Recovering trees from the cohomology ring of their configuration spaces

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Abstract

Given a tree T, the cohomology ring of its unordered configuration space $H^*(U\mathcal{D}^nT)$ is an exterior face algebra if T is a binary core tree (if by removing the leaves from T we obtain a binary tree), or if $n = 4$. This means that every cup product is determined by a simplicial complex K_nT. In this paper we show how to recover the tree T from the simplicial complex K_nT when $n = 4$.

1 Introduction

For a finite graph G and a positive integer n, the discretized unlabelled configuration space on n points of G is defined as

$$U\mathcal{D}^nG = \{\{x_1, \ldots, x_n\} : x_i \in V(G) \cup E(G), x_i \cap x_j = \emptyset \text{ if } i \neq j\}.$$

Configuration spaces of graphs have been widely studied, in particular, their cohomology ring is well known when G is a tree. Using discrete Morse theory techniques, D. Farley gave in [2] an efficient description of the additive structure of the cohomology ring of $U\mathcal{D}^nG$ when G is a tree T. Later, and in order to get to the multiplicative structure, the Morse theoretic methods were replaced in [3] by the use of a Salvetti complex $\mathcal{S}$ obtained by identifying opposite faces of cells in $U\mathcal{D}^nT$. Being a union of tori, $\mathcal{S}$ has a well understood cohomology ring.

Given a tree T we can construct another tree $F(T)$, where the vertex set of $F(T)$ is $V(F(T)) = \{x \in V(T) : d(x) > 2\}$ and two vertices are adjacent in $F(T)$ if the unique path joining them in T does not contain any other vertex of $F(T)$. We say that a tree T is a binary core tree if $F(T)$ is a binary tree. In [1], the following theorem was proven:

Theorem 1.1. *[1] Given an integer $n \geq 4$, the cohomology ring $H^*(U\mathcal{D}^nT)$ is an exterior face ring if either $n = 4$, or T is a binary core tree.*

This means that when T is a binary core tree or $n = 4$, all products in $H^*(UD^nT)$ are given by a simplicial complex K_nT, where the vertices of K_nT are the basis elements of

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dimension 1, and a set of k vertices forms a simplex in K_nT if and only if the cup product of the corresponding k elements is non zero. The simplicial complex K_nT is called the n-interaction complex of T.

The 1-skeleton of the space $U\mathcal{D}^nT$, coincides precisely with the token graph $F_n(T)$ of T on n tokens. In [4], Fabila-Monroy and Trujillo-Negrete proved that if a graph does not have induced cycles of length four, or induced diamonds (a graph isomorphic to $K_4 \setminus \{e\}$ the complete graph on four vertices without an edge), then the graph G is uniquely reconstructible from $F_k(G)$ and gave an algorithm to do so in polynomial time. One then might ask if there is a topological version of this, this is, if we can recover the tree T from a topological property of $U\mathcal{D}^nT$. We are going to prove that we can recover the tree T from the n-interaction complex K_nT in the case when $n = 4$, for any tree T.

We shall assume that the tree T is embedded in the plane and has as root a vertex of degree one $\star$. The edges inciding in a vertex x are enumerated by this embedding and we fix the edge that lies on the unique $x\star$ path to be the 0th edge. We shall say that a vertex y lies on x-direction i for $1 \leq i \leq d(x) - 1$ if x belongs to the unique path joining y and the root vertex, and if this path contains the ith edge inciding in x. This also implies that x lies on y-direction 0. If x does not belong to the $y\star$ path and y does not belong to the $x\star$ path we shall also say that x lies on y-direction 0 and y lies on x-direction 0, in which case we say that the vertices x and y are not stacked. Given two vertices of degree at least three x and y, we shall say that $x < y$ if they are stacked and x lies on y-direction zero, or if they are not stacked and there exists z such that x lies on z-direction i and y lies on z-direction j with $i < j$.

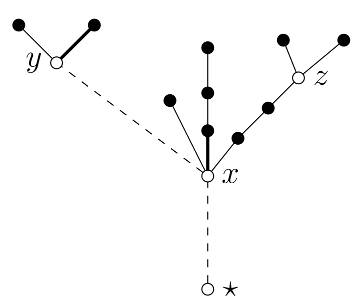

Figure 1: The tree T containing the vertices x, y and z.

Example 1.2. *Consider the tree shown in Figure 1, where the dashed edges represent paths of any length greater than two. Then the vertices x and y are stacked, the vertices z and x are also stacked but the vertices y and z are not stacked.*

We are going to define the interaction complex K_nT only for the case when $n = 4$, in which case it is a graph. For the the definition of the interaction complex in the general case we refer the reader to [1].

2 The graph K_4T

Definition 2.1. *The graph K_4T is defined as follows:*

- *The vertices are 4-tuples (k, x, p, q) such that x is a vertex of T with $d(x) \geq 3$, k is a non-negative integer and p and q are integer vectors having non-negative entries such that p has at least one positive entry, the sum of their lengths $l(p) + l(q)$ is $d(x) - 1$ and the sum of their entries is $3 - k$.*

- *Let $v = (k_1, x_1, p_1, q_1)$ and $w = (k_2, x_2, p_2, q_2)$ be two vertices, then*

  1. *if the vertices x_1 and x_2 are not stacked, then $(v, w) \in E(K_4T)$ if $k_1 + k_2 \geq 4$*
  2. *if x_2 lies on x_1-direction i with $l(p_1) \geq i$, then $(v, w) \in E(K_4T)$ if $p_{1,i} > 4 - k_2$ or $p_{1,i} + k_2 = 4$ and there exists a $j \neq i$ such that $p_{1,j} \neq 0$*
  3. *if x_2 lies on x_1-direction i with $l(p_1) < i$, then $(v, w) \in E(K_4T)$ if $q_{1,i-l(p_1)} > 4 - k_2$*

In particular, the graph K_4T has a lot of isolated vertices, as we can see in the following lemma.

Lemma 2.2. *A vertex $v = (k, x, p, q)$ is an isolated vertex if one of the following conditions hold:*

- *$k = 1$*

- *the sum of the entries of p is two.*

Proof. Assume first that $k = 1$. Notice that since $l \in \{0, 1, 2\}$, if a vertex $w = (l, y, r, s)$ is such that x and y are not stacked, then both k and l must be two for v and w to be adjacent. If x and y are stacked, every entry of $p = (p_1, \ldots, p_{l(p)})$ is at most two, thus $2 \geq p_i$ and $4 - l \geq 2$. Finally notice that every entry of $q = (q_1, \ldots, q_{l(q)})$ is at most one so $1 \geq q_{i-l(p_1)}$ and $4 - k_2 \geq 2$. This proves that every vertex with $k = 1$ is isolated. Assume now that the sum of the entries of p is two, and consider again $w = (l, y, r, s)$. This means that $k = 0$ and q has one unique entry with value one. Then again $2 \geq p_i$, $4 - l \geq 2$, $1 \geq q_{i-l(p_1)}$ and $4 - k_2 \geq 2$ thus v is an isolated vertex. □

For our purposes, the isolated vertices are not relevant, so we shall ignore them.

Proposition 2.3. *Given a vertex x of T, fix $i \in \{1, \ldots, d(x) - 1\}$ and let*

$$\Lambda_i = \{(k, x, p, q) \in K_4T : p_i \geq 2 \text{ if } i \leq l(p) \text{ or } q_{i-l(p)} \geq 2 \text{ if } i > l(p)\}.$$

Then the set Λ_i has cardinality $\frac{1}{2}(d(x)-1)(d(x)-2)$, and this is also the amount of vertices having $k = 2$.

Proof. Assume first that $k = 0$. Since the sum of the entries of $p = (p_1, \ldots, p_{l(p)})$ and q must be at most $3 - k = 3$, and one entry is fixed to be 2, there can only be one additional non zero entry. Recall also that p must have at least one zero entry, and since we are considering only vertices in which the sum of the entries of p is not two, p must have an additional non zero entry or $p_i = 3$. Then for every possible length of p, we have $l(p)$ options, one for each additional 1 as an entry. Since $l(p)$ varies between 1 and $d(x)-2$ we have $\sum_{i=1}^{d(x)-2} i = (d(x)-1)(d(x)-2)$. The same happens if $k=2$, since the remaining 1 must be an entry of p. □

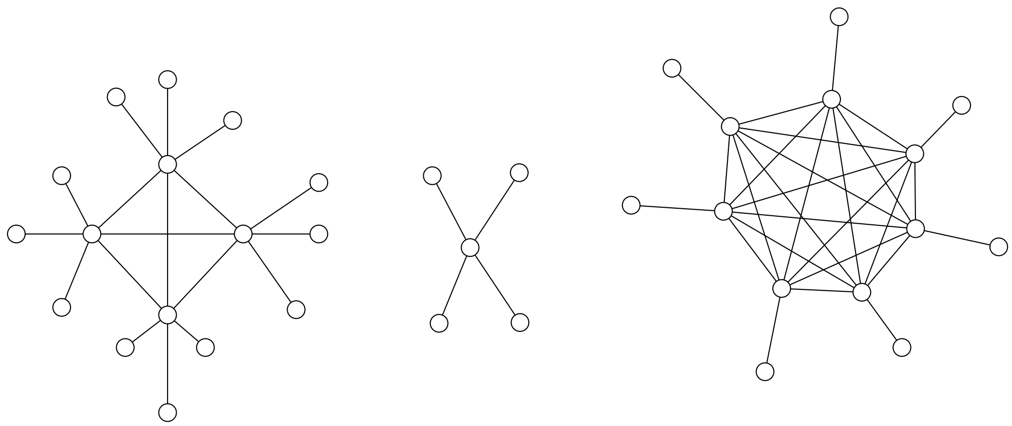

Figure 2: The graph K_4T, the tree $T = K_{1,4}$ and an example of the graph KP_7.

Example 2.4. *When $T = K_{1,m}$, the graph K_4T is the graph obtained from the complete graph K_m by adding to each vertex, $\frac{1}{2}(m-1)(m-2)$ new neighbours as in Figure 2.*

The following proposition follows from the definition of K_4T.

Proposition 2.5. *Let $v = (k, x, p, q)$ be a vertex in K_4T. Then the degree of v is the amount of vertices in $F(T)$ which lie on direction i, where*

- *$i = 0$ if $k = 2$,*
- *i is such that the ith entry of p is at least two if $i \leq l(p)$,*
- *i is such that the $(i + l(p))$th entry of q is two if $i > l(p)$.*

Corollary 2.6. *Given a binary tree T, the number of leaves in K_4T is the number of leaves in $F(T)$.*

Proof. Let w be a leaf in $F(T)$ different from the root, and let v be its unique neighbour. Assume w lies on v-direction i. Then the vertex u is a leaf in K_4T and has as unique neighbour the vertex $(2, v, 1, 0)$, where $u = (0, w, p, q)$ is such that $u = \begin{cases} (0, w, 3, 0) & \text{if } i = 1 \\ (0, w, 1, 2) & \text{if } i = 2. \end{cases}$ If w is the root vertex with unique neighbour v, then in K_4T the vertex $(2, v, 1, 0)$ is a leaf and its unique neighbour is $(0, w, 3, 0)$. □

To simplify notation, we shall often write 0 to denote the vector having every entry zero. Let $L(G)$ denote the set of leaves of a graph G.

Proposition 2.7. *If two vertices u and v in K_4T have the same neighbourhood then $u = (k, x, p, q)$ and $v = (l, x, r, s)$.*

Proof. Assume $u = (k, x, p, q)$ and $v = (l, y, r, s)$ have the same (non empty) neighbourhood for $x \neq y$. Notice first that u and v are not adjacent since K_4T has no loops. Notice also that u and v can not be both leaves, since that would mean that $u = (2, x, p, 0)$ and $v = (2, y, r, 0)$ thus $(u, v) \in E(K_4T)$, a contradiction.

Assume first that x and y are not stacked. This means that either k or l is different that 2. Assume without loss of generality that $k \neq 2$, thus $k = 0$. Since $N(u) \neq \emptyset$, there exists $w \in N(u)$, $w = (2, z, p', q')$ and z lies on x-direction i. This is a contradiction since $w \in N(v)$ and the vertices x and y are not stacked.

Now we may assume that the vertices x and y are stacked, and assume without loss of generality that y lies on x-direction i. This means that $k = 0$ and $l = 2$. If x is not a leaf in $F(T)$, the root vertex z of $F(T)$ is such that $(0, z, p', q')$ is adjacent to v for some vectors p' and q'. But this means that $(0, z, p', q')$ is also adjacent to $u = (0, x, p, q)$ which is a contradiction. If x is a leaf, since we are assuming that y lies on x-direction i, x must be the root vertex and $u = (0, x, p, 0)$. Then $v = (2, y, r, 0)$, and $w = (2, z, p', 0) \in N(u)$ with z the unique neighbour of x. Since z and y are stacked, $w \notin N(v)$ which is a contradiction. Hence $x = y$. □

Corollary 2.8. *If T is a binary tree, then there are no two vertices having the same non empty neighbourhood in K_4T.*

Proof. Assume v and w have the same neighbourhood. By Proposition 2.7, we must have that $v, w \in \{(0, x, 1, 2), (0, x, 3, 0), (2, x, 1, 0)\}$. If $v = (2, x, 1, 0)$ and $w \neq v$, then for every vertex y such that y lies on x-direction 0, we have that if x lies on y-direction i, then $u \in N(v)$ where $u = \begin{cases} (2, y, 1, 0) & \text{if } i = 0 \\ (0, y, 3, 0) & \text{if } i = 1 \\ (0, y, 1, 2) & \text{if } i = 2. \end{cases}$ But, since $w \neq v$, we have that u can not be adjacent to w. This means that there are no vertices lying on x-direction 0, but this implies that $N(v) = \emptyset$. So assume that $v = (0, x, 1, 2)$ and $w = (0, x, 3, 0)$. Then again, if there exist vertices $y, z \in T$ such that y lies on x-direction 1 and z lies on x-direction two, the vertex $(2, y, 1, 0)$ is adjacent to v but not to w and the vertex $(2, z, 1, 0)$ is adjacent to w but not to v. This means that x is a leaf in $F(T)$, but this implies that $N(v) = \emptyset = N(w)$. □

3 Recovering the tree T

We shall first consider the case when T is a binary tree before considering the general case, since this case is easier both notation-wise and mathematically.

3.1 Binary trees

Throughout this section we shall assume that T is a binary tree. The main reason for this is Proposition 2.7. If T is a binary tree, then T is completely determined by $F(T)$.

Proposition 3.1. *Assume T is a binary tree and let K_m be a complete subgraph of K_4T for $m \geq 3$. Then in $F(T)$ there exists an independent set I of m vertices.*

Proof. Let $v_i = (k_i, x_i, p_i, q_i)$ for $1 \leq i \leq m$ be the vertices of K_m in K_4T. Assume that x_j lies on x_i-direction $t \in \{1, 2\}$ for some $i \neq j$. Then $v_j = (2, x_j, 1, 0)$ and $v_i = (0, x_i, 3, 0)$

or $v_i = (0, x_i, 1, 2)$ (if $t = 1$ or $t = 2$ respectively). Since v_i multiplies every vertex v_l, this means that x_l lies on x_i-direction t for every $l \in \{1, \ldots, m\} \setminus \{i\}$. Recall that v_j, and hence every v_l, must be of the form $(2, x_l, 1, 0)$ for $l \in \{1, \ldots, m\} \setminus \{i\}$. This means that x_l lies on x_j-direction 0 and vice versa x_j lies on x_l-direction 0 for $j, l \in \{1, \ldots, m\} \setminus \{i\}$. This immediately implies that $\{x_1, \ldots, x_m\} \setminus \{x_i\}$ is an independent set of $m-1$ vertices. Now assume the edge (x_i, x_j) exists in $F(T)$ for some $j \in \{1, \ldots, m\} \setminus \{i\}$. Then x_j lies on x_i-direction 0 but this implies that x_i also lies on x_l-direction for $l \in \{1, \ldots, m\} \setminus \{i, j\}$ which is a contradiction. Hence $I = \{x_1, \ldots, x_m\}$ is an independent set in $F(T)$. □

Corollary 3.2. *Assume that T is a binary tree, and that $F(T)$ has m leaves. Then there exists in K_4T a subgraph isomorphic to K_m such that each vertex in K_m is adjacent to exactly one leaf.*

Let KP_m be the subgraph obtained by a attaching a leaf to each vertex of a complete subgraph K_m (see Figure 2 (right)).

Let Kp denote the subgraph of K_4T consisting of the complete subgraph mentioned in Corollary 3.2 together with the adjacent leaves. If we delete from K_4T the subgraph Kp, we obtain the subgraph of K_4T consisting only of vertices (k, x, p, q) such that x is not a leaf of $F(T)$. Moreover, the amount of leaves in $K_4T - Kp$ corresponds to the amount of leaves in $F(T) - L(F(T))$ and we can apply Corollary 3.2 again.

This means that if S_1 be the subgraph of K_4T isomorphic to KP_{m_1} for some m_1, then $K_4T - S_1$ contains S_2, a subgraph isomorphic to KP_{m_2}. We can continue this way until $K_4T - \cup_{i=1}^{n} S_i$ is isomorphic to either a KP_3, a $KP_2 = P_4$ or a $KP_1 = K_2$ (since T is binary), and notice that $m_1 \geq m_2 \geq \cdots \geq m_n$.

Definition 3.3. *Let $V(K_4T) = \cup_{i=1}^{n} S_i$ such that the induced subgraph generated by S_i is isomorphic to KP_{m_i} as in the previous discussion. We say that a vertex $v \in V(K_4T)$ belongs to level i if it is a vertex of S_i. Moreover, $V(S_i) = L_i \cup R_i$ where L_i is the set of leaves of S_i.*

We are now ready to recover the tree T from K_4T when T is a binary tree. First, notice that $V(K_4T) = \cup_{i=1}^{n} V(S_i)$ and since each S_i has $2m_i$ vertices, we are going to label each vertex of L_i and R_i with the labels $\{x_1^i, \ldots, x_{m_i}^i\}$ and $\{y_1^i, \ldots, y_{m_i}^i\}$ respectively, thus in S_i, y_j^i is the only neighbour of x_j^i. We are going to construct a tree F_T isomorphic to $F(T)$.

Definition 3.4. *The tree F_T is defined as follows:*

- *The vertices of F_T are $V(F_T) = z_0 \cup \left(\bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} z_j^i \right)$, and*
- $N(z_0) = \bigcup_{j=1}^{m_n} z_j^n$
- $(z_j^i, z_l^k) \in E(F_T)$ *if* $k = min\{m > i : (x_l^m, y_j^i) \in E(K_4T)\}$.

We now must prove that this tree is indeed isomorphic to $F(T)$.

Lemma 3.5. *The trees $F(T)$ and F_T are isomorphic.*

Proof. We proceed by induction over n, and assume $n = 1$. This means that $K_4T = S_1 \cong KP_{m_1}$ which means that $F(T)$ is a star $F(T) \cong K_{1,m_1}$. On the other hand, F_T consists only of the vertices z_0 together with $\cup_{i=1}^{m_1} z_i^1$, but since $n = 1$ we have that every vertex is adjacent to z_0, thus $F_T = K_{1,m_1}$.

Assume now, that the lemma is valid for values smaller than n. We have that $K_4T \backslash S_1 = K_4T'$ where T' is such that $F(T')$ is obtained from $F(T)$ by removing every leaf. Then $V(K_4T') = \cup_{i=2}^{n} S_i$ thus by induction hypothesis, $F_{T'} = F(T') = F(T) \setminus L(F(T))$. Notice that the amount of vertices in $F(T) \setminus F(T')$ is the amount of leaves in $F(T)$ which is m_1, which is also the amount of vertices in $F_T \setminus F_{T'}$. Take a vertex $w \in F(T)$, and let $v \in F(T)$ be a leaf adjacent to w. By Corollary 2.6, one of the following holds:

- the vertex $(0, v, 1, 2)$ is a leaf in K_4T and $(2, w, 1, 0)$ is its unique neighbour,
- the vertex $(0, v, 3, 0)$ is a leaf in K_4T and $(2, w, 1, 0)$ is its unique neighbour, or
- the vertex w is the root vertex of T, the vertex $(2, v, 1, 0)$ is a leaf in K_4T, and its unique neighbour is $(0, w, 3, 0)$.

Assume the first case holds. Then $(0, v, 1, 2) = x_j^1$ and $(2, w, 1, 0) = y_i^1$ for $1 \leq i, j \leq m_1$. Since v is not a leaf in $F(T)$, there exists $u \in V(F(T))$ such that $(v, u) \in E(F(T))$ and u lies on v-direction 0. This means that $(2, v, 1, 0)$ is adjacent to $(0, u, 3, 0)$ or $(0, u, 1, 2)$. We can assume without loss of generality that $(0, u, 3, 0)$ is adjacent to $(2, v, 1, 0)$, and thus it must also be adjacent to $(2, w, 1, 0) = y_i^1$. Since u is a neighbour of v, $(0, u, 3, 0)$ is a leaf in K_4T and hence is a leaf in S_2, thus $(0, u, 3, 0) = x_l^2$. Then $(x_l^2, y_i^1) \in E(K_4T)$ and thus $(z_i^1, z_l^2) \in E(F_T)$. This means that given an edge $(w, v) \in E(F(T))$ outside of $F(T')$, there exists an edge $(z_i^1, z_l^2) \in E(F_T)$. The other two cases are analogous.

Now consider an edge $(z_i^1, z_j^2) \in F_T$ outside of $F_{T'}$. This means that $(x_j^2, y_i^1) \in E(K_4T)$ with $x_j^2 \in L_2$ and $y_i^1 \in R_1$. Since $y_i^1 \in R_1$, again by Corollary 2.6 we have that $y_i^1 = (2, w, 1, 0)$ or $y_i^1 = (2, w, 1, 0)$ if w is the root vertex of $F(T)$. Assume that $y_i^1 = (2, w, 1, 0)$, then $x_i^1 = (0, v, 3, 0)$ or $(0, v, 1, 2)$ and v is the unique neighbour of the leaf w, thus $(v, w) \in F(T)$. If $y_i^1 = (2, w, 1, 0)$ with w the root of $F(T)$, we have that $x_i^1 = (0, v, 3, 0)$ where v is the unique neighbour of w. Hence for every edge in F_T outside of $F_{T'}$ there exists an edge in $F(T)$ outside of $F(T')$. Thus $F(T) = F_T$. □

3.2 The general case

Let RK_4T be the graph obtained from K_4T as follows. For every set of vertices $\{v_1, \ldots, v_m\}$ such that $N(v_i) = N(v_j)$ for $i \neq j$, we remove from K_4T the vertices $\{v_2, \ldots, v_m\}$. Thus in RK_4T there are no two vertices having the same neighbourhood. Notice that the vertices of the graph RK_4T can also be separated into m_k sets $S_1, \ldots, S_k$ such that each $S_i \cong KP_{m_i}$ for some m_i. Assume again that $\{x_1^i \ldots, x_{m_i}^i\}$ and $\{y_1^i \ldots, y_{m_i}^i\}$ denote the set of leaves and non-leaf vertices of S_i for $1 \leq i \leq k$ respectively. We then have the following definition.

Definition 3.6. *Assume $V(RK_4T) = \cup_{i=1}^{k} V(S_i)$ with $|V(S_i)| = m_i$. The tree RF_T is defined as follows.*

- *The vertices of RF_T are $V(F_T) = z_0 \cup \left(\cup_{i=1}^{n} \cup_{j=1}^{m_i} z_j^i\right)$, and*

- $N(z_0) = \cup_{j=1}^{m_n} z_j^n$,
- $(z_j^i, z_l^k) \in E(RF_T)$ *if* $k = min\{m > i : (x_l^m, y_j^i) \in E(RK_4T)\}$.

The proof of the following Corollary is analogous to the proof of Lemma 3.5.

Corollary 3.7. *The trees RF_T and $F(T)$ are isomorphic.*

Since T is not binary it is not fully determined by $F(T)$. Consider the set of vertices $LR = \{x_1, \ldots, x_{m_1}\}$ of K_4T, which have degree one in RK_4T. Assume that x_i has degree d_i in K_4T for $1 \le i \le m_1$. Then $d_i = \frac{1}{2}(k_i - 1)(k_i - 2)$ for some integer k_i.

Proposition 3.8. *If a leaf $x \in RK_4T$ has degree $d = \frac{1}{2}(k-1)(k-2)$ in K_4T for some k. Then the corresponding leaf $z \in F(T)$ has degree k in T.*

Proof. Let z be a leaf in $F(T)$, and let w be its unique neighbour in $F(T)$. Assume also that z lies on w direction i. If $z \neq \star$, then $i \neq 0$ and consider a vertex $v = (0, w, p, q) \in K_4T$ having 2 or 3 as the ith entry of p or as the $i - l(p)$th entry of q. They all have the same neighbourhood consisting of vertices of the form $(2, z, p, 0)$, and by Proposition 2.3, there are $d = \frac{1}{2}(k-1)(k-2)$ such vertices, where $d(z) = k$. Thus all the vertices of the form of v correspond to a leaf $x \in RK_4T$ which has degree $d = \frac{1}{2}(k-1)(k-2)$. If $z = \star$, we consider vertices of the form $v = (0, z, p, q)$ having 2 or 3 as the ith entry of p or as the $i - l(p)$th entry of q, and they all have the same neighbourhood consisting of vertices of the form $(2, w, p, 0)$. Thus these last vertices correspond to a leaf $x \in RK_4T$ which has the desired degree in K_4T.

□

This means that we have fully recovered T from the graph K_4T.

Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest. This manuscript has no associated data.

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