# Predefined-time convergence control for high-order integrator systems using time base generators (TBGs)

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***Abstract*—In this paper, we propose a control method for high-order integrator systems that achieves predefined-time convergence, i.e., the system is driven to the origin in a desired settling time that can be set as an explicit parameter of the controller and it is achieved independently of the initial conditions. Our method can be applied to any single-input single-output (SISO) controllable linear system, to any SISO nonlinear system that can be transformed to the normal form with stable zero dynamics and to multiple-input multiple-output systems that can be decoupled into SISO subsystems in the previously mentioned forms. The proposed approach is based on the so-called time base generators (TBGs), which are time dependent functions used to build time-varying control laws. The contribution of this paper is the generalization of the TBGs to develop predefined-time controllers for high-order systems, providing procedures to build the required time dependent functions. The performance of the proposed controllers is evaluated and compared to finite-time and fixed-time controllers in simulations and experiments. We show the applicability of the proposed approach to control electromechanical systems, in particular for the dynamic control of robotic systems.**

## I. INTRODUCTION

Convergence time of a control system is an important design parameter for real-time applications. Many applications require a strict time scheduling that constrains the time response of the control system. For example, a robot that must reach a desired position every specified time to grasp an object moving on a conveyor belt or the orbital stabilization of a biped robot to give a step in a predefined time. In this context is where finite-time convergence [1] and fixed-time convergence [2] of control systems make an important contribution. Nowadays, both finite-time and fixed-time controllers have been developed on the basis of sliding mode control (SMC) theory. Nevertheless, neither finite-time controllers nor fixed-time controllers guarantee convergence on a specified time (predefined time) for any initial condition.

Finite-time stability means that the solutions of a system starting in an open neighborhood of the origin converge to the origin in finite time [1]. Finite-time convergence of second order systems has been widely studied [1], [3], [4]. A bounded continuous time-invariant controller that globally finite-time stabilizes the double integrator system is introduced in [1]. In order to reduce the undesired chattering phenomenon of classical SMC, the theory of high-order SMC has been developed [3]. An example of high-order SMC with finite-time stability is the discontinuous twisting controller, which is specific for second order systems. The continuous version of the twisting controller is studied in [4] on a disturbed double integrator. Efforts have been made to develop tuning rules to set the control gains accordingly to a desired settling time in [5] for the discontinuous twisting controller and in [6] for the continuous version. The methodologies for pure twisting control allow to estimate an upper bound on the settling time that depends on the initial conditions.

Fixed-time stability demands uniform boundedness of the settling time for a globally finite-time stable system [7]. Thus, fixed-time stability guarantees convergence of the closed-loop system before an estimated bound on the settling time, independently of the initial conditions. However, the settling time changes for different initial conditions. Fixed-time controllers have been proposed on the basis of SMC. In [2], the control parameters are directly set as a function of the upper bound on the settling time, such that fixed-time stability for high-order systems is provided. A nonsingular fixed-time terminal SMC is presented in [8] for the particular case of second order nonlinear systems. A constructive scheme for tuning the control parameters based on implicit Lyapunov functions is introduced in [7], which achieves fixed-time stabilization of a chain of integrators. The results in [9] prove that any finite-time convergent homogeneous sliding mode controller can be transformed into a fixed-time convergent one by means of a discrete dynamic extension. A controller where the settling time can be specified has been introduced in [10], however, it is limited to the class of systems with the same number of control inputs as state variables. Thus, it is worth emphasizing that fixed-time convergence does not guarantee convergence in a constant predefined time for any initial condition.

Predefined-time convergence means that a desired settling time is a parameter that can be specified by the user, and this *constant convergence time* is achieved independently of the initial conditions. An approach to achieve predefined-time convergence for first order systems has been early introduced in [11] to induce attraction of force fields, in the context of movement neuroscience, by using functions named time base generators (TBGs). In the context of robot control, the TBG for first order systems has been used together with an adaptive SMC to yield finite-time tracking in [12]. The time-constraint property of the TBG for first order systems has been used for the parametrization of a hierarchical inverse kinematics control of robots [13] and to yield smooth control signals in visual control of humanoid robots [14]. Also, a TBG has been exploited in a hierarchical scheme of kinematic tasks for multirobot systems in [15]. A related work was presented in [16], where a linear protocol that uses time-varying control

The first author was supported in part by Conacyt [grant 220796].

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gains was proposed for reaching network consensus at a preset time. The approach is valid for first and second order integrator systems and no generalization is provided for high-order systems. To the best of our knowledge, predefined-time controllers have only been reported for first and second order integrator systems. Moreover, the robustness of those methods have not been analyzed.

In this work, we propose a novel control approach for high-order integrator systems that achieves predefined-time convergence, where the desired settling time is a user-defined parameter that is reached for any initial condition. The proposed approach can be applied to the class of $n$-order single-input-single-output (SISO) controllable linear systems and nonlinear systems that can be linearized to an integrator chain or transformed to the normal form with stable zero dynamics [17]. It is also applicable to those multiple-input-multiple-output (MIMO) systems that can be transformed to decoupled subsystems in the form of integrator chains or in the normal form with stable zero dynamics. Furthermore, we have introduced additional terms to the controllers to provide robustness against uncertainties and disturbances, which enlarge its applicability. The proposed approach allows the synthesis of controllers based on TBGs that are exploited in a trajectory tracking control scheme. The method includes procedures to build the TBGs as polynomial functions. Moreover, an optimization criterion is introduced to determine the best polynomial functions to accomplish the task. The resulting controllers have the additional advantage of yielding smoother control signals with considerable smaller control effort than fixed-time controllers reported in the literature. In particular, the controller for second order systems can benefit different applications in robotics at the level of acceleration control. Simulations and experiments demonstrate the applicability of the control method for systems like a simple rotational pendulum or a complex anthropomorphic manipulator.

The paper is organized as follows: Section II defines the class of systems for which the proposed method is applicable and defines the addressed problem. Section III introduces our general approach to synthesize predefined-time controllers for high-order systems. Section IV presents simulations and experimental results that illustrate the applicability and robustness of the proposed controllers. Moreover, the proposed control laws are compared to finite-time and fixed-time controllers reported in the literature. Section V remarks some conclusions.

## II. DEFINITIONS

In this work, single-input single-output (SISO) $n$-order systems are considered in an integrator-chain form as follows

$$\begin{aligned} &\dot{e}_j = e_{j+1}, \qquad j = 1, ..., n-1 \\ &\dot{e}_n(t) = u(t) + \rho(t), \\ &y = c_1 e_1 + c_2 e_2 + ... + c_n e_n, \end{aligned} \tag{1}$$

with $y$ being the system's output, $u(t)$ the control input, $\rho(t)$ a bounded matched disturbance and $e_1(t_i) = e_{1i}$,..., $e_n(t_i) = e_{ni}$ the initial conditions. The state vector is denoted as $\mathbf{e}(t) = [e_1(t), ..., e_n(t)]^T$ while the vector of initial conditions as $\mathbf{e}_i = [e_{1i}, ..., e_{ni}]^T$.

In the rest of the paper, the standard state space notation will be used

$$\begin{aligned} &\dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t) + \mathbf{B}(u(t) + \rho(t)), \\ &y = \mathbf{C}\mathbf{e}(t), \end{aligned} \tag{2}$$

for appropriate matrix $\mathbf{A}$ and vectors $\mathbf{B}$ and $\mathbf{C}$. Also, we will use the following notation for time-derivatives:

$$\dot{e}(t) = \tfrac{de(t)}{dt}, \; \ddot{e}(t) = \tfrac{d^2 e(t)}{dt^2}, ..., \; e^{(n)}(t) = \tfrac{d^n e(t)}{dt^n}. \tag{3}$$

Any SISO linear time-invariant system $\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}u$ can be transformed into an $n$-order integrator system (1) provided it is controllable, by transforming the system into the so-called *controllable canonical form* [18] and applying an input that cancels the open-loop dynamics of the $n^{\text{th}}$ state equation.

An analog procedure can be used for nonlinear systems that can be transformed to the *normal form* [17], with stable zero-dynamics. For instance, the following $(n + r)$ order system with relative degree $n$ is in the normal form,

$$\begin{aligned} &\dot{\xi}_j = \xi_{j+1}, \qquad j = 1, ..., n-1 \\ &\dot{\xi}_n(t) = \alpha(\boldsymbol{\xi}, \boldsymbol{\eta}) + \beta(\boldsymbol{\xi}, \boldsymbol{\eta})u + \rho(t), \\ &\dot{\eta}_1 = q_1(\boldsymbol{\xi}, \boldsymbol{\eta}), \\ &\quad \vdots \\ &\dot{\eta}_r = q_r(\boldsymbol{\xi}, \boldsymbol{\eta}), \\ &y = \xi_1, \end{aligned} \tag{4}$$

where $\boldsymbol{\xi} = [\xi_1, ..., \xi_n]^T$ and $\boldsymbol{\eta} = [\eta_1, ..., \eta_r]^T$.

By applying the input $u = (-\alpha(\boldsymbol{\xi}, \boldsymbol{\eta}) + v)/\beta(\boldsymbol{\xi}, \boldsymbol{\eta})$ (notice that $\beta(\boldsymbol{\xi}, \boldsymbol{\eta}) \neq 0$ so the relative degree is well defined), the input-output behavior of the closed-loop system evolves as the $n$-integrator chain (1), with $v$ being an auxiliary control input, together with the zero dynamics $\boldsymbol{\eta}$ which must be assumed to be stable.

Furthermore, the forthcoming analysis can also be applied to multiple-input-multiple-output (MIMO) systems that can be decoupled into SISO subsystems of the form (1) or (4) with stable zero-dynamics. An example of the application of our control approach to a nonlinear MIMO system will be shown in the subsection IV.A.2.

The problem to address in this work is stated as follows.

*Definition 2.1 (Problem Statement):* Given a $n$-order integrator chain (1), design a control law $u = \gamma(\mathbf{e}, t)$ such that the system is driven from any initial state $\mathbf{e}_i$, at an initial time $t_i$, to a neighborhood of the origin of the state space in a predefined finite time $t_f > t_i$, assuming knowledge of the state. In such case, the control law $u = \gamma(\mathbf{e}, t)$ is said to achieve *predefined-time convergence*.

## III. TBG CONTROL FOR HIGH-ORDER SYSTEMS

In this section, we present the main contribution of the paper: a methodology for the design of predefined-time controllers for $n$-order integrators (1). For that, let us firstly introduce a set of $n$ continuous $n$-differentiable time functions $h_1(t)$, $h_2(t)$, ... , $h_n(t)$; fulfilling the following conditions, at initial time $t_i$ and final time $t_f$,

$$\begin{array}{l} \forall k \in \{1,...,n\},\ \forall j \in \{0,...,n\} \\ h_k^{(j)}(t)|_{t=t_i} = \left\{ \begin{array}{ll} 1 & \text{if } j = k-1 \\ 0 & \text{otherwise} \end{array} \right. \\ h_k^{(j)}(t)|_{t \geq t_f} = 0. \end{array} \tag{5}$$

Next, consider the following matrix

$$\mathbf{H}(t) = \left[ \begin{array}{cccc} h_1(t) & h_2(t) & ... & h_n(t) \\ \dot{h}_1(t) & \dot{h}_2(t) & ... & \dot{h}_n(t) \\ \vdots & \vdots & & \vdots \\ h_1^{(n-1)}(t) & h_2^{(n-1)}(t) & ... & h_n^{(n-1)}(t) \end{array} \right]. \tag{6}$$

Notice that the condition (5) implies that $\mathbf{H}(t_i) = \mathbf{I}$ and $\mathbf{H}(t \geq t_f) = \mathbf{0}$. Now, let us introduce a first result.

*Lemma 3.1:* Consider the matrix $\mathbf{H}(t)$ as defined in (6), fulfilling the condition (5) and assuming that it is invertible in the time interval $[t_i, t_f)$. The time-variant feedback control law

$$\begin{array}{l} u = \mathbf{K}(t) \cdot \mathbf{e}(t), \\ \text{with} \\ \mathbf{K}(t) = \left[ h_1^{(n)}(t),\ h_2^{(n)}(t), ...,\ h_n^{(n)}(t) \right] \mathbf{H}(t)^{-1}, \end{array} \tag{7}$$

achieves predefined-time convergence for the $n$-order system (1) with ideal dynamics ($\rho(t) = 0$), i.e., $e_1(t)$ and their derivatives converge to zero in a desired settling time $\tau_s = t_f - t_i$. Furthermore, the resulting state trajectory in the interval $t \in [t_i, t_f)$ is

$$\mathbf{e}(t) = \mathbf{H}(t) \cdot \mathbf{e}_i \tag{8}$$

*Proof:* First notice that (8) is equivalent to

$$\begin{array}{l} e_1(t) = e_{1i}h_1(t) + e_{2i}h_2(t) + ... + e_{ni}h_n(t), \\ e_2(t) = e_{1i}\dot{h}_1(t) + e_{2i}\dot{h}_2(t) + ... + e_{ni}\dot{h}_n(t), \\ \quad \vdots \\ e_n(t) = e_{1i}h_1^{(n-1)}(t) + e_{2i}h_2^{(n-1)}(t) + ... + e_{ni}h_n^{(n-1)}(t). \end{array}$$

This expression constitutes a coherent solution for the system (1), i.e., the first derivative of the solution of $e_1(t)$ equals the solution of $e_2(t)$, the first derivative of the solution of $e_2(t)$ equals the solution of $e_3(t)$ and so on. Furthermore, the condition (5) implies that at time $t = t_i$, $\mathbf{e}(t) = \mathbf{e}_i$ (since $\mathbf{H}(t_i) = \mathbf{I}$). Moreover, the same condition implies that at time $t = t_f$, $\mathbf{e}(t) = \mathbf{0}$ (since $\mathbf{H}(t_f) = \mathbf{0}$).

In order to induce the solution (8), the input must fulfill $u(t) = \dot{e}_n(t)$, which can be easily computed as

$$u(t) = e_{1i}h_1^{(n)}(t) + e_{2i}h_2^{(n)}(t) + ... + e_{ni}h_n^{(n)}(t).$$

In vectorial notation,

$$u(t) = \left[ h_1^{(n)}(t),\ h_2^{(n)}(t), ...,\ h_n^{(n)}(t) \right] \cdot \mathbf{e}_i. \tag{9}$$

Now, (8) implies $\mathbf{e}_i = \mathbf{H}(t)^{-1} \cdot \mathbf{e}(t)$. Thus, (9) can be written as (7). In other words, the linear feedback (7) controls the system (1) in such way that (8) describes its solution. ■

The computation of the required functions $h_k(t)$, fulfilling the condition (5), is addressed in the following subsection.

In the Lemma 3.1, it is assumed that the matrix $\mathbf{H}(t)$ is invertible, which might be an issue of the proposed linear feedback (7). Moreover, the feedback (7) may not provide stability and robustness as a proper controller when disturbances and errors in the state measure are considered. For that reason, the following theorems combine the TBG of the Lemma 3.1 with feedback controllers, in such a way that the TBG is used as a reference trajectory in a tracking control scheme, avoiding numerical issues and providing stability and robustness.

*Theorem 3.2:* Consider the matrix $\mathbf{H}(t)$ as defined in (6), fulfilling the condition (5). Let $\mathbf{K}_f$ be a constant state-feedback gain matrix such that the eigenvalues of $(\mathbf{A} - \mathbf{B}\mathbf{K}_f)$ have negative real parts (considering the state-space notation (2)). The time-variant feedback control law

$$\begin{array}{l} u = \mathbf{K}_t(t)\mathbf{e}_i - \mathbf{K}_f\left(\mathbf{e}(t) - \mathbf{H}(t)\mathbf{e}_i\right), \\ \text{with} \\ \mathbf{K}_t(t) = \left[ h_1^{(n)}(t),\ h_2^{(n)}(t), ...,\ h_n^{(n)}(t) \right], \end{array} \tag{10}$$

achieves predefined-time convergence for the $n$-order system (1) with ideal dynamics ($\rho(t) = 0$), i.e., $e_1(t)$ and their derivatives converge to zero in a desired settling time $\tau_s = t_f - t_i$. Furthermore, global asymptotic stability of the tracking error $\varepsilon(t) = \mathbf{e}(t) - \mathbf{H}(t)\mathbf{e}_i$ is achieved.

*Proof:* According to the Lemma 3.1, if the initial state is set as $\hat{\mathbf{e}}(t_i) = \mathbf{e}_i$ and the control input $u = \mathbf{K}_t(t)\mathbf{e}_i$ (9) is applied then the system

$$\dot{\hat{\mathbf{e}}}(t) = \mathbf{A}\hat{\mathbf{e}}(t) + \mathbf{B}\mathbf{K}_t(t)\mathbf{e}_i,$$

evolves such that $\hat{\mathbf{e}}(t) = \mathbf{H}(t)\mathbf{e}_i$, i.e., $\hat{\mathbf{e}}(t)$ becomes the TBG reference trajectory.

In this way, using (2) for $\dot{\mathbf{e}}(t)$ and the complete control law (10), we have

$$\begin{array}{rcl} \dot{\varepsilon}(t) & = & \dot{\mathbf{e}}(t) - \dot{\hat{\mathbf{e}}}(t) \\ & = & \mathbf{A}\mathbf{e}(t) + \mathbf{B}\mathbf{K}_t(t)\mathbf{e}_i - \mathbf{B}\mathbf{K}_f\left(\mathbf{e}(t) - \mathbf{H}(t)\mathbf{e}_i\right) \\ & & -\mathbf{A}\hat{\mathbf{e}}(t) - \mathbf{B}\mathbf{K}_t(t)\mathbf{e}_i \\ & = & \left(\mathbf{A} - \mathbf{B}\mathbf{K}_f\right)\left(\mathbf{e}(t) - \hat{\mathbf{e}}(t)\right) \\ & = & \left(\mathbf{A} - \mathbf{B}\mathbf{K}_f\right)\varepsilon(t). \end{array} \tag{11}$$

Thus, since $\mathbf{K}_f$ is such that the eigenvalues of $(\mathbf{A} - \mathbf{B}\mathbf{K}_f)$ have negative real parts, the tracking error is asymptotically stable, meaning that $\mathbf{e}(t)$ follows $\mathbf{H}(t)\mathbf{e}_i$. Then, as proved in the Lemma 3.1, the reference $\mathbf{H}(t)\mathbf{e}_i$ vanishes for $t \geq t_f$ and $\mathbf{e}(t)$ converges to the origin of the state space in the predefined-time window $\tau_s$. Any small final error at time $t \geq t_f$ is corrected by the control input $u = -\mathbf{K}_f\mathbf{e}(t)$, which is applied for $t \geq t_f$, guaranteeing the global stability of the closed-loop system. ■

The Theorem 3.2 establishes that the closed-loop system not only achieves predefined-time convergence but also stability in the single control law (10). Although in (10) the information of $\mathbf{e}_i$ is used explicitly, it appears as a parameter, thus the control law (10) can be straightforwardly computed without the need of a tuning procedure for different initial conditions to obtained a preset convergence time. The linear control law (10) is just one among the different control techniques that can be used with the TBG.

In the following theorem, the TBG is combined with a super-twisting controller (STC), which is known to be able to compensate for matched uncertainties/disturbances $\rho(t)$ [19], leading to a more robust predefined-time controller. The STC is applicable to systems with relative degree one, however, it can be extended for high-order systems if the sliding surface is designed such that it has relative degree one [20].

*Theorem 3.3:* Consider the matrix $\mathbf{H}(t)$ as defined in (6), fulfilling the condition (5) and the tracking error vector $\boldsymbol{\varepsilon}(t) = \mathbf{e}(t) - \mathbf{H}(t)\mathbf{e}_i$. There exist gains $k_1, k_2 \in \mathbb{R}$ and $\mathbf{K}_{fr} \in \mathbb{R}^{n-1}$ such that the continuous time-variant sliding mode control law

$$\begin{aligned} & u = \mathbf{K}_t(t)\mathbf{e}_i - \mathbf{K}_{fr}\boldsymbol{\varepsilon}_{2:n} - k_1|s|^{1/2}\text{sign}(s) + v, \\ & \dot{v} = -k_2\text{sign}(s), \\ & \text{with} \\ & \mathbf{K}_t(t) = \left[h_1^{(n)}(t),\ h_2^{(n)}(t), ...,\ h_n^{(n)}(t)\right], \\ & \boldsymbol{\varepsilon}_{2:n} = [\varepsilon_2, \varepsilon_3, ..., \varepsilon_n]^T \text{ and} \\ & s = [\mathbf{K}_{fr}, 1]\,\boldsymbol{\varepsilon}(t), \end{aligned} \tag{12}$$

achieves predefined-time convergence for the disturbed $n$-order system (1). Furthermore, global asymptotic stability of the tracking error is achieved.

*Proof:* By using the $n$-order integrator dynamics (1), it can be shown that the tracking error system evolves as:

$$\begin{aligned} \dot{\varepsilon}_j &= \varepsilon_{j+1}, \qquad j = 1, ..., n-1 \\ \dot{\varepsilon}_n &= u + \rho - \mathbf{K}_t(t)\mathbf{e}_i. \end{aligned} \tag{13}$$

Thus, the dynamics of the sliding surface variable is given by:

$$\dot{s} = [\mathbf{K}_{fr}, 1]\,\dot{\boldsymbol{\varepsilon}}(t) = \mathbf{K}_{fr}\boldsymbol{\varepsilon}_{2:n} + u + \rho - \mathbf{K}_t(t)\mathbf{e}_i. \tag{14}$$

Using the control law (12), the closed-loop dynamics is:

$$\begin{aligned} \dot{s} &= -k_1|s|^{1/2}\text{sign}(s) + \varrho, \\ \dot{\varrho} &= -k_2\text{sign}(s) + \dot{\rho}, \end{aligned} \tag{15}$$

where $\varrho = v + \rho$. It has been proved in [19] that, for a bounded continuously differentiable disturbance, i.e., if $|\rho| < L$ and $|\dot{\rho}| < M$ for some constants $L > 0, M > 0$, the second order dynamics (15) converges globally to the origin ($s = 0$, $\varrho = 0$) in finite-time in spite of the disturbance if adequate positive control gains $k_1$ and $k_2$ are used. Moreover, the remaining dynamics of the tracking error system is constrained to the sliding surface, such that $s = \dot{s} = 0$. Thus, considering the sliding surface definition, the first $n-1$ equations of (13) represent the remaining dynamics:

$$\begin{aligned} \dot{\varepsilon}_j &= \varepsilon_{j+1}, \qquad j = 1, ..., n-2 \\ \dot{\varepsilon}_{n-1} &= -\mathbf{K}_{fr}\boldsymbol{\varepsilon}_{1:n-1}, \end{aligned} \tag{16}$$

which can be enforced to have global asymptotic stability through $\mathbf{K}_{fr}$, since the system is a controllable chain of $n-1$ integrators, meaning that $\mathbf{e}(t)$ follows $\mathbf{H}(t)\mathbf{e}_i$ and consequently, $\mathbf{e}(t)$ converges to the origin of the state space in the predefined-time window $\tau_s$. ■

Notice that the robust controller (12) allows the method to deal with parametric uncertainties, i.e., perfect knowledge of the system dynamics is not required, since some mismatches can be considered in the term $\rho(t)$, which is a disturbance fulfilling the matching condition [17] that can be rejected by the super-twisting controller using appropriate control gains [19].

Even if the initial conditions are not exactly known, the predefined-time convergence will be achieved according to the Theorems 3.2 and 3.3 provided that the corresponding tracking controller guarantees that the settling time of the tracking error will be lower than the predefined convergence time. In the case of the super-twisting controller, the settling time to the sliding surface –and hence to zero tracking error– is a finite value. Thus, predefined-time convergence with rough knowledge of the initial conditions can be guaranteed by the super-twisting controller using appropriate control gains [21].

### *A. Computation of functions $h_k(t)$.*

The computation of functions $h_k(t)$ means the computation of the reference trajectory (8) when either the applied control law is (10) or (12). In this subsection a couple of methods for computing such functions as polynomials are introduced.

First, let us introduce a simple possibility to compute functions $h_k(t)$, fulfilling (5), as polynomials of order $2n+1$. For that, consider the following notation:

$$\mathbf{G}(t) = \begin{bmatrix} \mathbf{g}(t) \\ \dot{\mathbf{g}}(t) \\ \vdots \\ \mathbf{g}^{(n)}(t) \end{bmatrix}, \tag{17}$$

where

$$\mathbf{g}(t) = \left[t^{2n+1},\ t^{2n}, ...,\ t,\ 1\right].$$

*Proposition 3.4:* Consider the matrix $\mathbf{G}(t)$ as in (17). For each index $k \in \{1, .., n\}$, compute the following vector of coefficients:

$$\mathbf{c}_k = \begin{bmatrix} \mathbf{G}(t_i) \\ \mathbf{G}(t_f) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix}, \tag{18}$$

where $\mathbf{I}_k$ is the $k^{\text{th}}$ column vector of the identity matrix with dimension $n+1$ and $\mathbf{0}$ is a column vector of zeros with dimension $n+1$. The functions $h_k(t)$ fulfilling (5) can be computed as

$$h_k(t) = \begin{cases} \mathbf{g}(t) \cdot \mathbf{c}_k & \text{if } t \in [t_i, t_f] \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

*Proof:* By definition, $\mathbf{g}(t)$ and its derivatives are independent vectors of dimension $2(n+1)$. Furthermore, the evaluation of $\mathbf{g}(t)$ and its derivatives at times $t_i$ and $t_f$ leads to independent vectors. Thus, the matrix built for the computation of $\mathbf{c}_k$ in (18) is square and has inverse, consequently $\mathbf{c}_k$ is well defined and unique. By defining $h_k(t) = \mathbf{g}(t) \cdot \mathbf{c}_k$, (18) implies

$$\begin{bmatrix} \mathbf{g}(t_i) \\ \dot{\mathbf{g}}(t)|_{t=t_i} \\ \vdots \\ \mathbf{g}^{(n)}(t)|_{t=t_i} \\ \mathbf{g}(t_f) \\ \dot{\mathbf{g}}(t)|_{t=t_f} \\ \vdots \\ \mathbf{g}^{(n)}(t)|_{t=t_f} \end{bmatrix} \mathbf{c}_k = \begin{bmatrix} h_k(t_i) \\ \dot{h}_k(t)|_{t=t_i} \\ \vdots \\ h_k^{(n)}(t)|_{t=t_i} \\ h_k(t_f) \\ \dot{h}_k(t)|_{t=t_f} \\ \vdots \\ h_k^{(n)}(t)|_{t=t_f} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix}. \tag{20}$$

This equation states that $h_k(t)$ and its derivatives are zero at time $t = t_f$. Furthermore, $h_k(t)$ and its derivatives are zero at time $t = t_i$ excepting for $h_k^{(k-1)}(t) = 1$. Thus, $h_k(t)$ fulfills the condition (5) at $t_i$ and $t_f$. Moreover, (20) implies that $h_k^{(n)}(t) = 0$ at $t = t_f$, meaning that $u = 0$ at $t = t_f$. Finally, by (19), each $h_k(t)$ and its derivatives are null for $t > t_f$. ■

Previous proposition proposes a particular way for computing functions $h_k(t)$. However, different possibilities may be explored. In the following proposition a higher degree polynomial vector is used, i.e., $\mathbf{g}(t) = [t^s, ..., t, 1]$ with $s > 2n + 1$, allowing more degrees of freedom during the computation of the coefficients $\mathbf{c}_k$, which can be adjusted by means of an optimization criterion. In particular, let us consider the quadratic cost function (21) defined in terms of the closed-loop state trajectory $\mathbf{e}(t)$ and the input $u(t)$, where the applied control action is computed as either (10) or (12) without disturbance $\rho(t)$, and the functions $h_1(t), ..., h_n(t)$ are defined as $h_k(t) = \mathbf{g}(t)\mathbf{c}_k$, fulfilling (5),

$$J(\mathbf{c}_1, ..., \mathbf{c}_n) = \int_{t_i}^{t_f} \mathbf{e}(t)^T \mathbf{Q} \mathbf{e}(t) + u(t) R u(t) dt, \tag{21}$$

where $\mathbf{Q}$ is a positive definite matrix of appropriate dimensions and $R > 0$.

*Proposition 3.5:* The coefficients $\mathbf{c}_1, ..., \mathbf{c}_n$, leading to the minimum cost TBG (21), can be computed as the following quadratic programming problem ($QPP$):

$$\min_{\mathbf{c}_1,...,\mathbf{c}_n} J(\mathbf{c}_1, ..., \mathbf{c}_n) = \left[\mathbf{c}_1^T, ..., \mathbf{c}_n^T\right] \mathbf{M}_{\mathbf{e_i}}{}^T \mathbf{T}(t_i, t_f) \mathbf{M}_{\mathbf{e_i}} \begin{bmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_n \end{bmatrix}, \tag{22}$$

subject to, $\forall k \in \{1, .., n\}$,

$$\begin{bmatrix} \mathbf{G}(t_i) \\ \mathbf{G}(t_f) \end{bmatrix} \mathbf{c}_k = \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix}, \tag{23}$$

where

$$\begin{aligned} \mathbf{T}(t_i, t_f) &= \int_{t_i}^{t_f} \mathbf{G}^T(t) \mathbf{Q} \mathbf{G}(t) + \mathbf{g}^{(n)}(t)^T R \mathbf{g}^{(n)}(t) dt \\ \mathbf{M}_{\mathbf{e_i}} &= \left[\mathbf{I} e_{i,1}, ..., \mathbf{I} e_{i,n}\right], \end{aligned} \tag{24}$$

with $e_{i,k}$ denoting the $k^{\text{th}}$ entry of $\mathbf{e}_i$ and $\mathbf{I}$ being the identity matrix of dimension $s + 1$.

*Proof:* First, assume that the coefficients $\mathbf{c}_1, ..., \mathbf{c}_n$ fulfill with (23). Then, by the Proposition 3.4 (defining $h_k(t) = \mathbf{g}(t)\mathbf{c}_k$), the condition (5) is fulfilled. Notice that

$$\begin{aligned} \left[h_1^{(n)}(t), ..., \ h_n^{(n)}(t)\right] &= \mathbf{g}^{(n)} \left[\mathbf{c}_1, ..., \mathbf{c}_n\right], \\ \mathbf{H}(t) &= \mathbf{G}(t) \left[\mathbf{c}_1, ..., \mathbf{c}_n\right]. \end{aligned} \tag{25}$$

Assuming that $\mathbf{e}(t_i) = \mathbf{e}_i$ and that there do not exist disturbances, the control laws (10) and (12) are equivalent to (9). Thus, considering (8) and (9), the cost function (21) can be written as

$$J(\mathbf{c}_1, ..., \mathbf{c}_n) = \int_{t_i}^{t_f} \mathbf{e}_i^T \mathbf{H}(t)^T \mathbf{Q} \mathbf{H}(t) \mathbf{e}_i + \mathbf{e}_i^T \left[h_1^{(n)}(t), ..., \ h_n^{(n)}(t)\right]^T R \left[h_1^{(n)}(t), ..., \ h_n^{(n)}(t)\right] \mathbf{e}_i dt.$$

Moreover, factorizing $\mathbf{e}_i$ and considering (25) it follows:

$$J(\mathbf{c}_1, ..., \mathbf{c}_n) = \int_{t_i}^{t_f} \mathbf{e}_i^T \Big( \left[\mathbf{c}_1, ..., \mathbf{c}_n\right]^T \mathbf{G}(t)^T \mathbf{Q} \mathbf{G}(t) \left[\mathbf{c}_1, ..., \mathbf{c}_n\right] + \left[\mathbf{c}_1, ..., \mathbf{c}_n\right]^T \mathbf{g}^{(n)}(t)^T R \mathbf{g}^{(n)}(t) \left[\mathbf{c}_1, ..., \mathbf{c}_n\right] \Big) \mathbf{e}_i dt.$$

Notice that $\mathbf{e}_i$ and $[\mathbf{c}_1, ..., \mathbf{c}_n]$ are time-independent. Thus, by using (24), the previous expression is equivalent to

$$J(\mathbf{c}_1, ..., \mathbf{c}_n) = \mathbf{e}_i^T \left[\mathbf{c}_1, ..., \mathbf{c}_n\right]^T \mathbf{T}(t_i, t_f) \left[\mathbf{c}_1, ..., \mathbf{c}_n\right] \mathbf{e}_i.$$

Finally, the previous expression is equivalent to (22) by using the fact that

$$\left[\mathbf{c}_1, ..., \mathbf{c}_n\right] \mathbf{e}_i = \mathbf{M}_{\mathbf{e_i}} \left[\mathbf{c}_1^T ... \mathbf{c}_n^T\right]^T.$$

■

The matrix $\mathbf{T}(t_i, t_f)$ is positive definite and the quadratic cost function $J$ is convex, hence, it has a finite global minimum. Moreover, $\mathbf{T}(t_i, t_f)$ can be computed by numerical integration, independently of the coefficients and the initial conditions. In general, the coefficients $\mathbf{c}_i$ depend on the initial condition, however, their computation can be efficiently achieved (even on-line) by using existing optimization algorithms for solving the $QPP$ (22). Furthermore, in robotics and electromechanical control applications, it frequently occurs that the output of the system is its position and the system is static at the initial time. In such case, the optimal coefficients in the Proposition 3.5 can be computed by assuming $\mathbf{e}_i = [1, 0, ..., 0]^T$, thus the $QPP$ becomes independent on the initial conditions.

## IV. SIMULATION AND EXPERIMENTAL RESULTS

This section is divided in two main parts. The first part presents simulation results and the second part presents some experiments on a real electromechanical system.

### A. Simulation results

*1) Comparison with finite-time and fixed-time controllers:* Let us first provide a comparison of the proposed approach with respect to finite-time and fixed-time controllers reported in the literature ( [1]–[4], [8]). Three of them ensure finite-time convergence: the discontinuous twisting controller [3], the continuous twisting controller [4], and the Bhat and Bernstein controller [1]. The rest of them guarantees fixed-time convergence: the Polyakov's controller [2], the Zuo's controller [8], and the proposed TBG controller. We use the double integrator system (1) with $n = 2$ for the comparative analysis of the settling time for the set of controllers.

The controllers were implemented in Matlab using the Euler forward method to approximate the time-derivatives with time step of 0.5 milliseconds. For each controller, we set the initial condition $x_{2_i} = 0.5$, and we varied the initial value $x_{1_i}$ from 0 to 100. For every initial condition, we measured the convergence time of the system when $\| \mathbf{e} \| < 1 \times 10^{-2}$ and the maximum absolute value of the control input $u$.

The five mentioned controllers were manually tuned to achieve a similar convergence time around 6 seconds for

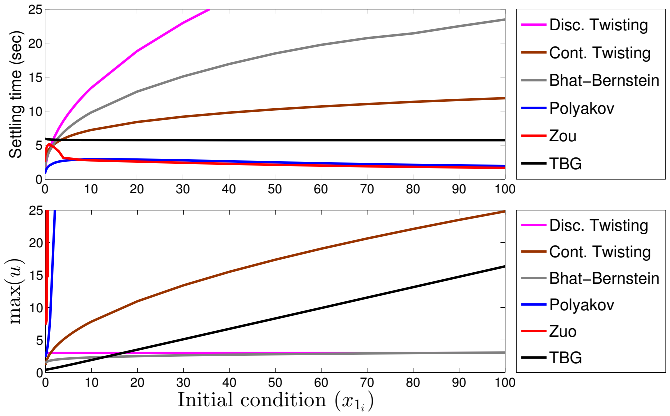



Fig. 1. Comparison of a proposed TBG controller (10) versus finite-time and fixed-time controllers. Top: Settling time as a function of the initial condition $x_{1_i}$. Bottom: Maximum absolute value of the control input as a function of the initial condition $x_{1_i}$.

initial conditions $x_{1_i} = 1$ and $x_{2_i} = 0.5$. Then, we kept the same control gains for the simulations with different initial conditions of $x_{1_i}$. We use the linear TBG controller defined in (10) for $n = 2$ and TBG functions (19) with $\tau_s = 6$ seconds and $\mathbf{K}_f = [2, 3]$.

The results are shown in Fig. 1. On the one hand, the settling time for the twisting and Bhat and Bernstein finite-time controllers increased unbounded as the initial conditions became bigger. On the other hand, in the Polyakov's and the Zuo's controllers the convergence time was kept bounded, below 6 seconds, however the maximum absolute value of the control input increased really fast with them (the corresponding profiles early escaped toward the top in Fig. 1).

The TBG controller was the only in which the convergence time was kept constant at 6 seconds for every initial condition, while the increment of the maximum control effort was slow and close to linear. Additionally, the TBG controller is not affected by the time step of the control loop, while the other compared controllers needed a high frequency control cycle since they are all based on the sliding modes paradigm.

*2) 7-DoF robot manipulator PA10:* Let us illustrate the application of the TBG method in a nonlinear MIMO system. Consider the manipulator of Fig. 2(a), with the equations of motion in terms of the Euler-Lagrange formalism as:

$$\mathcal{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathcal{C}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{u}, \tag{26}$$

where $\{\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\} \in \mathbb{R}^7$ are the joint configuration, velocity, and acceleration, respectively. $\mathcal{B}(\mathbf{q}) \in \mathbb{R}^{7\times 7}$ denotes the symmetric positive definite inertial matrix, $\mathcal{C}(\mathbf{q}, \dot{\mathbf{q}}) = C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + D(\cdot)\dot{\mathbf{q}} + g(\mathbf{q})$ contains the Coriolis vector $C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \in \mathbb{R}^7$, $D(\cdot)\dot{\mathbf{q}} \in \mathbb{R}^7$ is the damping vector that expresses the inner dissipative energy of the system, and $g(\mathbf{q}) \in \mathbb{R}^7$ is the gravity vector. The vector of control torques is $\mathbf{u} \in \mathbb{R}^7$. We applied the computed-torque control scheme in which the control torques are expressed as:

$$\mathbf{u} = \mathcal{B}(\mathbf{q})\mathbf{v} + \mathcal{C}(\mathbf{q}, \dot{\mathbf{q}}), \tag{27}$$

where $\mathbf{v} \in \mathbb{R}^7$ is an auxiliary control input. The computed-torque scheme decouples the joint dynamics and hence, $\mathbf{v}$ is applied to a set of seven decoupled subsystems of second order. Thus, $\mathbf{v}$ is a stack of seven equations (10) with

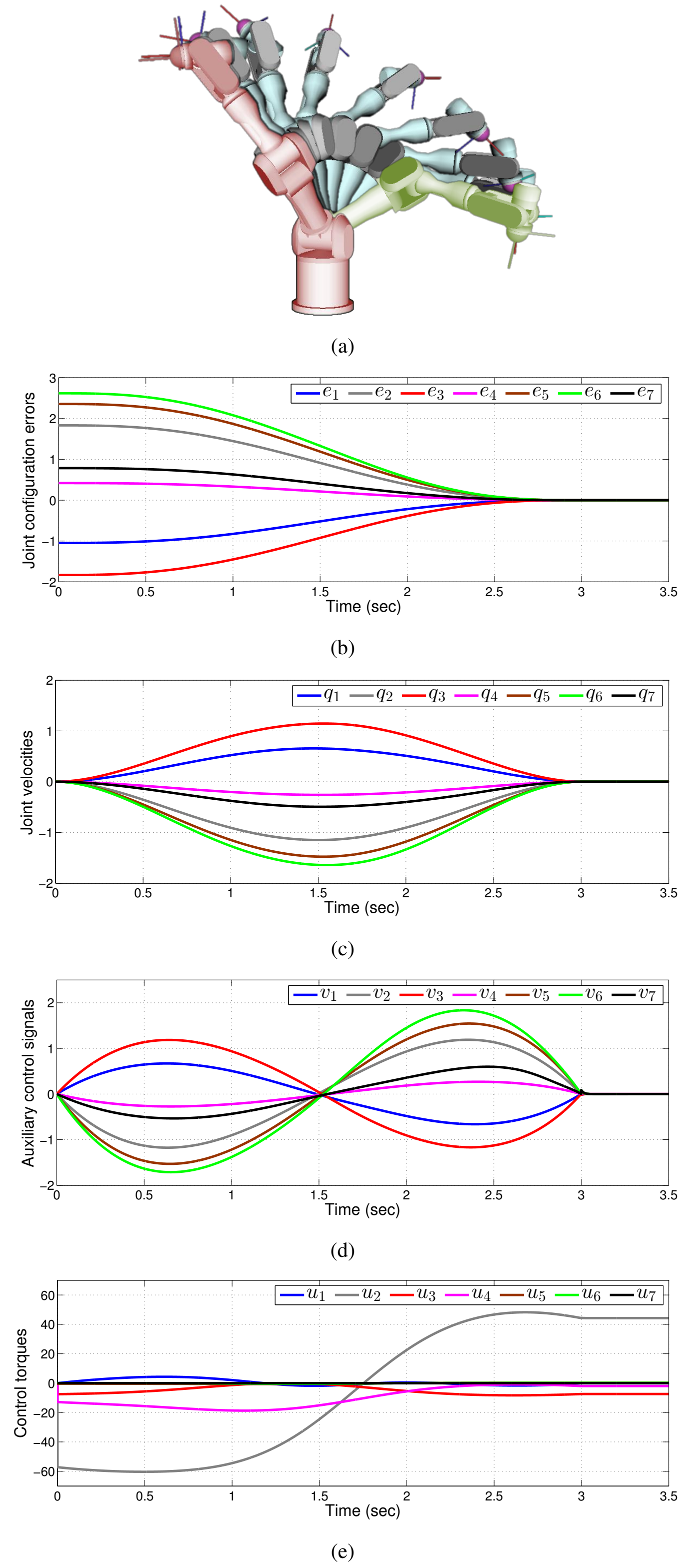



Fig. 2. PA10 manipulator with the TBG controller (10) for convergence time of 3 seconds. (a) illustrates the initial (green) and desired (red) configurations as well as some configurations along the computed trajectory. (b) shows the joint configuration errors. (c) shows the joint velocities profiles. (d) and (e) show the auxiliary control and the control inputs for each joint, respectively.

$\mathbf{e}_j = [q_j - q_j^d, \dot{q}_j]^T$ for $j = 1, ..., 7$ and the same $\mathbf{K}_f = [2, 3]$ for each subsystem. The matrices $\mathbf{K}_t$ and $\mathbf{H}$ are also the same for each subsystem and they are computed as in the Proposition 3.4 for $t_i = 0$ and $t_f = 3$ seconds.

The implementation of the complete control scheme required the instantaneous evaluation of $\mathcal{B}(\mathbf{q})$ and $\mathcal{C}(\mathbf{q}, \dot{\mathbf{q}})$. For this, we used the efficient recursive spatial algorithms described in [22]. In particular, we obtained $\mathcal{B}(\mathbf{q})$ by means of the Composite Rigid Body algorithm while $\mathcal{C}(\mathbf{q}, \dot{\mathbf{q}})$ was calculated with the spatial version of the Newton-Euler algorithm with $\ddot{\mathbf{q}} = \mathbf{0}$ as its argument. The simulation was implemented in Matlab using the *ode45* solver with time step of 1 millisecond. In Fig. 2, we show the results of regulating the joint configuration error of the PA10 manipulator toward zero. The initial and desired joint configurations were set to $\mathbf{q}_i = \left(0\ \frac{\pi}{3}\ \frac{-\pi}{4}\ \frac{\pi}{3}\ \frac{\pi}{4}\ \frac{\pi}{2}\ \frac{\pi}{4}\right)^T$ and $\mathbf{q}_d = \left(\frac{\pi}{3}\ \frac{-\pi}{4}\ \frac{\pi}{3}\ \frac{\pi}{5}\ \frac{-\pi}{2}\ \frac{-\pi}{3}\ 0\right)^T$. It can be seen in Fig. 2(b) that the joint configuration errors converge to zero in the predefined 3 seconds. In Fig. 2(c), it can be observed the smooth behavior of the joint velocities. The auxiliary control signals in Fig. 2(d) start at zero and converge to zero within the same time interval. Observe in Fig. 2(e) that the control inputs do not converge to zero; they have to compensate the nonlinear effects $\mathcal{C}(\mathbf{q}, \dot{\mathbf{q}})$ at the desired configuration. This is expected since $\mathbf{q}_d \neq 0$.

*3) Robust trajectory tracking control:* Let us illustrate the robust option for the TBG given by the Theorem 3.3 in a third order system. The simulation considers a time-varying disturbance $\rho(t) = 2(1 + \sin(2t))$ and the reference trajectory provided by the TBG uses different initial conditions to those in the system, i.e., the control law considers $\mathbf{e}_i = (5, -3, 3)^T$ while the real initial conditions of the system are $(10, -6, 6)^T$. The coefficients of the TBGs are obtained as in the Proposition 3.4 for a predefined-time of 6 seconds. The control gains are set as $k_1 = 5$, $k_2 = 3$ and $\mathbf{K}_{fr} = (6, 5)$.

Fig. 3 shows that the state trajectories converge to the references around 4 seconds and therefore, they are driven to zero in the desired predefined time. It can be seen that the control input remains oscillating at steady-state since it is effectively rejecting the disturbance $\rho(t)$.

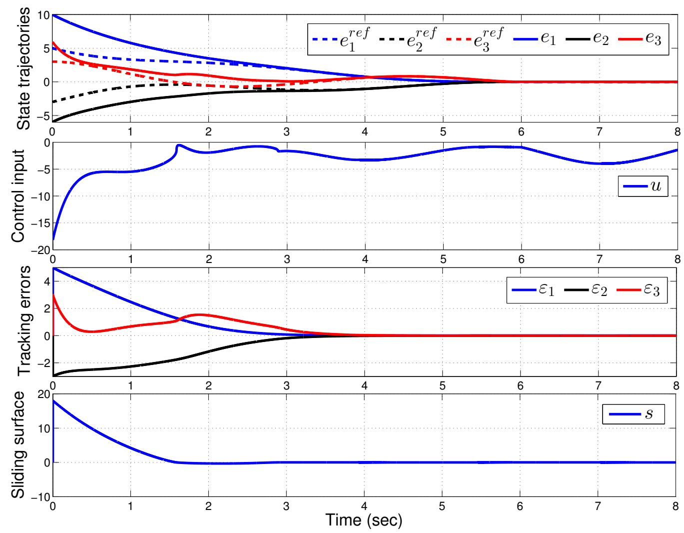



Fig. 3. Robust predefined-time TBG controller (12) for a disturbed third order integrator system. The convergence time is preset to 6 seconds.

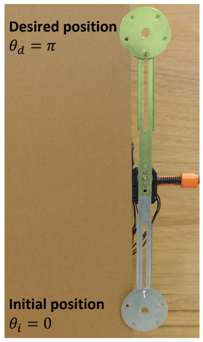
## B. Experimental results

The experimental setup consists of a simple rotational pendulum as shown in Fig. 4(a). The system parameters used in the controller were roughly estimated to the following values: length of the link $l = 20$ cm, mass $m = 0.11$ kg and damping constant $k = 0.11$ kg/sec. The torque-mode servo controller device is a Dynamixel MX-28T. The controller is connected to the computer for sending the instantaneous torque commands as well as for obtaining the instantaneous position and velocity profiles at 1 kHz. We verified the behavior of the TBG controller formulated in the Theorem 3.2 with $n = 2$ for different initial conditions and the same convergence predefined-time. In particular, Fig. 4(b) shows the profiles of the state of the system ($\theta$ and $\dot{\theta}$) along the time axis (top) and the control signal (bottom) for 4 initial conditions with a convergence predefined-time of 3 seconds. The desired position and velocity were $\pi$ and zero, respectively. The tracking control gain matrix was set to $\mathbf{K}_f = [11, 1.1]$. The polynomial functions of the TBGs were calculated by means of the optimization method given in the Proposition 3.5 with 3 additional degrees of freedom ($s = 8$).

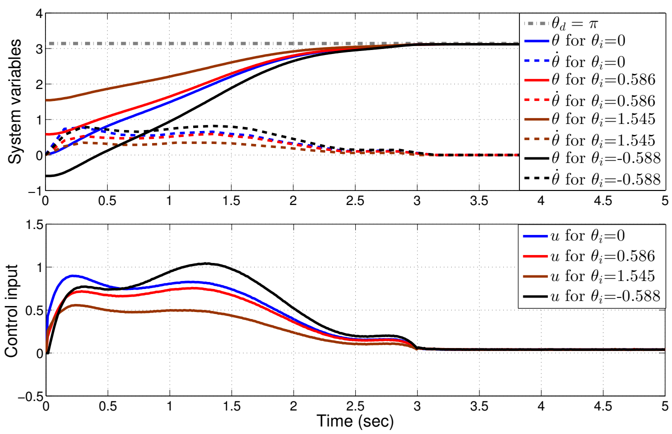
For comparison purposes, we applied the Polyakov's controller [2] with 4 initial conditions and 3 seconds as the predefined convergence time (see Fig. 4(c)). It is evident the difference in the controllers performance, the TBG-based controller is able to keep the predefined convergence time while the Polyakov's controller achieves the convergence in different times.

## V. CONCLUSIONS

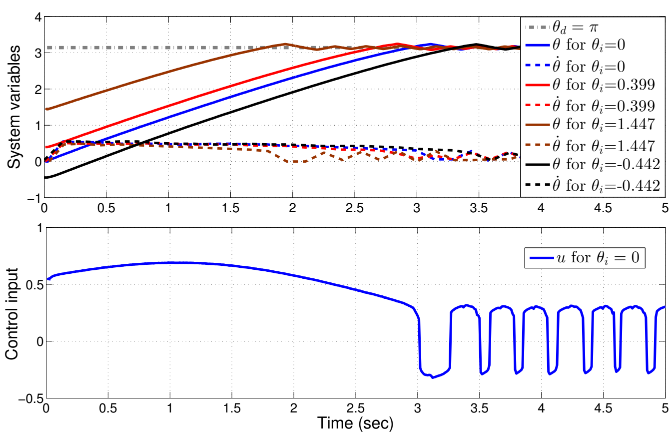
In this paper, we have proposed a general method to synthesize controllers whose desired closed-loop settling time can be set by the user and it is achieved independently of the initial conditions. This property is called predefined-time convergence. The kind of systems that can be controlled with this approach are SISO linear controllable systems and nonlinear systems that can be transformed to the normal form, with stable zero dynamics. Moreover, it is possible to control MIMO systems that can be decoupled into SISO subsystems in the previously mentioned forms. In our approach, a reference trajectory is firstly computed, named TBG. Later, the TBG is combined with feedback controllers to achieve closed-loop stability and robustness, in particular applying a linear feedback controller and a super-twisting controller with the TBG. Furthermore, we have proposed methods to build the TBGs as polynomial functions. We consider of special interest the applicability of the predefined-time controllers in the control of robotic systems. The performance of some controllers as study cases has been evaluated for regulation problems and compared with existing finite-time and fixed-time controllers in simulations. Experiments have also shown good performance of the control method in a real setup. Moreover, a good benefit of the proposed controllers is that they yield smoother control signals with considerable smaller magnitudes than finite-time and fixed-time controllers reported in the literature.




(a)




(b)




(c)

Fig. 4. Experimental setup: the behavior of a real pendulum with the TBG controller of the Theorem 3.2 and the Polyakov's controller [2]. (a) illustrates the real system at two different configurations. (b) shows the behavior of the TBG controller with four initial conditions and 3 seconds as the predefined time of convergence. (Top) shows the state profiles. (Bottom) shows the computed control inputs. (c) shows the behavior of Polyakov's controller with four initial conditions and 3 seconds as the fixed time of convergence. (Top) shows the state profiles. (Bottom) shows only the computed control input of the first initial condition for clarity of the visualization.

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