Asymptotic behaviour near extinction of continuous state branching processes

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Abstract

In this note, we study the asymptotic behaviour near extinction of (sub-) critical continuous state branching processes. In particular, we establish an analogue of Khintchin's law of the iterated logarithm near extinction time for a continuous state branching process whose branching mechanism satisfies a given condition and its reflected process at its infimum.

KEY WORDS AND PHRASES: Continuous state branching processes, Lamperti transform, Lévy processes, conditioning to stay positive, rate of growth.

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1   Introduction and main results.

A continuous-state branching process (or CB-process for short) is a non-negative valued strong Markov process with probabilities ($\mathbb{P}_x$, $x \geq 0$) such that for any $x, y \geq 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$. CB-processes may be thought of as the continuous (in time and space) analogues of classical Bienaymé-Galton-Watson branching processes. Such classes of processes have been introduced by Jirina [7] and studied by many authors included Bingham [2], Grey [5], Grimvall [6], Lamperti [10], to name but a few. More precisely, a CB-process $Y = (Y_t, t \geq 0)$ is a Markov process taking values in $[0, \infty]$, where 0 and ∞ are two absorbing states, and satisfying the branching property; that is to say, its Laplace transform satisfies

$$\mathbb{E}_x\left[e^{-\lambda Y_t}\right] = \exp\{-xu_t(\lambda)\}, \qquad \text{for } \lambda \geq 0, \tag{1}$$

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for some function $u_t(\lambda)$. According to Silverstein [14], the function $u_t(\lambda)$ is determined by the integral equation

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)} \mathrm{d}u = t \tag{2}$$

where ψ satisfies the celebrate Lévy-Khincthine formula

$$\psi(\lambda) = a\lambda + \beta\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}} \right) \Pi(\mathrm{d}x), \tag{3}$$

where $a \in \mathbb{R}$, $\beta \geq 0$ and Π is a σ-finite measure such that $\int_{(0,\infty)} \left(1 \wedge x^2\right)\Pi(\mathrm{d}x)$ is finite. The function ψ is known as the branching mechanism of Y.

Note that the first moment of Y_t can be obtained by differentiating (1) with respect to λ, i.e. $\mathbb{E}_x[Y_t] = xe^{-\psi'(0^+)t}$. Hence, in respective order, a CB-process is called supercritical, critical or subcritical depending on $\psi'(0^+) < 0$, $\psi'(0^+) = 0$ or $\psi'(0^+) > 0$. Moreover since

$$\mathbb{P}_x \left(\lim_{t\to\infty} Y_t = 0 \right) = e^{-\eta x},$$

where η is the largest root of the branching mechanism ψ, the sign of $\psi'(0+)$ yields the criterion for a.s. extinction. More precisely a CB-process Y with branching mechanism ψ has a finite time extinction almost surely if and only if

$$\int^{\infty} \frac{\mathrm{d}u}{\psi(u)} < \infty \quad \text{and} \quad \psi'(0+) \geq 0. \tag{4}$$

We denote by T_0 the extinction time of the CB-process Y, i.e. $T_0 := \inf\{t \geq 0 : Y_t = 0\}$.

In this paper, we are interested in a detailed description of how continuous state branching processes become extinct. Hence, in what follows we always assume that assumption (4) is satisfied. One of the starting points of this paper are the results of Kyprianou and Pardo [8] for the CB-process in the self-similar case, i.e. when the branching mechanism is given by $\psi(\lambda) = c_+\lambda^\alpha$, for $1 < \alpha \leq 2$ and $c_+ > 0$. Note that such branching mechanism clearly satisfies condition (4). The authors in [8] described the upper and lower envelopes of the self-similar CB-process near extinction via integral test. In particular they obtained the following laws of the iterated logarithm (LIL for short) for the upper envelope of Y and its version reflected at its running infimum

$$\limsup_{t\to 0} \frac{Y_{(T_0-t)^-}}{t^{1/(\alpha-1)} \log\log(1/t)} = c_+(\alpha-1)^{1/\alpha-1}, \qquad \mathbb{P}_x - \text{a.s.},$$

and

$$\limsup_{t\to 0} \frac{(Y - \underline{Y})_{(T_0-t)^-}}{t^{1/(\alpha-1)} \log\log(1/t)} = c_+(\alpha-1)^{1/\alpha-1}, \qquad \mathbb{P}_x - \text{a.s.},$$

where $\underline{Y}_t$ denotes the infimum of the CB-process $(Y, \mathbb{P}_x)$ over $[0,t]$.

In order to state our main results, we first introduce some basic notation. Let us define the mapping

$$\phi(t) := \int_t^{\infty} \frac{\mathrm{d}u}{\psi(u)}, \qquad \text{for} \quad t > 0,$$

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and note that $\phi : (0, \infty) \to (0, \infty)$ is a bijection and thus its inverse exist, here denoted by φ. From (2), it is straightforward to get

$$u_t(\lambda) = \varphi(t + \phi(\lambda)) \qquad \lambda, t > 0.$$

Since $\phi(\infty) = 0$, we clearly have $u_t(\infty) = \varphi(t)$, for $t > 0$. Hence we deduce that for every $x, t > 0$,

$$\mathbb{P}_x(T_0 \le t) = \mathbb{P}_x(Y_t = 0) = \lim_{\lambda \to \infty} \mathbb{E}_x\left[e^{-\lambda Y_t}\right] = e^{-x\varphi(t)}.$$

Finally, let us introduce the lower and upper exponents of ψ at infinity,

$$\gamma := \sup\left\{c \ge 0 : \lim_{\lambda \to \infty} \frac{\psi(\lambda)}{\lambda^c} = \infty\right\} \qquad \text{and} \qquad \eta := \inf\left\{c \ge 0 : \lim_{\lambda \to \infty} \frac{\psi(\lambda)}{\lambda^c} = 0\right\}.$$

Since ψ satisfies (3), we necessarily have $1 \le \gamma \le \eta \le 2$. Now, we introduce the following exponent,

$$\delta := \sup\left\{c \ge 0 : \exists\, Q \in (0, \infty) \text{ s.t. } Q\psi(u)u^{-c} \le \psi(v)v^{-c}, 1 \le u \le v\right\}. \tag{5}$$

Therefore, we necesarilly have $1 \le \delta \le \gamma \le \eta \le 2$. Note that in the case where ψ is regularly varying at ∞ with index $\alpha > 1$, then $\delta = \gamma = \eta = \alpha$.

Our first result consist in a law of the iterated logarithm (LIL for short) at 0 for the upper envelope of the time-reversal process $(Y_{(T_0-t)^-}, 0 \le t \le T_0)$, under $\mathbb{P}_x$.

Theorem 1. *Assume that $\delta > 1$, then*

$$\limsup_{t \to 0} \frac{Y_{(T_0-t)^-}\varphi(t)}{\log\log \varphi(t)} = 1 \qquad \mathbb{P}_x\text{-}a.s.,$$

for every $x > 0$.

Recall that $\underline{Y}$ and $((Y - \underline{Y})_{(T_0-t)^-}, 0 \le t < T_0)$ denote the running infimum of Y and the time-reversal process of Y reflected at its running minimum, respectively. We also introduce the so-called scale function $W : [0, \infty) \mapsto [0, \infty)$ which is the unique absolutely continuous increasing function whose Laplace transform is $1/\psi$. Let us suppose that for all $\beta < 1$, the scale function W satisfies the following hypothesis

$$(\mathbf{H}) \qquad\qquad \limsup_{x \to 0} \frac{W(\beta x)}{W(x)} < 1.$$

We remark that the above hypothesis is satisfied, in particular, when ψ is regularly varying at ∞.

Theorem 2. *Suppose that $\delta > 1$, then under the hypothesis $(\mathbf{H})$, we have*

$$\limsup_{t \to 0} \frac{(Y - \underline{Y})_{(T_0-t)^-}\varphi(t)}{\log\log \varphi(t)} = 1 \qquad \mathbb{P}_x\text{-}a.s.,$$

for every $x > 0$.

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It is important to note that in the self-similar case, i.e. when $\psi(\lambda) = c_+\lambda^\alpha$ for $1 < \alpha \le 2$ and $c_+ > 0$, we have

$$\frac{\log\log\varphi(t)}{\varphi(t)} = \frac{\log\log(c_+(\alpha-1)t)^{-\frac{1}{\alpha-1}}}{(c_+(\alpha-1)t)^{-\frac{1}{\alpha-1}}} \qquad \text{for} \quad t > 0.$$

We can replace the above function by

$$\frac{\log\log\frac{1}{t}}{\varphi(t)} = (c_+(\alpha-1)t)^{\frac{1}{\alpha-1}}\log\log\frac{1}{t} \qquad \text{for} \quad t > 0,$$

since they are asymptotically equivalent at 0. Similarly, we can take the previous function in the regularly varying case.

2 Proofs

Let $(\mathbf{P}_x,\ x \in \mathbb{R})$ be a family of probability measures on the space of càdlàg mappings from $[0,\infty)$ to $\mathbb{R}$, denoted by $\mathcal{D}$, such that for each $x \in \mathbb{R}$, the canonical process X is a Lévy process with no negative jumps issued from x. Set $\mathbf{P} := \mathbf{P}_0$, so $\mathbf{P}_x$ is the law of $X + x$ under $\mathbf{P}$. The Laplace exponent $\psi : [0,\infty) \to (-\infty,\infty)$ of X is specified by $\mathbf{E}[e^{-\lambda X_t}] = e^{t\psi(\lambda)}$, for $\lambda \in \mathbb{R}$, and can be expressed in terms of the Lévy-Khintchine formula (3).

Henceforth, we shall assume that $(X,\mathbf{P})$ is not a subordinator (recall that a subordinator is a Lévy process with increasing sample paths). In that case, it is known that the Laplace exponent ψ is strictly convex and tends to ∞ as λ goes to ∞. In this case, we define for $q \ge 0$

$$\Phi(q) = \inf\left\{\lambda \ge 0 : \psi(\lambda) > q\right\}$$

the right-continuous inverse of ψ and then $\Phi(0)$ is the largest root of the equation $\psi(\lambda) = 0$. Theorem VII.1 in [1] implies that condition $\Phi(0) > 0$ holds if and only if the process drifts to ∞. Moreover, almost surely, the paths of X drift to ∞, oscillate or drift to $-\infty$ accordingly as $\psi'(0+) < 0$, $\psi'(0+) = 0$ or $\psi'(0+) > 0$.

Lamperti [10] observed that continuous state branching processes are connected to Lévy processes with no negative jumps by a simple time-change. More precisely, consider a spectrally positive Lévy process $(X,\mathbf{P}_x)$ started at $x > 0$ and with Laplace exponent ψ. Now, we introduce the clock

$$A_t = \int_0^t \frac{\mathrm{d}s}{X_s} \qquad t \in [0,\tau_0),$$

where $\tau_0 = \inf\{t \ge 0 : X_t \le 0\}$, and its right-continuous inverse

$$\theta(t) = \inf\{s \ge 0 : A_s > t\}.$$

Then, the time changed process $Y = (X_{\theta(t)}, t \ge 0)$, under $\mathbf{P}_x$, is a continuous state branching process with initial population of size x. The transformation described above will henceforth be referred to as the *CB-Lamperti representation*.

Now, define $\widehat{X} := -X$, the dual process of X. Denote by $\widehat{\mathbf{P}}_x$ the law of $\widehat{X}$ when issued from x so that $(X, \widehat{\mathbf{P}}_x) = (\widehat{X}, \mathbf{P}_{-x})$. The dual process conditioned to stay positive is a Doob h-transform of $(X, \widehat{\mathbf{P}}_x)$ killed when it first exists $(0, \infty)$ with the harmonic function W. In this case, assuming that $\psi'(0+) \geq 0$, one has

$$\widehat{\mathbf{P}}_x^\uparrow(X_t \in \mathrm{d}y) = \frac{W(y)}{W(x)}\widehat{\mathbf{P}}_x(X_t \in \mathrm{d}y, t < \tau_0), \quad t \geq 0, \quad x, y > 0.$$

Under $\widehat{\mathbf{P}}_x^\uparrow$, X is a process taking values in $(0, \infty)$. It will be referred to as the dual Lévy process started at x and conditioned to stay positive. The measure $\widehat{\mathbf{P}}_x^\uparrow$ is always a probability measure and there is always weak convergence as $x \downarrow 0$ to a probability measure which we denote by $\widehat{\mathbf{P}}^\uparrow$.

Let

$$\sigma_x = \sup\{t > 0 : X_t \leq x\},$$

be the last passage time of X below $x \in \mathbb{R}$. The next proposition, whose proof may be found in [8], gives us a time-reversal property from extinction for the C.B. process (see Theorem 1). Recall that, under $\widehat{\mathbf{P}}^\uparrow$, the canonical process X drifts towards ∞ and also that $X_t > 0$ for $t > 0$.

Proposition 1. *If condition (4) holds, then for each $x > 0$*

$$\left\{(Y_{(T_0-t)^-} : 0 \leq t < T_0), \mathbb{P}_x\right\} \overset{d}{=} \left\{(X_{\theta(t)}, 0 \leq t < A_{\sigma_x}), \widehat{\mathbf{P}}^\uparrow\right\}.$$

Recall that ψ is a branching mechanism satisfying conditions (3) and (4); and whose exponent δ defined by (5) is strictly larger than 1. Therefore, since $\delta > 1$, there exist $c \in (1, \infty)$ and $C \in (0, \infty)$ such that

$$\psi(\lambda) \leq C\psi(\lambda u)u^{-c}, \tag{6}$$

for any $u, \lambda \in [1, \infty)$. Now, we introduce the so-called first and last passage times of the process $\{(X_{\theta(t)}, t \geq 0), \widehat{\mathbf{P}}^\uparrow\}$ by

$$S_y = \inf\left\{t \geq 0 : X_{\theta(t)} \geq y\right\} \quad \text{and} \quad U_y = \sup\left\{t \geq 0 : X_{\theta(t)} \leq y\right\},$$

for $y \geq 0$. Note that the processes $(S_y, y \geq 0)$ and $(U_y, y \geq 0)$ are increasing with independent increments since the process $\{(X_{\theta(t)}, t \geq 0), \widehat{\mathbf{P}}^\uparrow\}$ has no positive jumps.

Observe that A_{σ_x} and U_x are equal and that the latter, under $\widehat{\mathbf{P}}^\uparrow$, has the same law as T_0 under $\mathbb{P}_x$. This clearly implies that the distribution of U_x, under $\widehat{\mathbf{P}}^\uparrow$, satisfies

$$\widehat{\mathbf{P}}^\uparrow\left(U_x \leq t\right) = e^{-x\varphi(t)}, \tag{7}$$

for every $x, t > 0$.

Let $(J_t, t \geq 0)$ be the future infimum process of $(X_{\theta(t)}, t \geq 0)$ which is defined as follows

$$J_t = \inf_{s \geq t} X_{\theta(s)}, \quad \text{for} \quad t \geq 0.$$

Proposition 2. *Assume that $\delta > 1$, then*

$$\limsup_{t \to 0} \frac{J_t \varphi(t)}{\log \log \varphi(t)} = 1, \qquad \widehat{\mathbf{P}}^{\uparrow}\text{-}a.s.$$

Proof: For simplicity, we let

$$f(t) = \frac{\log \log \varphi(t)}{\varphi(t)} \qquad \text{for} \quad t > 0.$$

In order to prove this result, we need the following two technical lemmas. Recall that ϕ is the inverse function of φ.

Lemma 1. *For every integer $n \geq 1$ and $r > 1$, put*

$$t_n = \phi(r^n) \qquad and \qquad a_n = f(t_n).$$

(i) *The sequence $(t_n : n \geq 1)$ decreases.*

(ii) *The series $\sum_n \widehat{\mathbf{P}}^{\uparrow}\big(J_{t_n} > r a_n\big)$ converges.*

Proof of Lemma 1: The first assertion follows readily from the fact that ϕ is decreasing. In order to prove (ii), we note that for $n \geq 1$, $\varphi(t_n) = \varphi(\phi(r^n)) = r^n$. This entails

$$\log \log \varphi(t_n) = \log \log r^n. \tag{8}$$

Now, since the last passage times process is the right continuos inverse of the future infimum of $(X_{\theta(t)}, t \geq 0)$, we have

$$\widehat{\mathbf{P}}^{\uparrow}\Big(J_{t_n} > r a_n\Big) = \widehat{\mathbf{P}}^{\uparrow}\Big(U_{r a_n} < t_n\Big).$$

Hence (7) and (8) imply

$$\sum_n \widehat{\mathbf{P}}^{\uparrow}\Big(U_{r a_n} < t_n\Big) \;\leq\; \sum_n \left(\frac{1}{n \log r}\right)^r,$$

which converges, and our statement follows. $\qquad\qquad\square$

Lemma 2. *For every integer $n \geq 2$ and $r > 1$, put*

$$s_n = \phi(e^{n^r}) \qquad and \qquad b_n = f(s_n).$$

We have that the series $\sum_n \widehat{\mathbf{P}}^{\uparrow}\Big(U_{b_n/r} \leq s_n\Big)$ diverges.

Proof of Lemma 2: First we note that for $n \geq 1$, $\varphi(s_n) = \varphi(\phi(e^{n^r})) = e^{n^r}$. This entails $\log \log \varphi(s_n) = \log n^r$. Hence, the identity (7) implies

$$\sum_n \widehat{\mathbf{P}}^{\uparrow}\Big(U_{b_n/r} \leq s_n\Big) \;=\; \sum_n \frac{1}{n},$$

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which diverges, and our claim follows. □

We are now able to establish the law of the iterated logarithm. In order to prove the upper bound, we use Lemma 1. Take any $t \in [t_{n+1}, t_n]$, so, provided that n is large enough

$$f(t) \geq \frac{\log \log \varphi(t_n)}{\varphi(t_{n+1})},$$

since φ decreases. Note that the denumerator is equal to r^{n+1} and the numerator is equal to $\log \log r^n$. We thus have

$$\limsup_{t \to 0} \frac{f(t_n)}{f(t)} \leq r. \tag{9}$$

On the other hand, an application of the Borel-Cantelli Lemma to Lemma 1 shows that

$$\limsup_{n \to \infty} \frac{J_{t_n}}{f(t_n)} \leq r, \qquad \widehat{\mathbf{P}}^\uparrow\text{-a.s.},$$

and we deduce that

$$\limsup_{t \to 0} \frac{J_t}{f(t)} \leq \left(\limsup_{n \to \infty} \frac{J_{t_n}}{f(t_n)} \right) \left(\limsup_{t \to 0} \frac{f(t_n)}{f(t)} \right) \leq r^2, \qquad \widehat{\mathbf{P}}^\uparrow\text{-a.s.}$$

To prove the lower bound, we use Lemma 2 and observe that the sequence $(b_n, n \geq 2)$ decreases. First, from Lemma 2, we have

$$\sum_n \widehat{\mathbf{P}}^\uparrow \left(U_{b_n/r} - U_{b_{n+1}/r} \leq s_n \right) \geq \sum_n \widehat{\mathbf{P}}^\uparrow \left(U_{b_n/r} \leq s_n \right) = \infty,$$

so by the Borel-Cantelli Lemma for independent events, we obtain

$$\liminf_{n \to \infty} \frac{U_{b_n/r} - U_{b_{n+1}/r}}{s_n} \leq 1, \qquad \widehat{\mathbf{P}}^\uparrow\text{-a.s.}$$

If we admit for a while that

$$\limsup_{n \to \infty} \frac{U_{b_{n+1}/r}}{s_n} = 0, \qquad \widehat{\mathbf{P}}^\uparrow\text{-a.s.}, \tag{10}$$

we can conclude

$$\liminf_{n \to \infty} \frac{U_{b_n/r}}{s_n} \leq 1, \qquad \widehat{\mathbf{P}}^\uparrow\text{-a.s.}$$

This implies that the set $\{s : U_{f(s)/r} \leq s\}$ is unbounded $\widehat{\mathbf{P}}^\uparrow$-a.s. Plainly, the same then holds for $\{s : J_s \geq f(s)/r\}$, and as a consequence

$$\limsup_{t \to 0} \frac{J_t}{f(t)} \geq \frac{1}{r}, \qquad \widehat{\mathbf{P}}^\uparrow\text{-a.s.} \tag{11}$$

Next we establish the behaviour in (10). First, since $\delta > 1$ we observed from inequality (6) that there exist $c \in (1, \infty)$ and $C \in (0, \infty)$ such that

$$\varepsilon\phi(t) = \varepsilon \int_t^\infty \frac{du}{\psi(u)} \geq \left(\frac{C}{\varepsilon} \right)^{\frac{1}{c-1}} \int_t^\infty \frac{du}{\psi\left(\left(\frac{C}{\varepsilon} \right)^{\frac{1}{c-1}} u \right)} = \int_{\left(\frac{C}{\varepsilon} \right)^{\frac{1}{c-1}} t}^\infty \frac{du}{\psi(u)} = \phi\left((C/\varepsilon)^{\frac{1}{c-1}} t \right), \tag{12}$$

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for any $t \geq 1$ and $0 < \varepsilon < \min(C, 1)$. On the other hand, we see that for n large enough and $0 < \varepsilon < \min(C, 1)$

$$\widehat{\mathbf{P}}^{\uparrow}\left(U_{b_{n+1}/r} > \varepsilon s_n\right) = 1 - \exp\left\{-\frac{b_{n+1}}{r}\varphi(\varepsilon s_n)\right\} \leq \frac{b_{n+1}}{2}\varphi(\varepsilon s_n).$$

Hence from inequality (12), we have

$$\widehat{\mathbf{P}}^{\uparrow}\left(U_{b_{n+1}/r} > \varepsilon s_n\right) \leq \left(\frac{C}{\varepsilon}\right)^{\frac{1}{c-1}} \exp\left\{n^r - (n+1)^r\right\}\log(n+1) \leq \left(\frac{C}{\varepsilon}\right)^{\frac{1}{c-1}} e^{-(r-1)n^r+1},$$

where the las identity follows since $(n+1)^r - n^r \geq rn^{r-1}$. In conclusion, we have that the series $\sum_n \widehat{\mathbf{P}}^{\uparrow}\left(U_{b_{n+1}/r} > s_n\right)$ converges, and according to the Borel-Cantelli lemma,

$$\limsup_{n \to \infty} \frac{U_{b_{n+1}/r}}{s_n} \leq \varepsilon, \qquad \widehat{\mathbf{P}}^{\uparrow}\text{-a.s.},$$

which establishes (10) since $0 < \varepsilon < \min(C, 1)$ can be chosen arbitrarily small. The proof of (11) is now complete. The two preceding bounds show that

$$\frac{1}{r} \leq \limsup_{t \to 0} \frac{J_t}{f(t)} \leq r^2, \qquad \widehat{\mathbf{P}}^{\uparrow}\text{-a.s.}$$

Hence the result follows taking r close enough to 1. $\qquad\square$

Proof of Theorem 1: In order to establish our result, we first prove the following law of the iterated logarithm holds

$$\limsup_{t \to 0} \frac{X_{\theta(t)}}{f(t)} = 1, \qquad \widehat{\mathbf{P}}^{\uparrow} - \text{a.s.}, \tag{13}$$

and then use Proposition 1.

The lower bound of (13) is easy to deduce from Proposition 2. More precisely,

$$1 = \limsup_{t \to 0} \frac{J_t}{f(t)} \leq \limsup_{t \to 0} \frac{X_{\theta(t)}}{f(t)} \qquad \widehat{\mathbf{P}}^{\uparrow} - \text{a.s.}$$

Now, we prove the upper bound. Let $r > 1$ and denote by $\overline{X}_{\theta(\cdot)}$ for the supremum process of $\{(X_{\theta(s)}, s \geq 0), \widehat{\mathbf{P}}^{\uparrow}\}$, which is defined by $\overline{X}_{\theta(t)} = \sup_{0 \leq s \leq t} X_{\theta(s)}$ for $t \geq 0$.

Lemma 3. *Let $c' > 1$, $0 < \epsilon < 1 - 1/c'$ and $r > 1$. For $t_n = \phi(r^n)$, $n \geq 1$, then there exist a positive real number K such that*

$$\widehat{\mathbf{P}}^{\uparrow}\left(J_{t_n} > (1-\epsilon)c'f(t_n)\right) \geq \epsilon K^2 \widehat{\mathbf{P}}^{\uparrow}\left(\overline{X}_{\theta(t_n)} > c'f(t_n)\right).$$

Proof: From the Markov property, we have

$$\widehat{\mathbf{P}}^{\uparrow}\left(J_{t_n} > (1-\epsilon)c'f(t_n)\right) \geq \widehat{\mathbf{P}}^{\uparrow}\left(\overline{X}_{\theta(t_n)} > c'f(t_n), J_{t_n} > (1-\epsilon)c'f(t_n)\right)$$

$$= \int_0^{t_n} \widehat{\mathbf{P}}^{\uparrow}\left(S_{c'f(t_n)} \in dt\right)\widehat{\mathbf{P}}^{\uparrow}_{c'f(t_n)}\left(J_{t_n} > (1-\epsilon)c'f(t_n)\right)$$

$$\geq \widehat{\mathbf{P}}^{\uparrow}\left(\overline{X}_{\theta(t_n)} > c'f(t_n)\right)\widehat{\mathbf{P}}^{\uparrow}_{c'f(t_n)}\left(\inf_{0 \leq s} X_{\theta(s)} > (1-\epsilon)c'f(t_n)\right).$$

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Now from the Lamperti transform and Lemma VII.12 in [1], we have

$$\widehat{\mathbf{P}}^{\uparrow}_{c'f(t_n)}\left(\inf_{0\leq s} X_{\theta(s)} > (1-\epsilon)c'f(t_n)\right) = \widehat{\mathbf{P}}^{\uparrow}_{c'f(t_n)}\left(\tau_{[0,(1-\epsilon)c'f(t_n))} = \infty\right) = \frac{W\big(c'\epsilon f(t_n)\big)}{W\big(c'f(t_n)\big)},$$

where $\tau_{[0,z)} = \inf\{t \geq 0 : X_t \in [0,z)\}$.

On the other hand an application of Proposition III.1 in [1] gives that there exist a positive real number K such that

$$K\frac{1}{x\psi(1/x)} \leq W(x) \leq K^{-1}\frac{1}{x\psi(1/x)}, \qquad \text{for all} \quad x > 0, \tag{14}$$

then it is clear

$$\frac{W\big(c'\epsilon f(t_n)\big)}{W\big(c'f(t_n)\big)} \geq K^2\epsilon^{-1}\frac{\psi\big(1/c'f(t_n)\big)}{\psi\big(\epsilon^{-1}/c'f(t_n)\big)}.$$

From the above inequality and Lemma 3 in [13], we deduce

$$\widehat{\mathbf{P}}^{\uparrow}_{c'f(t_n)}\left(\inf_{0\leq s} X_{\theta(s)} > (1-\epsilon)c'f(t_n)\right) \geq \epsilon K^2,$$

which clearly implies our result. $\qquad\qquad\qquad\square$

Now, we prove the upper bound for the LIL of $((X_{\theta(t)}, t \geq 0), \widehat{\mathbf{P}}^{\uparrow})$. Let $c' > 1$ and fix $0 < \epsilon < 1 - 1/c'$. Recall from (7) that

$$\widehat{\mathbf{P}}^{\uparrow}\left(J_{t_n} > (1-\epsilon)c'f(t_n)\right) = \widehat{\mathbf{P}}^{\uparrow}\left(U_{(1-\epsilon)c'f(t_n)} < t_{n-1}\right) = (n\log r)^{-(1-\epsilon)c'}.$$

Hence from Lemma 3, we deduce

$$\sum_{n\geq 1}\widehat{\mathbf{P}}^{\uparrow}\left(S_{t_n} > c'f(t_n)\right) \leq K^{-2}\epsilon^{-1}\sum_n (n\log r)^{-(1-\epsilon)c'} < \infty,$$

since $(1-\epsilon)c' > 1$. Therefore an application of the Borel-Cantelli Lemma shows

$$\limsup_{n\to\infty}\frac{\overline{X}_{\theta(t_n)}}{f(t_n)} \leq c', \quad \widehat{\mathbf{P}}^{\uparrow}\text{-a.s.}$$

Using (9), we deduce

$$\limsup_{t\to 0}\frac{\overline{X}_{\theta(t)}}{f(t)} \leq \left(\limsup_{n\to\infty}\frac{\overline{X}_{\theta(t_n)}}{f(t_n)}\right)\left(\limsup_{t\to 0}\frac{f(t_n)}{f(t)}\right) \leq rc', \quad \widehat{\mathbf{P}}^{\uparrow}\text{-a.s.}$$

Hence from the first part of the proof and taking r close enought to 1 above, we deduce

$$\limsup_{t\to 0}\frac{X_{\theta(t)}}{f(t)} \in [1, c'], \quad \widehat{\mathbf{P}}^{\uparrow}\text{-a.s.},$$

By the Blumenthal zero-one law, it must be a constant number k, $\widehat{\mathbf{P}}^{\uparrow}$-a.s.

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Now we prove that the constant k equals 1. Fix $\epsilon \in (0, 1/2)$ and define

$$R_n = \inf \left\{ \frac{1}{n} \leq s : \frac{J_s}{kf(s)} \geq (1 - \epsilon) \right\}.$$

Note that for n sufficiently large $1/n < R_n < \infty$ and that R_n converge to 0, $\widehat{\mathbf{P}}^\uparrow$-a.s., as n goes to ∞. From Lemma VII.12 in [1], the strong Markov property and since the process $\{(X_{\theta(t)}, t \geq 0), \widehat{\mathbf{P}}^\uparrow\}$ has no positive jumps, we have

$$\widehat{\mathbf{P}}^\uparrow \left(\frac{J_{R_n}}{kf(R_n)} \geq (1 - 2\epsilon) \right) = \widehat{\mathbf{P}}^\uparrow \left(J_{R_n} \geq \frac{(1 - 2\epsilon)X_{\theta(R_n)}}{(1 - \epsilon)} \right)$$

$$= \widehat{\mathbf{E}}^\uparrow \left(\widehat{\mathbf{P}}^\uparrow \left(J_{R_n} \geq \frac{(1 - 2\epsilon)X_{\theta(R_n)}}{(1 - \epsilon)} \Big| X_{\theta(R_n)} \right) \right)$$

$$= \widehat{\mathbf{E}}^\uparrow \left(\frac{W(\ell(\epsilon)X_{\theta(R_n)})}{W(X_{\theta(R_n)})} \right),$$

where $\ell(\epsilon) = \epsilon/(1 - \epsilon)$. Applying (14) and Lemma 3 in [13], give us

$$\widehat{\mathbf{E}}^\uparrow \left(\frac{W(\ell(\epsilon)X_{\theta(R_n)})}{W(X_{\theta(R_n)})} \right) \geq K^2 \ell(\epsilon),$$

wich implies

$$\lim_{n \to \infty} \widehat{\mathbf{P}}^\uparrow \left(\frac{J_{R_n}}{kf(R_n)} \geq (1 - 2\epsilon) \right) > 0.$$

Since $R_n \geq 1/n$,

$$\widehat{\mathbf{P}}^\uparrow \left(\frac{J_t}{kf(t)} \geq (1 - 2\epsilon), \text{ for some } t \geq 1/n \right) \geq \widehat{\mathbf{P}}^\uparrow \left(\frac{J_{R_n}}{kf(R_n)} \geq (1 - 2\epsilon) \right).$$

Therefore, for all $\epsilon \in (0, 1/2)$

$$\widehat{\mathbf{P}}^\uparrow \left(\frac{J_t}{kf(t)} \geq (1 - 2\epsilon), \text{ i.o., as } t \to 0 \right) \geq \lim_{n \to \infty} \widehat{\mathbf{P}}^\uparrow \left(\frac{J_{R_n}}{kf(R_n)} \geq (1 - 2\epsilon) \right) > 0.$$

The event on the left hand side is in the lower-tail sigma-field of $(X, \widehat{\mathbf{P}}^\uparrow)$ which is trivial from Bertoin's contruction (see for instance Section 8.5.2 in [4]). Hence

$$\limsup_{t \to 0} \frac{J_t}{f(t)} \geq k(1 - 2\epsilon), \qquad \widehat{\mathbf{P}}^\uparrow\text{-a.s.,}$$

and since ϵ can be chosen arbitraily small, we deduce that $1 \geq k$. $\qquad\square$

Proof of Theorem 2: Here we follow similar arguments as those used in the last part of the previous result. Assume that the hypothesis (**H**) is satisfied. From Theorem 1, it is clear that

$$\limsup_{t \to 0} \frac{X_{\theta(t)} - J_t}{f(t)} \leq \limsup_{t \to 0} \frac{X_{\theta(t)}}{f(t)} = 1, \qquad \widehat{\mathbf{P}}^\uparrow\text{-a.s.}$$

Fix $\varepsilon \in (0, 1/2)$ and define

$$R_n = \inf \left\{ \frac{1}{n} \leq s : \frac{X_{\theta(s)}}{f(s)} \geq (1 - \epsilon) \right\}.$$

First note that for n sufficiently large $1/n < R_n < \infty$ $\widehat{\mathbf{P}}^{\uparrow}$-a.s. Moreover, from Theorem 1 we have that R_n converge to 0 as n goes to ∞, $\widehat{\mathbf{P}}^{\uparrow}$-a.s.

From Lemma VII.12 in [1], then strong Markov property and since $\{(X_{\theta(t)}, t \geq 0), \widehat{\mathbf{P}}^{\uparrow}\}$ has no positive jumps, we have

$$\widehat{\mathbf{P}}^{\uparrow} \left(\frac{X_{\theta(R_n)} - J_{R_n}}{f(R_n)} \geq (1 - 2\epsilon) \right) = \widehat{\mathbf{P}}^{\uparrow} \left(J_{R_n} \leq \frac{\epsilon}{1 - \epsilon} X_{\theta(R_n)} \right)$$

$$= \widehat{\mathbf{E}}^{\uparrow} \left(\widehat{\mathbf{P}}^{\uparrow} \left(J_{R_n} \leq \frac{\epsilon}{1 - \epsilon} X_{\theta(R_n)} \Big| X_{\theta(R_n)} \right) \right)$$

$$= 1 - \widehat{\mathbf{E}}^{\uparrow} \left(\frac{W(\ell(\epsilon) X_{\theta(R_n)})}{W(X_{\theta(R_n)})} \right),$$

where $\ell(\epsilon) = (1 - 2\epsilon)/(1 - \epsilon)$. Since the hypothesis (**H**) is satisfied, an application of Fatou-Lebesgue Theorem shows that

$$\limsup_{n \to +\infty} \widehat{\mathbf{E}}^{\uparrow} \left(\frac{W(\ell(\epsilon) X_{\theta(R_n)})}{W(X_{\theta(R_n)})} \right) \leq \widehat{\mathbf{E}}^{\uparrow} \left(\limsup_{n \to \infty} \frac{W(\ell(\epsilon) X_{\theta(R_n)})}{W(X_{\theta(R_n)})} \right) < 1,$$

which implies that

$$\lim_{n \to \infty} \widehat{\mathbf{P}}^{\uparrow} \left(\frac{X_{\theta(R_n)} - J_{R_n}}{f(R_n)} \geq (1 - 2\epsilon) \right) > 0.$$

Next, we note

$$\widehat{\mathbf{P}}^{\uparrow} \left(\frac{X_{\theta(R_p)} - J_{R_p}}{f(R_p)} \geq (1 - 2\epsilon), \text{ for some } p \geq n \right) \geq \widehat{\mathbf{P}}^{\uparrow} \left(\frac{X_{\theta(R_n)} - J_{R_n}}{f(R_n)} \geq (1 - 2\epsilon) \right).$$

Since R_n converges to 0, $\widehat{\mathbf{P}}^{\uparrow}$-a.s. as n goes to ∞, it is enough to take limits in both sides. Therefore for all $\epsilon \in (0, 1/2)$

$$\widehat{\mathbf{P}}^{\uparrow} \left(\frac{X_{\theta(t)} - J_t}{f(t)} \geq (1 - 2\epsilon), \text{ i.0., as } t \to 0 \right) \geq \lim_{n \to +\infty} \widehat{\mathbf{P}}^{\uparrow} \left(\frac{X_{\theta(R_n)} - J_{R_n}}{f(R_n)} \geq (1 - 2\epsilon) \right) = 1.$$

Then,

$$\limsup_{t \to 0} \frac{X_{\theta(t)} - J_t}{f(t)} \geq 1 - 2\epsilon, \quad \widehat{\mathbf{P}}^{\uparrow} - \text{a.s.},$$

and since ϵ can be chosen arbitrarily small, we get the result. $\qquad \square$

3 Concluding remarks on quasi-stationarity

We conclude this paper a brief remark about a kind of conditioning of CB-process which result in a so-called quasi-stationary distribution. Specifically we are interested in establishing the existence of a normalization constant $\{c_t, t \geq 0\}$ such that the weak limit

$$\lim_{t \to \infty} \mathbb{P}_x(Y_t/c_t \in dz | T_0 > t),$$

exist for $x > 0$ and $z \geq 0$.

Results of this kind have been established for CB-processes for which the underlying spectrally positive Lévy process has a second moment in [9]; see also [11], and for the α-stable CB-process with $\alpha \in (1, 2]$ in [8]. In the more general setting, [12] formulates conditions for the existence of such a limit and characterizes the resulting quasi-stationary distribution. The result below shows, when the branching mechanism is regularly varying at ∞, an explicit formulation of the normalization sequence $\{c_t : t \geq 0\}$ and the limiting distribution is possible.

Lemma 4. *Suppose that the branching mechanism ψ is regularly varying at ∞ with index $\alpha \in (1, 2]$. Then, for all $x \geq 0$, with $c_t = 1/\varphi(t)$*

$$\lim_{t \to \infty} \mathbb{E}_x\left[e^{-\lambda Y_t/c_t} \Big| T_0 > t\right] = 1 - \frac{1}{[1 + \lambda^{-(\alpha-1)}]^{1/(\alpha-1)}}.$$

Proof: The proof pursues a similar line of reasoning to the the aforementioned references [8, 9, 11, 12]. From (1) it is straightforward to deduce that

$$\lim_{t \to \infty} \mathbb{E}_x\left[1 - e^{-\lambda Y_t/c_t} \Big| T_0 > t\right] = \lim_{t \to \infty} \frac{u_t(\lambda/c_t)}{u_t(\infty)} = \lim_{t \to \infty} \frac{\varphi(t + \phi(\lambda/c_t))}{\varphi(t)},$$

if the limit on the right hand side exists. However, since ψ is regularly varying at ∞ with index α, we have that $1/\psi$ is regularly varying at ∞ with index $-\alpha$. On the other hand an application of Karamata's Theorem (see for instance Bingham et al [3]) gives

$$\phi(t) \sim -\frac{1}{1 - \alpha} \frac{t}{\psi(t)} \quad \text{as} \quad t \to \infty,$$

and therefore,

$$\lim_{t \to \infty} \frac{\phi(\lambda t)}{\phi(t)} = \lambda^{1-\alpha} \quad \text{for} \quad \lambda > 0.$$

In other words ϕ is regularly varying at ∞ with index $1 - \alpha$ and then its inverse φ is regularly varying at ∞ with index $\frac{1}{1-\alpha}$, thus for all $\lambda, \epsilon > 0$, and t sufficiently large, we have

$$\left(\frac{\lambda}{1 - \epsilon}\right)^{-(\alpha-1)} t \leq \phi(\lambda\varphi(t)) \leq \left(\frac{\lambda}{1 + \epsilon}\right)^{-(\alpha-1)} t.$$

Therefore since φ is decreasing, we deduce

$$\frac{\varphi\left(\left(1 + (\lambda/(1 + \epsilon))^{-(\alpha-1)}\right) t\right)}{\varphi(t)} \leq \frac{\varphi(t + \phi(\lambda/c_t))}{\varphi(t)} \leq \frac{\varphi\left(\left(1 + (\lambda/(1 - \epsilon))^{-(\alpha-1)}\right) t\right)}{\varphi(t)},$$

for t sufficiently large. On the other hand, since φ is regular varying we deduce

$$\frac{1}{\left(1 + \left(\frac{\lambda}{1+\epsilon}\right)^{-(\alpha-1)}\right)^{\frac{1}{\alpha-1}}} \leq \frac{\varphi(t + \phi(\lambda/c_t))}{\varphi(t)} \leq \frac{1}{\left(1 + \left(\frac{\lambda}{1-\epsilon}\right)^{-(\alpha-1)}\right)^{\frac{1}{\alpha-1}}},$$

for t sufficiently large. Hence the result follows taking the limit in the above inequality as ϵ goes to 0. $\qquad\square$

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