Distance-k graphs of random d-regular graphs

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Plan

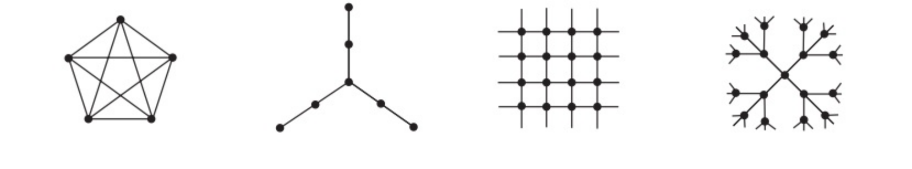
Definition

A *graph* is a pair $G = (V, E)$, where V is the set of *vertices* and E the set of *edges*. We write $x \sim y$ (adjacent) if they are connected by an edge.

Definition

We call a graph *undirected* if $x \sim y$ implies $y \sim x$. A *loop* is an edge of the form $x \sim x$, we say a graph is *simple* if it has not loops.

Examples



General Framework

Graphs and Spectra

$G = (V, E)$: a finite graph, i.e. $|V| < \infty$

Definition

The *adjacency matrix* of a graph $G = (V, E)$ is defined by

$$A = [A_{xy}]_{x,y \in V} \qquad A_{xy} = \begin{cases} 1, & x \sim y \\ 0, & \text{otherwise.} \end{cases}$$

The *spectrum* of G is defined by $\mathrm{Spec}(G) = \mathrm{Spec}(A)$.

- $(A^k)_{ij}$ =# paths of length k from i to j.
- $(A^k B^l)_{ij}$

General Framework

Formulation of Problem

- Let $\mathcal{A}$ be the $*$-algebra generated by A.
- Let $\varphi(\cdot)$ be a state.
- The adjacency matrix A as a random variable of $(\mathcal{A}, \varphi)$.

General Framework

Formulation of Problem (Main Problem)

Let $G_{(\nu)} = (V_{(\nu)}, E_{(\nu)})$ be a growing graph and let $\varphi_\nu(\cdot)$ be a state on $\mathcal{A}(G_{(\nu)})$. Find a probability distribution μ on $\mathbb{R}$ satisfying

$$\varphi_\nu\left(\left(\frac{A_{(\nu)} - \varphi(A_{(\nu)})_\nu}{(A_{(\nu)} - \varphi(A_{(\nu)})_\nu)^2)_\nu^{1/2}}\right)^m\right) \to \int_{-\infty}^{\infty} x^m \mu(dx), \quad m = 1, 2, \cdots .$$

The above μ is called the *asymptotic spectral distribution* of $G_{(\nu)}$ in the states $\varphi(\cdot)_\nu$.

General Framework

Two States

1. $\varphi_{tr}(A) = \frac{Tr(A)}{|V|} = \frac{\text{\# closed paths of size } k}{|V|}$.
   The *spectral distribution* μ of A i determined by

$$\varphi(A^m)_{tr} = \int_{-\infty}^{\infty} x^m \mu(dx), \quad m = 1, 2, \ldots.$$

   μ coincides with the *eigenvalue distribution* of A:

$$\mu = \frac{1}{|V|} \sum_i m_i \delta_{\lambda_i}.$$

2. $\varphi_1(A) = (A)_{11} =$ # closed paths from the root of size k.

Direct Product

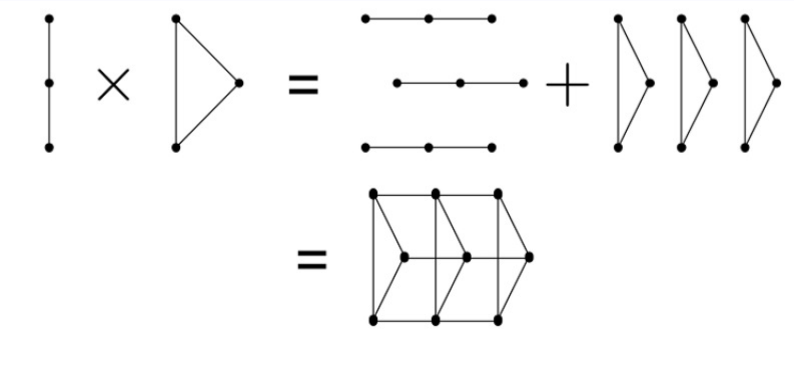
Definition

For $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two finite graphs, the *direct product graph* of G_1 with G_2 is the graph $G_1 \times G_2 = (V_1 \times V_2, E)$ such that for $(v_1, w_1), \ (v_2, w_2) \in V_1 \times V_2$ the edge $e = (v_1, w_1) \sim (v_2, w_2) \in E$ if and only if one of the following holds:

1. $v_1 = v_2$ and $w_1 \sim w_2$
2. $v_1 \sim v_2$ and $w_1 = w_2$.

Direct Product

Example



Direct Product

Classical Central Limit Theorem

Teorema

Let $G = (V, E)$ be a finite connected graph. Let G^N de N-fold direct power of G, and let A_{G^N} be its adjacency matrix. Then we have

$$\lim_{N\to\infty} \varphi_{tr}\left(\left(\frac{A_{G^N}}{N^{1/2}\left(\frac{|V|}{2|E|}\right)^{1/2}}\right)^m\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^m e^{-x^2/2}dx, \quad m = 1, 2, \ldots.$$

Boolean Product

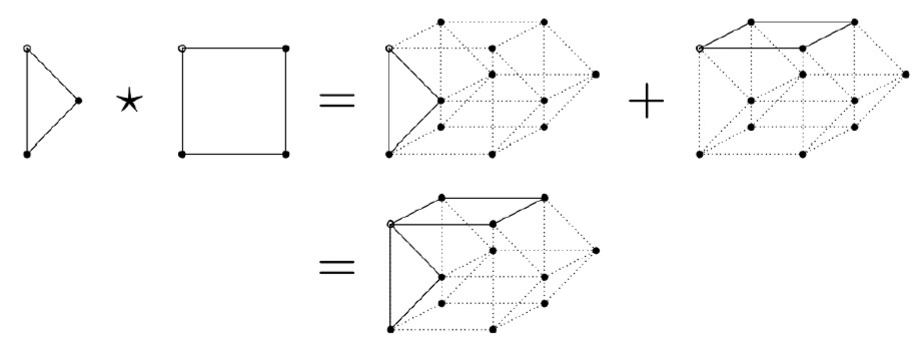
Definition

For $G_1 = (V_1, E_1, r_1)$ and $G_2 = (V_2, E_2, r_2)$ be two finite rooted graphs, the *Boolean product graph* of G_1 with G_2 is the graph $G_1 \star G_2 = (V_1 \times V_2, E)$ such that for $(v_1, w_1), \ (v_2, w_2) \in V_1 \times V_2$ the edge $e = (v_1, w_1) \sim (v_2, w_2) \in E$ if and only if one of the following holds:

$$
\begin{aligned}
1.\ v_1 &= v_2 = r_1 \text{ and } w_1 \sim w_2 \\
2.\ v_1 &\sim v_2 \text{ and } w_1 = w_2 = r_2.
\end{aligned}
$$

Boolean Product

Example



Boolean Product

Boolean Central Limit Theorem

Teorema

Let $G = (V, E, r)$ be a finite connected graph. Let $G^{\star N}$ de N-fold Boolean power of G, and let $A_{G^{\star N}}$ be its adjacency matrix. Then we have

$$\lim_{N\to\infty} \varphi_1\left(\left(\frac{A_{G^{\star N}}}{N^{1/2}deg(r)}\right)^m\right) = \frac{1}{2}\int_{-\infty}^{\infty} x^m(\delta_{-1}+\delta_1)dx, \quad m = 1, 2, \ldots.$$

Monotone Product

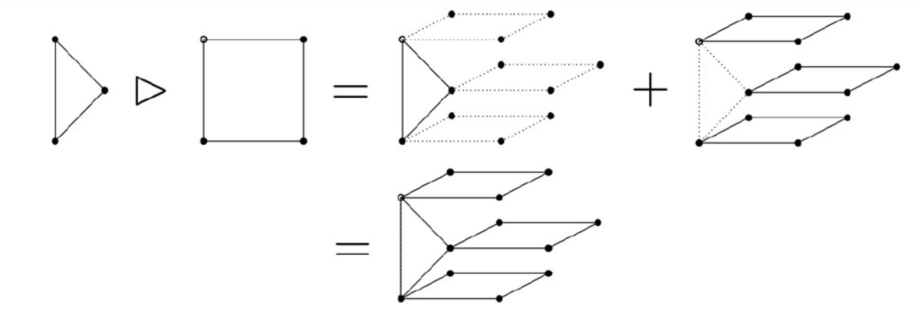
Definition

For $G_1 = (V_1, E_1, r_1)$ and $G_2 = (V_2, E_2, r_2)$ be two finite rooted graphs, the *monotone (comb) product graph* of G_1 with G_2 is the graph $G_1 \triangleright_{r_2} G_2 = (V_1 \times V_2, E)$ such that for $(v_1, w_1),\ (v_2, w_2) \in V_1 \times V_2$ the edge $e = (v_1, w_1) \sim (v_2, w_2) \in E$ if and only if one of the following holds:

$$\begin{aligned} 1.\ v_1 &= v_2 \text{ and } w_1 \sim w_2 \\ 2.\ v_1 &\sim v_2 \text{ and } w_1 = w_2 = r_2. \end{aligned}$$

Monotone Product

Example



Monotone Product

Monotone Central Limit Theorem

Teorema

Let $G = (V, E, r)$ be a finite connected graph. Let $G^{\triangleright N}$ de N-fold monotone power of G, and let $A_{G^{\triangleright N}}$ be its adjacency matrix. Then we have

$$\lim_{N\to\infty} \varphi_1\left(\left(\frac{A_{G^{\triangleright N}}}{N^{1/2}deg(r)}\right)^m\right) = \frac{1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} \frac{x^m}{\sqrt{2-x^2}}dx, \quad m = 1, 2, \ldots.$$

Free Product

$V^0 = V \backslash \{e\}$. Let (V_i, e_i) rooted vertex sets $i \in I$.

$$*_{i \in I} V_i = \{e\} \cup \{v_1 v_2 \cdots v_m : v_k \in V^0_{i_k}, \text{ and } i_1 \neq i_2 \neq \cdots \neq i_m,\ m \in \mathbb{N}\},$$

and e is the empty word.

Definition

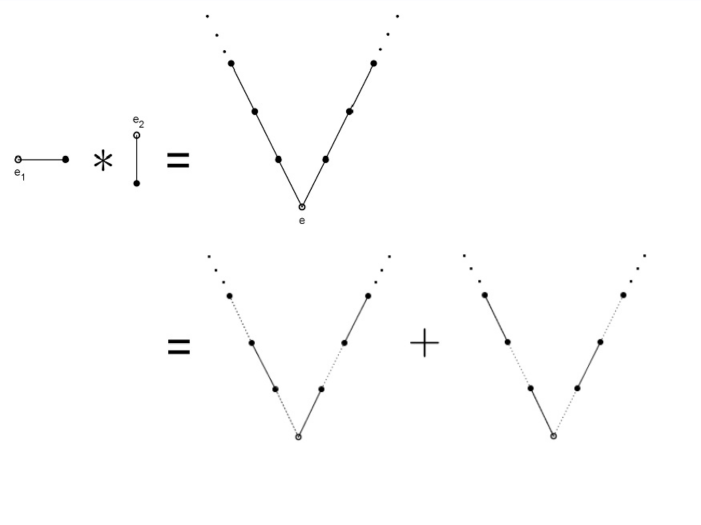
The *free product of rooted graph* $(G_i, e_i),\ i \in I$, is define by the rooted graph $(*_{i \in I} G_i, e)$ with vertex set $*_{i \in I} V_i$ and the edges set $*_{i \in I} E_i$, define by

$$*_{i \in I} E_i := \{(vu, v'u) : (v, v') \in \bigcup_{i \in I} E_i \text{ and } u,\ vu,\ v'u \in *_{i \in I} V_i\}.$$

We denote this product by $*_{i \in I}(G_i, e_i)$ or $*_{i \in I} G$ if no confusion arises.

Free Product

Example



Free Product

Free Central Limit Theorem

Teorema

Let A be the adjacency matrix of (G, e) *and let* A^{*N} *denote the adjacency matrix of* $(G, e)^{*N}$*. Then*

$$\lim_{N \to \infty} \varphi_1 \left(\left(\frac{A^{*N}}{\sqrt{N deg(e)}} \right)^{2m} \right) = c_m$$

where c_m *is the m-th Catalan number for* $m \in \mathbb{N}$, $c_0 = 1$*. The odd moments vanish.*

Distance-k Graphs

Definition

For a given graph $G = (V, E)$ and a positive integer k the *distance k-graph* is defined to be a graph $G^{[k]} = \left(V, E^{[k]}\right)$ with

$$E^{[k]} = \{(x, y) : x,\ y \in V,\ \partial_G (x, y) = k\},$$

where $\partial_G (x, y)$ is the graph distance.

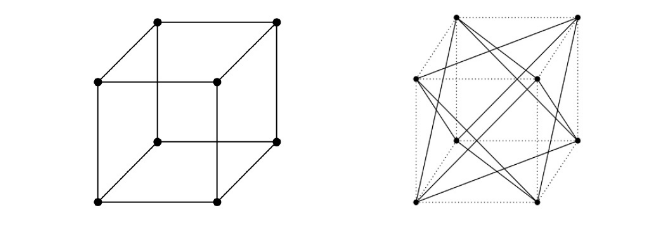


Figure: 3-Cube and its distance 2-graph

Distance-k Graphs

Direct Product

Teorema (Hibino, Lee and Obata (2012))

Let $G = (V, E)$ be a finite connected graph with $|V| \geq 2$. For $N \geq 1$ and $k \geq 1$ let $G^{[N,k]}$ be the distance k-graph of $G^N = G \times \cdots \times G$ (N-fold Cartesian power) and $A^{[N,k]}$ its adjacency matrix. Then, for a fixed $k \geq 1$, the eigenvalue distribution of $N^{-k/2} A^{[N,k]}$ converges in moments as $N \to \infty$ to the probability distribution of

$$\left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!} \tilde{H}_k(g), \tag{1}$$

where $\tilde{H}_k$ is the monic Hermite polynomial of degree k and g is a random variable obeying the standard normal distribution $N(0, 1)$.

Distance-k Graphs

Boolean Product

Teorema (Arizmendi, G. (2014))

Let $G = (V, E, e)$ be a locally finite connected graph and let $k \in \mathbb{N}$ be such that $G^{[k]}$ is not trivial. For $N \geq 1$ and $k \geq 1$ let $G^{[\star N,k]}$ be the distance k-graph of $G^{\star N} = G \star \cdots \star G$ (N-fold star power) and $A^{[\star N,k]}$ its adjacency matrix. Furthermore, let $\sigma = V_e^{[k]}$ be the number of neighbours of e in the distance k-graph of G, then the distribution with respect to the vacuum state of $(N\sigma)^{-1/2} A^{[\star N,k]}$ converges in distribution as $N \to \infty$ to a centered Bernoulli distribution. That is,

$$\frac{A^{[\star N,k]}}{\sqrt{N\sigma}} \longrightarrow \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1,$$

weakly.

Proof.- Fourth Boolean Moment Lemma.

Lemma (Fourth Boolean Moment)

Let $\{X_n\}_{n\geq 1} \subset (\mathcal{A}, \varphi)$ *, be a sequence of self-adjoint random variables in some non-commutative probability space, such that* $\varphi(X_n) = 0$ *and* $\varphi(X_n^2) = 1$*. If* $\varphi\left(X_n^4\right) \to 1$*, as* $n \to \infty$*, then* μ_{X_n} *converges in distribution to a symmetric Bernoulli random variable* **b***.*

Let A be the adjacency matrix of the d-regular tree.

Lemma

Let $d \geq 1$ fixed, then it follows, $A^{(1)} = A$, $A^{(2)} = A^2 - dI$, and

$$AA^{(k)} = A^{(k+1)} + (d-1)A^{(k-1)} \quad k = 1,\ 2,\ \ldots, d-1.$$

Distance-k Graphs

d-regular Trees

PROOF

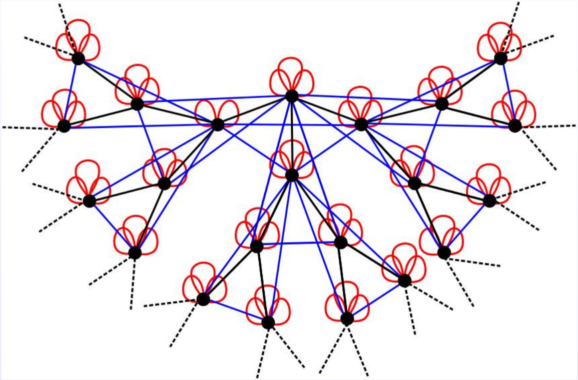


Figure: Graph of A^2 split in two parts $A^2 = A_d^{(2)} + dI$

If $k \geq 3$.

Case 1

$$\delta(i,j) = k+1 \Rightarrow (A^{(k)}A)_{ij} = 1.$$

Case 2

$$\delta(i,j) = k-1 \Rightarrow (A^{(k)}A)_{ij} = d-1.$$

Case 3

$$|\delta(i,j) - k| \neq 1 \Rightarrow (A^{(k)}A)_i j = 0.$$

Distance-k Graphs

Proposition

For $d \geq 2$, let $A_d^{(k)}$ be the adjacency matrix of distance-k graph of the d-regular tree. Then the distribution with respect to the vacuum state of $A_d^{(k)}$ is given by the probability distribution of

$$T_k\left(\frac{b}{2\sqrt{d-1}}\right), \tag{2}$$

with

$$T_k(x) = \begin{cases} 1 & \text{if } k = 0, \\ \sqrt{\frac{d-1}{d}} P_k(x) - \frac{1}{\sqrt{d(d-1)}} P_{k-2}(x) & \text{if } k = 1, 2, \ldots, \end{cases}$$

where P_k is the Chebychev polynomial of degree k and b is a random variable with distribution μ_d.

$$d\mu_d = \frac{d}{2\pi}\frac{\sqrt{4(d-1)-x^2}}{d^2-x^2}dx$$

Distance-k Graphs

d-regular Trees

Teorema (Arizmendi, G. In progress)

For $d \geq 2$, let $A_d^{(k)}$ be the adjacency matrix of distance-k graph of the d-regular tree. Then the distribution with respect to the vacuum state of $d^{k/2} A_d^{(k)}$ converges in moments as $d \to \infty$ to the probability distribution of

$$P_k(s), \tag{3}$$

where $P_k(s)$ is the Chebychev polynomial of degree k and s is a random variable obeying the semicircle law.

$$\frac{A_d}{d^{1/2}}\frac{A_d^{(k)}}{d^{k/2}} = \frac{A_d^{(k+1)}}{d^{(k+1)/2}} + \frac{A_d^{(k-1)}}{d^{(k-1)/2}} - \frac{1}{d}\frac{A_d^{(k-1)}}{d^{(k-1)/2}}$$

If $d \to \infty$ and $X = \frac{A_d}{d^{1/2}}$ then

$$P^{(1)}(X) = X, \quad P^{(2)}(X) = X^2 - I,$$

$$XP^{(k)}(X) = P^{(k+1)}(X) + P^{(k-1)}(X) - \frac{1}{d}P^{(k-1)}(X).$$

Random d-regular Graphs

$X_1, X_2, \ldots$: sequence of d-regular graphs.
$n(X)$: number of vertices of the graph X.
$c_j(X)$: number of cycles of length j of X.
$A^{(k)}(X)$: the adjacency matrix of the distance-k graph of X.

Proposition

If $c_j(X_i)/n(X_i) \to 0$ as $i \to \infty$, then

$$A^{(k)}(X_i) \xrightarrow{m} A_d^{(k)}.$$

PROOF

- $n_r(X_i)$: # vertices of X_i s.t. the subgraph induced by the vertices at most $r = mk$ from each ones has no cycles.
- By hypothesis $n_r(X_i)/n(X_i) \to 1$ as $i \to \infty$.
- $\theta_m(X_i)$: # closed walks of length m for the remaining vertices. Then $0 \leq \theta_m(X_i) \leq d^r$.
- Hence

$$\varphi_{tr}\left(A^{(k)}(X_i)\right) = \frac{\varphi_1(A_d^{(k)})n_r(X_i)}{n(X_i)} + \frac{(n(X_i) - n_r(X_i))\theta_m(X_i)}{n(X_i)}$$
$$\longrightarrow \varphi_1(A_d^{(k)}) \quad \text{as } i \to \infty.$$

THANK YOU!