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IFAC PapersOnLine 51-13 (2018) 246–253

Predefined-time consensus using a time base generator (TBG)

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Abstract: Predefined-time convergence means that a system is driven to the origin of its state space in a desired settling time that can be set as an explicit parameter of the controller and it is achieved independently of the initial conditions. In this paper, we propose distributed control protocols that enforce predefined-time convergence in the consensus problem for first order systems. The proposed control method is based on the so-called time base generator (TBGs), which are time-dependent functions used to build time-varying control laws. Our method can be applied to any first-order linearizable nonlinear system. In particular, a robust nonlinear control protocol is proposed to deal with perturbed systems. Stability of the closed-loop protocols is proved for connected undirected communication topologies of the network system and for directed topologies having a spanning tree. The performance of the proposed consensus protocols are evaluated and compared in simulation with finite and fixed-time controllers and other previous scheme that also reaches consensus at a preset time.

Keywords: Consensus, predefined-time convergence, time base generator (TBG).

1. INTRODUCTION

Consensus algorithms concerns a net of agents which interact in order to achieve a common objective by considering only local information. In particular, consensus algorithms may force a network of agents to agree on a common value for its internal state (see e.g. Jiang and Wang (2009); Olfati-Saber and Murray (2004)), by using only communication among “neighbors”. The neighboring relation is frequently described by a graph (which could be directed), in which nodes are the agents and arcs represent communication among agents.

Several works have been published proposing consensus algorithms for different types of systems. Regarding first-order agents, the standard protocol (the input of an agent is a linear combination of the errors between the agent's state and those of his neighbors) achieves consensus if the graph topology is strongly connected (Olfati-Saber et al. (2007); Cai (2012); Ren and Beard (2008)). This algorithm achieves consensus also for strongly-connected dynamic topologies (Olfati-Saber et al. (2007); Cai and Ishii (2014)). For directed graphs topologies (the information may flow from one agent to another but not in the opposite sense), a common requirement is that the graph contains a spanning-tree (e.g., Li and Duan (2014)). All these algorithms reach consensus asymptotically, since the consensus protocol is linear.

A number of finite-time protocols have been proposed for consensus of first-order agents. In Tu et al. (2017); Cao and Ren (2014); Lu et al. (2013) finite-time protocols were proposed. Frequently, finite-time consensus protocols are defined as functions $sign(\bullet)$ evaluated on each of the neighbors' errors (Franceschelli et al. (2013)) or as functions $|\bullet|^{\alpha} sign(\bullet)$ evaluated either on each of the neighbors' errors (Cao and Ren (2014)) or on the sum of the neighbors' errors (Tu et al. (2017); Lu et al. (2013)). In Chen et al. (2011) a finite-time protocol is proposed as a sum of the signs of each neighbor's error. Authors called this protocol “binary”. Some algorithms have been extended to achieve finite-time consensus for strongly-connected dynamic topologies (Shang (2012); Franceschelli et al. (2013)). In Franceschelli et al. (2013), the topology can switch among disconnected graphs, but additional conditions are imposed to ensure finite-time convergence.

Fixed-time protocols have been proposed in Parsegov et al. (2013); Zuo et al. (2014). In these, there exists a bound for the convergence time that is independent of the initial conditions (Cruz-Zavala et al. (2010); Polyakov (2012)). To the best of our knowledge, the only work dealing with consensus in predefined-time was presented by Yong et al. (2012), where a linear protocol that uses time-varying control gain was proposed for reaching network consensus at a preset time. The convergence time is a parameter defined by the user and being independent of the initial conditions. The approach is valid for first- and second-

⋆ The first and third authors have been partially supported by Conacyt [grant 220796]. The authors would also like to acknowledge the partial support of Intel Corporation for the development of this work.

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2018.07.286

order integrator systems and no generalization is provided for high order systems. Moreover, the robustness of those methods has not been analyzed.

In this work, a novel predefine-time consensus protocol for first order agents is proposed. This protocol takes advantage of time base generators (TBG), which are parametric time signals that converge to zero in a predefined-time (Becerra et al. (2017)), and can be tracked by means of feedback controllers. The proposed approach can be applied to the class of first-order controllable linear systems and nonlinear systems that can be linearized (Khalil and Grizzle (2002)). In particular, a couple of consensus algorithms, one using a linear-feedback and the other using a super-twisting controller, are introduced for the tracking of the TBG signal. The application of these algorithms is shown through simulations. In comparison with Yong et al. (2012), the main contribution of this paper is three-fold: First, the proposed controllers yield smoother auxiliary control signals (without considering the linearization terms) with smaller magnitudes. Second, the proposed time-varying feedback gain to solve the consensus problem does not depend on the algebraic connectivity of the graph considered. Third, a super-twisting controller is presented in order to deal with perturbed dynamics of the agents of the network.

This paper is organized as follows. Section 2 defines the class of systems for which the proposed method is applicable and defines the addressed problem. Section 3 introduces the TBG as a time-varying control gain in the consensus problem. Section 4 presents a linear-feedback and a super-twisting controller for the tracking of the TBG signal. Section 5 presents simulation results of the proposed controllers. Section 6 remarks some conclusions.

2. PRELIMINARIES AND PROBLEM DEFINITION

2.1 Notations

Let $\mathbb{R}$ (or $\mathbb{C}$) denote the set of all real (or complex) numbers. $\mathbf{1}_N := [1, \ldots, 1]^T$, the $N \times 1$ column vector of ones. i represents the imaginary unit. Given a complex number $\lambda = \alpha + \mathrm{i}\beta \in \mathbb{C}$, $Re(\lambda)$ represents the real part of λ. Notation diag$\{a_1, \ldots, a_N\}$ represents a diagonal matrix with components $\{a_1, \ldots, a_N\}$ strung along the diagonal.

2.2 Algebraic graph theory

Agents' communication is represented by a directed graph $\mathcal{G} = (V, E, A)$, which consists of a set of vertices (nodes or agents in this work) $V = \{1, \ldots, N\}$, a set of edges $E \subseteq V \times V$, and a weighted adjacency matrix $A = [a_{ij}]$ with nonnegative adjacency elements a_{ij}. In an undirected graph the pairs of nodes are unordered and then $A = A^T$. An entry of A that fulfills $a_{ij} > 0$, i.e., $(i, j) \in E$, represents that agent j has the information of the state value x_j of the agent i, otherwise $a_{ij} = 0$, i.e., $(j, i) \notin E$. The set of neighbors of agent i is denoted by $N_i = \{j \in V : (j, i) \in E\}$.

A *directed path* (resp. *undirected path*) is a sequence of distinct edges in a directed graph (resp. *undirected graph*). A graph is *connected* if there exists a path between any two distinct vertices.

We say that a directed graph $\mathcal{G}$ has a *directed spanning tree* if there exists a node v_i (a root) such that all other nodes can be linked to v_i via a directed path.

The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ of a general graph $\mathcal{G}$ is defined by

$$l_{ij} = \begin{cases} -a_{ij}, & \text{if } i \neq j, \\ \sum_{k=1,k\neq i}^{N} a_{ik}, & \text{if } i = j. \end{cases}$$

We will assume the following well-known properties of the Laplacian matrix:

- P1 For the types of considered graphs, at most one eigenvalue can be zero, i.e., $\lambda_1 = 0$ and $Re(\lambda_i) > 0, \forall i = \{2, \ldots, N\}$ (Ren and Beard (2008); Li and Duan (2014)).
- P2 For a connected undirected graph, L has a left eigenvector $\gamma = \mathbf{1}_N$, i.e., $\gamma^T L = 0$ (Olfati-Saber and Murray (2004)).
- P3 For a directed graph with a directed spanning tree, L has a left eigenvector γ satisfying $\gamma^T L = 0$ and $\gamma^T \mathbf{1}_N = 1$ (Li and Duan (2014)).

2.3 Problem definition

Consider a multi-agent system composed of N agents that are connected through a network with first-order nonlinear dynamics given by:

$$\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i(t) + \rho_i(t),\ i \in \{1, \ldots, N\} \quad (1)$$

where $x_i \in \mathbb{R}$ is the system's state, $u_i(t) \in \mathbb{R}$ is the control input, $f_i(x_i)$ and $g_i(x_i)$ are smooth nonlinear functions, and $\rho_i(t) \in \mathbb{R}$ is a bounded disturbance of agent i.

Denote $x = [x_1, x_2, \ldots, x_N]^T$, $u = [u_1, u_2, \ldots, u_N]^T$, $f = [f_1, f_2, \ldots, f_N]^T$, $g = [g_1, g_2, \ldots, g_N]^T$, $\rho = [\rho_1, \rho_2, \ldots, \rho_N]^T \in \mathbb{R}^N$. Then, the whole dynamics (1) can be written as:

$$\dot{x}(t) = f(x) + g(x)u(t) + \rho(t). \quad (2)$$

By applying the control input $u_i = (-f_i(x_i)+v_i)/g_i(x_i)$ for each agent in (1), where $g_i(x_i) \neq 0$ so the relative degree is well defined (Khalil and Grizzle (2002)) and the agent is controllable and v_i being an auxiliary control input, the i-th agent evolves as:

$$\dot{x}_i(t) = v_i(t) + \rho_i(t),\ i \in \{1, \ldots, N\} \quad (3)$$

and the whole remaining dynamics is linear and given by:

$$\dot{x}(t) = v(t) + \rho(t) \quad (4)$$

where $v = [v_1, v_2, \ldots, v_N]^T \in \mathbb{R}^N$.

The weighted error of agent i with respect to its neighbors (Olfati-Saber and Murray (2004)) is defined as:

$$e_i(t) = \sum_{j\in N_i} a_{ij}(x_j(t) - x_i(t)),\ i \in \{1, \ldots, N\}. \quad (5)$$

The weighted error function (5) can be expressed in a compact form as:

$$e(t) = [e_1(t), e_2(t), \ldots, e_N(t)]^T = -Lx(t). \quad (6)$$

Definition 1. Problem statement: Given a network of agents with first order nonlinear dynamics (1) and with an associated graph $\mathcal{G}$, for each agent, design a control protocol $u_i = \gamma(e_i, x_i, t)$ such that the network system achieves a consensus value x^* in a *predefined time* t_f from any initial state $x(0)$, i.e., $x_i(t) \to x^*, \forall i = \{1, 2, \dots, N\}$, as $t \to t_f$. Consequently, the weighted error (6) converges to zero at time t_f.

3. TBG AS TIME-VARYING CONTROL GAIN

Let us first introduce a continuous and differentiable time function $h(t)$ as proposed in Becerra et al. (2017) that will be used through this paper, fulfilling the following conditions, at zero initial time and final time t_f:

$$h(t) = \begin{cases} 1, & \text{if } t = 0 \\ 0, & \text{if } t \geq t_f \end{cases}$$
$$\dot{h}(t) = 0, \text{ if } t = 0 \text{ or } t \geq t_f$$
$$\dot{h}(t) < 0, \ \forall t \in (0, t_f). \quad (7)$$

The kind of time-dependent functions fulfilling conditions (7) are called *time base generators (TBGs)*. They have been previously used to achieve predefined-time convergence of first and higher order dynamics in Becerra et al. (2017). Taking advantage of those results, the following control protocol is proposed to solve the problem stated in Definition 1:

$$u_i(t) = \frac{1}{g_i(x_i)}(-f_i(x_i) + v_i), \ i \in \{1, \dots, N\}$$
$$v_i = -k(t) \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)). \quad (8)$$

Then, for each agent i, the dynamics (1) is linearized to:

$$\dot{x}_i(t) = -k(t) \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)) + \rho_i(t)$$

which can be expressed using (6) in matrix form as:

$$\dot{x}(t) = -k(t)e(t) + \rho(t) = k(t)Lx(t) + \rho(t) \quad (9)$$

where $k(t) \in \mathbb{R}$ is a time-varying feedback gain that will be defined later in terms of the TBG and L is the Laplacian matrix of the graph $\mathcal{G}$. In the sequel, we will analyze the behavior of the control protocol (8) in the consensus problem for two kind of networks: connected undirected graphs (Ren and Beard (2008)) or directed graphs with a directed spanning tree (Li and Duan (2014)).

Let $J \in M_N(\mathbb{C})$ be the Jordan form associated with L (Horn and Johnson (2012)). Then, there exists a nonsingular matrix $S \in M_N(\mathbb{C})$ such that $S^{-1}LS = J$.

Applying the similarity transformation $\eta(t) = S^{-1}x(t)$ to the system (9) with ideal dynamics ($\rho(t) = 0$), we have

$$\dot{\eta}(t) = S^{-1}(k(t)Lx(t)) = k(t)J\eta(t). \quad (10)$$

Notice that for undirected graphs, L is a symmetric matrix with real eigenvalues (Ren and Beard (2008)). Hence, $J = \text{diag}\{0, \lambda_2, \dots, \lambda_N\}$ for undirected graphs. The similarity transformation will be used in the proof of the following proposition.

Proposition 2. Assuming that $\eta_i(t) \to 0$ as $t \to t_f$ for $i = \{2, 3, \dots, N\}$ when the control protocol (8) is applied to each agent, then the system (2) with $\rho(t) = 0$ achieves a consensus value x^* in time t_f.

Proof. Without loss of generality, we assume that the first column vector of the matrix S is $\mathbf{1}_N$ (Yong et al. (2012)), which is associated to the null eigenvalue of L. Moreover, assume that $\eta_i(t) \to 0$ as $t \to t_f$ for $i = \{2, \dots, N\}$. Since

$$\lim_{t \to t_f} x(t) = S \lim_{t \to t_f} \eta(t) \quad (11)$$

it follows that

$$\lim_{t \to t_f} x(t) = S[\eta_1(0), 0, \dots, 0]^T = \eta_1(0)\mathbf{1}_N. \quad (12)$$

It means that

$$x_i(t) \to x^* = \eta_1(0) \text{ as } t \to t_f, \text{ for } i = \{1, 2, \dots, N\}.$$

Thus, a consensus value is achieved at time t_f. Now, given properties P2 and P3 for the two types of considered graphs, then, $y(t) = \gamma^T x(t)$ is an invariant quantity, since

$$\dot{y}(t) = \gamma^T \dot{x}(t) = k(t)\gamma^T Lx(t) = 0, \ \forall x(t).$$

Thus, $\lim_{t \to t_f} y(t) = \lim_{t \to t_f} \gamma^T x(t) = y(0)$ or

$$\lim_{t \to t_f} \gamma^T x(t) = \gamma^T x(0). \quad (13)$$

Then, using (12) in (13) for a connected undirected graph ($\gamma = \mathbf{1}_N$) and solving for $\eta_1(0)$, we obtain

$$\eta_1(0) = \frac{\sum_{i=1}^{N} x_i(0)}{N} = Ave(x_i(0)). \quad (14)$$

Hence, the consensus value x^* is the average of the initial state $x(0)$ for connected undirected graphs. Now, using (12) in (13) for a directed graph with a directed spanning, we have

$$\eta_1(0) = \frac{\gamma^T x(0)}{\gamma^T \mathbf{1}_N} = \gamma^T x(0). \quad (15)$$

Therefore, the consensus value is given by (15) for a directed graph with a directed spanning tree. □

In the next lemma, we will define the form of the time-varying control gain $k(t)$ as a function of the TBG (7) and we will show that the assumption of the convergence of $\eta_i(t) \to 0$ as $t \to t_f$ for $i = \{2, 3, \dots, N\}$ in Proposition 2 is accomplished.

Lemma 3. Given any finite time $t_f > 0$, the time-varying feedback control protocol

$$u_i(t) = \frac{1}{g_i(x_i)}(-f_i(x_i) + v_i), \ i \in \{1, \dots, N\} \quad (16)$$

$$v_i = -k(t) \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)), \text{ with } k(t) = \frac{\dot{h}(t)}{h(t) + \delta}$$

$h(t)$ as in (7) and δ a small positive value, drives the system (2) with $\rho(t) = 0$ to reach a state in a neighborhood of the consensus value x^* in a predefined time t_f from any initial state $x(0)$. Moreover, the weighted error (6) converges to a neighborhood of the origin in the predefined time t_f.

Proof. Now, applying the similarity transformation $\eta(t) = S^{-1}x(t)$, from (10), we obtain

$$\dot{\eta}(t) = k(t) \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & J_{m_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{m_q}(\lambda_N) \end{pmatrix} \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \\ \vdots \\ \eta_N(t) \end{pmatrix}$$

where q is the number of Jordan blocks for eigenvalue multiplicities $m_2, ..., m_q$, and each Jordan block $J_m(\lambda)$ is a $m \times m$ upper triangular matrix (Horn and Johnson (2012)). Rewriting the previous expression, we have

$$\dot{\eta}_1(t) = 0$$
$$\begin{pmatrix} \dot{\eta}_2(t) \\ \vdots \\ \dot{\eta}_N(t) \end{pmatrix} = k(t) \begin{pmatrix} J_{m_2}(\lambda_2) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_{m_q}(\lambda_N) \end{pmatrix} \begin{pmatrix} \eta_2(t) \\ \vdots \\ \eta_N(t) \end{pmatrix}. \tag{17}$$

Denote, $\zeta(t) = [\eta_2(t), \ldots, \eta_N(t)]^T \in \mathbb{C}^{N-1}$ and then (17) can be written as:

$$\dot{\zeta}(t) = \frac{\dot{h}(t)}{h(t)+\delta}\hat{J}\zeta(t) \tag{18}$$

where $\hat{J}$ is the Jordan matrix associated with L for eigenvalues $\{\lambda_2, \ldots, \lambda_N\}$. It can be verified that the solution of the differential equation (18) is the following:

$$\zeta(t) = \exp(z\hat{J})\zeta(0),\ t \in [0, t_f]. \tag{19}$$

where $z = \ln\left(\frac{h(t)+\delta}{1+\delta}\right)$, $\ln(.)$ is the natural logarithm and $\exp(z\hat{J}) \in \mathbb{R}^{N-1\times N-1}$ is the exponential matrix, i.e.,

$$\lim_{t\to t_f} \zeta(t) = [\eta_2(t), \ldots, \eta_N(t)]^T = [0, \ldots, 0_N]^T \text{ as } \delta \to 0.$$

Hence, by Proposition 2, it follows that the control protocol (16) solves the consensus problem in the predefined time t_f provided that $\delta \to 0$. Since, δ is a fixed parameter of the control protocol that must be different of zero to avoid numerical issues in the computation of the time-varying control gain $k(t)$, the convergence is to a neighborhood of the consensus value x^*.

Additionally, by Proposition 2, it follows that, if the graph $\mathcal{G}$ is connected and undirected, the consensus value is $x^* = \frac{\sum_{i=1}^{N} x_i(0)}{N}$. Similarly, if the graph $\mathcal{G}$ is directed and has a directed spanning tree, it follows from Proposition 2 that the consensus value is $x^* = \gamma^T x(0)$. In both cases it is known that $\mathbf{1}_N x^*$ belongs to the null-space of L (Ren and Beard (2008); Li and Duan (2014)). Then, taking the limit of the weighted error (6) at t_f

$$\lim_{t\to t_f} e(t) = \lim_{t\to t_f} -Lx(t) = -L\mathbf{1}_N x^* = 0. \tag{20}$$

Therefore, the error (6) converges to zero for any initial condition at a predefined time t_f provided that $\delta \to 0$. Otherwise, if $\delta \neq 0$, the error converges to a neighborhood of the origin. However, the final error can be arbitrarily small by making $\delta \to 0$. □

It is worth noting that the auxiliary control inputs $v_i(t)$ are smooth signals even at $t = 0$, which is not the case of the previous work Yong et al. (2012). This is due to the properties of the TBG. Noticed that $k(0) = k(t \geq t_f) = 0$, according to the definition of $h(t)$ as in (7), then we have that $v_i(0) = v_i(t \geq t_f) = 0, \forall i \in \{1, \ldots, N\}$.

4. TBG AS REFERENCE TRAJECTORY TO TRACK

The control protocol (8) ensures convergence to a neighborhood of the consensus value in time t_f. However, after convergence, it is convenient to switch to a control law $u = ke$ with $k \in \mathbb{R}$ a constant stabilizing gain, which maintains the stability of the system for $t > t_f$. To ensure an accurate convergence to the consensus value and provide closed-loop stability even for $t > t_f$, the following theorem extends the previous result by addressing the predefined-time consensus problem as a trajectory tracking problem where the TBG is the reference trajectory. Let $x(0) = x_0$ being the initial state, $e(0) = e_0 \in \mathbb{R}^N$ the initial weighted error of (6), and $\bar{x} \in \mathbb{R}$ will be a desired consensus value.

Before to present a trajectory tracking controller, we first propose an open-loop controller that enforces predefined-time convergence to a consensus value provided that the network dynamics have ideal conditions (i.e., pure integrator dynamics with known initial state).

Lemma 4. Given any finite time $t_f > 0$, the time-varying feedback control law

$$u_i(t) = \frac{1}{g_i(x_i)}(-f_i(x_i) + v_i),\ i \in \{1, \ldots, N\}$$
$$v_i = \dot{h}(t)(x_i(0) - \bar{x}) \tag{21}$$

with $h(t)$ as in (7) and $\bar{x} \in \mathbb{R}$, achieves predefined-time convergence to a consensus value x^* at t_f for the system (2) with ideal dynamics $\rho(t) = 0$ and from any initial state x_0. Furthermore, the consensus achieved x^* is given by the desired consensus value $\bar{x}$ and the resulting weighted error trajectory in the interval $t \in [0, t_f]$ is

$$\hat{e}(t) = h(t)e_0. \tag{22}$$

Proof. Now, using the control (21), the dynamics (1) with $\rho_i(t) = 0, \forall i = \{1, \ldots, N\}$ can be linearized and expressed in matrix form as $\dot{x}(t) = \dot{h}(t)(x(0) - \bar{x}\mathbf{1}_N)$, and its solution is given by

$$x(t) = h(t)(x_0 - \bar{x}\mathbf{1}_N) + c \tag{23}$$

where $c \in \mathbb{R}^N$. Using the initial condition $h(0) = 1$, we have that $c = \bar{x}\mathbf{1}_N$. Therefore, for the final time $h(t_f) = 0$:

$$x(t_f) = h(t_f)(x_0 - \bar{x}\mathbf{1}_N) + c = \bar{x}\mathbf{1}_N. \tag{24}$$

It means that the system (2) with ideal dynamics $\rho(t) = 0$ achieves predefined-time convergence at t_f from any initial condition x_0 using the control (21), i.e.:

$$x_i(t) \to x^* = \bar{x} \text{ as } t \to t_f, i = \{1, 2, \ldots, N\}$$

and $\bar{x}$ can be set as a desired consensus value. Now, taking the time derivative of the weighted error function (6)

$$\dot{e}(t) = -L\dot{x}(t) = -\dot{h}(t)Lx_0 \tag{25}$$

where $L\mathbf{1}_N = 0$ because $\mathbf{1}_N$ belongs to the null-space of L (Ren and Beard (2008); Li and Duan (2014)).

Note that (22) constitutes a coherent solution for dynamics (25), i.e., taking the time derivative of (22)

$$\dot{\hat{e}}(t) = \dot{h}(t)e_0 = -\dot{h}(t)Lx_0.$$

Furthermore, condition (7) implies that at time $t = 0$, $\hat{e}(t) = e_0$ (since $h(0) = 1$). Moreover, condition (7) implies that at time $t = t_f$, $\hat{e}(t) = 0$ (since $h(t_f) = 0$). In other words, the control law (21) controls the system (25) in such a way that (22) describes its solution. □

The following theorem introduces a feedback-based controller to track the trajectories given by the TBG.

Theorem 5. Let $k_f \in \mathbb{R}^+$ be a constant state-feedback gain such that the eigenvalues $\lambda_q, \forall q = \{2, \ldots, N\}$ of $-k_f L$ have negative real parts. Then, the time-variant feedback control law

$$u_i(t) = \frac{1}{g_i(x_i)}(-f_i(x_i) + v_i),\ i \in \{1, \ldots, N\}$$
$$v_i = \dot{h}(t)(x_i(0) - \bar{x}) + k_f(e_i(t) - h(t)e_i(0)) \tag{26}$$

with $h(t)$ as in (7) and $\bar{x} \in \mathbb{R}$, achieves predefined-time convergence to a consensus value x^* at t_f for the system (2) with ideal dynamics $\rho(t) = 0$ and from any initial state x_0. Furthermore, the consensus achieved x^* is given by a desired consensus value $\bar{x}$ and global asymptotic stability of the tracking error $\xi(t) = e(t) - h(t)e_0$ is achieved.

Proof. The dynamics (1) with $\rho_i(t) = 0, \forall i = \{1, \ldots, N\}$ can be linearized using (26) and can be written in vectorial notation as

$$\dot{x}(t) = \dot{h}(t)(x_0 - \bar{x}\mathbf{1}_N) + k_f(e(t) - h(t)e_0). \quad (27)$$

Using the properties P2 and P3, define $y(t) = \gamma^T x(t)$ and taking its time derivative, we get the following differential equation

$$\dot{y}(t) = \gamma^T \dot{x}(t) = \dot{h}(t)(\gamma^T x_0 - \bar{x}\gamma^T \mathbf{1}_N) \quad (28)$$

where $\gamma^T L = 0$, as stated before (Olfati-Saber and Murray (2004); Li and Duan (2014)). Thus, the equivalent solution of (28) is given by

$$y(t) = \gamma^T x(t) = h(t)(\gamma^T x_0 - \bar{x}\gamma^T \mathbf{1}_N) + c$$

where $c \in \mathbb{R}$. Using the initial condition $h(0) = 1$, we have that $c = \bar{x}\gamma^T \mathbf{1}_N$. Therefore, for the final time $h(t_f) = 0$:

$$y(t_f) = \gamma^T x(t_f) = c = \gamma^T \mathbf{1}_N \bar{x}.$$

Then, $\gamma^T x(t_f) = \gamma^T \mathbf{1}_N \bar{x}$. It follows that

$$x(t) \to x(t_f) = \mathbf{1}_N x^* = \mathbf{1}_N \bar{x} \text{ as } t \to t_f.$$

Hence, we have that the system (2) with $\rho(t) = 0$ achieves predefined-time convergence at t_f from any initial state x_0 using the control (26), and $\bar{x}$ can be set arbitrarily as a desired consensus value.

Now, let us show the stability of the tracking error. According to Lemma 4, if the initial weighted error is set as $\hat{e}(0) = e_0$ and the control input $u_i(t) = \frac{1}{g_i(x_i)}(-f_i(x_i)+v_i)$ (21) is applied to each agent, then the system

$$\dot{\hat{e}}(t) = -\dot{h}(t)Lx_0$$

evolves such that $\hat{e}(t) = h(t)e_0$, i.e., $\hat{e}(t)$ becomes the TBG reference trajectory. In this way, computing $\dot{e}(t)$ from (6) and using the linearized dynamics (27), we have

$$\dot{\xi}(t) = \dot{e}(t) - \dot{\hat{e}}(t) = -L\dot{x}(t) - \dot{h}(t)e_0 = -k_f L\xi(t) \quad (29)$$

where $L\mathbf{1}_N = 0$ because $\mathbf{1}_N$ belongs to the null-space of L (Ren and Beard (2008); Li and Duan (2014)).

For a connected undirected graph G, it follows from Olfati-Saber et al. (2007) (Theorem 3 and Corollary 1) that the dynamics (29) achieves global asymptotic stability for all initial tracking error $\xi(0)$. The convergence speed is determined by the eigenvalue λ_2 of L and the control gain k_f. Similarly, for a directed graph containing a directed spanning tree, Ren and Beard (2008) (Theorem 2.8) showed that the dynamics (29) achieves global asymptotic stability for any initial error $\xi(0)$.

Therefore, the globally asymptotically stability of the tracking error implies that $e(t)$ follows $h(t)e_0$. Then, as proved in Lemma 4, the reference $h(t)e_0$ vanishes for $t \geq t_f$ and $e(t)$ converges to the origin of the state space in the predefined-time t_f. Any small final error at time $t \geq t_f$ is corrected by the control input $u_i = \frac{1}{g_i(x_i)}(-f_i(x_i) + k_f e_i(t))$, which is applied to each agent for $t \geq t_f$, guaranteeing the stability of the closed-loop system (2). □

Notice that the smooth behavior of the auxiliary control inputs is also achieved in this trajectory tracking scheme. Assuming to know the initial conditions and given the properties of the TBG, v_i start in zero and returns to zero at time t_f.

All previous results consider ideal integrator dynamics ($\rho(t) = 0$). One approach to deal with model uncertainty is to use a robust controller like the super-twisting controller (STC), which is known to be able to compensate for matched uncertainties/disturbances $\rho(t)$ (Moreno and Osorio (2012)). In our last result, the TBG will be combined with a STC leading to a robust predefined-time controller.

In the following, we will consider that there is a leader $x_l(t) \in \mathbb{R}$ in the network system, whose dynamics can be modeled as in Mondal et al. (2017) by

$$\dot{x}_l(t) = u_l(t) \quad (30)$$

where $u_l(t) \in \mathbb{R}$ is the leader's control input. N agents are now the followers and for each agent $b_i \in \mathbb{R}$ represents the leader's adjacency with $b_i > 0$ if agent i is a neighbor of the leader, otherwise $b_i = 0$. Then, the consensus error for each follower is given by (Mondal et al. (2017))

$$e_i^f(t) = \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)) - b_i(x_i(t) - x_l(t)) \quad (31)$$

and the tracking error is defined as follows:

$$\xi_i(t) = e_i^f(t) - \hat{e}_i(t) \quad (32)$$

where $\hat{e}_i(t) = h(t)e_{i_0}$ is the TBG reference trajectory for the i-th agent as in (22), and the tracking error will be used as the sliding surface:

$$s_i(t) = \xi_i(t). \quad (33)$$

Theorem 6. There exist gains $k_1 > 0$ and $k_2 > 0$ such that for each agent the following nonlinear controller

$$\begin{aligned} u_i(t) &= (\beta_i g_i)^{-1}\left(u_i^{nom} + u_i^{stc}\right), \ i \in \{1, \ldots, N\} \\ u_i^{nom} &= b_i u_l + \sum_{j \in N_i} a_{ij}(f_j + g_j u_j) - \beta_i f_i - \dot{h}(t)e_i(0) \\ u_i^{stc} &= k_1 |s_i|^{1/2} sign(s_i) - w_i \\ \dot{w}_i &= -k_2 sign(s_i), \end{aligned} \quad (34)$$

where $\beta_i = (\sum_{j \in N_i} a_{ij} + b_i)$ and $h(t)$ as in (7), achieves predefined-time convergence to the consensus value x^* at t_f for each perturbed agent (1) with $\rho_i(t) \neq 0$ and from any initial state $x_i(0)$. Furthermore, x^* is given by the leader's state x_l and the tracking error $\xi_i(t)$ for each agent converges to zero.

Proof. By the consensus error (31) and using the control law (34), the time-derivative of the sliding surface (33) can be written as

$$\begin{aligned} \dot{s}_i &= -k_1 |s_i|^{1/2} \text{sign}(s_i) + w_i + \rho_{d_i} \\ \dot{w}_i &= -k_2 \text{sign}(s_i) \end{aligned} \quad (35)$$

where $\rho_{d_i} = \sum_{j \in N_i} a_{ij}\rho_j - \beta_i \rho_i$. Let $z_i = w_i + \rho_{d_i}$, then (35) can be rewritten as

$$\begin{aligned} \dot{s}_i &= -k_1 |s_i|^{1/2} \text{sign}(s_i) + z_i \\ \dot{z}_i &= -k_2 \text{sign}(s_i) + \dot{\rho}_{d_i}. \end{aligned} \quad (36)$$

Moreno and Osorio (2012) have proved that, for a bounded continuously differentiable disturbance, i.e., if $|\rho_{d_i}| < L$ and $|\dot{\rho}_{d_i}| < M$ for some constants $L > 0, M > 0$, the second-order dynamics (36) converges globally to the origin ($s_i = 0, z_i = 0$) in finite time in spite of the disturbance if adequate positive control gains k_1 and k_2 are used. Moreover, the remaining dynamics of the tracking error system is constrained to the sliding surface, such that $s_i = \dot{s}_i = 0$, meaning that for each agent $e_i^f(t)$ follows $\hat{e}_i(t)$. Consequently, the consensus error converges to zero in the predefined-time t_f and the consensus value x^* is given by the leader's state (Mondal et al. (2017)). □

The result in Theorem 6 can be extended to deal with nonlinear dynamics of high order by joining the general results of Mondal et al. (2017) and Becerra et al. (2017).

In a practical implementation of the proposed protocols, the synchronization of all clocks of the agents in the network is an important issue. Also, physical constraints of the systems must be considered to set t_f. It is clear that a small t_f will generate larger control inputs. Thus, the maximum allowable input of each agent must be take into consideration to set t_f. Finally, notice that the fixed control gains of the tracking controllers need to be re-tuned for different convergence times; larger gains are required for small times t_f.

5. SIMULATION RESULTS

The proposed protocols were implemented in MATLAB using the Euler forward method to approximate the time-derivatives with a time step of 0.1 ms and considering a multi-agent system with $N = 8$. Agents 1 to 4 are similar to each other, whose dynamics equations are described as $\dot{x}_i = x_i^2 + 5u_i + \rho_i(t), i \in \{1, \ldots, 4\}$ and the dynamics of the agents 5 to 8 are similar to each other, which are chosen as $\dot{x}_i = \sin(x_i) + 5u_i + \rho_i(t), i \in \{5, \ldots, 8\}$. We use the same communication graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ presented in Yong et al. (2012), where $\mathcal{G}_1$ is an undirected connected graph and $\mathcal{G}_2$ is a directed graph containing a directed spanning tree (see Fig. 1).

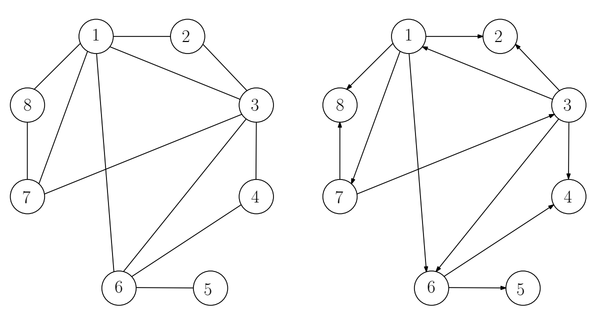

Fig. 1. Communication graphs. Left: undirected graph $\mathcal{G}_1$. Right: directed graph $\mathcal{G}_2$.

First, we provide a comparison of the proposed approach with respect to finite-time and fixed-time protocols reported in the literature. Two of them ensure finite-time convergence: Franceschelli et al. (2013) and Cao and Ren (2014). The protocol of Zuo et al. (2014) guarantees fixed-time convergence. The one of Yong et al. (2012) and ours guarantee predefined-time convergence. The finite-time and fixed-time controllers were manually tuned to achieve a similar convergence time around 5s for the same initial conditions x_0. Then, we kept the same control gains for the simulations and for each controller, we varied the mean of the initial state x_0 from -50 to 50. For every initial condition, we measured the convergence time of the system when $||e|| < 1 \times 10^{-4}$ and the maximum absolute value of the auxiliary control input v. In this comparison, we used the linear TBG-tracking controller defined in (26) with $k_f = 3$ and the TBG function (7) with $t_f = 5$s.

The results of the comparison are shown in Figs. 2 and 3 for undirected and directed graphs respectively. It can be seen the constant convergence time of the predefined-time control with TBG in contrast to the finite and fixed-time controllers. Regarding the auxiliary control effort, the TBG-based control and the one of Franceschelli et al. (2013) were those in which the maximum auxiliary control efforts were lower.

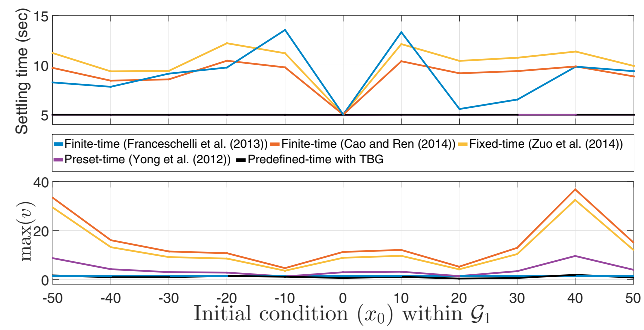

Fig. 2. Comparison of the TBG-tracking controller (26) versus finite-time and fixed-time controllers for $\mathcal{G}_1$. Top: settling time as a function of the initial condition x_0. Bottom: maximum value of the auxiliary control input as a function of the initial condition x_0.

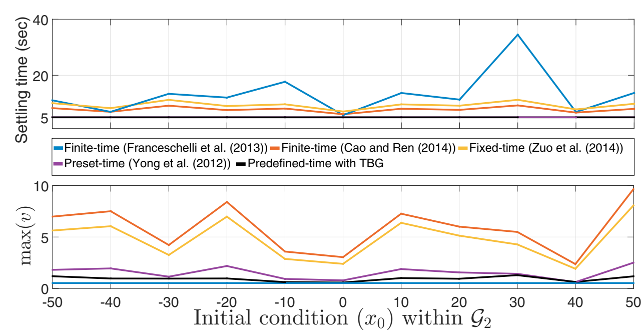

Fig. 3. Comparison of the TBG-tracking controller (26) versus finite-time and fixed-time controllers for $\mathcal{G}_2$. Top: settling time as a function of the initial condition x_0. Bottom: maximum value of the auxiliary control input as a function of the initial condition x_0.

Now, in Figs. 4 to 7, a comparison of the preset-time controller (Yong et al. (2012)) with respect to the proposed predefined-time TBG controllers is presented. It can be seen that the preset-time controller (Figs. 4 and 6), as well as the TBG-direct (16) and the TBG-tracking (26) controllers (Figs. 5 and 7) are able to keep constant the convergence time at 5s. However, the auxiliary control inputs for the predefined-time TBG controllers (16) and (26) were smoother and of lower magnitude than the preset-time controller (Yong et al. (2012)). Notice particularly in Fig. 5(middle) that the auxiliary control inputs of the TBG-based controllers start in zero at difference of the large initial control effort of the controller in Yong et al. (2012).

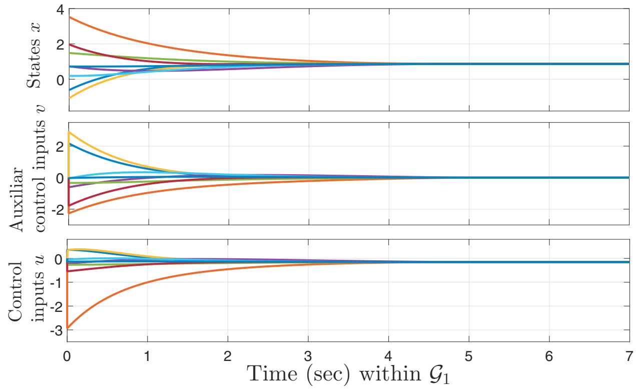

Fig. 4. Preset-time controller (Yong et al. (2012)) for $\mathcal{G}_1$.

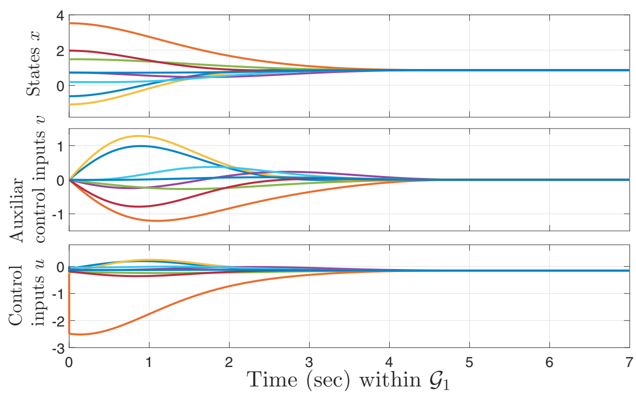

Fig. 5. Predefined-time TBG-direct controller (16) for $\mathcal{G}_1$.

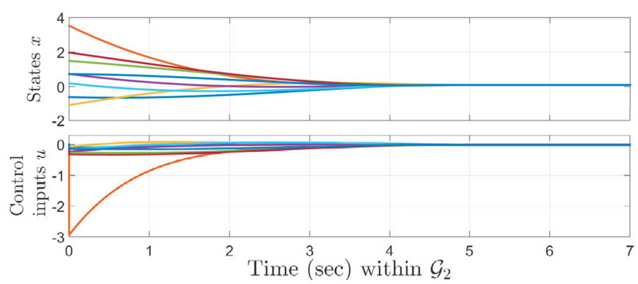

Fig. 6. Preset-time controller (Yong et al. (2012)) for $\mathcal{G}_2$.

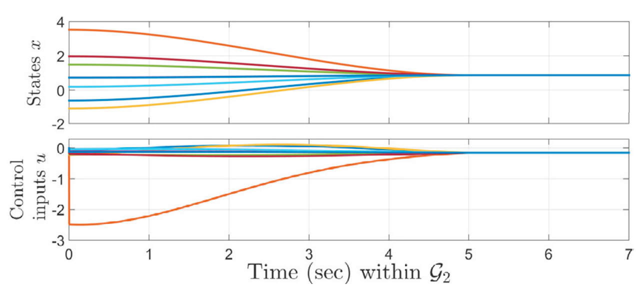

Fig. 7. Predefined-time TBG-tracking controller (26) for $\mathcal{G}_2$.

Before to present the results for the proposed robust controller (34), we first show the results of the TBG-tracking controller (26) for perturbed dynamics with $\mathcal{G}_2$, in order to show the improvement with the robust action. The simulations consider a time-varying disturbance $\rho_i(t) = \alpha_i(1+1\sin(5t))$, where α_i was randomly selected in $(0, 0.5)$, and the convergence time is preset to 5s. It is shown in Fig.

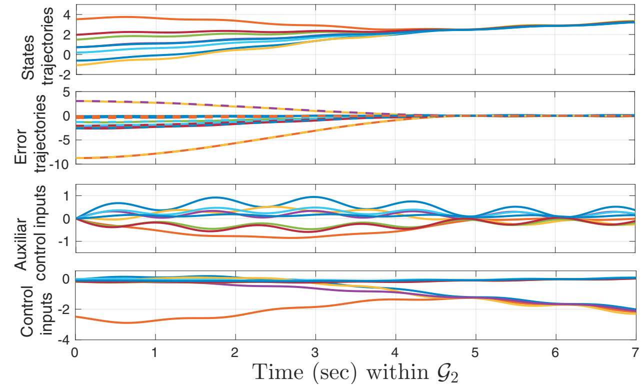

Fig. 8. Predefined-time TBG-tracking controller (26) for perturbed dynamics and $\mathcal{G}_2$. In the error trajectories, the continue lines represent the evolution of the errors whereas the reference is drawn with dash line.

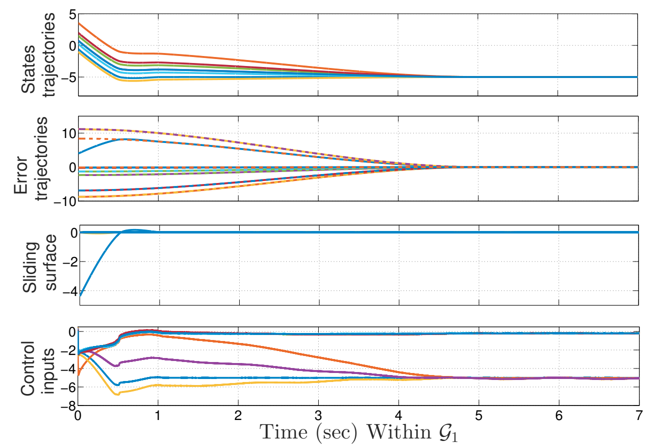

Fig. 9. Robust predefined-time TBG controller (34) for perturbed single integrator systems and $\mathcal{G}_1$. In the error trajectories, the continue lines represent the evolution of the errors whereas the reference is drawn with dash line.

8 that under the time-varying disturbance, the proposed TBG-tracking controller (26) cannot reach convergence state to the average of their initial conditions and they keep oscillating after the preset time is reached. A similar behavior is obtained with the preset time controller of Yong et al. (2012). Hence, these controllers are not able to deal with matched disturbances.

Finally, let us illustrate the proposed robust predefined-time TBG controller (34) for topologies $\mathcal{G}_1$ and $\mathcal{G}_2$ with eight followers and one leader. The control gains are set as $k_1 = 5$ and $k_2 = 5$. The leader's values is considered as $x_l = -5$ with control input $u_l = 0$. For the followers, we set $b_1 = 1$ and $b_i = 0$, $\forall i = \{2, \ldots, 8\}$. Figs. 9 and 10 show that the state trajectories converge to the leader's value $x_l = -5$ and the error trajectories are driven to zero in the desired predefined time t_f. It can be seen that the control input remains oscillating at steady state since it is effectively rejecting the disturbance $\rho(t)$.

6. CONCLUSIONS

In this paper, we have proposed distributed control protocols for networks of first order systems whose desired

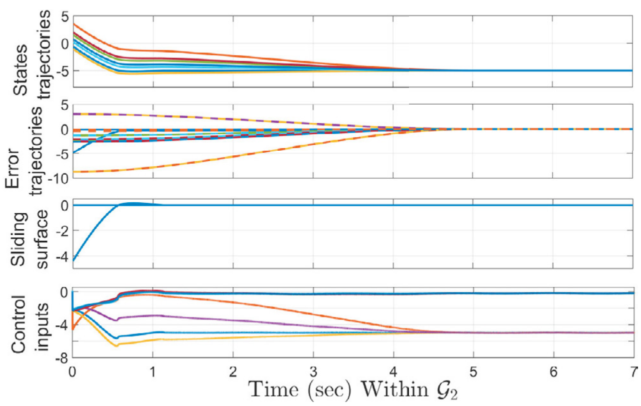

Fig. 10. Robust predefined-time TBG controller (34) for perturbed single integrator systems and $\mathcal{G}_2$. In the error trajectories, the continue lines represent the evolution of the errors whereas the reference is drawn with dash line.

settling time to achieve consensus can be set by the user and it is achieved independently of the initial conditions. This property is called predefined-time consensus. In our approach, a reference trajectory is first computed, named TBG (time base generator). Later, the TBG is combined with feedback controllers to achieve closed loop stability and robustness, in particular applying a super-twisting controller in order to deal with perturbed dynamics of the agents of the network. The performance of the proposed controllers has been compared with existing finite-time and fixed-time controllers in simulations. The results have shown good performance of the proposed control method for connected undirected communication topologies of the network system and for directed topologies having a directed spanning tree. Moreover, a good benefit of the proposed controllers is that they yield smoother control signals with smaller magnitudes than finite-time and fixed-time controllers reported in the literature.

As future work, we will extend our results to deal with nonlinear dynamics of high order for the agents of the network by following the generalization of the works of Becerra et al. (2017) and Mondal et al. (2017).

REFERENCES

Becerra, H.M., Vázquez, C.R., Arechavaleta, G., and Delfin, J. (2017). Predefined-time convergence control for high-order integrator systems using time base generators. *IEEE Trans. on Control Systems Technology*.

Cai, K. (2012). Averaging over general random networks. *IEEE Trans. on Automatic Control*, 57(12), 3186–3191.

Cai, K. and Ishii, H. (2014). Average consensus on arbitrary strongly connected digraphs with time-varying topologies. *IEEE Trans. on Automatic Control*, 59(4), 1066–1071.

Cao, Y. and Ren, W. (2014). Finite-time consensus for multi-agent networks with unknown inherent nonlinear dynamics. *Automatica*, 50, 2648–2656.

Chen, G., Lewis, F.L., and Xie, L. (2011). Finite-time distributed consensus via binary protocols. *Automatica*, 47, 1962–1968.

Cruz-Zavala, E., Moreno, J.A., and Fridman, L. (2010). Uniform second-order sliding mode observer for mechanical systems. In *Variable Structure Systems (VSS), 2010 11th International Workshop on*, 14 –19.

Franceschelli, M., Giua, A., Pisano, A., and Usai, E. (2013). Finite-time consensus for switching network topologies with disturbances. *Nonlinear Analysis: Hybrid Systems*, 10, 83 – 93.

Horn, R.A. and Johnson, C.R. (2012). *Matrix Analysis*. Cambridge University Press, 2 edition.

Jiang, F. and Wang, L. (2009). Finite-time information consensus for multi-agent systems with fixed and switching topologies. *Physica D: Nonlinear Phenomena*, 238(16), 1550 – 1560.

Khalil, H.K. and Grizzle, J.W. (2002). *Nonlinear Systems*, volume 3. Prentice Hall Upper Saddle River.

Li, Z. and Duan, Z. (2014). *Cooperative control of multi-agent systems: a consensus region approach*. CRC Press.

Lu, X., Lu, R., Chen, S., and L, J. (2013). Finite-time distributed tracking control for multi-agent systems with a virtual leader. *IEEE Trans. on Circuits and Systems*, 60, 352–362.

Mondal, S., Su, R., and Xie, L. (2017). Heterogeneous consensus of higher-order multi-agent systems with mismatched uncertainties using sliding mode control. *Int. J. of Robust and Nonlinear Control*, 27(13), 2303–2320.

Moreno, J.A. and Osorio, M. (2012). Strict lyapunov functions for the super-twisting algorithm. *IEEE Trans. on Automatic Control*, 57(4), 1035–1040.

Olfati-Saber, R., Fax, J.A., and Murray, R.M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1), 215–233.

Olfati-Saber, R. and Murray, R.M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. on Automatic Control*, 49(9), 1520–1533.

Parsegov, S., Polyakov, A., and Shcherbakov, P. (2013). Fixed-time consensus algorithm for multi-agent systems with integrator dynamics. In *IFAC Workshop on Distributed Estimation and Control in Networked Systems*, 110–115. IFAC.

Polyakov, A. (2012). Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Trans. on Automatic Control*, 57(8), 2106–2110.

Ren, W. and Beard, R.W. (2008). *Distributed consensus in multi-vehicle cooperative control*. Springer.

Shang, Y. (2012). Finite-time consensus for multi-agent systems with fixed topologies. *Int. J. of Systems Science*, 43(3), 499–506.

Tu, Z., Yu, H., and Xia, X. (2017). Decentralized finite-time adaptive consensus of multiagent systems with fixed and switching network topologies. *Neurocomputing*, 219, 59–67.

Yong, C., Guangming, X., and Huiyang, L. (2012). Reaching consensus at a preset time: Single-integrator dynamics case. In *31st Chinese Control Conference (CCC)*, 6220–6225. IEEE.

Zuo, Z., Yang, W., Tie, L., and Meng, D. (2014). Fixed-time consensus for multi-agent systems under directed and switching interaction topology. In *American Control Conference*, 5133–5138.