The topological complexity of non-k-equal spaces

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Topological Complexity

Definition

The topological complexity of a space X equals the minimum n such that $X \times X$ admits an open cover $U_0, \ldots, U_n$ such that each U_i has a motion planning algorithm s_i.

TC bounds

Theorem

Let X be a c-connected space having the homotopy type of a CW complex, then

$$\operatorname{zcl}(X) \leq \operatorname{TC}(X) \leq \frac{2\operatorname{hdim}(X)}{c+1}.$$

Here $\operatorname{hdim}(X)$ *stands for the cellular homotopy dimension of* X, *and* $\operatorname{zcl}(X)$ *is the zero-divisors-cup-length for* X.

Non-k equal spaces

Definition (The k-equal arrangement)

$$A_d^{(k)}(n) = \left\{(x_1, \dots, x_n) \in \left(\mathbb{R}^d\right)^n \,\middle|\, x_{i_1} = \cdots = x_{i_k}\right\}$$

Definition (The non-k-equal space)

$$M_d^{(k)}(n) = \left(\mathbb{R}^d\right)^n$$

Non-k equal coordinates

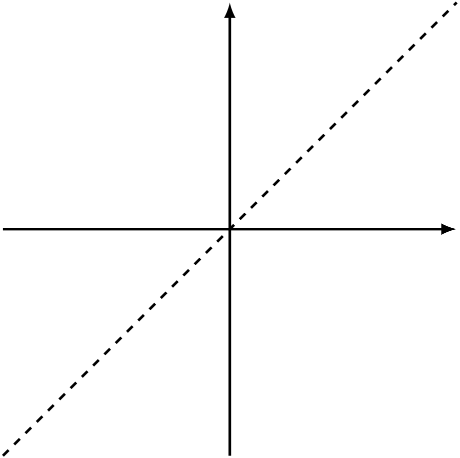
Spaces of n-tuples

$$M_d^{(k)}(n)$$

entries in $\mathbb{R}^d$

Trivial example

$\mathbb{R}^2 \setminus \Delta$

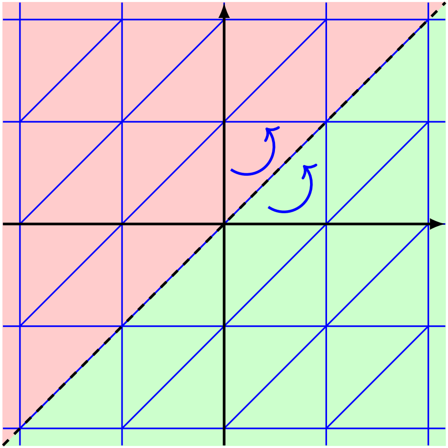


$$H^n(M_2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{si } n = 0, \\ 0 & \text{o. c.} \end{cases}$$

?

Trivial example

$\mathbb{R}^2 \setminus \Delta$



$$H^n(M_2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{si } n = 0, \\ 0 & \text{o. c.} \end{cases}$$

$\updownarrow$

$$H_n^{BM}(\mathbb{R}^2, \Delta) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{si } n = 2, \\ 0 & \text{o. c.} \end{cases}$$

The cohomology of $M_1^{(k)}(n)$

Example

One of the **elementary** cells generating cohomology in $M_1^{(3)}(8)$ are of the form

$$(1)[2,3](4,7,8)$$

where the notation means:

$$\{(x_1, x_2, \ldots, x_8) \mid x_1 \le x_2 = x_3 \le x_4, x_5, x_6, x_7, x_8\}.$$

Cup products in $M_1^{(k)}(n)$

Example

$$(1)[2,3](4,7,8) \smile (1,2,3,4)[5,6](7,8) = (1)[2,3](4)[5,6](7,8).$$

is the intersection product corresponding to

$$\{(x_1, x_2, \ldots, x_8) \,|\, x_1 \leq x_2 = x_3 \leq x_4 \leq x_5 = x_6 \leq x_7, x_8\}.$$

Cohomology ring of $M_1^{(k)}(n)$

Theorem (Baryshnikov (19991))

For $k \geq 3$, the cohomology ring $H^*(M_1^{(k)}(n))$ is isomorphic to the (anti)commutative free exterior algebra generated in dimension $k-2$ by the elementary preorders subject to the following relations:

1. $\sum_{\iota\in I}(-1)^{g(\iota)}(I-\iota)[J+\iota](K) = \sum_{\kappa\in K}(-1)^{g(\kappa)}(I)[J+\kappa](K+\kappa)$ *whenever* $\mathbf{n}$ *can be written as a disjoint union* $\mathbf{n} = I \coprod J \coprod K$ *with* $\mathrm{card}(J) = k-2$.
2. $(I)[J](K)\cdot(I')[J'](K') = 0$, *for elementary preorders* $(I)[J](K)$ *and* $(I')[J'](K')$ *whose intersection has a* $[\ ]$*-block of cardinality larger than* $k-1$.

Consecuences

- Severs & White (2012) shown the existence of a minimal CW model for $M_1^{(k)}(n)$.
- Minimum non-trivial cohomological dimension is $k-2$ (elementary terms). Hence $M_1^{(k)}(n)$ is $k-3$ connected.
- Maximum dimension is $\left\lfloor \frac{n}{k} \right\rfloor (k-2)$. If $q = \left\lfloor \frac{n}{k} \right\rfloor$, we can write

$$[1,\ldots,k-1]\,(k)\,[k+1,\ldots,2k-1]\,(2k)\cdots[(q-1)k+1,\ldots,qk-1]\,(qk)$$

Therefore, $\mathrm{hdim}(M_1^{(k)}(n)) = \left\lfloor \frac{n}{k} \right\rfloor (k-2)$ y $\dfrac{\mathrm{hdim}(M_1^{(k)}(n))}{k-2} = \left\lfloor \frac{n}{k} \right\rfloor$.

Upper bound

Corollary

La categoría y complejidad topológica (secuencial) para $M_1^{(k)}(n)$ satisface las siguientes desigualdades

$$\mathrm{cl}(M_1^{(k)}(n)) \leq \mathrm{cat}(M_1^{(k)}(n)) \leq \left\lfloor \frac{n}{k} \right\rfloor,$$

$$\mathrm{zcl}(M_1^{(k)}(n)) \leq \mathrm{TC}(M_1^{(k)}(n)) \leq 2\left\lfloor \frac{n}{k} \right\rfloor,$$

$$\mathrm{zcl}_s(M_1^{(k)}(n)) \leq \mathrm{TC}_s(M_1^{(k)}(n)) \leq s\left\lfloor \frac{n}{k} \right\rfloor.$$

Zero-divisors

- To produce zero divisors we use $x_m, x'_m \in H^{k-2}(M_1^{(k)}(n))$ dados por

$$x_m = (1, \ldots, m-2, m-1)\,[m, m+1, \ldots, m+k-2]\,(m+k-1, \ldots, n)\,,$$

$$x'_m = (1, \ldots, m-2, m)\,[m-1, m+1, \ldots, m+k-2]\,(m+k-1, \ldots, n)\,,$$

Then $y_m = x_m \otimes 1 + 1 \otimes x_m$ is a zero divisor

Zero-divisors-cup-length

Theorem

If i, k, n are such that $2 \leq i$ and $ik \leq n$, then

$$\prod_{j=1}^{i} y_{(j-1)k+1} y_{(j-1)k+2} \in H^*(M_1^{(k)}(n))^{\otimes 2}$$

is non-zero.

Corollary

$2 \left\lfloor \frac{n}{k} \right\rfloor \leq \mathrm{zcl}(M_1^{(k)}(n))$

cat y TC para $d = 1$

Theorem (J. González, J. L. León-Medina, and C. Roque-Márquez (2019))

In summary

$$\mathrm{TC}(M_1^{(k)}(n)) = 2\left\lfloor \frac{n}{k} \right\rfloor,$$

Khovanov proved that $M_1^{(3)}(n)$ is the classifying space of PP_n, the pure planar pure braid group.

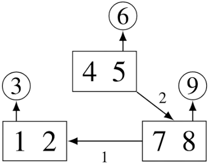
$$\mathrm{TC}(PP_n) = 2\left\lfloor \frac{n}{3} \right\rfloor$$

The case $M_d^{(k)}(n)$ with $d \geq 2$

The combinatorial description of the cohomology ring of $M_d^{(k)}(n)$ given by Dobrinskaya and Turchin is similar to the case $d = 1$. In this case, the cohomology is encoded by combinatorial objects called admissible k-forests.

Example

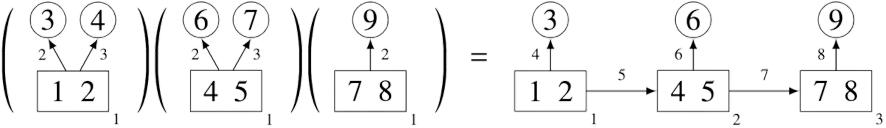
The next figure is a forest in $M_2^3(9)$ is an oriented 3-forest of degree 11. That 3-forest corresponds to a manifold in $M_2^3(9)$ as it is indicated in the right-hand side.



$$\Rightarrow \quad \begin{array}{ll} x_1 = x_2, & x_1^{(1)} = x_2^{(1)} \leq x_3^{(1)} \\ x_4 = x_5, & x_4^{(1)} = x_5^{(1)} \leq x_6^{(1)} \\ x_7 = x_8, & x_7^{(1)} = x_8^{(1)} \leq x_9^{(1)} \end{array}$$

Cup products

Hence, since the *k*-forests encode manifolds that in turn are cycles in Borel-Moore homology we have nice cup products dictated in terms of the intersection of the corresponding submanifolds or superposition of their graphs. An example of such a product is the following:



Bounds for TC

Unfortunately, the bounds do not coincide for the case $d > 1$. The inequalities determined by Lemma 2 for non-k-equal spaces for $d > 2$ are:

Corollary (J. Gonzalez, J.L. León, Theorem 3.3)

The LS category and TC_s *for* $M_d^{(k)}(n)$ *is bounded by*

$$s\left\lfloor\frac{n}{k}\right\rfloor \leq \mathrm{TC}_s(M_d^{(k)}(n)) \leq s\left(\left\lfloor\frac{n}{k}\right\rfloor + \left\lfloor\frac{\left(\left\lfloor\frac{n}{k}\right\rfloor + b - 1\right)(d-1)}{a}\right\rfloor\right).$$

where $a = d(k-1) - 1$ *and* $b = n - k\left\lfloor\frac{n}{k}\right\rfloor$ *(so* $0 \leq b < k$*).*

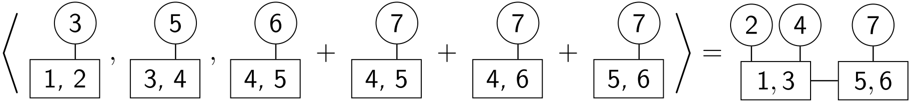
For $d = 2$

$n\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$k+1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+2$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+3$	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+4$	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+5$	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+6$	3	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+7$	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+8$	?	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+9$	4	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+10$	?	3	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
$k+11$	?	3	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1
$k+12$	?	4	3	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1
$k+13$	?	4	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
$k+14$	?	4	3	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1
$k+15$	?	?	4	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1
$k+16$	?	5	4	3	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1
$k+17$	?	5	4	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1	1
$k+18$	?	?	4	4	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1
$k+19$	?	?	4	4	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1
$k+20$	?	6	5	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1
$k+21$	?	?	5	4	4	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1
$k+22$	?	?	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2

Massey products

Theorem

Let $n \geq 7$, $d \geq 2$, then the following is a Massey product in $M_d^{(3)}(n)$



is non zero.

Ongoing/Future work

- See if Massey products can help improve the lower bound.
- Work with other complements of configuration spaces similar to k equal arrangements.
- For example $D_{n,k}$ (non-k-equal spaces up to signs).

Thank you for your attention!