On distance-k graphs of star and free products of graphs

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Preliminaries

For a given graph $G=(V,E)$ and a positive integer k the *distance-k graph* is defined to be a graph $G^{[k]}=\left(V,E^{[k]}\right)$ with

$$E^{[k]}=\{(x,y):x,\ y\in V,\ \partial_G(x,y)=k\},$$

where $\partial_G(x,y)$ is the graph distance. Figure below shows the distance-2 graph induced by the 3 dimentional cube.

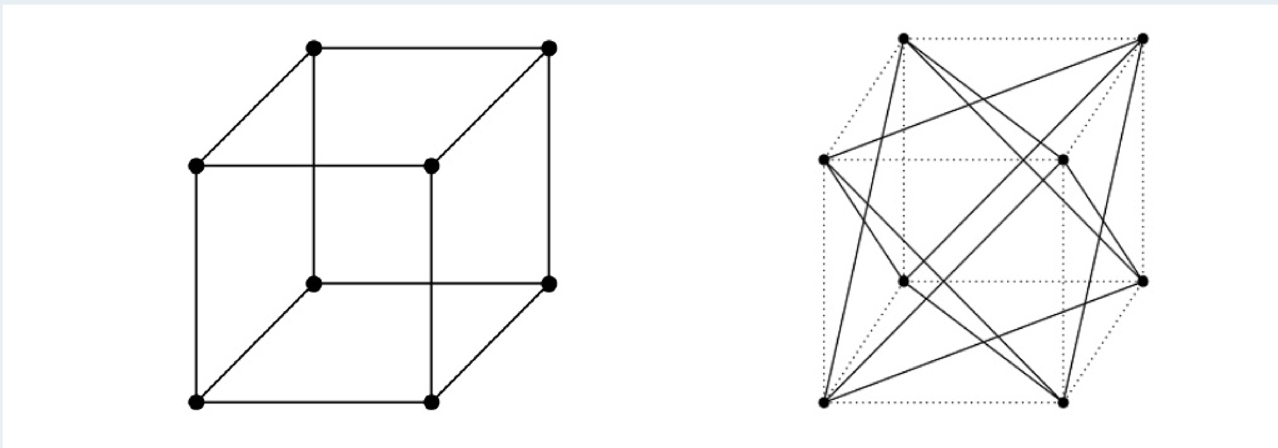

Figura
3-Cube and its distance-2 graph

For $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graph with distinguished vertices $o_1\in V_1$ and $o_2\in V_2$, the *star product graph* of G_1 with G_2 is the graph $G_1\star G_2=(V_1\times V_2,E)$ such that for $(v_1,w_1),\ (v_2,w_2)\in V_1\times V_2$ the edge $e=(v_1,w_1)\sim(v_2,w_2)\in E$ if and only if one of the following holds:

1. $v_1=v_2=o_1$ and $w_1\sim w_2$
2. $v_1\sim v_2$ and $w_1=w_2=o_2$.

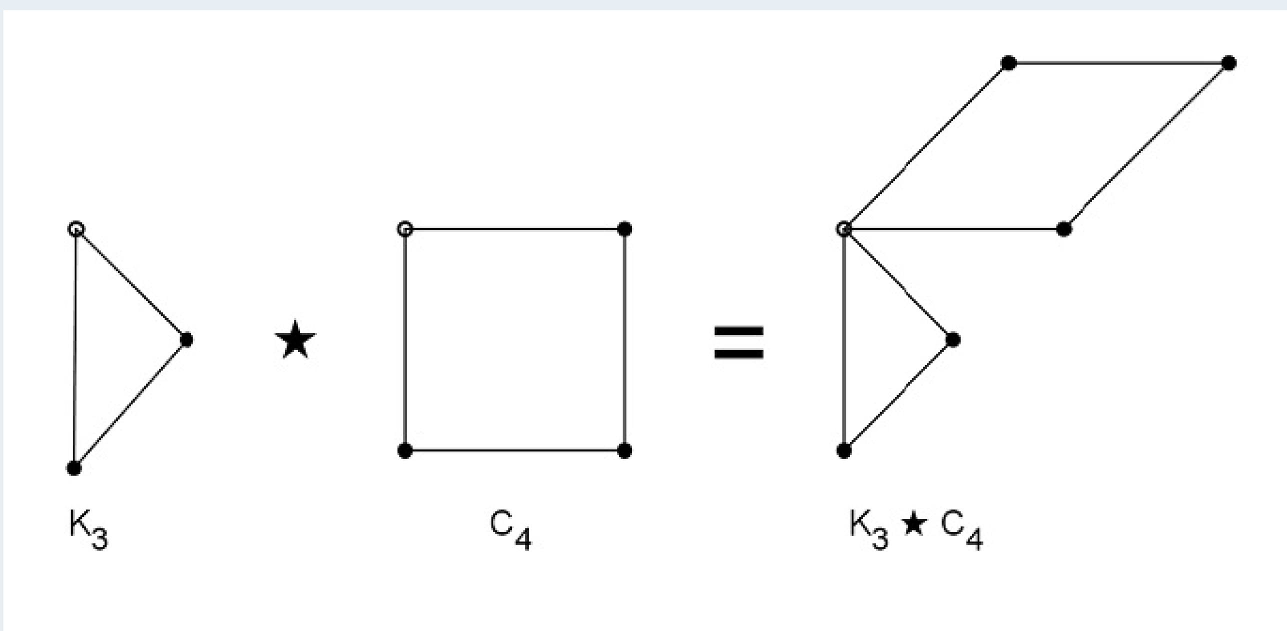

Figura
Star product of two cycles

We define the *free product* of the rooted vertex sets (V_i,e_i), $i\in I$, where I is a countable set, by the rooted set $(*_{i\in I}V_i,e)$, where

$$*_{i\in I}V_i=\{e\}\cup\{v_1v_2\cdots v_m:v_k\in V^0_{i_k},\text{ and } i_1\neq i_2\neq\cdots\neq i_m,\ m\in\mathbb{N}\},$$

and e is the empty word.

Definition

The *free product of rooted graph* $(\mathscr{G}_i,e_i)$, $i\in I$, is defined by the rooted graph $(*_{i\in I}\mathscr{G}_i,e)$ with vertex set $*_{i\in I}V_i$ and edge set $*_{i\in I}E_i$, defined by

$$*_{i\in I}E_i:=\{(vu,v'u):(v,v')\in\bigcup_{i\in I}E_i\text{ and } u,\ vu,\ v'u\in *_{i\in I}V_i\}.$$

We denote this product by $*_{i\in I}(\mathscr{G}_i,e_i)$ or $*_{i\in I}\mathscr{G}$ if no confusion arises. If $I=[n]$, we denote by $G^{*n}=(*_{i\in I}G,e)$.

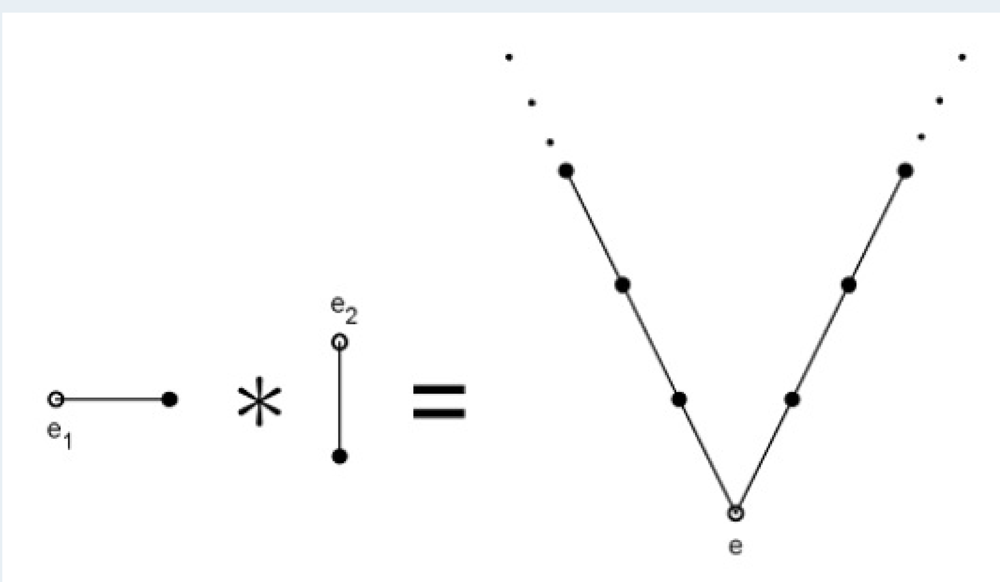

Figura
Free product K_2 with itself

Distance-k graph of star product of graphs

Unlike Cartesian (and also free) product, the asymptotic distribution of distance-k graph of star product does not depend on k, just as stated in the following theorem.

Theorem (O. Arizmendi & T. Gaxiola (2015))

Let $G=(V,E,e)$ be a locally finite connected graph and let $k\in\mathbb{N}$ be such that $G^{[k]}$ is not trivial. For $N\geq 1$ and $k\geq 1$ let $G^{[\star N,k]}$ be the distance k-graph of $G^{\star N}=G\star\cdots\star G$ (N-fold star power) and $A^{[\star N,k]}$ its adjacency matrix. Furthermore, let $\sigma=V_e^{[k]}$ be the number of neighbours of e in the distance k-graph of G, then the distribution with respect to the vacuum state of $(N\sigma)^{-1/2}A^{[\star N,k]}$ converges in distribution as $N\to\infty$ to a centered Bernoulli distribution. That is,

$$\frac{A^{[\star N,k]}}{\sqrt{N\sigma}}\longrightarrow\frac{1}{2}\delta_{-1}+\frac{1}{2}\delta_1,$$

weakly.

Distance-k graph of d-regular trees

For $d\geq 2$, let $A_d^{(k)}$ be the adjacency matrix of distance-k graph of d-regular tree. We write $A^{(1)}=A$. Then we can express

$$A^2=A_d^{(2)}+dI, \tag{1}$$

(see Figure 4). Since $A_d^{[2]}=A^2-dI$ then the distribution is given by the law of x^2-d, where x is a random variable obeying the Kesten-McKay distribution of parameter d.

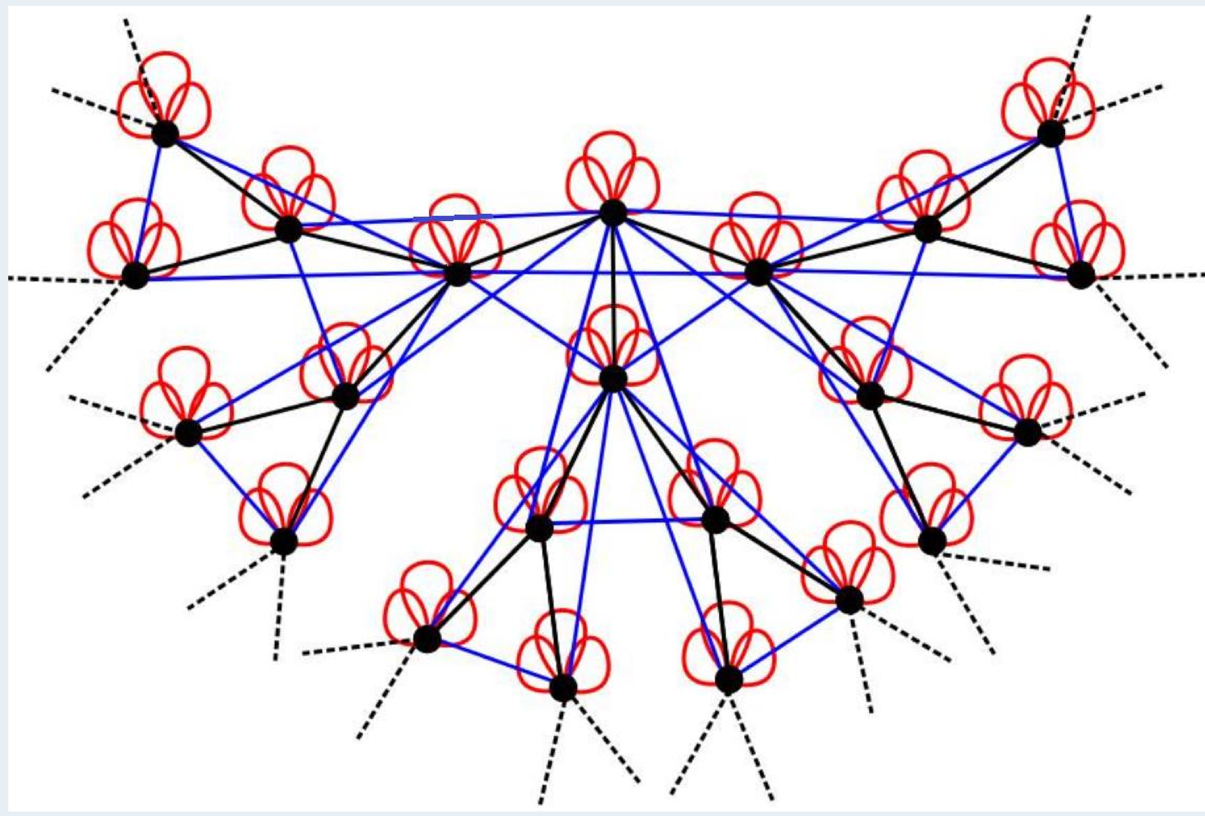

Figura
Graph of A^2 split in two parts $A^2=A_d^{(2)}+dI$

In general, for $k\geq 3$ we have the following recurrence formula.

Lemma

Let $d\geq 1$ fixed, then it follows, $A^{(1)}=A$, $A^{(2)}=A^2-dI$, and

$$AA^{(k)}=A^{(k+1)}+(d-1)A^{(k-1)}\quad k=1,\ 2,\ \ldots,d-1.$$

Since $A_d^{[k]}$ fulfills a recurrence formula, and A is distributed as the Kesten-McKay distribution μ_d, then we arrive to the following theorem.

Theorem

For $d\geq 2$, $k\geq 1$, let $A_d^{[k]}$ be the adjacency matrix of distance-k graph of the d-regular tree. Then the distribution with respect to the vacuum state of $A_d^{[k]}$ is given by the probability distribution of

$$T_k(b)$$

where T_k is the orthogonal of order k with respect to μ_d and b is a random variable with Kesten-McKay distribution, μ_d.

d-Regular Random Graphs

Let X be a d-regular graph with vertex set $\{1,2,\ldots,n(X)\}$. For each $i\geq 3$ let $c_i(X)$ be the number of cycles of length i. Let $A^{[k]}(X)$ be the adjacency matrix of the distance-k graph of X. The following is an analogous of main theorem in McKay (1981), but in distance-k framework.

Theorem

Let d, k be fixed integers and, for each n, let $F_n(x)$ be the expected eigenvalue distribution of the distance-k graph of a random regular graph with degree d and order $2n$. Then, as n tends to infinity, $F_n(x)$ converges to the distribution of $A_d^{[k]}$ with respect to the vacuum state.

Distance-k graph of free products

Theorem (O. Arizmendi & T. Gaxiola (2016))

Let $G=(V,E,e)$ be a finite connected graph and let $k\in\mathbb{N}$. For $N\geq 1$ and $k\geq 1$ let $G^{[*N,k]}$ be the distance-k graph of $G^{*N}=G*\cdots*G$ (N-fold free power) and $A^{[*N,k]}$ its adjacency matrix. Furthermore, let σ be the number of neighbors of e in the graph G. Then the distribution with respect to the vacuum state of $(N\sigma)^{-k/2}A^{[*N,k]}$ converges in moments (and then weakly) as $N\to\infty$ to the probability distribution of

$$P_k(s), \tag{2}$$

where P_k is the Chebychev polynomial of order k and s is a random variable obeying the semicircle law.

References

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