The Cohomology of Real Parabolic Arrangements

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Abstract

Parabolic arrangements obtained as the result of deleting faces of the Coxeter complex of a reflection group appear as natural objects related to subspace arrangements. By reconciling geometric methods for real subspace arrangements—such as those proposed by Baryshnikov, Dobrinskaya, and Turchin—with the combinatorial-algebraic viewpoint of k-parabolic arrangements proposed by Barceló, Severs and White, we derive explicit formulas for the cohomology ring of complements of parabolic arrangements. Our main results introduce a chain complex generated by parabolic cosets, whose homology is isomorphic to the cohomology of the complement, and provide an intersection-based cup product formula. This approach recovers and extends the known Betti numbers of classic families of arrangements (including the k-equal, type B, and type D analogues) by recovering their cohomological product. This theory can be applied as well to large families of real subspace arrangements by interpreting linear conditions as unions of parabolic cones.

Introduction and Motivation

A subspace arrangement $\mathscr{A}$ is a collection of linear, finite-dimensional subspaces of a vector space V. One of the central questions is to determine the (co)homological structure of the complement $\mathscr{M}(\mathscr{A}) = V \setminus \bigcup \mathscr{A}$.

The description of the cohomology ring for the complex case $V = \mathbb{C}$ was established by the Arnold–Brieskorn and Orlik–Solomon theorems. However an extension of the theory for the case $V = \mathbb{R}$ has not been established.

Goresky–MacPherson's stratified Morse theory gave combinatorial formulas for Betti numbers, with applications by Björner–Welker and Björner–Sagan. Yet the intersection lattices can be difficult and do not recover the cohomological product for real arrangements (Ziegler).

Baryshnikov described the cohomology ring of real non-k-equal spaces; Dobrinskaya–Turchin generalized this to $(\mathbb{R}^d)^n$. Severs–White described Betti numbers of k-parabolic subspace arrangements via CW models. We reconcile both viewpoints through cones of parabolic cosets, obtaining explicit ring formulas.

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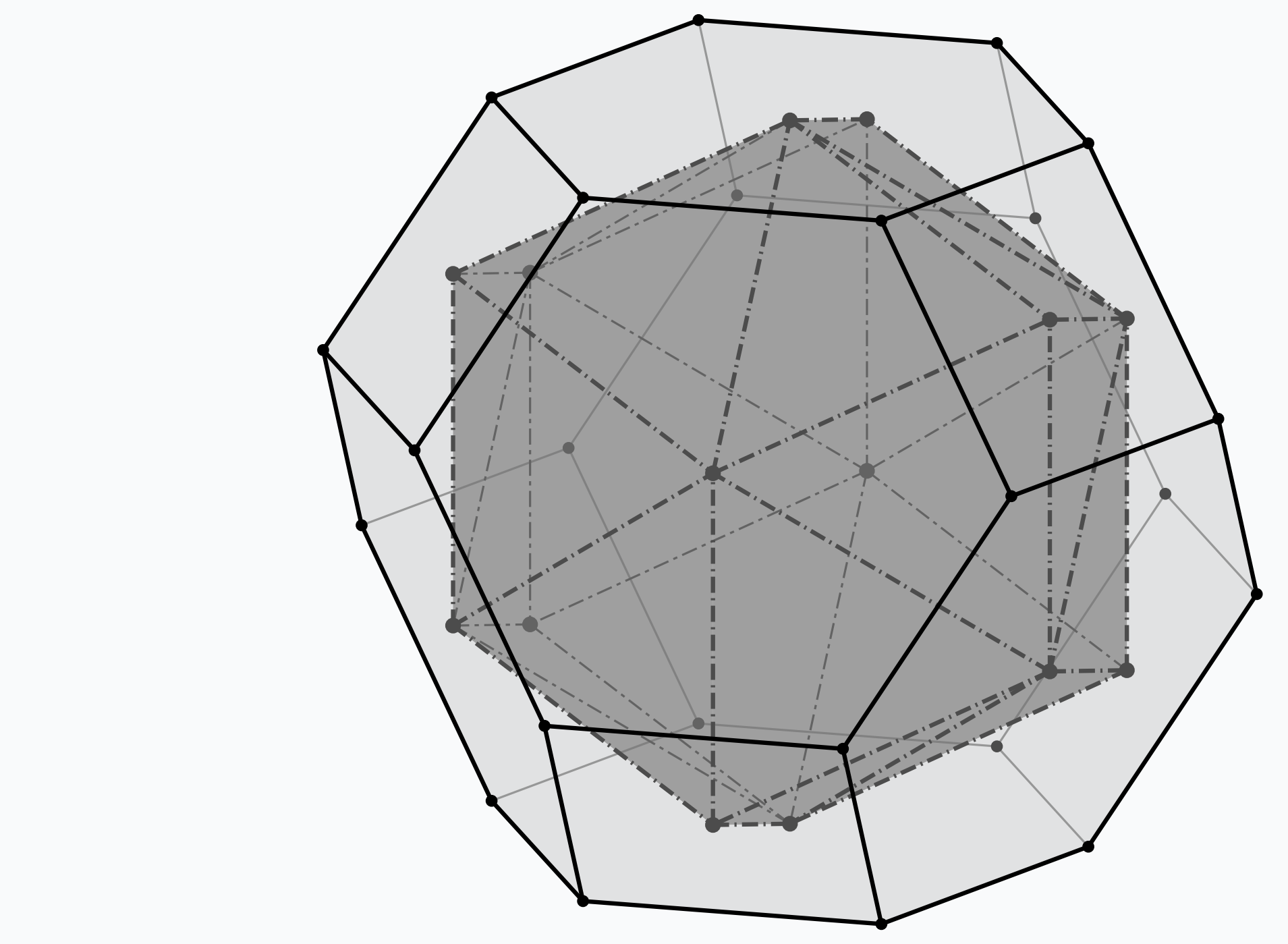
Cones and Coxeter Complex

Consider a finite real reflection group W of rank n with a set of simple reflections S. For each $I \subseteq S$, we will denote by C_I the (closed) cone associated to I, that is,

$$C_I = \{x \in \mathbb{R}^n \mid (x, \alpha) = 0,\ \ \forall \alpha \in I,\ (x, \beta) \geq 0,\ \forall \beta \in S \setminus I\},$$

which is a convex cone in $\mathbb{R}^n$. In particular, $C_\emptyset$ is called the fundamental domain of W. Given $w \in W$, the translated cone $wC_I = \{wx \mid x \in C_I\}$ is again a convex cone in $\mathbb{R}^n$ and the collection of all such cones will be denoted by $\mathscr{H}(W)$. It constitutes a polyhedral decomposition of $\mathbb{R}^n$, and the Coxeter complex $\Delta(W)$ is defined as the intersection $\mathscr{H}(W) \cap \mathbb{S}^{n-1}$.

Permutahedron and its dual



Coxeter complex of D_3 (dashed) and its dual permutahedron (solid).

Complement and CW Model (via Permutahedron)

Given a parabolic arrangement $\mathscr{A}$ of type W, let $\mathcal{M}(\mathscr{A}) = \mathbb{R}^n \setminus \bigcup_{X \in \mathscr{A}} X$. Since each $X \in \mathscr{A}$ is parametrized by a parabolic coset, a natural cell structure on $\mathcal{M}(\mathscr{A})$ is inherited from the Coxeter permutahedron $\mathrm{Perm}(W)$. Set

$$\Delta_{\mathscr{A}}(W) = \{F \in \Delta(W) \mid \exists X \in \mathscr{A} \text{ with } F \subseteq X\}.$$

Radial projection gives $\mathcal{M}(\mathscr{A}) \simeq |\Delta(W)| \setminus |\Delta_{\mathscr{A}}(W)|$. As this difference need not be simplicial, pass to the dual cell structure: delete the ideal $\mathcal{F}(\Delta_{\mathscr{A}}(W))$ inside $\mathcal{F}(\Delta(W))$ and take the opposite poset. The resulting subcomplex

$$\mathrm{Perm}_{\mathscr{A}}(W) := \left|\left(\mathcal{F}(\Delta(W)) \setminus \mathcal{F}(\Delta_{\mathscr{A}}(W))\right)^{\mathrm{op}}\right| \subset \mathrm{Perm}(W)$$

is a regular CW model, and $\mathcal{M}(\mathscr{A}) \simeq \mathrm{Perm}_{\mathscr{A}}(W)$.

Cohomology via Parabolic Cosets

Since we have a parametrization of $\Delta(W)$ by parabolic cosets and its face poset is precisely the face poset of parabolic cosets (minus top and bottom), we control the dual poset underlying $\mathrm{Perm}_{\mathscr{A}}(W)$. Computing the cohomology of $\mathrm{Perm}_{\mathscr{A}}(W)$ is equivalent to computing the homology of $(\Delta(W), \Delta_{\mathscr{A}}(W))$, and we proceed by declaring orientations and boundaries via parabolic cosets.

Orientation. Consider the ordered simple system $S = \{s_1, \dots, s_n\}$ and assign to each cone wC_I the *algebraic orientation block* $[s_{i_1}, \dots, s_{i_k}]$, where each s_{i_ℓ} is a simple root not fixing wC_I, ordered following S. This yields

$$\partial[s_{i_1}, \dots, s_{i_k}] = \sum_{\ell=1}^{k} (-1)^{\ell-1} [s_{i_1}, \dots, \widehat{s_{i_\ell}}, \dots, s_{i_k}],$$

and equivariantly

$$\partial(wW_I) = \sum_{j \notin I} (-1)^{o(j)-1} wW_{I \cup \{s_j\}}, \tag{1}$$

where $o(j)$ is the relative position of the jth element of $S \setminus I$ in S.

Theorem (Additive). Let $\mathscr{A}$ be a parabolic arrangement of type W. Let C be the chain complex where C_k is the free abelian group generated by parabolic cosets (w, I) with $|I| = k$, modulo those lying in $\mathscr{A}$, with boundary as in (1). Then

$$H^k(\mathscr{M}(\mathscr{A}); \mathbb{Z}) \cong H_k(C).$$

In particular we have the following commutative diagram.

$$\begin{tikzcd}
C^k(\mathrm{Perm}_{\mathscr{A}}(W)) \arrow[r, "\psi", "\cong"'] \arrow[d, "g^{\#}"'] & C_{n-k}(\Delta(W), \Delta_{\mathscr{A}}(W)) \arrow[d, "j_{\#}", "\cong"'] \\
C^k(\mathrm{sd}(\mathrm{Perm}_{\mathscr{A}}(W))) \arrow[r, "\cap\, \mathrm{sd}\gamma"] & C_{n-k}(\mathrm{sd}(\Delta(W)), \mathrm{sd}(\Delta_{\mathscr{A}}(W)))
\end{tikzcd}$$

Multiplicative Structure

The Lefschetz' duality isomorphism used in the Additive Theorem can be understood in terms of a simplicial cap product as in the Poincaré–Lefschetz duality theorem. The common subdivision of both $\Delta(W)$ and $\mathrm{Perm}(W)$ gives a canonical isomorphism $\mathrm{sd}(\Delta(W)) \cong \mathrm{sd}(\mathrm{Perm}(W))$, inducing the natural duality

$$\_ \cap \Gamma \;:\; H^k(\mathrm{Perm}_{\mathscr{A}}(W)) \longrightarrow H_{n-k}(\Delta(W), \Delta_{\mathscr{A}}(W)).$$

Fundamental class. Let Γ be the fundamental class of $\Delta(W)$. Then $\Gamma_{\mathscr{A}} = \Gamma/\mathscr{A}$ is a fundamental class for the pair $(\Delta(W), \Delta_{\mathscr{A}}(W))$.

Theorem (Multiplicative). Let $\alpha \in H^p(\mathcal{M}(\mathscr{A}))$ and $\beta \in H^q(\mathcal{M}(\mathscr{A}))$. Then

$$\alpha \cup \beta = [A \pitchfork B],$$

where A and B are chains of parabolic cosets representing α and β in $H_p(C)$ and $H_q(C)$, respectively.

$$\begin{tikzcd}
H^p(\mathscr{M}(\mathscr{A})) \otimes H^q(\mathscr{M}(\mathscr{A})) \arrow[r, "\smile"] \arrow[d, "\mathrm{twist} \circ (D \otimes D)"'] & H^{p+q}(\mathscr{M}(\mathscr{A})) \arrow[d, "D"] \\
H^{\mathrm{BM}}_{n-q}(\mathscr{H}, \mathscr{A}) \otimes H^{\mathrm{BM}}_{n-p}(\mathscr{H}, \mathscr{A}) \arrow[r, "\bullet"] & H^{\mathrm{BM}}_{n-p-q}(\mathscr{H}, \mathscr{A})
\end{tikzcd}$$