Likelihood-confidence intervals for quantiles in Extreme Value Distributions

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Abstract

Profile likelihood intervals of large quantiles in Extreme Value distributions provide a good way to estimate these parameters of interest since they take into account the asymmetry of the likelihood surface in the case of small and moderate sample sizes; however they are seldom used in practice. In contrast, maximum likelihood asymptotic (mla) intervals are commonly used without respect to sample size. It is shown here that profile likelihood intervals actually are a good alternative for the estimation of quantiles for sample sizes $n \geq 25$ of block maxima, since they presented adequate coverage frequencies in contrast to the poor coverage frequencies of mla for these sample sizes, which also tend to underestimate the quantile and therefore might be a dangerous statistical practice.

In addition, maximum likelihood estimation can present problems when Weibull models are considered for moderate or small sample sizes due to singularities of the corresponding density function when the shape parameter is smaller than one. These estimation problems can be traced to the commonly used continuous approximation to the likelihood function and could be avoided by using the exact or correct likelihood function, at least for the settings considered here. A rainfall data example is presented to exemplify the suggested inferential procedure based on the analyses of profile likelihoods.

Key words: Exact likelihood function, extreme value distributions, maximized likelihood, profile likelihood, likelihood-confidence intervals, rainfall data.

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1 Introduction

According to the Fisher-Tippet theorem [2], only three families of distributions are the limits for the distribution of normalized maxima of i.i.d. random variables: Weibull, Gumbel, and Fréchet. These three families of Extreme Value distributions (EV) are submodels of a single family of distributions proposed independently by Von Mises [7] and Jenkinson [3] which is now known as the Generalized Extreme Value distribution (GEV).

Usually large quantiles Q_α of probability α of these distributions are of interest. Different confidence intervals for these quantiles can be obtained depending on the model used, the GEV or a specific subfamily of models–Fréchet, Gumbel or Weibull. Under the selected model, the usual procedure is to obtain asymptotic maximum likelihood (aml) confidence intervals which are symmetric about the maximum likelihood estimate (mle) and usually do not take into account the commonly marked asymmetry of the likelihood surface of large quantiles in the case of small or moderate samples and thus tend to underestimate the true value of the quantile.

Profile likelihood intervals for quantiles have not been fully explored in statistical literature for Extreme Value Theory and neither have their coverage properties in the cases of small and moderate samples. In this work, the coverage frequencies and lengths of likelihood intervals for quantiles are explored and compared to those of aml confidence intervals through a simulation study.

In addition, the profile likelihood intervals for the shape parameter of the GEV were also considered and shown to have good coverage frequencies. These intervals are of special importance since they can be used as an aid for submodel selection.

The use of the exact likelihood function, described in the following section, is recommended for the case of small sample sizes where a Weibull model might be reasonable, in order to avoid maximum likelihood estimation problems due to singularities of the corresponding density function.

As an example, a data set of yearly rain maxima collected at a monitoring station in Michoacán, México is presented to exemplify the likelihood based estimation procedures.

2 Relevant Related Statistical Concepts

The relative and profile or maximized likelihood functions of a parameter of interest will be presented here. In addition, the exact or correct likelihood function is defined as well. These functions contribute to simplify and improve the estimation of parameters of interest such as quantiles of Extreme Value distributions. Also, expressions for the probability densities and distribution functions of all the models involved are here provided, as well as for their corresponding quantiles, which are the main parameters of interest.

The densities of the three EV families for maxima are

$$\text{Gumbel:}\quad \lambda(x;\mu,\sigma) = \frac{1}{\sigma}\exp\left\{-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right] - \frac{x-\mu}{\sigma}\right\} I_{(-\infty,\infty)}(x)\,, \tag{1}$$

$$\text{Fréchet:}\quad \varphi(x;\mu,\sigma,\beta) = \frac{\beta}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{-\beta-1}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{-\beta}\right] I_{[\mu,\infty)}(x)\,,\ \text{and} \tag{2}$$

$$\text{Weibull:}\quad \psi(x;\mu,\sigma,\beta) = \frac{\beta}{\sigma}\left(\frac{\mu-x}{\sigma}\right)^{\beta-1}\exp\left[-\left(\frac{\mu-x}{\sigma}\right)^{\beta}\right] I_{(-\infty,\mu]}(x)\,, \tag{3}$$

with location, scale and shape parameters $\mu \in \mathbb{R}$, $\sigma > 0$ and $\beta > 0$, respectively. For the Weibull and Fréchet densities, μ is also a threshold parameter, since it represents an upper or lower bound, respectively, for the support of the corresponding random variable. Note that for $\beta < 1$, the Weibull density has a singularity at $x = \mu$.

The Generalized Extreme Value distribution (GEV) density function is

$$g(z;a,b,c)=\begin{cases}\frac{1}{b}\left[1+c\left(\frac{x-a}{b}\right)\right]^{-1-1/c}\exp\left\{-\left[1+c\left(\frac{x-a}{b}\right)\right]^{-\frac{1}{c}}\right\}I_{(-\infty,a-\frac{b}{c})}(x) & \text{if } c<0,\\ \exp\left\{-\exp\left[-\left(\frac{x-a}{b}\right)\right]\right\}I_{(-\infty,\infty)}(x), & \text{if } c=0,\\ \frac{1}{b}\left[1+c\left(\frac{x-a}{b}\right)\right]^{-1-1/c}\exp\left\{-\left[1+c\left(\frac{x-a}{b}\right)\right]^{-\frac{1}{c}}\right\}I_{(a-\frac{b}{c},\infty)}(x) & \text{if } c>0,\end{cases} \tag{4}$$

where a,b,c are location, scale and shape parameters, respectively, $b>0$ and $a,c\in\mathbb{R}$. The GEV corresponds to the Weibull, Gumbel, or Fréchet distributions according to whether c is negative, zero, or positive, respectively. Note that the expression given for $c=0$ is the limit of $g(z;a,b,c)$ when c tends to zero. The parameters of the EV models and the corresponding GEV are connected through a one to one relationship given in Table 1.

	Parameter:	Threshold/Location	Scale	Form
Weibull	$c<0$	$\mu=a-b/c$	$\sigma=-b/c$	$\beta=-1/c$
Gumbel	$c=0$	$\mu=a$	$\sigma=b$	—
Fréchet	$c>0$	$\mu=a-b/c$	$\sigma=b/c$	$\beta=1/c$

Table 1. Parameters for the EV and GEV distributions

In the case of the Weibull and Fréchet models for maxima, the threshold is isolated in a single parameter μ that may have a clear physical interpretation. Inferences in terms of estimation intervals for this parameter are simpler with an EV distribution in contrast to the corresponding threshold for the GEV, which is a function of all three parameters a,b,c.

It is important to note that there exist Weibull and Fréchet models that are very close and practically indistinguishable from a Gumbel model. That is, the Gumbel distribution is a limit of Weibull distributions with parameters related as shown in Table 1. The Gumbel model is embedded in the Weibull family of models in this sense, as well as in the Fréchet family (Cheng and Iles [1]).

All these models can be parametrized in terms of a quantile of interest by direct algebraic substitution in (1), (2) and (3) since any quantile can be expressed as a function of the other parameters as shown in Table 2. Therefore, the model can be expressed in terms of the quantile of interest which substitutes one of the remaining parameters. For example, the Weibull model can be reparametrized in terms of (Q_α,σ,β) instead of (μ,σ,β).

	Quantile of probability α
Weibull	$Q_\alpha=\mu-\sigma\left(-\log\alpha\right)^{1/\beta}$
Gumbel	$Q_\alpha=\mu-\sigma\log\left(-\log\alpha\right)$
Fréchet	$Q_\alpha=\mu+\sigma\left(-\log\alpha\right)^{-1/\beta}$
GEV	$Q_\alpha=\begin{cases}a-b\log\left(-\log\alpha\right), & \text{if } c=0,\\ a-\frac{b}{c}\left[1-\left(-\log\alpha\right)^{-c}\right], & \text{if } c\neq 0.\end{cases}$

Table 2. Quantiles for the EV and GEV distributions.

The asymptotic properties of maximum likelihood estimators are invoked in order to obtain confidence intervals for the parameters of interest. Usually the **continuous approximation to the likelihood** function as defined in Kalbfleisch [4] is the one used in most statistical textbooks to define the likelihood function for continuous random variables, without taking notice that it is an approximation. For an observed sample of n independent continuous random variables identically distributed, the continuous approximation to the likelihood function is

$$L(\theta; x_1, ..., x_n) = \prod_{i=1}^{n} f(x_i; \theta), \tag{5}$$

where θ is the vector of parameters, and f is the density function of the selected model.

This continuous approximation to the likelihood is only valid if the density functions do not have singularities (see Montoya et al [6]). For example, for a given observed sample, the joint Weibull density has a singularity when the threshold parameter equals the largest observation, $\mu = x_{(n)}$, if the shape parameter β is smaller than one, $\beta < 1$.

However, the data are *always* discrete since all measuring instruments have finite precision. Therefore, the data can only be recorded to a finite number of decimals. Thus the observation $X = x$ can be interpreted as $x - \frac{1}{2}h \leq X \leq x + \frac{1}{2}h$, where h is the precision of the measuring instrument, and so is a fixed positive number. For independent observations $x = (x_1, ..., x_n)$, the **exact or correct likelihood function** L_E is defined to be proportional to the joint probability of the sample,

$$\begin{aligned} L_E(\theta; y) &\propto \prod_{i=1}^{n} P(y_i - \tfrac{1}{2}h \leq Y_i \leq y_i + \tfrac{1}{2}h) \\ &= \prod_{i=1}^{n} \left[F\left(y_i + \tfrac{1}{2}h; \theta\right) - F\left(y_i - \tfrac{1}{2}h; \theta\right)\right], \end{aligned} \tag{6}$$

where F is the corresponding distribution function of the continuous model in consideration.

Allowing $h = 0$ implies that the measuring instrument has infinite precision and that the observations can be recorded to an infinite number of decimals. Since for a continuous random variable X, $P(X = x; \theta) = 0$ for all x and θ, this cannot be the basis for obtaining a likelihood function. If in contrast, one assumes that the precision of the measuring instrument is $h > 0$, then conditions are required for the density function $f(y; \theta)$ to be used as an approximation to the likelihood function (6), as required by the Mean Value Integral Theorem of Calculus. But if the density function has a singularity at any given value of θ, then these conditions are violated and $f(y; \theta)$ cannot be used to approximate the likelihood function at that value of θ ([4], Section 9.4).

As Meeker and Escobar ([5], p. 275) mention, there is a path in the parameter space for which the continuous approximation to the likelihood (5) goes to infinity, in particular for the Weibull case, when $\beta < 1$ and $\mu \to x_{(n)}$. It should be stressed that the likelihood approaches infinity not necessarily because the probability of the data is large in that region of the parameter space, but instead because of a breakdown in the density approximation to the likelihood function. There is usually, as happened with all simulations considered here, though not necessarily always, a local maximum for this likelihood surface corresponding to the maximum of the exact likelihood based on the probability of the data shown in (6).

A useful standardized version of a likelihood function $L(\theta; x)$ that will be used here, is the relative likelihood function that has a value of 1 at its maximum, the mle $\hat{\theta}$, and is defined as

$$R(\theta;x) = \frac{L(\theta;x)}{L\left(\hat{\theta};x\right)}, \tag{7}$$

so that $0 \leq R(\theta;x) \leq 1$. Values of θ with $R(\theta;x)$ close to one are more plausible than values close to zero. A relative likelihood is easy to plot and to interpret. Likelihood intervals or regions of $k\%$ likelihood level are obtained by cutting horizontally this likelihood function; that is

$$\{\theta : R(\theta;x) \geq k\}, \quad 0 \leq k \leq 1. \tag{8}$$

For example, if $k = 0.15$, under some regularity conditions, the corresponding likelihood interval has an asymptotic approximate 95% confidence level, using the Chi-squared limit distribution for the likelihood ratio statistic ([4] Section 11.3). However this result may also hold for moderate samples, and even small samples, if the likelihood surface is symmetric about the mle. In these cases the interval in (8) is called a likelihood-confidence interval.

If the GEV model is parametrized in terms of a quantile of interest, then the profile or maximized likelihood function of Q_α (Kalbfleisch, 1985, Section 10.3) is defined for sample $x = (x_1, ..., x_n)$ as

$$L_p(Q_\alpha;x) = \max_{b,c|Q_\alpha} L(Q_\alpha, b, c; x).$$

The corresponding relative likelihood can be calculated as in (7) . Profile relative likelihoods and their plots are very informative about plausible ranges for the parameter of interest, in the light of the observed sample.

In the case of the profile likelihood of the GEV shape parameter c, the relative likelihood at $c = 0$ is indicative of the support given by the sample to the Gumbel model, which corresponds to $c = 0$. For example if $R_p(c = 0) \geq 0.5$, the Gumbel model has moderate or high plausibility and should definitely be considered as a possible model; its fit to the sample should be compared with the fit of the best member of the family of EV models suggested by the sign and value of the mle $\hat{c}$.

Summarizing, in order to make inferences about a parameter of interest, for example a quantile, the corresponding plot of the relative profile likelihood should be analyzed because it is very informative. Inferences about the parameter of interest should be presented in terms of likelihood-confidence intervals, especially in the case of small or moderate samples. These intervals calculated for two large quantiles, $Q_{.95}$, $Q_{.99}$, and for the GEV shape parameter c showed through simulations, reported in the following sections, to have adequate coverage frequencies for moderate sample sizes ($n \geq 50$), and even for $n = 25$ in the case of Gumbel and Fréchet models.

3 Simulations

For the simulation study, the samples of maxima were chosen to come from one of the EV distributions, (or equivalently a GEV distribution) and not from a distribution belonging to the domain of attraction of an EV. Samples were simulated from the GEV with parameters $a = 1$, $b = 1$ and

$$c \in \{-0.5, -0.4, -0.3, -0.2, -0.1, -0.05, 0, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5\},$$

for sample sizes of $n = 25$ and 50. Additional values of c, ± 0.01 and ± 0.001 were considered as well as the previous ones, for $n = 100$ in order to explore the cases around $c = 0$. These cases are such that there are models from the three subfamilies of EV that are very close to each other.

Size 50 is frequently found in samples coming from meteorological applications, and sample size 100 was chosen to explore the effect of increasing sample size. For each value of c and sample size, 10,000 samples were generated in Matlab 7.

For each sample of maxima, the mle's of the parameters (a, b, c) of the GEV distribution were calculated using the continuous approximation to the likelihood function. This is the current procedure in Extreme Value literature. The cases where the singularities of this density caused numerical problems for finding the local maximum (the mle) were registered and the exact likelihood function was used then to obtain the mle's.

For each simulated sample, the corresponding EV model was selected automatically according as $\hat{c} < -10^{-5}$ (Weibull), $|\hat{c}| < 10^{-5}$ (Gumbel) or $\hat{c} > 10^{-5}$ (Fréchet). The mle's of the corresponding parameters were obtained by maximizing the likelihood derived from (1), (2) or (3), accordingly, reparametrized in terms of the quantile of interest, which worked well in most of the cases. Only when $\hat{c} < -1$ and $\hat{\beta} < 1$, it was necessary to use the corresponding exact Weibull likelihood function, as mentioned above. These cases were registered, since they represent cases where the continuous approximation to the likelihood function would not have been able to produce an mle with these EV distributions.

Using the invariance property of the likelihood function, the mle's of quantiles $Q_{.95}$ and $Q_{.99}$ can be obtained from the mle's of the parameters of the EV or GEV, though they were obtained directly from the corresponding likelihood function parametrized in terms of these quantiles. From their corresponding relative likelihoods, 15% likelihood intervals were obtained for c, $Q_{.95}$, and $Q_{.99}$. As mentioned above, these intervals may have an approximate 95% confidence level in the case of moderate sample sizes, using the Chi-squared limit distribution for the likelihood ratio statistic ([4] Section 11.3). For each of these intervals it was checked whether they included the true value of the corresponding parameter in order to calculate the associated coverage frequency. For those intervals that excluded the true value of the parameter of interest, the number of times that the interval underestimated or overestimated was registered. Also the lengths of the intervals that covered the true value of the parameter were registered and compared as shown in the following section. In addition, the asymptotic maximum likelihood (aml) confidence intervals were obtained for $Q_{.95}$ and $Q_{.99}$ and their coverage frequencies were registered.

4 Results

Tables 3 and 4 present the coverage frequencies for $Q_{.95}$ and $Q_{.99}$ of 15% relative profile likelihood intervals and their corresponding aml intervals in the case of samples of size $n = 25, 50,$ and 100. Asymptotically these 15% likelihood intervals should have 95% coverage frequencies. Table 5 gives the coverage frequencies of 15% relative profile likelihood intervals for the parameter c of the GEV model for samples of size 100 and 50. The last two columns of this table report for each scenario the number of samples that selected the correct EV model according to the sign of the mle $\hat{c}$ and the number of samples where the product of the interval endpoints was negative. These are cases where the three EV models are plausible, since the value of $c = 0$ is included in the interval.

Figure 1 shows the coverage frequencies of the quantiles of interest contained in Tables 3 to 5 in a graphical way. Figures 2 and 3 show the ratios of the lengths of the relative profile likelihood intervals under the selected EV model compared to those under the GEV model and Figures 4 and 5 give the length of profile likelihood intervals for the GEV using boxplots in which the box corresponds to the interquartile range and the whiskers have a maximum length of 1.5 times the interquartile range. Points beyond the end of the whiskers are represented individually and the line inside the box is the median. Only samples for which all intervals covered the true value of the quantile were considered in these graphs.

Some remarks about the tables and figures are given below. Note that EV submodels are selected automatically, based only on the sign and size of $\hat{c}$, so the reported coverage frequencies correspond to a 'worst case' scenario. With a real data set, additional external information from experts would be taken into account for choosing an adequate submodel, and consequently the statistical modeling would be more efficient.

1. **Coverage frequencies of GEV profile likelihood intervals and number of samples with estimation problems.** Coverage frequencies of relative profile likelihood intervals for the GEV were very stable throughout the range of values of c for both quantiles. They tend to decrease as c moves towards more negative values. For $n = 100$ there were no numerical problems when calculating the mle's. For $n = 50$ the number of samples with numerical problems was insignificant. However for $n = 25$, more samples presented problems in the case of Weibull models with values of c smaller than -0.2. The number of problematic cases grows as c goes to -0.5 and is above 1.8% for $c = -0.4$ and above 5% for $c = -0.5$. The number of samples that had numerical problems was the same for both quantiles considered. Therefore, numerical problems are associated to small sample sizes and Weibull models with large negative values of c.

2. **Coverage frequencies of EV profile likelihood intervals.** Coverage frequencies of relative profile likelihood function intervals for the EV were not so stable, and in all cases there is a region of decrease, mainly in the Fréchet domain, where frequencies drop, as shown in Figure 1. This region grows wider as the sample size gets smaller, and the value where the minimum occurs shifts to the right from around 0.1 for $n = 100$ to around 0.2 for $n = 25$. The drop is always more pronounced for $Q_{.99}$ than for $Q_{.95}$. This can be explained by the fact that for the samples that did not cover the true value of the quantile, the mle $\hat{c}$ was negative in most cases and the whole interval lay below this true value and therefore underestimated it (see the second columns in Tables 3 and 4). In the Fréchet cases, these problems were associated to estimating a large Fréchet quantile with a Weibull model that has a bounded right tail.

3. **Coverage frequencies of aml intervals.** Aml intervals always had poorer coverage frequencies than relative profile likelihood intervals for the GEV for all the sample sizes considered here. Coverage frequencies for aml intervals calculated for the GEV and EV distributions are almost identical. Although coverage frequencies for these intervals improve as the sample size grows, as predicted by asymptotic theory, they

can be very poor for $n = 25$ and 50, and still unsatisfactory even for $n = 100$. This indicates that samples of greater size are required for these intervals to have suitable coverage frequencies. In all cases the intervals that failed to cover the true values tended to underestimate them.

4. **Asymmetry of proportions of intervals that exclude the true value.** Except for one single case ($n = 50,\ Q_{.99},\ c = 0.5$) there were always more relative profile likelihood intervals that underestimated than overestimated the true value of the quantile. This asymmetry is more pronounced for smaller sample sizes, $n = 25$. The asymmetry also increases as c becomes smaller and is very marked in the Weibull case. This may be due to the fact that the Weibull distribution has a finite upper limit and intervals tend to increase in size with c. Therefore estimating a large quantile from a sample with $\hat{c} << c$ will tend to underestimate the true value while in the case $\hat{c} >> c$ the interval will be larger and more likely to include the true value. However, even if this asymmetry is not desirable, the asymmetry of aml intervals is certainly much more marked than the one for profile likelihood intervals.

5. **Interval lengths.** Almost always intervals obtained with the GEV models are larger than those obtained with EV distributions as shown in Figures 2 and 3. Only samples where both intervals included the true value of the parameter were considered. The length of the intervals tended to be alike for large values of $|c|$, although there is some asymmetry in this, with Fréchet intervals being closer in length than the corresponding Weibull cases. Also, the ratio of lengths is closer to one for $Q_{.95}$ than for $Q_{.99}$. For both quantiles the largest difference occurs at $c = -0.05$ for $n = 100$ and 50, and at $c = -0.1$ for $n = 25$. In Figures 2 and 3, the region where the interquartile boxes are visible (i.e. where the length differences are more important) coincides roughly with the region where there is a drop in the coverage frequencies for the EV distributions. This shows that there is a trade off between coverage and precision in the choice of a model: There is the possibility of gaining precision in the estimation but a the risk of reducing the confidence level of the interval. It is important to note that for the same quantile and sample size, the lengths of confidence intervals grow with c, as shown by Figures 4 and 5. This is to be expected since Weibull distributions are bounded above while Gumbel and Fréchet are not. Figure 6 shows the length between the true values of $Q_{.01}$ and $Q_{.99}$ of the corresponding distribution, as the parameter c increases.

6. **Effect of sample size on interval length.** As one would expect, the length of the intervals decreases as the sample size increases, but not uniformly. Halving the sample size from $n = 50$ to 25 increases interval length by a factor between 1.84 to 2.65, depending on the value of c, and by a factor of 1.56 to 1.78 when decreasing from $n = 100$ to 50. Also, for a fixed sample size the length of intervals for $Q_{.99}$ is always larger than those of $Q_{.95}$, as shown in Figure 5.

7. **Coverage frequencies of GEV shape parameter** c. The coverage frequencies of the profile likelihood intervals of this parameter, shown in Table 5, are stable throughout the range of values of c, with a slight decrease for the more negative values of c. The proportion of intervals that underestimate is much larger than those that overestimate

the true value of c, especially in the Weibull cases. This asymmetry diminishes as c takes larger positive values.

8. **Asymmetry in the correct automatic selection of a model.** The number of simulated samples where the estimator $\hat{c}$ has the same sign as the true value of c, as the column “correct” shows in Table 5, depends on the value of c. Although the difference is not pronounced, it is always more likely for the same value of $|c|$ that the signs coincide in a Weibull case than in the corresponding Fréchet case. On the other hand, it is more likely that intervals in the Fréchet case cover the origin, and therefore make plausible a Gumbel model, as the “negative” column shows in Table 5.

5 Rain Data Example

In the state of Michoacán, México, near its capital city Morelia, there is a monitoring meteorological station located at the Cointzio dam. This station is representative of rainfall patterns in this area. Yearly maxima of daily rainfall were obtained for 58 years in a period between 1940 and 2002. In this area, there is a marked rainy season from May to September. This data set will serve to illustrate the statistical modelling procedures suggested here. As a first step, the relative profile likelihood of the GEV shape parameter c shown in Figure 7(a) assigns plausibility only to positive values of c and the mle is $\hat{c} = 0.21$. therefore suggesting a Fréchet model. Since rain data are necessarily non-negative, for physical reasons it is important to consider a Fréchet model with a non-negative lower threshold parameter $\mu \geq 0$ that could very well simplify to a two parameter Fréchet model, where $\mu = 0$. The relative profile likelihood of μ under the three parameter Fréchet model shown in Figure 7(b), clearly assigns a very high plausibility to the value of $\mu = 0$, so that the data appear to support strongly a two parameter Fréchet model. Under this model, the maximum likelihood estimates are

$\hat{\sigma}$	$\hat{\beta}$	$\hat{Q}_{.95}$	$\hat{Q}_{.99}$
36.99	4.57	70.87	101.25

.

Figures 8(a) and 8(b) present together, for the sake of comparison, the corresponding relative profile likelihoods of these large quantiles of interest under the two parameter Fréchet model and also under the GEV model without any restrictions to its parameters. The GEV model without restrictions for its threshold corresponds as well to a three parameter Fréchet model without restriction to its threshold parameter; the corresponding Fréchet mle’s are

$\hat{\mu}$	$\hat{\sigma}$	$\hat{\beta}$	$\hat{Q}_{.95}$	$\hat{Q}_{.99}$
-1.55	38.57	4.76	70.64	100.44

.

In terms of the GEV distribution’s parameters, the mle’s are given by

$\hat{a}$	$\hat{b}$	$\hat{c}$
37.02	8.1	0.21

The likelihood intervals obtained for these quantiles with the GEV model are larger and imply that larger values of these quantiles are plausible. Also in these graphs, the aml GEV intervals are marked and show that their right endpoints tend to coincide with the right endpoints of the profile likelihood intervals of the two Fréchet model for these quantiles; nevertheless the left points are much smaller than the other likelihoods endpoints and therefore include small values of the quantiles that are implausible under both models (two parameter Fréchet and the GEV). That is, the aml intervals tend to underestimate the values of the quantiles.

The likelihood ratio statistic of these two models for this data set is

$$\frac{L_{\text{Fréchet}}\left(\mu = 0, \hat{\sigma}, \hat{\beta}; x\right)}{L_{\text{Fréchet}}\left(\hat{\mu}, \hat{\sigma}, \hat{\beta}; x\right)} = 0.9983.$$

Since these models are nested, this likelihood ratio has a chi squared distribution with one degree of freedom. The observed value of 0.9983 with a p-value of 0.32, indicates that the two Fréchet parameter model makes the observed sample equally probable. However since the two Fréchet parameter model is simpler and fits adequately the data set as shown in Figure 9(a), this model should be preferred. Figure 9(a) shows the corresponding quantile-quantile plot with pointwise likelihood bands that includes all observed values. Moreover, this model should be taken into account due to the physical considerations stated above.

Likelihood-confidence intervals of 15% likelihood level and approximate 95% confidence level for the quantiles of interest $Q_{.95}$ and $Q_{.99}$ under the two parameter Fréchet model are $(61.6, 85.06)$ and $(83.02, 131.66)$ respectively. Finally Figure 9(b) shows the return periods plot with profile likelihood 15% level bands marked for both the GEV model and the two Fréchet model. Since rainfall levels higher than 200ml are associated with floodings of Morelia, and since a return period of a 100 years is associated to quantile $Q_{.99}$, then the probability is extremely low that the city of Morelia gets flooded within 100 years.

6 Conclusions

Overall, profile likelihood intervals of large quantiles of Extreme Value distributions and of the GEV shape parameter c performed well and had adequate coverage frequencies for moderate and small sample sizes. In contrast, the corresponding aml intervals are symmetric about the mle and had lower and poor coverage frequencies in the case of samples of size $n \leq 100$. Moreover, a large proportion of the aml intervals that excluded the true value tended to underestimate it. The aml intervals are frequently used in Extreme Value Theory applications without notice of these issues.

Profile likelihood intervals of EV submodels tend to be shorter than the corresponding GEV profile likelihood intervals when the true value of c is close to zero, that is when $c \in (-.05, .05)$ if the sample size is $n \leq 50$. Nevertheless, their coverage frequencies are adequate so that they should be preferred when the model selection of an EV is clear. However, if there is no additional external information on a given preferred EV model suggested by the theory behind the specific phenomenon of interest, then using GEV profile likelihood intervals is a conservative procedure since they also had good coverage frequencies, even though these intervals tended to be larger.

Profile likelihood intervals of c may serve as an aid in model selection. They also had adequate coverage frequencies. For values of c in a region around zero $(-0.01, 0.01)$ approximately 95% of the likelihood intervals for the simulated samples included the value of zero. These are cases where the three EV models are plausible for the given sample, and also where the Gumbel model usually has a moderate or high plausibility given by the relative profile likelihood of c at zero. This is indicative of the need of additional external information of experts and other diagnostic methods to select adequately the best and most simple model for the phenomenon of interest. This will improve the estimating precision, and will prevent underestimating the quantile of interest.

Finally, for sample sizes smaller than 50 and in the case that a Weibull model might be an appropriate choice, then the use of the exact likelihood function is suggested in order to make inferences about the parameters of interest through profile likelihood intervals.

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n=100, Q_{95}													
	SUBMODEL						GEV						
	Profile Likelihood Ints.			AML			Profile Likelihood Ints.				AML		
c	<	C. F.	>	<	C. F.	>	<	C. F.	>	SNP	<	C. F.	>
-0.5	543	9314	143	984	8877	139	543	9314	143	0	984	8877	139
-0.4	438	9408	154	884	9003	113	438	9408	154	0	884	9003	113
-0.3	442	9374	184	888	9023	89	442	9374	184	0	887	9024	89
-0.2	379	9449	172	872	9076	52	378	9450	172	0	872	9076	52
-0.1	347	9457	196	844	9120	36	344	9469	187	0	842	9124	34
-0.05	392	9423	185	805	9169	26	363	9464	173	0	805	9170	25
-0.01	400	9395	205	755	9215	30	315	9496	189	0	755	9216	29
-0.001	423	9374	203	778	9191	31	338	9469	193	0	777	9192	31
0.0	394	9415	191	751	9221	28	325	9490	185	0	751	9221	28
0.001	416	9380	204	792	9180	28	332	9473	195	0	792	9182	26
0.01	443	9396	161	785	9194	21	341	9509	150	0	784	9196	20
0.05	446	9358	196	735	9240	25	309	9501	190	0	731	9245	24
0.1	496	9302	202	781	9195	24	321	9477	202	0	781	9195	24
0.2	285	9495	220	727	9262	11	267	9513	220	0	727	9262	11
0.3	280	9506	214	757	9232	11	280	9506	214	0	757	9232	11
0.4	298	9477	225	782	9214	04	298	9477	225	0	782	9214	04
0.5	296	9472	232	740	9260	0	296	9472	232	0	740	9260	0
n=50													
c	<	C. F.	>	<	C. F.	>	<	C. F.	>	SNP	<	C. F.	>
-0.5	576	9288	136	1382	845	168	572	9287	136	5	1379	8448	168
-0.4	552	9309	139	1346	8547	107	551	9308	139	2	1343	8548	107
-0.3	505	9345	15	1271	8663	66	505	9345	15	0	1271	8663	66
-0.2	433	9435	132	1195	8763	42	433	9435	132	0	1195	8763	42
-0.1	412	9432	156	1105	8881	14	383	9466	151	0	1105	8882	13
-0.05	447	9379	174	1111	8875	14	388	9449	163	0	1112	8876	12
0.0	503	9327	17	1047	8945	8	351	9486	163	0	1045	8947	8
0.05	567	9243	19	1067	8927	6	346	9466	188	0	1063	8931	6
0.1	64	9168	192	1069	8929	2	359	9449	192	0	1068	893	2
0.2	461	9333	206	1001	8997	2	304	9491	205	0	1001	8997	2
0.3	333	9458	209	978	9022	0	297	9494	209	0	977	9023	0
0.4	309	9492	199	985	9015	0	307	9494	199	0	985	9015	0
0.5	283	9468	249	993	9007	0	283	9468	249	0	993	9007	0
n=25													
c	<	C. F.	>	<	C. F.	>	<	C. F.	>	SNP	<	C. F.	>
-0.5	757	9107	136	2165	755	285	583	8787	115	515	1826	7469	19
-0.4	639	9244	117	2037	7834	129	55	9153	109	188	1886	7815	111
-0.3	599	9287	114	1839	8097	64	545	9264	112	79	1773	8089	59
-0.2	556	9313	131	1759	8209	32	526	9328	129	17	1744	8208	31
-0.1	547	9315	138	1658	833	12	499	9356	136	9	1648	8331	12
-0.05	595	9259	146	1659	8334	7	485	9365	145	5	1656	8332	7
0	586	9252	162	1535	8457	8	398	9439	158	5	1528	8459	8
0.05	714	9103	183	1576	8421	3	43	9385	182	3	1571	8423	3
0.1	734	9109	157	1481	8518	1	382	9458	157	3	1478	8518	1
0.2	819	9008	173	1457	8543	0	392	9435	173	0	1457	8543	0
0.3	625	9195	18	1434	8566	0	338	9482	18	0	1433	8567	0
0.4	398	9414	188	1311	8689	0	286	9526	188	0	1311	8689	0
0.5	353	9434	213	1385	8615	0	307	948	213	0	1385	8615	0

Table 3. Coverage frequencies for $Q_{.95}$ with sample sizes 100, 50 and 25. C.F. stands for Coverage Frequencies, '<' is the number of intervals that fell below the true value, '>' the number that fell above and SNP represents the number of samples with numerical problems.

n=100, Q_{99}													
	SUBMODEL						GEV						
	Profile Likelihood Ints.			AML			Profile Likelihood Ints.				AML		
c	<	C. F.	>	<	C. F.	>	<	C. F.	>	SNP	<	C. F.	>
-0.5	590	9332	78	1730	8258	12	590	9332	78	0	1731	8257	12
-0.4	448	9460	92	1376	8622	2	448	9460	92	0	1375	8623	2
-0.3	418	9440	142	1227	8770	3	418	9440	142	0	1227	8770	3
-0.2	401	9444	155	1128	8870	2	401	9448	151	0	1128	8870	2
-0.1	335	9376	289	1054	8940	6	335	9493	172	0	1053	8943	4
-0.05	353	9325	322	967	9028	5	341	9479	180	0	966	9031	3
-0.01	431	9286	283	913	9083	4	299	9490	211	0	913	9084	3
-0.001	494	9242	264	923	9074	3	330	9468	202	0	923	9075	2
0.0	490	9264	246	923	9077	0	324	9476	200	0	924	9076	0
0.001	510	9247	243	930	9065	5	349	9458	193	0	930	9069	1
0.01	570	9223	207	907	9091	2	337	9485	178	0	905	9094	1
0.05	863	8913	224	875	9124	1	293	9490	217	0	869	9130	1
0.1	857	8925	218	887	9113	0	315	9467	218	0	883	9117	0
0.2	282	9486	232	845	9155	0	263	9505	232	0	845	9155	0
0.3	269	9496	235	857	9143	0	269	9496	235	0	857	9143	0
0.4	291	9472	237	887	9113	0	291	9472	237	0	888	9112	0
0.5	288	9477	235	865	9135	0	288	9477	235	0	864	9136	0
n=50													
	SUBMODEL						GEV						
-0.5	628	9306	66	2297	7690	13	623	9306	66	5	2293	7689	13
-0.4	576	9355	69	1915	8083	2	575	9354	69	2	1913	8083	2
-0.3	521	9356	123	1702	8298	0	521	9359	120	0	1702	8298	0
-0.2	429	9412	159	1475	8525	0	429	9452	119	0	1475	8525	0
-0.1	555	9177	268	1399	8598	3	399	9440	161	0	1398	8602	0
-0.05	421	9320	259	1326	8672	2	383	9460	157	0	1325	8675	0
0.0	624	9157	219	1249	8750	1	361	9459	180	0	1249	8751	0
0.05	923	8872	205	1261	8739	0	348	9455	197	0	1257	8743	0
0.1	1131	8661	208	1232	8768	0	333	9460	207	0	1231	8769	0
0.2	577	9190	233	1138	8862	0	289	9478	233	0	1137	8863	0
0.3	313	9459	228	1127	8873	0	281	9491	228	0	1127	8873	0
0.4	286	9479	235	1172	8828	0	286	9479	235	0	1173	8827	0
0.5	264	9461	275	1137	8863	0	264	9461	275	0	1136	8864	0
n=25													
	SUBMODEL						GEV						
-0.5	772	9166	62	3069	6896	35	530	8893	62	515	2600	6872	13
-0.4	668	9247	85	2687	7309	4	549	9199	64	188	2509	7299	4
-0.3	624	9221	155	2321	7678	1	564	9257	100	79	2244	7677	0
-0.2	531	9290	179	2078	7922	0	516	9354	113	17	2061	7922	0
-0.1	535	9250	215	1916	8083	1	497	9349	145	9	1907	8084	0
-0.05	572	9234	194	1901	8099	0	458	9389	148	5	1896	8099	0
0.0	659	9149	192	1755	8245	0	405	9437	153	5	1748	8247	0
0.05	1015	8790	195	1822	8178	0	418	9396	183	3	1817	8180	0
0.1	1252	8555	193	1639	8361	0	377	9431	189	3	1636	8361	0
0.2	1363	8435	202	1618	8382	0	389	9409	202	0	1616	8384	0
0.3	788	8999	213	1607	8393	0	330	9457	213	0	1606	8394	0
0.4	411	9357	232	1492	8508	0	286	9482	232	0	1492	8508	0
0.5	326	9414	260	1593	8407	0	298	9442	260	0	1593	8407	0

Table 4. Coverage frequencies for $Q_{.99}$ with sample sizes 100, 50 and 25. C.F. stands for Coverage Frequencies, '<' is the number of intervals that fell below the true value, '>' the number that fell above and SNP represents the number of samples with numerical problems.

15% Profile Likelihood Intervals for c with n=100					
c	<	Cov. Freq.	>	Correct	Negative
-0.5	564	9328	108	10000	0
-0.4	396	9479	125	10000	0
-0.3	410	9425	165	10000	45
-0.2	371	9451	178	9993	1721
-0.1	348	9465	187	9375	6988
-0.05	297	9458	245	7787	8766
-0.01	272	9490	238	5772	9416
-0.001	296	9465	239	5264	9458
0.0	303	9460	237	0	9460
0.001	309	9461	230	4735	9454
0.01	286	9483	231	5357	9392
0.05	255	9482	263	7299	8605
0.1	299	9444	257	8866	6773
0.2	244	9463	293	9902	2232
0.3	246	9465	289	9992	265
0.4	236	9483	281	10000	5
0.5	259	9467	274	10000	0

15% Profile Likelihood Intervals for c with n=50					
c	<	Cov. Freq.	>	Correct	Negative
-0.3	467	9394	139	9989	1653
-0.2	388	9441	171	9821	5015
-0.1	371	9416	213	8515	8206
-0.05	327	9466	207	7075	8996
0.0	317	9442	241	0	9442
0.05	287	9456	257	6445	8987
0.1	321	9419	260	7939	7917
0.2	255	9460	285	9426	5022
0.3	256	9463	281	9833	2276
0.4	271	9458	271	9969	767
0.5	246	9434	320	9988	158

Table 5. Coverage frequencies for c with sample sizes 100 and 50: ‘<’ is the number of intervals that fell below the true value, ‘>’ the number that fell above, ‘Correct’ stands for the number of samples with correct choice of EV and ‘Negative’ stands for the number of samples with negative product of interval endpoints.

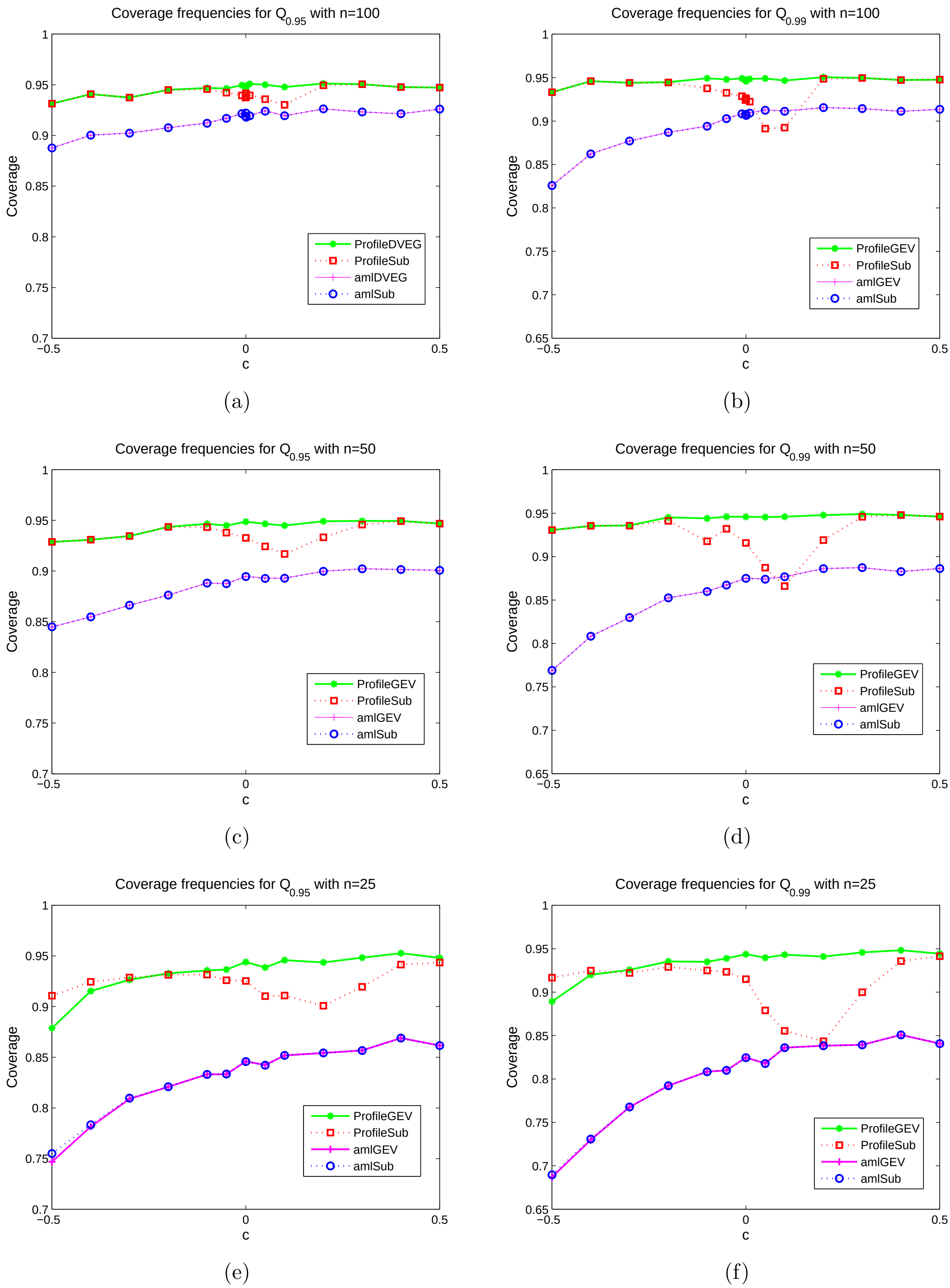


Figure 1: Coverage frequencies. The left column corresponds to Q_{95}, the right to Q_{99}. The first row corresponds to a sample size of 100, the middle row to sample size 50 and the bottom row to sample size 25.

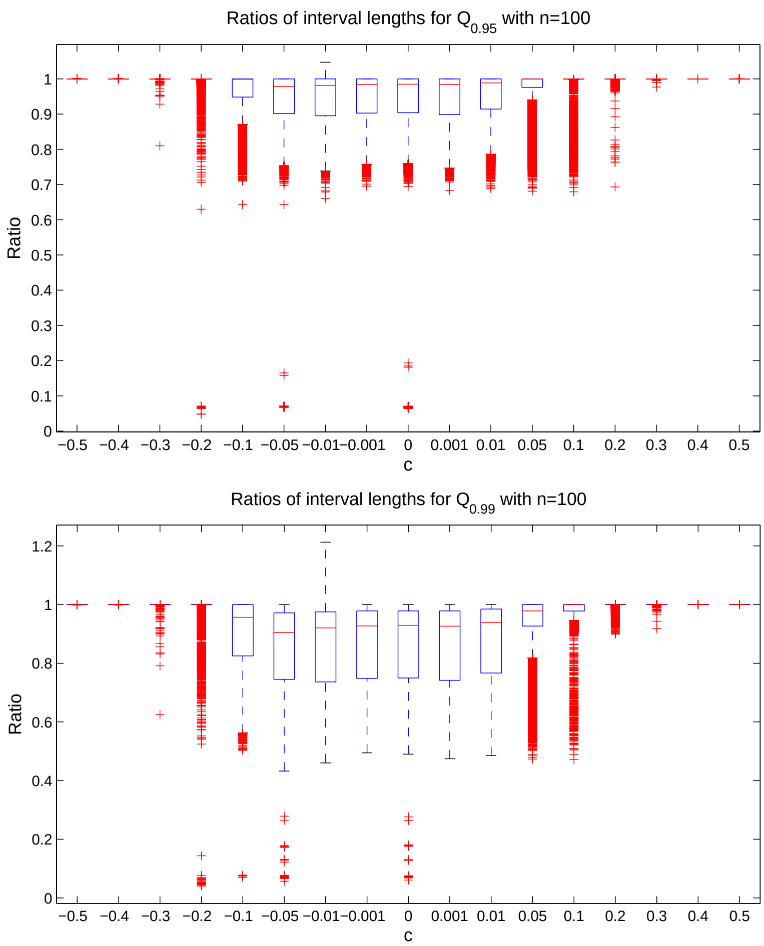


Figure 2: Ratio of length of likelihood-confidence intervals for Q_{95} (top) and Q_{99} (bottom) for the submodel over length of intervals for the GEV, sample size 100.

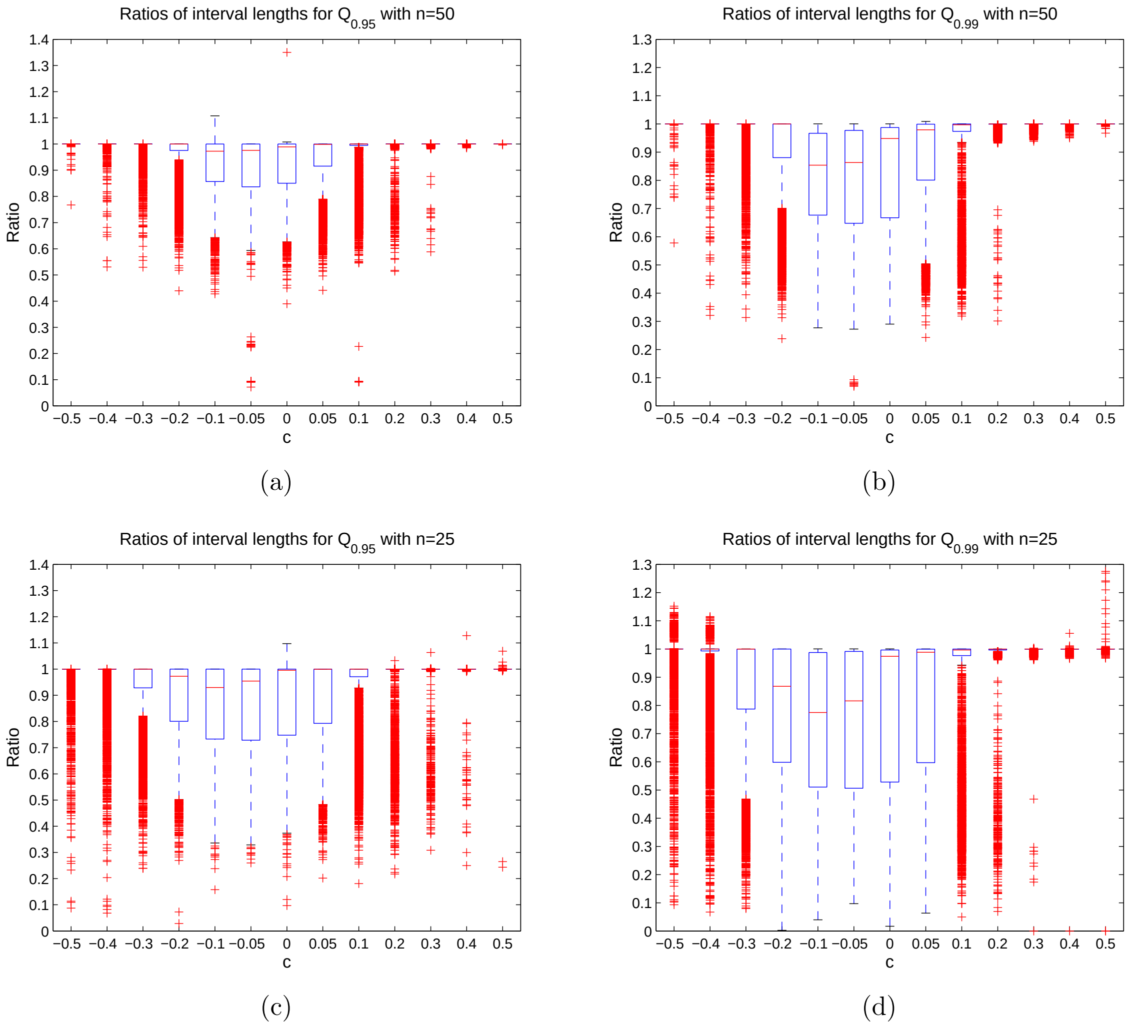


Figure 3: Ratio of length of likelihood-confidence intervals for Q_{95} (left) and Q_{99} (right) for the submodel over length of intervals for the GEV, sample sizes 50 (top) and 25 (bottom).

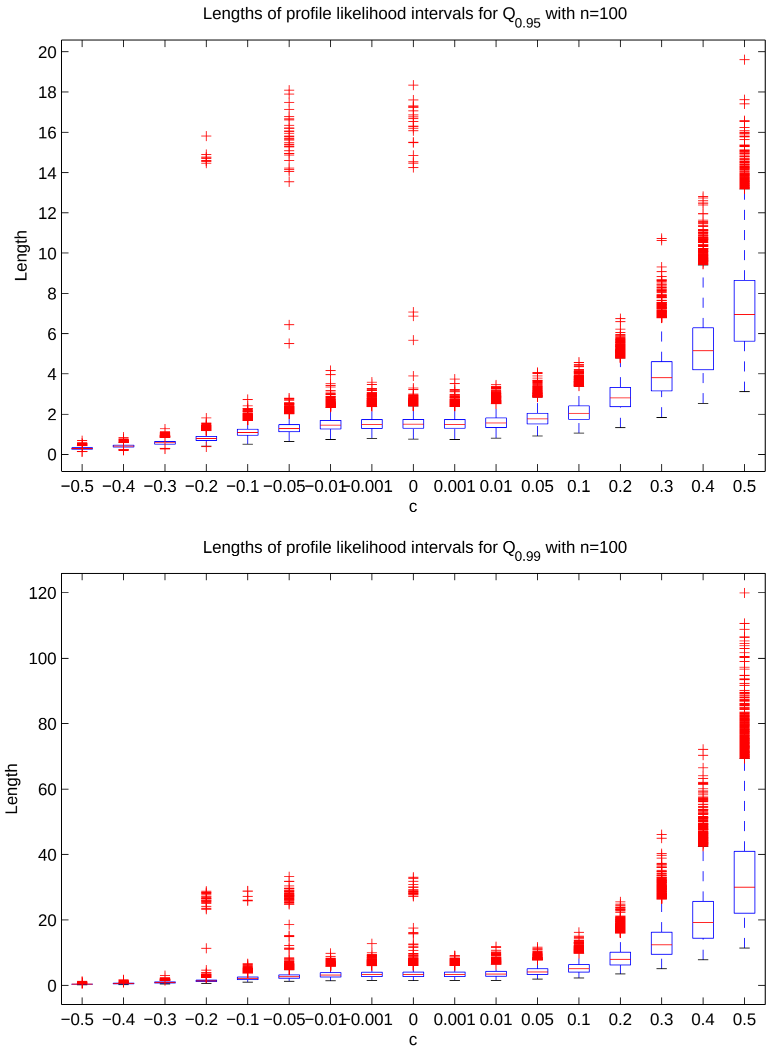


Figure 4: Length of profile likelihood-confidence intervals for Q_{95} (top) and Q_{99} (bottom) for the GEV, sample size 100.

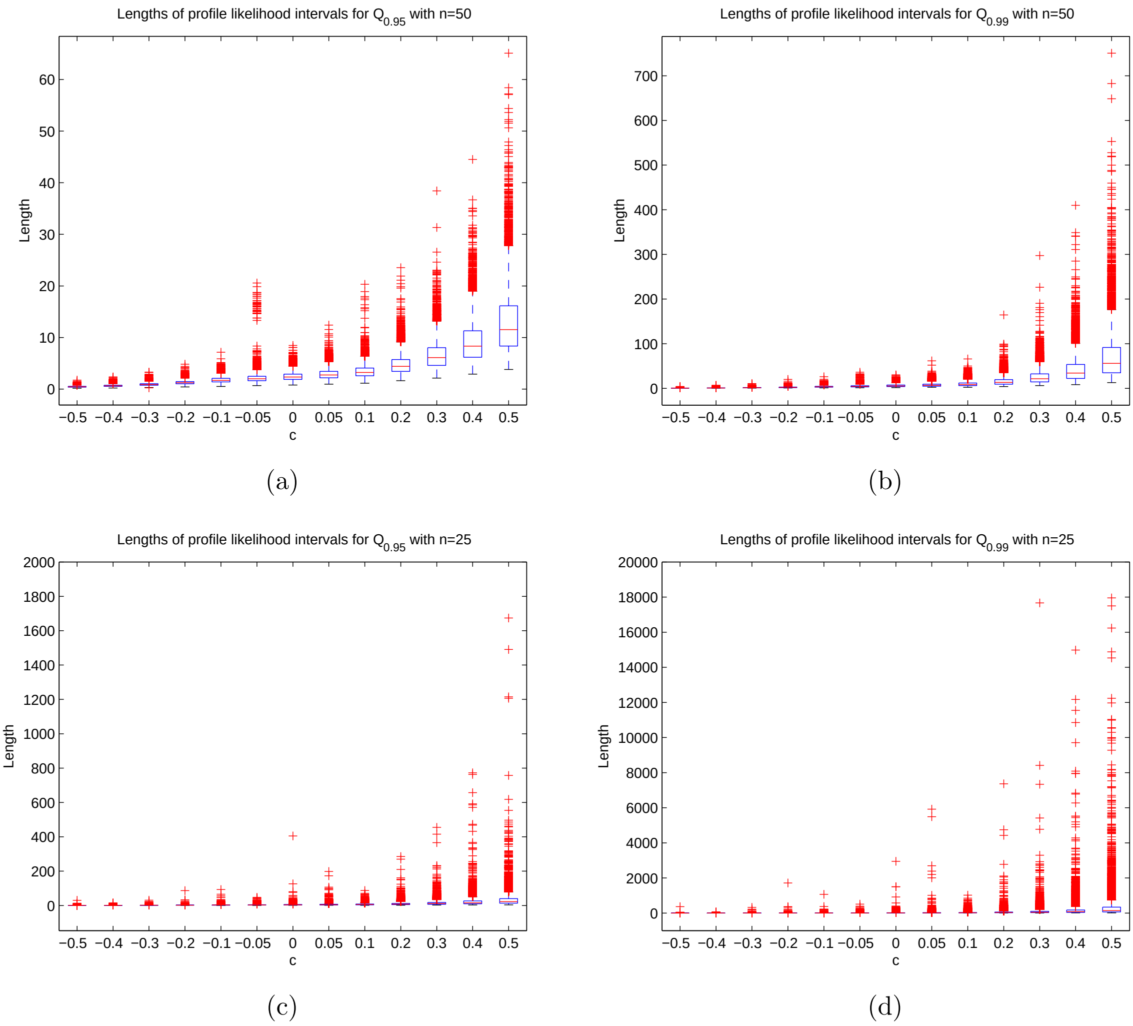


Figure 5: Length of profile likelihood-confidence intervals for Q_{95} (left) and Q_{99} (right), sample sizes $n = 50$ (top) and $n = 25$ (bottom) for the GEV. One outlying sample was excluded from plots (c) and (d).

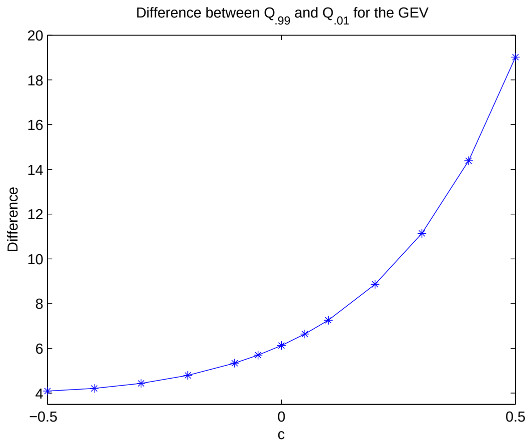


Figure 6: Difference between Q_{01} and Q_{99} for the GEV.

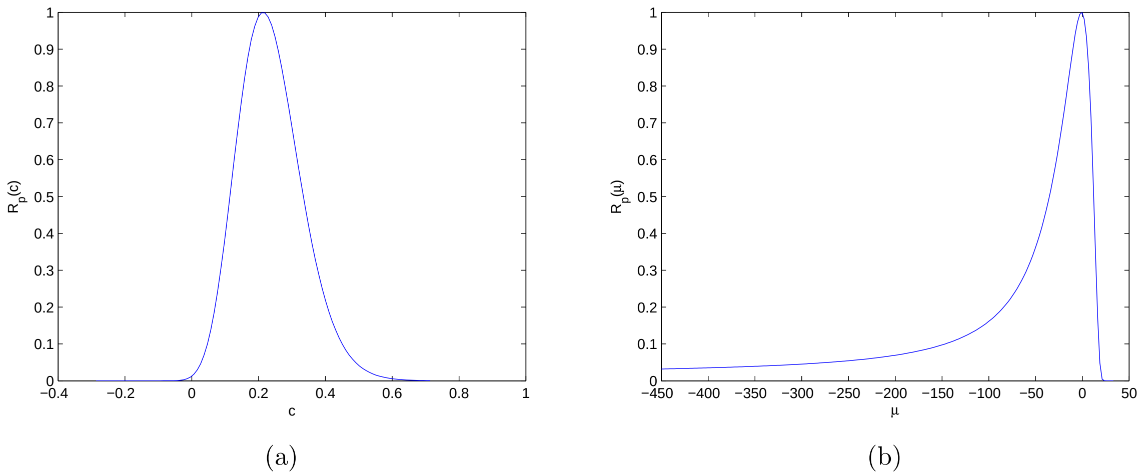
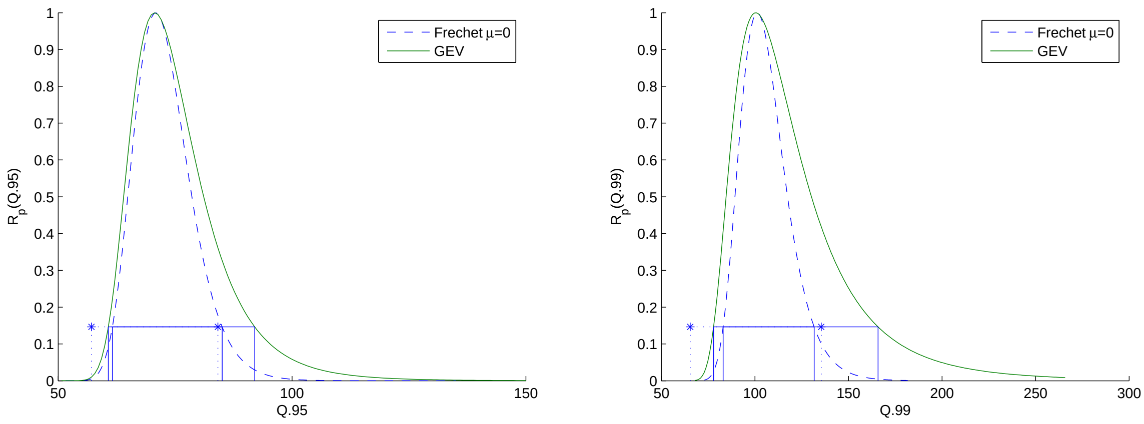
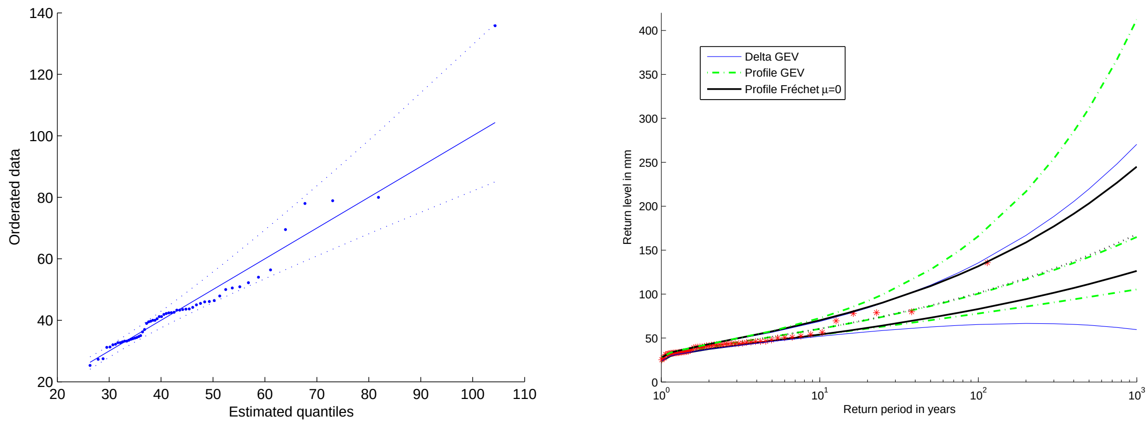


Figure 7: Rain data example: (a) Relative profile likelihood of GEV shape parameter c. (b) Relative profile likelihood of threshold parameter in three parameter Fréchet model.



(a) (b)

Figure 8: Rain data example: Relative profile likelihood of (a) $Q_{.95}$, (b) $Q_{.99}$.



(a) (b)

Figure 9: Rain data example: (a) Q-Q plot for the two parameter Fréchet model. (b) Return period plot.