-weight representations. $\square$

6.3 Highest-Weight Supports of Conical Representations

In this section we explore some of the properties of the highest-weight trees associated with conical representations. These trees form an invariant for conical representations, but as we shall see it is possible for two distinct conical representations to possess the same highest-weight tree.

First we show that the tree set of a conical representation is independent of the choice of conical vector:

Theorem 6.4. *Let $(\pi, \mathcal{H})$ be a unitary conical representation of U_∞. Then $\Gamma_n(\pi, v) = \Gamma_n(\pi, w)$ for any conical vectors v, w in $\mathcal{H}$.*

Proof. Suppose that both v and w are conical vectors in $\mathcal{H}$ and that $\mu \in \Gamma_n(\pi, w)$ but $\mu \notin \Gamma_n(\pi, v)$. Write w_μ for the projection of w onto the μ-isotypic vectors in $\mathcal{H}$. Since $\mu \in \Gamma_n(\pi, w)$, it follows that $w_\mu \neq 0$. Define $W = \langle \pi(U_\infty)w_\mu \rangle$ and $V = \langle \pi(U_\infty)v \rangle$. We claim that $W \perp V$, which will be a contradiction since V is dense in $\mathcal{H}$.

Note that $W = \bigcup_{m \geq n} \langle \pi(U_m)w_\mu \rangle$ and $V = \bigcup_{m \geq n} \langle \pi(U_m)v \rangle$. It is sufficient to show that $\langle \pi(U_m)w_\mu \rangle \perp \langle \pi(U_m)v \rangle$ for $m \geq n$. As before, we see from Lemma 3.35

and Theorem 5.20 that

$$\langle \pi(U_m)v\rangle = \bigoplus_{\lambda\in\Gamma_m(\pi,v)} \langle \pi(U_m)v_\lambda\rangle \cong_{U_m} \bigoplus_{\lambda\in\Gamma_m(\pi,v)} \mathcal{H}_\lambda$$

and

$$\langle \pi(U_m)w_\mu\rangle = \bigoplus_{\nu\in\Gamma_m^\mu(\pi,w)} \langle \pi(U_m)w_\nu\rangle \cong_{U_m} \bigoplus_{\nu\in\Gamma_m(\pi,w)} \mathcal{H}_\nu,$$

where $\Gamma_m^\mu(\pi,w) = \{\nu \in \Gamma_m(\pi,w) \text{ s.t. } \nu|_{\mathfrak{a}_n} = \mu\}$.

Fix $m \geq n$. Since $\mu \notin \Gamma_n(\pi,v)$, it follows that $\lambda|_{\mathfrak{a}_n} \neq \mu$ for all $\lambda \in \Gamma(\pi,v)$. Thus $\Gamma_m(\pi,v)$ and $\Gamma_m^\mu(\pi,w)$ are disjoint. This means that $\langle \pi(U_m)v_\lambda\rangle \perp \langle \pi(U_m)w_\nu\rangle$ for each $\lambda \in \Gamma_m(\pi,v)$ and $\nu \in \Gamma_m^\mu(\pi,w)$. Hence $\langle \pi(U_m)v\rangle \perp \langle \pi(U_m)w_\mu\rangle$ for all m, as we wanted to show. □

From now on, we write $\Gamma_n(\pi) \equiv \Gamma_n(\pi,v)$ and $\Gamma(\pi) = \varprojlim \Gamma_n(\pi,v)$, where v is any conical vector of a conical representation π of U_∞.

Corollary 6.5. *Let $(\pi,\mathcal{H})$ and $(\rho,\mathcal{K})$ be unitary conical representations of (U_∞, K_∞). If there is $n \in \mathbb{N}$ such that $\Gamma_n(\pi) \neq \Gamma_n(\rho)$, then $\pi \ncong \rho$.*

In particular, we have shown that having the same highest-weight tree is a necessary condition for two conical representations to be equivalent. Later we will provide examples of inequivalent conical representations with the same highest-weight trees. However, two conical representations with the same highest-weight trees are nonetheless *almost* equivalent in a certain sense, as the following theorem shows.

Theorem 6.6. *Let $(\pi,\mathcal{H})$ and $(\rho,\mathcal{K})$ be conical representations of (U_∞, K_∞) with respective conical vectors v and w such that $\Gamma_n(\pi) = \Gamma_n(\rho)$ for each n. Consider $V = \langle \pi(U_\infty)v\rangle$ and $W = \langle \rho(U_\infty)w\rangle$. Write π_V and ρ_W for the representations of U_∞ given by restricting π and ρ to the dense invariant subspaces V and W of $\mathcal{H}$ and $\mathcal{K}$, respectively. Then*

1. $\pi_V \cong \rho_W$
2. $\pi|_{U_n} \cong \rho|_{U_n}$ *for each n.*

Proof. We begin by proving (1). We claim that the map $L : V \to W$ induced by $\pi(g)v \mapsto \rho(g)w$ is a well-defined invertible U_∞-intertwining operator.

As before, write $V_m = \langle \pi(U_m)v\rangle$ and $W_m = \langle \pi(U_m)w\rangle$, so that $V = \bigcup_{m\geq n} V_m$ and $W = \bigcup_{m\geq n} W_m$. Then

$$V_m = \bigoplus_{\lambda\in\Gamma_m} \langle \pi(U_m)v_\lambda\rangle \cong_{U_m} \bigoplus_{\lambda\in\Gamma_m} \mathcal{H}_\lambda$$

and

$$W_m = \bigoplus_{\lambda\in\Gamma_m} \langle \rho(U_m)w_\lambda\rangle \cong_{U_m} \bigoplus_{\lambda\in\Gamma_m} \mathcal{H}_\lambda,$$

where $\Gamma_m = \Gamma_m(\pi) = \Gamma_m(\rho)$. Thus V_m and W_m are U_m-isomorphic. We must show that there is an invertible U_m-intertwining operator $L^m : V_m \to W_m$ that maps v to w.

In fact, we note that for each $\lambda \in \Gamma_m$ there is a (not necessarily unitary) U_m-intertwining operator $L_\lambda : \langle \pi(U_m)v_\lambda \rangle \to \langle \rho(U_m)w_\lambda \rangle$ given by $\pi(g)v_\lambda \mapsto \rho(g)w_\lambda$. We can then define

$$L^m = \bigoplus_{\lambda \in \Gamma_m} L_\lambda : V_m = \bigoplus_{\lambda \in \Gamma_m} \langle \pi(U_m)v_\lambda \rangle \to \bigoplus_{\lambda \in \Gamma_m} \langle \rho(U_m)w_\lambda \rangle = W_m.$$

Hence $L^m v = L^m(\sum_{\lambda \in \Gamma_m} v_\lambda) = \sum_{\lambda \in \Gamma_m} w_\lambda = w$.

Since v and w are cyclic vectors in V_m and W_m, respectively, L^m is in fact uniquely determined as an intertwining operator by the fact that it maps v to w. In particular, $L^m | V_n = L^n$ for all $n \le m$. Thus the family $\{L^m\}_{m \in \mathbb{N}}$ is a direct system of intertwining operators that induces a continuous U_∞-intertwining operator

$$L : V = \varinjlim V_m \to \varinjlim W_m = W$$

such that $Lv = w$.

Next we prove (2). Fix $n \in \mathbb{N}$. Define $\widetilde{V}_n = V_n$ and $\widetilde{V}_m = V_m \ominus V_{m-1}$ for $m > n$, where the orthogonal complement is taken with respect to the Hilbert space structure inherited by V_n as a closed subspace of $\mathcal{H}$. Notice that $\widetilde{V}_m$ is a finite-dimensional U_n-invariant subspace of $\mathcal{H}$ for each $m \ge n$. We define U_n-invariant spaces $\widetilde{W}_m \subseteq \mathcal{K}$ for each $m \ge n$ in exactly the same way.

Recall that V_m and W_m give equivalent representations of U_n for each $m \ge n$ under the intertwining operator L^m. It follows that $\widetilde{V}_m = V_m \ominus V_{m-1}$ and $\widetilde{W}_m = W_m \ominus W_{m-1}$ are U_n-isomorphic for all $m > n$. Note that

$$\mathcal{H} = \bigoplus_{m \ge n} \widetilde{V}_m \text{ and } \mathcal{K} = \bigoplus_{m \ge n} \widetilde{W}_m,$$

where the direct sums are orthogonal. Since there is a unitary U_n intertwining operator between $\widetilde{V}_m$ and $\widetilde{W}_m$ for all $m \ge n$, it follows that there is a unitary U_n-intertwining operator between $\mathcal{H}$ and $\mathcal{K}$. □

6.4 Smooth Conical Representations

Next we consider smooth conical representations of U_∞. These are of interest because they are precisely the conical representations which extend to smooth conical representations of the c-dual G_∞. Our next theorem classifies the smooth representations.

Theorem 6.7. *Suppose that $(\pi, \mathcal{H})$ is a smooth conical representation of U_∞. Then π decomposes into a direct sum of irreducible smooth highest-weight representations.*

Proof. Let v be a conical vector for π. For each U_n, write

$$v = \sum_{\lambda \in \Gamma_n(\pi)} v_\lambda$$

as before. As in Section 4.4, we recursively construct a countable basis $\{e_i\}_{n \in \mathbb{N}}$ for $\mathfrak{a}_\infty$ such that $\{e_1, \ldots, e_{r_n}\}$ is a basis for $\mathfrak{a}_n$ for each n. For each $\lambda \in \mathfrak{a}_n^*$, write

$$||\lambda||_\infty = \max_{1 \leq i \leq n} |\lambda(e_i)|.$$

In fact, if $\lambda \in \Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$ and $\lambda = \sum_{i=1}^{r_n} a_i e_i$, then we see from the data in Section 4.4 that $a_i \leq a_j$ when $i \leq j$; thus $||\lambda||_\infty = a_{r_n}$.

For each $\mu \in \Gamma_n(\pi)$, let $\Gamma_{n+1}^\mu(\pi) = \{\lambda \in \Gamma_{n+1}(\pi) : \lambda|_{\mathfrak{a}_n} = \mu\}$. Hence we have $||\lambda||_\infty \geq ||\mu||_\infty$ for each $\lambda \in \Gamma_{n+1}^\mu,$.

Now suppose that $\mu \in \Gamma_n(\pi)$ and that there are distinct weights $\lambda_1, \lambda_2 \in \Gamma_{n+1}^\mu(\pi)$. In this case we say that μ *splits* with respect to π. Because λ_1 and λ_2 in $\Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$ are by assumption distinct and agree on the first r_n coordinates, we see that they must differ on a coordinate i with $r_n < i \leq r_{n+1}$. Since the coefficients of dominant weights form an increasing sequence, we see that either $||\lambda_1||_\infty > ||\lambda_2||_\infty \geq ||\mu||_\infty$ or $||\lambda_2||_\infty > ||\lambda_1||_\infty \geq ||\mu||_\infty$

In other words, if a highest weight $\mu \in \Gamma_n(\pi)$ splits, then there is a U_{n+1}-highest weight in $\Gamma_{n+1}^\mu(\pi)$ with a coefficient which is strictly greater than all the coefficients in μ. It follows that unless there is a weight $\mu_n \in \Gamma_n(\pi)$ for some n which does not split and such that each $\lambda \in \Gamma_m^\mu(\pi)$ for any $m \geq n$ does not split, then we can repeat this process to obtain arbitrarily large coefficients of highest weights of representations appearing in π, contradicting Lemma 5.12. Hence, there is some highest weight $\mu \in \Gamma_n(\pi)$ such that, for each $m \geq n$, the vector v_μ is a U_m-highest-weight vector. Thus $\overline{\langle \pi(U_\infty) v_\mu \rangle}$ gives a highest-weight representation of U_∞.

Furthermore, we see that

$$v - v_\mu = \sum_{\lambda \in \Gamma_n(\pi) \setminus \mu} v_\lambda$$

generates a conical representation by Theorem 6.2 and that

$$\mathcal{H} = \overline{\langle \pi(U_\infty) v_\mu \rangle} \oplus \overline{\langle \pi(U_\infty)(v - v_\mu) \rangle}.$$

We have shown that every smooth unitary conical representation possesses an irreducible subrepresentation and that the orthogonal complement is also a smooth unitary conical representation. A standard Zorn's Lemma argument then shows that $\mathcal{H}$ decomposes into an orthogonal direct sum of irreducible smooth conical representations. □

It follows from Theorems 5.12 and 6.7 that every smooth unitary conical representation $(\pi, \mathcal{H})$ of U_∞ is an orthogonal direct sum of smooth highest-weight representations:

$$\pi \cong \bigoplus_{i \in \mathcal{A}} \pi_{\mu_i},$$

where $\mu_i \in \Lambda^+$ for each $i \in \mathcal{A}$. Write each highest weight μ_i in terms of fundamental weights as in Section 5.4:

$$\mu_i = \sum_{n=1}^{k_i} a_n^i \xi_i,$$

where $a_n^i \in \mathbb{N}$ for each i and n (each μ_i is a finite sum over the fundamental weights is finite because π_{μ_i} is a smooth highest-weight representation). By Theorem 5.12, the smoothness of π is equivalent to the existence of a bound $M > 0$ such that $\sum_{n=1}^{k_i} a_n^i < M$ for all $i \in \mathcal{A}$.

6.5 Disintegration of Conical Representations

If we remove the assumption in Theorem 6.7 that the conical representation $(\pi, \mathcal{H})$ is smooth, then we can no longer be assured that π has an irreducible subrepresentation. However, we would still like to describe general conical representations in terms of the irreducible ones. This sort of description is possible with a direct-integral decomposition.

Recall that

$$\Lambda^+ \equiv \Lambda^+(\mathfrak{u}_\infty, \mathfrak{a}_\infty) \equiv \varprojlim \Lambda^+(\mathfrak{u}_n, \mathfrak{a}_n) \subseteq \mathfrak{a}_\infty^*$$

denotes the set of dominant integral weights for the root system $\Sigma(\mathfrak{u}_\infty, \mathfrak{a}_\infty)$. We start by putting a topology on Λ^+. Each lattice $\Lambda^+(\mathfrak{u}_n, \mathfrak{a}_n)$ carries the discrete topology. We then consider the projective limit topology on Λ^+, which we shall refer to as the **tree topology**. This topology is defined by a basis consisting of the cylinder sets $B_\lambda = \{\mu \in \Lambda^+ | \mu_{|\mathfrak{a}_n} = \lambda\}$, where λ is a dominant integral weight on $\mathfrak{a}_n$. We refer to these cylinder sets as **node sets** for reasons that will become apparent later. Note that any two node sets are disjoint or else one contains the other, so that our basis is closed under intersections. Furthermore, Λ^+ is second-countable under this topology, since there are only countably many dominant integral weights on $i\mathfrak{a}_n$, for each fixed $n \in \mathbb{N}$, so that our basis is a countable union of countable sets.

Because it is second-countable, this topology is described entirely by sequences. Note that a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in Λ^+ converges to μ exactly when for each $m \in \mathbb{N}$ there is N such that $\mu_n|_{\mathfrak{a}_m} = \mu|_{\mathfrak{a}_m}$ for all $n \geq N$.

This topology is also Hausdorff; if μ and λ are distinct elements of Λ^+, then there is m such that $\mu|_{\mathfrak{a}_m} \neq \lambda|_{\mathfrak{a}_m}$. Hence $B_{\mu|_{\mathfrak{a}_m}}$ and $B_{\lambda|_{\mathfrak{a}_m}}$ are disjoint open sets containing μ and λ, respectively.

In fact, Λ^+ is highly disconnected; every node set is both open and closed. To see this, if we consider B_λ for some $\lambda \in \Lambda_n^+$, then we note that

$$\Lambda^+ \backslash B_\lambda = \{\mu \in \Lambda^+ | \mu_{|\mathfrak{a}_n} \neq \lambda\} = \bigcup_{\mu \in \Lambda_n^+ \backslash \{\lambda\}} B_\mu,$$

and hence $\Lambda^+ \backslash B_\lambda$ is open.

Next consider closed subsets Γ of Λ^+ with the property that, for each $n \in \mathbb{N}$, we have $\Gamma \bigcap B_\lambda = \emptyset$ for all but finitely many λ in Λ_n^+. We will refer to such sets as **tree sets** because, as we shall soon see, they are in one-to-one correspondence with trees of a certain type. We give each tree set Γ the subspace topology, so that it inherits the second-countability and Hausdorff properties from Λ^+. Write $\Gamma^\lambda = B_\lambda \bigcap \Gamma = \{\mu \in \Gamma | \mu_{|\mathfrak{a}_n} = \lambda\}$ for each n and each $\lambda \in \Lambda_n^+$. We refer to these sets as **node sets** for Γ. If $\lambda \in \Lambda_n^+$ and $\Gamma^\lambda \neq 0$ (that is, there is $\mu \in \Gamma$ such that $\mu|_{\mathfrak{a}_n} = \lambda$), then we say that λ is a **node** of the tree set Γ. We write $\Gamma_n = \{\mu_{|\mathfrak{a}_n} | \mu \in \Gamma\}$ for the set of all nodes of Γ that lie in Λ_n^+.

Now we spend a few moments explaining our tree-centric choice of terminology. For each tree set Γ, we can construct a tree as follows. Each element of Γ_n for each $n \in \mathbb{N}$ forms a node of the tree. Draw an edge from a node λ in Γ_n to a node μ in Γ_{n+1} if $\mu|_{\mathfrak{a}_n} = \lambda$. There is a correspondence between infinite paths in this tree and elements of Γ. Each infinite path $\{\lambda_n \in \Gamma_n\}_{n\in\mathbb{N}}$ of nodes of the tree defines a dominant weight $\lambda \in \Lambda^+$, since $\lambda_m|_{\mathfrak{a}_n} = \lambda_n$ for $m > n$. Because Γ is closed in the projective limit topology on Λ^+, it follows that $\lambda \in \Gamma$. Similarly, each dominant weight λ in Γ defines a path $\{\lambda|_{\mathfrak{a}_n} \in \Gamma_n\}_{n\in\mathbb{N}}$ in the tree. Hence, if λ is a node of Γ, then the node set Γ^λ corresponds to the set of all infinite paths in the tree which pass through the node λ.

It may also be readily seen that if π is a conical representation of U_∞, then the highest-weight tree $\Gamma(\pi) \subseteq \Lambda^+$ is a tree set.

Every tree set Γ is sequentially compact (and hence compact, since Λ^+ is second-countable). In fact, suppose that $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence in Γ. Now $\Gamma_n = \{\mu|_{a_n} | \mu \in \Gamma\}$ is finite for each n. In particular, there is a subsequence $\mu_{k_m^1}$ such that $\mu_{k_m^1}|_{\mathfrak{a}_1} = \mu_{k_n^1}|_{\mathfrak{a}_1}$ for each m and n. Repeating the process on this subsequence, we form a nested family of subsequences $\{\mu_{k_n^s}\}_{n\in\mathbb{N}}$ such that $\mu_{k_m^s}|_{\mathfrak{a}_s} = \mu_{k_n^s}|_{\mathfrak{a}_s}$ for each m and n. Then $\{\mu_{k_n^n}\}_{n\in N}$ is a subsequence that converges in the tree topology on Γ. Similarly, every node set in Γ is compact.

The complement of a node set in Γ is a *finite* union of node sets since Γ_n is finite for each n. The collection $\mathfrak{F}$ of finite unions of node sets for Γ thus forms an algebra of sets which generates the Borel σ-algebra $\mathfrak{B}$ for the tree topology on Γ.

We can use Γ to define a measurable family of Hilbert spaces $\lambda \mapsto \mathcal{H}_\lambda$ over $\lambda \in \Gamma$. For each $\lambda \in \Gamma$, consider the representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of U_∞ with highest-weight λ. For each such representation, pick out a unit highest-weight vector $v_\lambda \in \mathcal{H}_\lambda$.

To tie these Hilbert spaces together in a measurable way, we consider the family $\{s_g | g \in U_\infty\}$ of maps $s_g : \Gamma \to \dot{\bigcup}_{\lambda\in\Gamma}\mathcal{H}_\lambda$ given by $s_g(\lambda) = \pi_\lambda(g)v_\lambda$. Now choose a countable dense subset $E \subseteq U_\infty$ (recall that $U_\infty = \varinjlim U_n$ is separable) and consider the countable family

$$\{s_g | g \in E\}$$

of sections. We shall use this family as a measurable frame for our family of Hilbert spaces. Hence, we need to show that

$$\lambda \mapsto \langle s_g(\lambda), s_h(\lambda)\rangle = \langle \pi_\lambda(g)v_\lambda, \pi_\lambda(h)v_\lambda\rangle \tag{6.2}$$

is $\mathfrak{B}$-measurable for each $g, h \in E$. Suppose that $g, h \in U_n$ for some $n \in \mathbb{N}$. Then the representation of U_n on $\langle \pi_\lambda(U_n)v_\lambda\rangle$ is equivalent to $\pi_{\lambda|_{\mathfrak{a}_n}}$ for each λ. Thus the map in (6.2) is constant on each node set $\Gamma^{\lambda|_{\mathfrak{a}_n}}$ where $\lambda \in \Gamma$ and is hence $\mathfrak{B}$-measurable. Finally, note that $\langle\{s_g(\lambda) = \pi_\lambda(g)v_\lambda | g \in E\}\rangle$ is dense in $\mathcal{H}_\lambda$ since π_λ is irreducible and E is dense in U_∞. Thus, $\lambda \mapsto \mathcal{H}_\lambda$ is a measurable field of Hilbert spaces.

Next, we note that s_g is a measurable section for all $g \in U_\infty$. In fact, every $g \in U_\infty$ is a limit of a sequence $\{g_i\}_{i \in N} \subseteq E$. Hence, we have that

$$\lambda \mapsto \langle s_g(\lambda), s_h(\lambda)\rangle = \lim_{i\to\infty} \langle s_{g_i}(\lambda), s_h(\lambda)\rangle$$

is a measurable function for all $h \in E$, so that s_g is a measurable section.

In order to construct a direct integral of representations $(\pi_\lambda, \mathcal{H}_\lambda)$ over $\lambda \in \Gamma$, we still need a suitable choice of measure on $(\Gamma, \mathfrak{B})$. In particular, we need to choose a finite measure whose support is all of Γ (we will refer to such measures as having **full support**). The compactness of the node sets makes this easy because any finitely additive measure on $(\Gamma, \mathfrak{F})$ extends uniquely to a countably additive measure on $(\Gamma, \mathfrak{B})$.

This last claim follows from the E. Hopf Extension Theorem from measure theory, which states that a finitely additive measure μ on an algebra $\mathfrak{F}$ of subsets of X extends to a countably additive measure on the σ-algebra $\mathfrak{B}$ generated by $\mathfrak{F}$ if the measure is countably additive *on* $\mathfrak{F}$. That is, we must show that if $A = \dot{\bigcup}_{n\in\mathbb{N}} A_n$, where $A \in \mathfrak{F}$ and $A_n \in \mathfrak{F}$ for each n, then

$$\mu(A) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

However, in our case, the algebra $\mathfrak{F}$ consists of finite disjoint unions of node sets, and since every set in $\mathfrak{F}$ is compact, it follows that there is no decomposition of a set in $\mathfrak{F}$ into an infinite disjoint union of node sets.

Hence, all that we need to do is specify a (finitely additive) measure on the algebra of finite disjoint unions of node sets. We can do this rather easily. Start with the "top-level" node sets; that is, the node sets Γ^ν for $\nu \in \Gamma_1$. We can assign a measure $\mu(\Gamma^\nu)$ to each set in any way such that $\mu(\Gamma^\nu) > 0$ for each $\nu \in \Gamma_1$ and $\sum_{\nu\in\Gamma_1} \mu(\Gamma^\nu) = 1$. Next, for each $\lambda \in \Gamma_1$, consider

$$\Gamma_2^\lambda = \{\nu \in \Gamma_2 \big| \nu|_{\mathfrak{a}_1} = \lambda\}.$$

We can then assign $\mu(\Gamma^\nu)$ for each $\nu \in \Gamma_2^\lambda$ in any way such that $\mu(\Gamma^\nu) > 0$ and $\sum_{\nu\in\Gamma^\lambda} \mu(\Gamma^\nu) = \mu(\Gamma^\lambda)$. We can repeat this process, defining

$$\Gamma_{n+1}^\lambda = \{\nu \in \Gamma_{n+1} \big| \nu|_{\mathfrak{a}_n} = \lambda\}$$

for each $\lambda \in \Gamma_n$. Then we can assign $\mu(\Gamma^\nu)$ for all $\nu \in \Gamma_{n+1}^\lambda$ in such a way that $\mu(\Gamma^\nu) > 0$ for each ν and $\sum_{\nu\in\Gamma_{n+1}^\lambda} \mu(\Gamma^\nu) = \mu(\Gamma^\lambda)$. Doing this for all $\lambda \in \Gamma_n$ defines the measures of all node sets for wrights in Γ_{n+1}. This procedure always produces a Borel measure on Γ, and every finite Borel measure of full support on Γ can be constructed this way.

For instance, we can assign $\mu(\Gamma^\nu) = \frac{1}{\#\Gamma_1}$ for each ν in Γ_1. Then, for $\nu \in \Gamma_{n+1}$, recursively define $\mu(\Gamma^\nu) = \frac{1}{\#\Gamma^\lambda_{n+1}}\mu(\Gamma^\lambda)$ if $\lambda \in \Gamma_n$ and $\nu \in \Gamma^\lambda$. We have now defined the measures of all node sets from weights in Γ_2. This same method can be repeated recursively to define the measures of every node set in Γ. We will refer to this example method of assignment as giving the **recursively uniform measure**.

Given a finite Borel measure μ on Γ of full support, we may consider the direct integral $\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}_\mu d\mu(\lambda)$. Elements of this direct integral consist of measurable sections $x : \lambda \mapsto x(\lambda)$ of the field $\lambda \mapsto \mathcal{H}_\lambda$ such that the norm given by $||x||^2 = \int_\Gamma ||x(\lambda)||^2_{\mathcal{H}_\lambda} d\mu(\lambda)$ is finite.

Our next task is to show that $\lambda \to \pi_\lambda$ is a μ-measurable family of representations. Let $x \in \mathcal{H}$, and fix g in U_∞. We need to show that $\lambda \overset{\pi(x)}{\mapsto} \pi_\lambda(g)x(\lambda)$ is in $\mathcal{H}$. Now

$$\begin{aligned} \lambda \mapsto \langle \pi_\lambda(g)x(\lambda), s_h(\lambda)\rangle &= \langle \pi_\lambda(g)x(\lambda), \pi_\lambda(h)v_\lambda\rangle \\ &= \langle x(\lambda), \pi_\lambda(g^{-1}h)v_\lambda\rangle \\ &= \langle x(\lambda), s_{g^{-1}h}(\lambda)\rangle \end{aligned}$$

is measurable for all h in U_∞ since x is a measurable section of $\lambda \mapsto \mathcal{H}_\lambda$. Thus $\lambda \overset{\pi(g)x}{\mapsto} \pi_\lambda(g)x(\lambda)$ is a measurable section of $\lambda \mapsto \mathcal{H}_\lambda$. Furthermore, since each π_λ is unitary, it follows that $||\pi(g)x||_\mathcal{H} = ||x||_\mathcal{H} < \infty$. Hence $\pi = \int_\Gamma^\oplus \pi_\lambda d\mu(\lambda)$ is a unitary representation of U_∞. Our next task is to show that π is conical and classify all of its conical vectors.

The **essential support** of a function $f : \Gamma \to \mathbb{C}$ is defined to be the complement in Γ of the union of all open sets on which f vanishes μ-almost everywhere. That is, $\text{ess supp}\, f = \Gamma \setminus \bigcup\{A \subseteq \Gamma | A \text{ is open and } f|A = 0 \text{ a.e.}\}$.

Theorem 6.8. *Let Γ be a tree set and let μ be a finite Borel measure of full support on Γ. Consider the representation*

$$(\pi, \mathcal{H}) \equiv \left(\int_\Gamma^\oplus \pi_\lambda d\mu(\lambda), \int_\Gamma^\oplus \mathcal{H}_\lambda d\mu(\lambda) \right)$$

and suppose that w is any nonzero vector in $\mathcal{H}$. Then w generates a unitary conical representation of U_∞ if and only if there is $f \in L^2(\Gamma, \mu)$ such that $w = \int_\Gamma^\oplus f(\lambda)v_\lambda d\mu(\lambda)$. Furthermore, in that case w generates a conical representation with highest-weight support ess supp f *and*

$$\overline{\langle \pi(U_\infty)w\rangle} = \int_{\Gamma\setminus f^{-1}(0)}^\oplus \mathcal{H}_\lambda d\mu(\lambda)$$

In particular, π is a conical representation with conical vector $v = \int_\Gamma^\oplus v_\lambda d\mu(\lambda)$.

Proof. ($\Rightarrow$) Suppose that w is a conical vector for a subrepresentation of π and fix n in $\mathbb{N}$. Then $V_n \equiv \langle \pi(U_n)w\rangle$ is finite-dimensional, say with dimension d. We must show that $w(\lambda)$ is a conical vector in $\mathcal{H}_\lambda$ for almost all $\lambda \in \Gamma$. Our first task is to show that $V_n(\lambda) = \langle \pi(U_n)w \rangle$ is finite-dimensional for almost all $\lambda \in \Gamma$. It is intuitively obvious that $\dim V_n(\lambda) \leq \dim V_n$ for almost all λ. The next three paragraphs contain the technical details necessary to prove this statement..

Write $d = \dim V_n$. Fix an orthonormal basis $w_1, \ldots w_d$ for V_n and write $W(\lambda) = \langle w_1(\lambda), \ldots w_d(\lambda) \rangle$. We will show that $W(\lambda) = V_n(\lambda)$ (and hence $\dim V_n(\lambda) \leq d$) for almost all λ. Apply a Gram-Schmidt orthonormalization process to the collection $w_1(\lambda), \ldots, w_d(\lambda)$ for each λ. We then obtain a collection $\widetilde{w}_1(\lambda), \ldots, \widetilde{w}_d(\lambda)$ with the property that $\langle \widetilde{w}_i(\lambda), \widetilde{w}_j(\lambda) \rangle = 0$ for $i \neq j$ and $\langle \widetilde{w}_i(\lambda), \widetilde{w}_i(\lambda) \rangle \in \{0, 1\}$. One can show that $\lambda \mapsto \widetilde{w}_i(\lambda)$ is measurable and thus that $\widetilde{w}_i \in \mathcal{H}$ for each i.

Now $W(\lambda) = V_n(\lambda)$ if and only if $\pi(g)w(\lambda) \in W(\lambda)$ for all g in U_∞. Choose a countable dense subset $\{g_n\}_{n \in \mathbb{N}}$ in U_∞ (one notes that U_∞ is separable because it is a countable direct union of separable spaces). By the strong continuity of π, we see that $W(\lambda) = V_n(\lambda)$ if and only if $\pi(g_m)w(\lambda) \in W(\lambda)$ for all m in $\mathbb{N}$ (recall that $W(\lambda)$ is closed because it is finite-dimensional). In turn, this happens exactly when $\pi(g_m)w(\lambda)$ is equal to its orthogonal projection onto $W(\lambda)$. In other words, $W(\lambda) = V_n(\lambda)$ if and only if $F_m(\lambda) = 0$ for all $m \in \mathbb{N}$, where F_m is the non-negative measurable function on Γ defined by

$$F_m : \lambda \mapsto ||\pi(g_m)w(\lambda)||^2 - \sum_{i=1}^{d} |\langle \pi(g_m)w(\lambda), \widetilde{w}_i(\lambda) \rangle|^2.$$

for all $m \in \mathbb{N}$.

Write $A = \{\lambda \in \Gamma | W(\lambda) \neq V_n(\lambda)\}$ and $A_m = \{\lambda \in \Gamma | \pi(g_m)w(\lambda) \notin W(\lambda)\}$. Then $A = \bigcup_{m \in \mathbb{N}} A_m$. Furthermore, A_m is measurable for each m since $A_m = F_m^{-1}(0)$ and F_m is a measurable function.

Suppose that it is not true that $W(\lambda) = V_n(\lambda)$ for almost all λ in Γ. Then $\mu(A) > 0$. Since $A = \bigcup_{m \in \mathbb{N}} A_m$, it follows that $\mu(A_m) > 0$ for some m. Since $\pi(g_m)w(\lambda) \notin W(\lambda)$ for all $\lambda \in A_m$, we see that $\pi(g_m)w \notin \langle w_1, \ldots, w_d \rangle$, which contradicts the assumption that $w_1, \ldots, w_d$ is a basis for $V_n = \langle \pi(g_m)w \rangle$. Therefore, $W(\lambda) = V_n(\lambda)$ (and, in particular, $\dim V_n(\lambda) \leq d$) for almost all λ. In particular, $w(\lambda)$ is U_n-finite for almost all $\lambda \in \Gamma$.

Fix $n \in \mathbb{N}$. Since $\pi(M_n)w = w$, it follows that $\pi(M_n)w(\lambda) = w(\lambda)$ for almost all λ. Next, $\pi(\mathfrak{n}_n)w = w$ because $\pi(N_n)w = w$. In fact, $\pi(X)w = \int_\Gamma^\oplus \pi(X)w(\lambda)d\mu(\lambda)$ for $X \in \mathfrak{u}_n^\mathbb{C}$ by [1]. Thus $\pi(\mathfrak{n}_n)w(\lambda) = w(\lambda)$ for almost all λ, from which it follows that $\pi(N_n)w(\lambda) = w(\lambda)$ for almost all λ.

Since $\pi(M_n N_n)w(\lambda) = w(\lambda)$ for all n and almost all $\lambda \in \Gamma$, it follows from part (4) of Theorem 6.2 that for almost all λ there is $f(\lambda) \in \mathbb{C}$ such that $w(\lambda) = f(\lambda)v_\lambda$. Since $\lambda \mapsto f(\lambda) = \langle w(\lambda), v_\lambda \rangle$ is measurable and

$$||f||^2 = \int_\Gamma |f(\lambda)|^2 d\mu(\lambda) = \int_\Gamma ||w(\lambda)||^2 d\mu(\lambda) = ||w||^2,$$

we see that $f \in L^2(\Gamma, \mu)$, as was to be shown.

($\Leftarrow$) Now suppose that $w = \int_\Gamma^\oplus f(\lambda) v_\lambda d\mu(\lambda)$, where $f \in L^2(\Gamma, \mu)$. We show that w generates a conical representation of U_∞ with highest-weight support ess supp f.

Consider $V_n = \langle \pi(U_n) w \rangle$. We will show that V_n is finite-dimensional. As before,

$$\pi \cong \bigoplus_{\mu \in \Gamma_n} \left(\int_{\Gamma^\mu}^{\oplus} \pi_\lambda d\mu(\lambda) \right).$$

Write $w = \sum_{\mu \in \Gamma_n} w_\mu$, where $w_\mu = 1_{N_\mu} w \in \int_{\Gamma^\mu}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda) \subseteq \mathcal{H}_\Gamma$ for each μ.

Of course, if $f|_{\Gamma^\mu} = 0$, then $w_\mu = 0$. On the other hand, we claim that if $f|_{\Gamma^\mu} \neq 0$, then $\langle \pi(U_n) w_\mu \rangle \cong_{U_n} \pi_\mu$. In fact,

$$\sum_{i=1}^{k} c_i \pi(g_i) w_\mu = \int_{\Gamma^\mu} \sum_{i=1}^{k} c_i \pi(g_i) f(\lambda) v_\lambda d\mu(\lambda).$$

where $c_i \in \mathbb{C}$ and $g_i \in U_n$. Fix $\lambda \in \Gamma^\mu$ such that $f(\lambda) \neq 0$. Since $\lambda|_{\mathfrak{a}_n} = \mu$, we see that $\langle \pi(U_n) f(\lambda) v_\lambda \rangle$ is U_n-isomorphic to π_μ.

Now $\sum_{i=1}^k c_i \pi(g_i) w_\mu = 0$ in $\mathcal{H}$ if and only if $\sum_{i=1}^k c_i \pi(g_i) f(\lambda) v_\lambda = 0$ in $\mathcal{H}_\lambda$ for μ-almost all λ in Γ^μ. For any λ in Γ^μ such that $f(\lambda) = 0$, it follows automatically that $\sum_{i=1}^k c_i \pi(g_i) f(\lambda) v_\lambda = 0$. But for any fixed λ in Γ^μ such that $f(\lambda) \neq 0$, we see that $\sum_{i=1}^k c_i \pi(g_i) f(\lambda) v_\lambda = 0$ in $\mathcal{H}_\lambda$ if and only if $\sum_{i=1}^k c_i \pi(g_i) v_\mu = 0$ in $\mathcal{H}_\mu$.

Since f is not almost-everywhere zero on Γ^μ, we see that $\sum_{i=1}^k c_i \pi(g_i) w_\mu = 0$ in $\mathcal{H}$ if and only if $\sum_{i=1}^k c_i \pi(g_i) v_\mu = 0$ in $\mathcal{H}_\mu$. Hence there is an injective U_n-intertwining operator $L : \langle \pi(U_n) w_\mu \rangle \to \mathcal{H}_\mu$ with the property that $L w_\mu = v_\mu$. Since π_μ is irreducible, it follows that $\langle \pi(U_n) w_\mu \rangle \cong_{U_n} \pi_\mu$, as we wanted to show.

It follows from Lemma 5.8 that

$$\langle \pi(U_n) w \rangle \cong_{U_n} \bigoplus_{\mu \in \Gamma_n \text{ s.t. } w_\mu \neq 0} \langle \pi(U_n) w_\mu \rangle.$$

Furthermore, since $w = \sum_{\mu \in \Gamma_n} w_\mu$ and each w_μ is $M_n N_n$-invariant, we see that w is $M_n N_n$-invariant. Since this holds for all n, it follows that w generates a conical subrepresentation of π. The fact that this subrepresentation has highest-weight support ess supp f follows from the fact that $w_\mu = 0$ if and only if $f|_{\Gamma^\mu} = 0$ (recall that w_μ is the projection of w onto the μ-isotypic vectors in $\mathcal{H}$).

Our final task is to prove the statement about the subrepresentations generated by conical vectors. Next suppose that $f \in L^2(\Gamma, \mu)$ such that $w = fu : \lambda \to f(\lambda) v_\lambda$ is a conical vector in $\mathcal{H}_\Gamma$. We need to show that

$$\overline{\langle \pi(U_\infty) w \rangle} = \int_{\Gamma \setminus f^{-1}(0)}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda).$$

It suffices to show that

$$\overline{\langle \pi(U_\infty) w\rangle}^\perp = \int_{f^{-1}(0)}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda).$$

One direction of containment is clear: for any $x \in \overline{\langle \pi(U_\infty)w\rangle}$, we see that $x(\lambda) = 0$ for almost all λ such that $f(\lambda) = 0$ (since $w(\lambda) = 0$ if and only if $f(\lambda) = 0$). Hence, if $y \in \mathcal{H}$ such that $y|_{\Gamma \setminus f^{-1}(0)} = 0$, then $\langle x, y\rangle = \int_\Gamma \langle x(\lambda), y(\lambda)\rangle d\mu(\lambda) = 0$. In other words, $\int_{f^{-1}(0)}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda) \subseteq \overline{\langle \pi(U_\infty)w\rangle}^\perp$.

To prove the other containment, we first show that $hw \in \overline{\langle \pi(U_\infty)w\rangle}$ for all $h \in L^\infty(\Gamma, \mu)$. We begin by showing that $1_{\Gamma^\mu} w \in \langle \pi(U_\infty)w\rangle$ for every node set Γ^μ. As before, we choose $c_1, \dots, c_d \in \mathbb{C}$ and $g_1, \dots, g_d \in U_\infty$ such that $\sum_{i=1}^k c_i \pi_\mu(g_i) v_\mu = v_\mu$ and $\sum_{i=1}^k c_i \pi_\nu(g_i) v_\nu = 0$ for all $\nu \neq \mu$ in Γ_n. We claim that $1_{\Gamma^\mu} w = \sum_{i=1}^k c_i \pi_\mu(g_i) w$. If $f(\lambda) = 0$, then $w(\lambda) = 0$ and hence equality holds automatically. On the other hand, if $f(\lambda) \neq 0$, then recall that $\langle \pi(U_n)w\rangle$ is equivalent to $\pi_{\lambda|_{\mathfrak{a}_n}}$ by identifying $w(\lambda) = f(\lambda)v_\lambda$ with $v_{\lambda|_{\mathfrak{a}_n}}$. Hence $\sum_{i=1}^k c_i \pi_\mu(g_i) v_\lambda = v_\mu$ if $\lambda|_{\mathfrak{a}_n} = \mu$ (i.e., if $\lambda \in \Gamma^\mu$) and $\sum_{i=1}^k c_i \pi_\mu(g_i) v_\lambda = 0$ otherwise. Thus $1_{\Gamma^\mu} w = \sum_{i=1}^k c_i \pi_\mu(g_i) w$ and so $1_{\Gamma^\mu} w \in \langle \pi(U_\infty)w\rangle$.

Next we see that $1_A w \in \overline{\langle \pi(U_\infty)w\rangle}$ for all open sets A in Γ. Every open set A can be written as a disjoint union $A = \bigcup_{i=1}^\infty N_i$ of node sets. Write $A_n = \bigcup_{i=1}^n N_i$ for each n and note that $1_{A_n} = \sum_{i=1}^k 1_{N_i}$ is in $\langle \pi(U_\infty)v\rangle$ by the previous paragraph. One then sees that

$$\int_\Gamma^\oplus 1_{A_n}(\lambda) f(\lambda) v_\lambda d\mu(\lambda) = 1_{A_n} w \to 1_A w = \int_\Gamma^\oplus 1_A(\lambda) f(\lambda) v_\lambda d\mu(\lambda)$$

in $\mathcal{H}$ since $1_{A_n} f \to 1_A f$ in $L^2(\Gamma, \mu)$. Thus $1_A v \in \overline{\langle \pi(U_\infty)v\rangle}$.

Next we show that $1_B v \in \overline{\langle \pi(U_\infty)v\rangle}$ for every Borel set B in Γ. This follows since

$$\begin{aligned} \mu(B) &= \inf\left\{ \mu\left(\bigcup_{i=1}^\infty F_i\right) \middle| B \subseteq \bigcup_{i=1}^\infty F_i \text{ and } F_i \in \mathfrak{F} \right\} \\ &= \inf\{\mu(A) | B \subseteq A \text{ and } A \text{ open}\}. \end{aligned}$$

Thus $1_B f$ can be approximated in $L^2(\Gamma, \mu)$ by a sequence $1_{A_n} f$ given by open sets A_n, so that $1_{A_n} w \to 1_B w$ in $\mathcal{H}$. Hence $1_B w \in \overline{\langle \pi(U_\infty)w\rangle}$.

Finally, note that if $h_n \to h$ in $L^\infty(\Gamma, \mu)$, then $h_n f \to hf$ in $L^2(\Gamma, \mu)$ and hence $h_n w \to hw$ in $\mathcal{H}_\Gamma$. Because the measurable simple functions are dense in $L^\infty(\Gamma, \mu)$ (recall that μ is a finite measure), we see that $hw \in \overline{\langle \pi(U_\infty)w\rangle}$ for all $h \in L^\infty(\Gamma, \mu)$.

Now suppose that $x \perp \overline{\langle \pi(U_\infty)w\rangle}$. Define $h \in L^\infty(\Gamma, \mu)$ by

$$h(\lambda) = \frac{\overline{\langle x(\lambda), \pi_\lambda(g) f(\lambda) v_\lambda\rangle}}{|\langle x(\lambda), \pi_\lambda(g) f(\lambda) v_\lambda\rangle|}.$$

Then

$$0 = \langle x, \pi(g)hw \rangle = \int_\Gamma |\langle x(\lambda), \pi_\lambda(g) f(\lambda) v_\lambda \rangle| d\mu(\lambda).$$

for all g. Hence, for almost all λ, $\langle x(\lambda), \pi_\lambda(g) f(\lambda) v_\lambda \rangle = 0$ for all $g \in U_\infty$. It follows that, for almost all λ, either $x(\lambda) = 0$ or $f(\lambda) = 0$. Hence, $x(\lambda) = 0$ for almost all λ such that $f(\lambda) \neq 0$. In other words, $x \in \int_{f^{-1}(0)}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)$, and we are therefore done.

□

Corollary 6.9. *Every unitary conical representation of U_∞ is multiplicity-free and hence of Type I.*

Proof. Let $(\pi, \mathcal{H}) \equiv \left(\int_\Gamma^\oplus \pi_\lambda d\mu(\lambda), \int_\Gamma^\oplus \mathcal{H}_\lambda d\mu(\lambda) \right)$ be a conical representation and suppose that $L : \mathcal{H} \to \mathcal{H}$ is a U_∞-intertwining operator. Consider the conical vector $v = \int_\Gamma^\oplus v_\lambda d\mu(\lambda)$. Then Lv is a conical vector for a subrepresentation of π and can thus be written $Lv = fv$ for some $f \in L^2(\Gamma, \mu)$. It follows that

$$L(\pi(g)v) = \pi(g)(fv) = \int_\Gamma^\oplus \pi(g) f(\lambda) v_\lambda d\mu(\lambda) = f\pi(g)v$$

for all $g \in U_\infty$ and hence $Ly = fy$ for all $y \in \mathcal{H}$. In other words, intertwining operators for π may be identified with multiplier operators, and thus the ring of intertwining operators for π is commutative. Hence π is multiplicity-free. □

We now show that every unitary conical representation of U_∞ disintegrates into highest-weight representations as in the last theorem.

Theorem 6.10. *Suppose that $(\pi, \mathcal{H})$ is a unitary conical representation of U_∞ and $w \in \mathcal{H} \backslash \{0\}$ is a conical vector. Then there is a unique Borel measure μ on its highest-weight support $\Gamma(\pi)$ such that there is a unitary intertwining operator*

$$U : \mathcal{H} \to \int_{\Gamma(\pi)}^\oplus \mathcal{H}_\lambda d\mu(\lambda)$$

such that $Uw = \int_{\Gamma(\pi)}^\oplus v_\lambda d\mu(\lambda)$.

Proof. Without loss of generality, suppose that $||w|| = 1$. We begin by constructing a suitable measure μ. For each λ in $\Gamma_n(\pi)$, define $\mu(\Gamma^\lambda) = ||w_\lambda||^2$. Observe that $w_\lambda = \sum_{\nu \in \Gamma_m^\lambda} w_\nu$ and hence

$$\mu(\Gamma^\lambda) = ||w_\lambda||^2 = \sum_{\nu \in \Gamma_m^\lambda(\pi)} ||w_\nu||^2 = \sum_{\nu \in \Gamma_m^\lambda(\pi)} \mu(\Gamma^\lambda).$$

Similarly,

$$\sum_{\nu \in \Gamma_n(\pi)} \mu(\Gamma^\nu) = \sum_{\nu \in \Gamma_n(\pi)} ||w_\nu||^2 = ||w||^2 = 1$$

Thus μ extends uniquely to a Borel measure on $\Gamma(\pi)$.

Consider the representation $(\widetilde{\pi}, \widetilde{\mathcal{H}}) \equiv \left(\int^{\oplus}_{\lambda\in\Gamma(\pi)} \pi_\lambda d\mu(\lambda), \int^{\oplus}_{\lambda\in\Gamma(\mathcal{H})} \mathcal{H}_\lambda d\mu(\lambda)\right)$ and let $\widetilde{w} \equiv \int_{\Gamma(\pi)} v_\lambda d\mu(\lambda)$. Then $\widetilde{\pi}$ is conical with conical vector $\widetilde{w}$ and highest-weight support $\Gamma(\pi)$. We construct a unitary intertwining operator $U : \mathcal{H} \to \widetilde{\mathcal{H}}$ such that $Uw = \widetilde{w}$.

By Theorem 6.6 (i), there is a U_∞-intertwining operator $L : \langle\pi(U_\infty)w\rangle \to \langle\widetilde{\pi}(U_\infty)\widetilde{w}\rangle$ given by $Lw = \widetilde{w}$. For each n and each $\nu \in \Gamma_n(\pi)$, L restricts to an intertwining operator between $\langle\pi(U_n)w_\nu\rangle$ and $\langle\widetilde{\pi}(U_n)\widetilde{w}_\nu\rangle$ such that $L(w_\nu) = \widetilde{w}_\nu$. Furthermore,

$$||\widetilde{w}_\nu||^2 = \int_{\Gamma^\nu} ||\widetilde{w}_\lambda||^2 d\mu(\lambda) = \int_{\Gamma^\nu} 1 d\mu(\lambda) = \mu(\Gamma^\nu) = ||w_\nu||^2.$$

Hence, L restricts to a unitary operator on $\langle\pi(U_n)w_\nu\rangle$ for every n and every $\nu \in \Gamma_n(\pi)$. Because $\langle\pi(U_\infty)w_\nu\rangle$ and $\langle\pi(U_\infty)\widetilde{w}_\nu\rangle$ are dense in $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, respectively, L extends to a unitary intertwining operator from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$.

Now suppose that μ' is any Borel measure on $\Gamma(\pi)$ such that the representation $(\pi', \mathcal{H}') \equiv \left(\int^{\oplus}_{\lambda\in\Gamma(\pi)} \pi_\lambda d\mu'(\lambda), \int^{\oplus}_{\lambda\in\Gamma(\mathcal{H})} \mathcal{H}_\lambda d\mu'(\lambda)\right)$ is equivalent to $(\pi, \mathcal{H})$ via a unitary intertwining operator $U : \mathcal{H} \to \mathcal{H}'$ such that $Uw = w'$, where $w' = \int_{\Gamma(\pi)} v_\lambda d\mu'(\lambda)$. Then $Uw_\nu = w'_\nu$ for all $\nu \in \Gamma_n(\pi)$ and all $n \in \mathbb{N}$ by Theorem 6.6. In particular, $||w_\nu|| = ||w'_\nu||$ and so we have that

$$\mu'(\Gamma^\nu) = \int_{\Gamma^\nu} ||v_\lambda||^2 d\mu'(\lambda) = ||w'_\nu||^2 = ||w_\nu||^2 = \mu(\Gamma^\nu).$$

Since μ and μ' agree on all node sets, it follows that $\mu = \mu'$. □

As promised before, we now show that there are typically a very large number of inequivalent conical representations of U_∞ with a given highest-weight support Γ. By Theorem 6.8, this problem is equivalent to finding a large number of Borel measures with full support on Γ that are absolutely discontinuous with respect to each other.

We have already discussed the recursively-uniform measure μ_{rec} on Γ. One can see quite easily that the atoms of μ_{rec} are precisely the isolated points of the topological space Γ. All other singleton sets have measure zero under μ_{rec}. We now show that for any point x in Γ we can construct a Borel measure μ_x of full support on Γ whose atoms are precisely the isolated points of Γ and x. Thus, if $x \neq y$ are non-isolated points in Γ, then μ_x, μ_y, and μ_{rec} lie in distinct measure classes since their null sets do not agree:

$$\begin{array}{ll} \mu_x(\{x\}) > 0, & \mu_x(\{y\}) = 0 \\ \mu_y(\{x\}) = 0, & \mu_y(\{y\}) > 0 \\ \mu_{\text{rec}}(\{x\}) = 0, & \mu_{\text{rec}}(\{y\}) = 0 \end{array}$$

There are many ways to construct μ_x given $x \in \Gamma$, but we shall use the following method, which involves a simple modification to the recursively uniform measure.

For $\lambda \in \Gamma_1$, define $\mu_x(\Gamma^\lambda) = \frac{3}{4}$ if $x|_{\mathfrak{a}_n} = \lambda$ and $\mu_x(\Gamma^\lambda) = \left(\frac{1}{\#\Gamma_1 - 1}\right)\frac{1}{4}$ otherwise. Next suppose that $\mu_x(\Gamma^\nu)$ has been defined for all $\nu \in \Gamma_n$. For $\lambda \in \Gamma_{n+1}$, we define

$$\mu_x(\Gamma^\lambda) = \begin{cases} \frac{1}{2} + \frac{1}{2^{n+1}} & \text{if } x \in \Gamma^\lambda \\ \left(\frac{1}{2} - \frac{1}{2^{n+1}}\right)\frac{1}{(\#\Gamma_n^{\lambda|_{\mathfrak{a}_n}}) - 1} & \text{if } x \notin \Gamma^\lambda \text{ and } x \in \Gamma^{\lambda|_{\mathfrak{a}_n}} \\ \frac{1}{\#\Gamma_n^{\lambda|_{\mathfrak{a}_n}}}\mu(\Gamma^{\lambda|_{\mathfrak{a}_n}}) & \text{otherwise,} \end{cases}$$

where, as before, $\Gamma_n^\nu = \{\gamma \in \Gamma_n | \ \gamma|_{\mathfrak{a}_n} = \nu\}$. We have thus recursively defined a countably additive Borel measure μ_x on Γ. Note that μ_x has full support on Γ because $\mu_x(\Gamma^\lambda) > 0$ for every open basis set $\Gamma^\lambda \subseteq \Gamma$. Furthermore, one can easily check that $\mu_x(\{x\}) = \frac{1}{2}$ and that $\mu_x(\{y\}) = 0$ if $y \neq x$ and y is not an isolated point of Γ.

Chapter 7
Closing Remarks and Further Research

We have managed to prove several results for the unitary conical representations of U_∞, including the classification of unitary smooth conical representations, which generalize the finite-dimensional conical representations of finite-dimensional symmetric spaces. However, the question remains of whether it is possible to construct unitary conical representations of G_∞, in the sense of the following definition:

Definition 7.1. *A unitary representation* $(\pi, \mathcal{H})$ *of* G_∞ *is **conical** if there is a cyclic distribution vector* $v \in \mathcal{H}^{-\infty}$ *such that* $\pi(mn)v = v$ *for all* $m \in M_\infty$ *and* $n \in N_\infty$.

The most likely approach would be to construct a sort of unitary spherical principal series representation, perhaps by a direct limit of unitary principal series representations. See also [57] for one approach to constructing an analogue of the principal series for direct-limit groups.

Several questions about harmonic analysis on the symmetric space G_∞/K_∞ and $G_\infty/M_\infty N_\infty$ remain. While neither of these infinite-dimensional spaces possess G_∞-invariant measures, there is a possibility of constructing G_∞-invariant measures on larger spaces. We briefly overview this construction now.

Consider a direct system $\{G_n\}_{n\in\mathbb{N}}$ of Lie groups and suppose that there are measurable (not necessarily continuous) projections $p_n : G_{n+1} \to G_n$ such that p_n is G_n-equivariant and $p_n(g) = g$ for $g \in G_n$. In other words, one has a projective system of σ-algebras dual to the direct system of groups. The resulting projective-limit space $\overline{G_\infty} = \varprojlim G_n$ is acted on by the direct-limit group $G_\infty = \varinjlim G_n$. Each group G_n possesses a G_n-quasi-invariant probability measure μ_n.

It is then possible to define a projective-limit probability measure $\mu_\infty = \varprojlim \mu_n$ on $\overline{G_\infty}$ using Kolmogorov's theorem. If this measure is quasi-invariant under the action of G_∞ on $\overline{G_\infty}$ then it is possible to define a unitary "regular representation" of G_∞ on $L^2(\overline{G_\infty}, \mu_\infty)$. This "regular representation" can then be decomposed into irreducible representations.

In fact, precisely this scheme was used by Doug Pickrell in [46] to study analysis on an infinite-dimensional Grassmannian space and later by Olshanski and Borodin in [4] to develop a theory of harmonic analysis on the infinite-dimensional unitary group $U(\infty)$. The role played by probability theory in the latter context was crucial. In fact, the problem was shown to be related to the study of infinite point processes. Most intriguingly, probabilistic models from statistical mechanics appeared.

It would be interesting to consider a similar analysis on the infinite-dimensional symmetric space G_∞/K_∞ and the horocycle space $G_\infty/M_\infty N_\infty$. That is, one would construct projective-limit spaces $\overline{G_\infty/K_\infty}$ and $\overline{G_\infty/M_\infty N_\infty}$ which possess G_∞-quasi-invariant measures. The problem, then, would be to decompose the corresponding unitary representations of G_∞ on $L^2(\overline{G_\infty/K_\infty})$ and $L^2(\overline{G_\infty/M_\infty N_\infty})$ into irreducible subrepresentations. One interesting question is whether those representations decompose into direct integrals of unitary spherical and conical representations of G_∞, respectively.

Also of interest is whether a sort of Radon transform may be constructed between functions on G_∞/K_∞ and functions on $G_\infty/M_\infty N_\infty$. In fact, for spaces of regular functions this has been done in the recent paper [24]. However, it would be interesting if it were possible to develop a Hilbert space analogue of the Radon transform, perhaps mapping between functions in $L^2(\overline{G_\infty/K_\infty})$ and functions in $L^2(\overline{G_\infty/M_\infty N_\infty})$.

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Vita

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