LEFT-INVARIANT CR STRUCTURES ON 3-DIMENSIONAL LIE GROUPS

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Abstract. The systematic study of CR manifolds originated in two pioneering 1932 papers of Élie Cartan. In the first, Cartan classifies all homogeneous CR 3-manifolds, the most well-known case of which is a one-parameter family of left-invariant CR structures on $\mathrm{SU}_2 = S^3$, deforming the standard 'spherical' structure. In this paper, mostly expository, we illustrate and clarify Cartan's results and methods by providing detailed classification results in modern language for four 3-dimensional Lie groups, with the emphasis placed on $\mathrm{SL}_2(\mathbb{R})$. $\mathrm{SL}_2(\mathbb{R})$ admits two one-parameter families of left-invariant CR structures, called the elliptic and hyperbolic families, characterized by the incidence of the contact distribution with the null cone of the Killing metric. Low dimensional complex representations of $\mathrm{SL}_2(\mathbb{R})$ provide CR embedding or immersions of these structures. The same methods apply to all other three-dimensional Lie groups and are illustrated by descriptions of the left-invariant CR structures for SU_2, the Heisenberg group, and the Euclidean group.

Contents

1. Introduction

A real hypersurface M^3 in a 2-dimensonal complex manifold (such as $\mathbb{C}^2$) inherits an intrinsic geometric structure from the complex structure of its ambient space. This is called a CR structure and can be thought of as an odd-dimensional version of a complex structure. A main feature of CR structures, already noted by H. Poincaré [12], is that, unlike complex structures, they possess *local* invariants, similar to the well-known curvature invariants of Riemannian metrics. Consequently, a generic CR manifold admits *no* CR symmetries, even locally. The seminal work in this field is Élie Cartan's 1932 papers [5, 6], later extended by Tanaka [14], Chern and Moser [8] and many others to higher dimensions. In this article we restrict attention to the 3-dimensional case.

Building on Poincaré's observation that local CR invariants exist, Cartan used his method of equivalence and moving frames to determine these invariants. Using a more algebraic approach,

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Cartan classified in Chapter II of [5] *homogeneous* CR 3-manifolds, i.e., 3-dimensional CR manifolds admitting a transitive action of a Lie group by CR automorphisms, and finds that, up to a cover, every such CR structure is a left-invariant CR structure on a 3-dimensional Lie group [5, p. 69]. The items on this list form a rich source of natural examples of CR geometries which, in our opinion, has been hardly explored and mostly forgotten. In this article we present some of the most interesting items on Cartan's list. We outline Cartan's approach, in particular, the relation between the adjoint representation of the group and global realizability (the embedding of a CR structure as a hypersurface in a complex 2-dimensional manifold).

The *spherical CR structure* on the 3-sphere S^3 is the one induced from its embedding in $\mathbb{C}^2$ as the hypersurface $|z_1|^2 + |z_2|^2 = 1$. Any CR structure on a 3-manifold locally equivalent to this structure is called *spherical*. The symmetry group of the spherical CR structure on S^3 is the 8-dimensional non-compact Lie group $\mathrm{PU}_{2,1}$, the maximum dimension possible for a CR 3-manifold. The standard linear action of the unitary group U_2 on $\mathbb{C}^2$ provides an 'obvious' 4-dimensional group of symmetries; to see the full symmetry group, one needs to embed $\mathbb{C}^2$ as an affine chart in $\mathbb{C}\mathrm{P}^2$, in which S^3 appears as the space of complex 1-dimensional null directions in $\mathbb{C}^3$ with respect to a pseudo-hermitian inner product of signature $(2, 1)$.

The spherical CR structure on S^3 can be thought of as the unique left-invariant CR structure on the group $\mathrm{SU}_2 \simeq S^3$ which is also invariant by right translations by the standard diagonal circle subgroup $\mathrm{U}_1 \subset \mathrm{SU}_2$. There is a well-known and much studied 1-parameter family of deformations of this structure on SU_2 to structures whose only symmetries are left translations by SU_2 (see, for example, [2], [4], [7], [13]). An interesting feature of this family of deformations is that none of the structures, except the spherical one, can be globally realized as a hypersurface in $\mathbb{C}^2$ (although they can be realized as finite covers of hypersurfaces in $\mathbb{C}\mathrm{P}^2$, the 3-dimensional orbits of the projectivization of the conjugation action of SU_2 on $\mathfrak{sl}_2(\mathbb{C})$). This was first shown in [13] and later in [2] by a different and interesting proof; see Remark 5.2 for a sketch of the latter proof.

A left-invariant CR structure on a 3-dimensional Lie group G is given by a 1-dimensional complex subspace of its complexified Lie algebra $\mathfrak{g}_\mathbb{C}$, that is, a point in the 2-dimensional complex projective plane $\mathrm{P}(\mathfrak{g}_\mathbb{C}) \simeq \mathbb{C}\mathrm{P}^2$, satisfying a certain regularity condition (Definition 3.1 below). The automorphism group of G, $\mathrm{Aut}(G)$, acts on the space of left-invariant CR structures on G, so that two $\mathrm{Aut}(G)$-equivalent left-invariant CR structures on G correspond to two points in $\mathrm{P}(\mathfrak{g}_\mathbb{C})$ in the same $\mathrm{Aut}(G)$-orbit. Thus the classification of left-invariant CR structures on G, up to CR-equivalence by the action of $\mathrm{Aut}(G)$, reduces to the classification of the $\mathrm{Aut}(G)$-orbits in $\mathrm{P}(\mathfrak{g}_\mathbb{C})$. This leaves the possibility that two left-invariant CR structures on G which are not CR equivalent under $\mathrm{Aut}(G)$ might be still CR-equivalent, locally or globally. Using Cartan's equivalence method, as introduced in [5], we show in Proposition 3.1 that for *aspherical* left-invariant CR structures this possibility does not occur. Namely: two left-invariant aspherical CR structures on two 3-dimensional Lie groups are CR equivalent if and only if the they are CR equivalent via a Lie group isomorphism. See also [3] for a global invariant that distinguishes members of the left-invariant structures on SU_2 and Theorem 2.1 of [9, p. 246], from where our Proposition 3.1 is essentially taken. The asphericity condition in Proposition 3.1 is essential (see Remark 4.5).

Contents of the paper. In the next section, §2, we present the basic definitions and properties of CR manifolds. In §3 we introduce some tools for studying homogenous CR manifolds which will be used in later sections.

In §4 we study our main example of $G = \mathrm{SL}_2(\mathbb{R})$, where we find that up to $\mathrm{Aut}(G)$, there are two 1-parameter families of left-invariant CR structures, one *elliptic* and one *hyperbolic*, depending on the incidence relation of the associated contact distribution with the null cone of the Killing metric, see Proposition 4.1. Realizations of these structures are described in Proposition 4.3: the elliptic spherical structure can be realized as any of the generic orbits of the standard representation in $\mathbb{C}^2$, or the complement of $z_1 = 0$ in $S^3 \subset \mathbb{C}^2$. The rest of the structures are finite covers of orbits of the adjoint action in $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C})) = \mathbb{C}\mathrm{P}^2$. The question of their global realizability in $\mathbb{C}^2$ remains open, as far as we know.

In §5 we treat the simpler case of $G = \mathrm{SU}_2$, where we recover the well-known 1-parameter family of left-invariant CR structures mentioned above, all with the same contact structure, containing a single spherical structure.

The remaining two sections present similar results for the Heisenberg and Euclidean groups.

In the Appendix we state the main differential geometric result of [5] and the specialization to homogeneous CR structures.

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How 'original' is this paper? We are certain that Élie Cartan knew most the results we present here. Some experts in his methods could likely extract the *statements* of these results from his paper [5], where Cartan presents a classification of homogeneous CR 3-manifolds in Chapter II. As for finding the *proofs* of these results in [5], or anywhere else, we are much less certain. The classification of homogeneous CR 3-manifolds appears on p. 70 of [5], summing up more than 35 pages of general considerations followed by case-by-case calculations. We found Cartan's text justifying the classification very hard to follow. The general ideas and techniques are quite clear, but we were unable to justify many details of his calculations and follow through the line of reasoning. Furthermore, Cartan presents the classification in Chap. II of [5] before solving the equivalence problem for CR manifolds in Chap. III, so the CR invariants needed to distinguish the items on his list are not available, nor can he use the argument of our Proposition 3.1. In spite of extensive search and consultations with several experts, we could not find anywhere in the literature a detailed and complete statement in modern language of Cartan's classification of homogeneous CR manifolds, let alone proofs. We decided it would be more useful for us, and our readers, to abstain from further deciphering of [5] and to rederive his classification.

As for [9], apparently the authors shared our frustration with Cartan's text, as they redo parts of the classification in a style similar to ours. But we found their presentation sketchy and at times inadequate. For example, the reference on pp. 248 and 250 of [9] to the 'scalar curvature R of the CR structure' is misleading. There is no 'scalar curvature' in CR geometry. Cartan's invariant called R is coframe dependent and so the formula given by the authors is meaningless without specifying the coframe used. Also, the realizations they found for their CR structures are rather different from ours.

In summary, we lay no claim for originality of the results of this paper. Our main purpose here is to give a new treatment of an old subject. We hope the reader will find it worthwhile.

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2. Basic definitions and properties of CR manifolds

A *CR structure* on a 3-dimensional manifold M is a rank 2 subbundle $D \subset TM$ together with an almost complex structure J on D, i.e., a bundle automorphism $J : D \to D$ such that $J^2 = -Id$. The structure is *non-degenerate* if D is a contact structure, i.e., its sections bracket generate TM. We shall henceforth assume this non-degeneracy condition for all CR structures. We stress that in this article all CR manifold are assumed 3-dimensional and have an underlying contact structure.

A CR structure is equivalently given by a complex line subbundle $V \subset D_{\mathbb{C}} := D \otimes \mathbb{C}$, the $-i$ eigenspace of $J_{\mathbb{C}} := J \otimes \mathbb{C}$, denoted also by $T^{(0,1)}M$. Conversely, given a complex line subbundle $V \subset T_{\mathbb{C}}M := TM \otimes \mathbb{C}$ such that $V \cap \overline{V} = \{0\}$ and $V \oplus \overline{V}$ bracket generates $T_{\mathbb{C}}M$, there is a unique CR structure (D, J) on M such that $V = T^{(0,1)}M$. A section of V is a *complex vector field of type* $(0, 1)$ and can be equally used to specify the CR structure, provided it is non-vanishing.

A dual way of specifying a CR structure, particularly useful for calculations, is via an *adapted coframe.* This consists of a pair of 1-forms (ϕ, ϕ_1) where ϕ is a real contact form, i.e., $D = \mathrm{Ker}(\phi)$, ϕ_1 is a complex valued form of type $(1, 0)$, i.e., $\phi_1(Jv) = i\phi_1(v)$ for every $v \in D$, and such that $\phi \wedge \phi_1 \wedge \bar\phi_1$ is non-vanishing. $V \subset T_{\mathbb{C}}M$ can then be recovered from ϕ, ϕ_1 as their common kernel. The non-degeneracy of (D, J) is equivalent to the non-vanishing of $\phi \wedge \mathrm{d}\phi$. We will use in the sequel any of these equivalent definitions of a CR structure.

If M is a real hypersurface in a complex 2-dimensional manifold N there is an induced CR structure on M defined by $D := TM \cap \tilde J(TM)$, where $\tilde J$ is the almost complex structure on N, with the almost complex structure J on D given by the restriction of $\tilde J$ to D. Equivalently, $V = T^{(0,1)}M := (T_{\mathbb{C}}M) \cap \big(T^{(0,1)}N\big)$. A CR structure (locally) CR equivalent to a hypersurface in a complex 2-manifold is called (locally) *realizable.*

Two CR manifolds (M_i, D_i, J_i), $i = 1, 2$, are *CR equivalent* if there exists a diffeomorphism $f : M_1 \to M_2$ such that $\mathrm{d}f(D_1) = D_2$ and such that $(\mathrm{d}f|_{D_1}) \circ J_1 = J_2 \circ (\mathrm{d}f|_{D_1})$. Equivalently, $(\mathrm{d}f)_{\mathbb{C}}(V_1) = V_2$. A *CR automorphism* of a CR manifold is a CR self-equivalence, i.e., a diffeomorphism $f : M \to M$ such that $\mathrm{d}f$ preserves D and $\mathrm{d}f|_D$ commutes with J. Local CR equivalence and automorphism are defined similarly, by restricting the above definitions to open subsets. An *infinitesimal CR automorphism* is a vector field whose (local) flow acts by (local) CR automorphisms. Clearly, the set $\mathrm{Aut}_{\mathrm{CR}}(M)$ of CR automorphisms forms a group under composition and the set $\mathfrak{aut}_{\mathrm{CR}}(M)$ of infinitesimal CR vector fields forms a Lie algebra under the Lie bracket of vector fields. In fact, $\mathrm{Aut}_{\mathrm{CR}}(M)$ is naturally a Lie group of dimension $\leq \dim(\mathfrak{aut}_{\mathrm{CR}}(M)) \leq 8$, see Corollary A.1 in the Appendix.

The basic example of CR structure is the unit sphere $S^3 = \{|z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$ equipped with the CR structure induced from $\mathbb{C}^2$. Its group of CR automorphisms is the 8-dimensional simple Lie group $\mathrm{PU}_{2,1}$. The action of the latter on S^3 is seen by embedding $\mathbb{C}^2$ as an affine chart in $\mathbb{CP}^2$, $(z_1, z_2) \mapsto [z_1 : z_2 : 1]$, mapping S^3 unto the hypersurface given in homogeneous coordinates by $|Z_1|^2 + |Z_2|^2 = |Z_3|^2$, the projectivized null cone of the hermitian form $|Z_1|^2 + |Z_2|^2 - |Z_3|^2$ in $\mathbb{C}^3$ of signature $(2, 1)$. The group $\mathrm{U}_{2,1}$ is the subgroup of $\mathrm{GL}_3(\mathbb{C})$ leaving invariant this hermitian form and its projectivized action on $\mathbb{CP}^2$ acts on S^3 by CR automorphism. It is in fact its *full* automorphism group. This is a consequence of the Cartan's equivalence method, see Corollary A.1.

Here are two standard results of the general theory of CR manifolds.

Proposition 2.1 ('Finite type' property)**.** *Let M, M' be two CR manifolds with M connected and $f : M \to M'$ a local CR-equivalence. Then f is determined by its restriction to any open subset of M. In fact it is determined of its 2-jet at a single point of M.*

Proof. The Cartan equivalence method associates canonically with each CR 3-manifold M a certain principal bundle $B \to M$ with 5-dimensional fiber, a reduction of the bundle of second order frames on M, together with a canonical coframing of B (an e-structure, or 'parallelism'; see the Appendix for more details). Consequently, $f : M \to M'$ lifts to a bundle map $\tilde f : B \to B'$ between the associated bundles (in fact, the 2-jet of f, restricted to B), preserving the coframing. Now any coframe preserving map of coframed manifolds with a connected domain is determined by its value at a single point. Thus $\tilde f$ is determined by its value at a single point in B. It follows that f is determined by its 2-jet at single point in M. □

Proposition 2.2 ('Unique extension' property)**.** *Let $f : U \to U'$ be a CR diffeomorphism between open connected subsets of S^3. Then f can be extended uniquely to an element $g \in \mathrm{Aut}_{\mathrm{CR}}(S^3) = \mathrm{PU}_{2,1}$.*

Proof. Let $B \to S^3$ be the Cartan bundle associated with the CR structure, as in the proof of the previous proposition, and $\tilde f : B|_U \to B|_{U'}$ the canonical lift of f. Since $\mathrm{Aut}_{\mathrm{CR}}(S^3)$ acts transitively on B (in fact, freely, see Corollary A.1), for any given $p \in B|_U$ there is a unique $g \in \mathrm{Aut}_{\mathrm{CR}}(S^3)$ such that $\tilde{f}(p) = \tilde{g}(p)$. It follows, by the previous proposition, that $f = g|_U$. See also [1], Proposition 2.1, for a different proof. $\square$

Here is a simple consequence of the last two propositions that will be useful for us later.

Corollary 2.1. *Let M be a connected 3-manifold and $\phi_i : M \to S^3$, $i = 1, 2$, be two immersions. Then the two induced spherical CR structures on M coincide if and only if $\phi_2 = g \circ \phi_1$ for some $g \in \mathrm{Aut}_{\mathrm{CR}}(S^3) = \mathrm{PU}_{2,1}$.*

Proof. Let $U \subset M$ be a connected open subset for which each restriction $\phi_i|_U$ is a diffeomorphism unto its image $V_i := \phi_i(U) \subset S^3$, $i = 1, 2$. Then $(\phi_2|_U)\circ(\phi_1|_U)^{-1} : V_1 \to V_2$ is a CR diffeomorphism. By Proposition 2.2, there exists $g \in \mathrm{PU}_{2,1}$ such that $\phi_2|_U = (g \circ \phi_1)|_U$. It follows, by Proposition 2.1, that $\phi_2 = g \circ \phi_1$. $\square$

3. Left-invariant CR structures on 3-dimensional Lie groups

A natural class of CR structures are the *homogeneous* CR manifolds, i.e., CR manifolds admitting a transitive group of automorphisms. Up to a cover, every such structure is given by a left-invariant CR structure on a 3-dimensional Lie group (see, e.g., [5, p. 69]). Each such Lie group is determined, again, up to a cover, by its Lie algebra. The list of possible Lie algebras is a certain sublist of the list of 3-dimensional real Lie algebras (the ‘Bianchi classification’), and was determined by É. Cartan in Chapter II of his 1932 paper [5]. In this section we first make some general remarks about such CR structures, then state an easy to apply criterion for sphericity. Our main references here are Chapter II of É. Cartan’s paper [5] and §2 of Ehlers et al. [9].

3.1. Preliminaries.

Let G be a 3-dimensional Lie group G with identity element e and Lie algebra $\mathfrak{g} = T_eG$. To each $g \in G$ is associated the *left translation* $G \to G$, $x \mapsto gx$. A CR structure on G is *left-invariant* if all left translations are CR automorphisms. Clearly, a left-invariant CR structure (D, J) is given uniquely by its value (D_e, J_e) at e. Equivalently, it is given by a *non-real* 1-dimensional complex subspace $V_e \subset \mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$; i.e., $V_e \cap \overline{V_e} = \{0\}$. By the non-degeneracy of the CR structure, $D_e \subset \mathfrak{g}$ is not a Lie subalgebra; equivalently, $V_e \oplus \overline{V_e} \subset \mathfrak{g}_{\mathbb{C}}$ is not a Lie subalgebra. In other words, *left-invariant CR structures are parametrized by the non-real and non-degenerate elements of* $\mathrm{P}(\mathfrak{g}_{\mathbb{C}}) \simeq \mathbb{C}\mathrm{P}^2$.

Definition 3.1. *An element $[L] \in \mathrm{P}(\mathfrak{g}_{\mathbb{C}})$ is* real *if $[L] = [\overline{L}]$ and* degenerate *if $L, \overline{L}$ span a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. $[L]$ is* regular *if it is not real nor degenerate. The locus of regular elements in $\mathrm{P}(\mathfrak{g}_{\mathbb{C}})$ is denoted by $\mathrm{P}(\mathfrak{g}_{\mathbb{C}})_{\mathrm{reg}}$.*

Equivalently, if $[L] = [L_1 + iL_2] \in \mathrm{P}(\mathfrak{g}_{\mathbb{C}})$, where $L_1, L_2 \in \mathfrak{g}$, then $[L]$ is non-real if and only if L_1, L_2 are linearly independent and is regular if and only if $L_1, L_2, [L_1, L_2]$ are linearly independent.

Let $\mathrm{Aut}(G)$ be the group of Lie group automorphisms of G and $\mathfrak{aut}(\mathfrak{g})$ the group of Lie algebra automorphisms of $\mathfrak{g}$. For each $f \in \mathrm{Aut}(G)$, $\mathrm{d}f(e) \in \mathfrak{aut}(\mathfrak{g})$, and if G is connected then f is determined uniquely by $\mathrm{d}f(e)$, so $\mathrm{Aut}(G)$ embeds naturally as a subgroup $\mathrm{Aut}(G) \subset \mathrm{Aut}(\mathfrak{g})$. Every Lie algebra homomorphism of a *simply connected* Lie group lifts uniquely to a Lie group homomorphism, hence for simply connected G, $\mathrm{Aut}(G) = \mathrm{Aut}(\mathfrak{g})$. The adjoint representation of G defines a homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(G)$. Its image is a normal subgroup $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$, the group of *inner* automorphisms (also called ‘the adjoint group’). The quotient group, $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$, is the group of *outer* automorphisms. For a simple Lie group, $\mathrm{Out}(G)$ is a finite group. For example, $\mathrm{Out}(\mathrm{SU}_2)$ is trivial and $\mathrm{Out}(\mathrm{SL}_2(\mathbb{R})) \simeq \mathbb{Z}_2$, given by conjugation by any matrix $g \in \mathrm{GL}_2(\mathbb{R})$ with negative determinant, e.g., $g = \mathrm{diag}(1, -1)$.

Now $\mathrm{Aut}(G)$ clearly acts on the set of left-invariant CR structures on G. It also acts on $\mathrm{P}(\mathfrak{g}_{\mathbb{C}})_{\mathrm{reg}}$ by the projectivized complexification of its action on $\mathfrak{g}$. The map associating with a left-invariant CR structure $V \subset T_{\mathbb{C}}G$ the point $z = V_e \in \mathrm{P}(\mathfrak{g}_{\mathbb{C}})_{\mathrm{reg}}$ is clearly $\mathrm{Aut}(G)$-equivariant, hence if $z_1, z_2 \in \mathrm{P}(\mathfrak{g}_{\mathbb{C}})_{\mathrm{reg}}$ lie on the same $\mathrm{Aut}(G)$-orbit then the corresponding left-invariant CR structures on G are CR equivalent via an element of $\mathrm{Aut}(G)$. As mentioned in the introduction, the converse is true for *aspherical* left-invariant CR structures.

Proposition 3.1. *Consider two left-invariant aspherical CR structures $V_i \subset T_{\mathbb{C}}G_i$ on two connected 3-dimensional Lie groups G_i, with corresponding elements $z_i := (V_i)_{e_i} \in \mathrm{P}((\mathfrak{g}_i)_{\mathbb{C}}))_{\mathrm{reg}}$, where e_i is the identity element of G_i, $i = 1, 2$. If the two CR structures are equivalent, then there exists a group isomorphism $G_1 \to G_2$ which is a CR equivalence, whose derivative at e_1 maps $z_1 \mapsto z_2$. If the two CR structures are locally equivalent, then there exists a Lie algebra isomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$, mapping $z_1 \mapsto z_2$.*

Proof. Let $f : G_1 \to G_2$ be a CR equivalence. By composing f with an appropriate left translation, either in G_1 or in G_2, we can assume, without loss of generality, that $f(e_1) = e_2$. Since f is a CR equivalence, $(\mathrm{d}f)_{\mathbb{C}}V_1 = V_2$. In particular, $(\mathrm{d}f)_{\mathbb{C}}$ maps $z_1 \mapsto z_2$. We next show that f is a group isomorphism.

For any 3-dimensional Lie group G, the space $\mathfrak{R}(G)$ of right-invariant vector fields is a 3-dimensional Lie subalgebra of the space of vector fields on G, generating left-translations on G. Hence if G is equipped with a left-invariant CR structure then $\mathfrak{R}(G) \subset \mathfrak{aut}_{\mathrm{CR}}(G)$. If the CR structure is aspherical, then the Cartan equivalence method implies that $\dim(\mathfrak{aut}_{\mathrm{CR}}(M)) \leq 3$, see Corollary A.1 of the Appendix. Thus $\mathfrak{R}(G) = \mathfrak{aut}_{\mathrm{CR}}(G)$.

Now since $f : G_1 \to G_2$ is a CR equivalence, its derivative defines a Lie algebra isomorphism $\mathfrak{aut}_{\mathrm{CR}}(G_1) \simeq \mathfrak{aut}_{\mathrm{CR}}(G_2)$. It follows, by the last paragraph, that $\mathrm{d}f(\mathfrak{R}(G_1)) = \mathfrak{R}(G_2)$. This implies that f is a group isomorphism by a result from the theory of Lie groups: If $f : G_1 \to G_2$ is a diffeomorphism between two connected Lie groups such that $f(e_1) = e_2$ and $\mathrm{d}f(\mathfrak{R}(G_1)) = \mathfrak{R}(G_2)$ then f is a group isomorphism.

We could not find a reference for the (seemingly standard) last statement so we sketch a proof here. Let $G = G_1 \times G_2$ and $H = \{(x, f(x)) | x \in G_1\}$ (the graph of f). Then f is a group isomorphism if and only if $H \subset G$ is a subgroup. Let $\mathfrak{h} := T_eH$, where $e = (e_1, e_2) \in G$, and let $\mathcal{H} \subset TG$ the extension of $\mathfrak{h}$ to a right-invariant sub-bundle. Then, since $\mathrm{d}f : \mathfrak{R}(G_1) \to \mathfrak{R}(G_2)$ is a Lie algebra isomorphism, $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, $\mathcal{H}$ is integrable and H is the integral leaf of $\mathcal{H}$ through $e \in G$ (a maximal connected integral submanifold of $\mathcal{H}$). It follows that Hh is also an integral leaf of $\mathcal{H}$ for every $h \in H$. But $e \in H \cap Hh$, hence $H = Hh$ and so H is closed under multiplication and inverse, as needed.

To prove the last statement of the proposition, suppose $f : U_1 \to U_2$ is a CR equivalence, where $U_i \subset G_i$ are open subsets, $i = 1, 2$. By composing f with appropriate left translations in G_1 and G_2, we can assume, without loss of generality, that U_i is a neighborhood of $e_i \in G_i$, $i = 1, 2$, and that $f(e_1) = e_2$. Since f is a CR equivalence, its complexified derivative $(\mathrm{d}f)_{\mathbb{C}} : T_{\mathbb{C}}U_1 \to T_{\mathbb{C}}U_2$ maps $V_1|_{U_1}$ isomorphically onto $V_2|_{U_2}$; in particular, it maps $z_1 \mapsto z_2$. It remains to show that $\mathrm{d}f(e_1) : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra isomorphism.

For any Lie group G, the Lie bracket of two elements $X_e, Y_e \in \mathfrak{g} = T_eG$ is defined by evaluating at e the commutator $XY - YX$ of their unique extensions to *left*–invariant vector fields X, Y on G. If we use instead *right*–invariant vector fields, we obtain the negative of the standard Lie bracket. Now right-invariant vector fields generate left translations, hence if G is a 3-dimensional Lie group equipped with a left-invariant CR structure, there is a natural inclusion of Lie algebras $\mathfrak{g}_- \subset \mathfrak{aut}_{\mathrm{CR}}(G)$, where $\mathfrak{g}_-$ denotes $\mathfrak{g}$ equipped with the negative of the standard bracket. For any aspherical CR structure on a 3-manifold M we have $\dim(\mathfrak{aut}_{\mathrm{CR}}(M)) \leq 3$, hence for any open subset $U \subset G$ the restriction of a left-invariant aspherical CR structure on G to U satisfies $\mathfrak{aut}_{\mathrm{CR}}(U) = \mathfrak{R}(G)|_U \simeq \mathfrak{g}_-$.

Next, since $f : U_1 \to U_2$ is a CR equivalence, its derivative $\mathrm{d}f$ defines a Lie algebra isomorphism $\mathfrak{aut}_{\mathrm{CR}}(U_1) \to \mathfrak{aut}_{\mathrm{CR}}(U_2)$. By the previous paragraph, $\mathrm{d}f(e)$ is a Lie algebra isomorphism $(\mathfrak{g}_1)_- \to (\mathfrak{g}_2)_-$, and thus is also a Lie algebra isomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$. □

3.2. **A sphericity criterion via well-adapted coframes.** We formulate here a simple criterion for deciding whether a left-invariant CR structure $z \in \mathrm{P}(\mathfrak{g}_{\mathbb{C}})_{\mathrm{reg}}$ on a Lie group G is spherical or not. The basic tools are found in the seminal papers of Cartan [5],[6]. We defer a more complete discussion to the Appendix.

Definition 3.2. *Let M be a 3-manifold with a CR structure $V \subset T_{\mathbb{C}}M$. An* adapted coframe *is a pair of 1-forms (ϕ, ϕ_1) with ϕ real and ϕ_1 complex, such that $\phi|_V = \phi_1|_V = 0$ and $\phi \wedge \phi_1 \wedge \bar{\phi}_1$ is non-vanishing. The coframe is* well-adapted *if $\mathrm{d}\phi = i\phi_1 \wedge \bar{\phi}_1$.*

Adapted and well-adapted coframes always exist, locally. Starting with an arbitrary non-vanishing local section L of V (a complex vector field of type $(0,1)$) and a contact form θ (a non-vanishing local section of $D^{\perp} \subset T^*M$), define the complex $(1,0)$-form ϕ_1 by $\phi_1(L) = 0$, $\bar{\phi}_1(L) = 1$. Then (ϕ, ϕ_1) is an adapted coframe and any other adapted coframe is given by $\tilde{\phi} = |\lambda|^2\phi$, $\tilde{\phi}_1 = \lambda(\phi + \mu\phi_1)$ for arbitrary complex functions μ, λ, with λ non-vanishing. It is then easy to verify that for any λ and $\mu = i\,L(u)/u$ where $u = |\lambda|^2$, the resulting coframe $(\tilde{\phi}, \tilde{\phi}_1)$ is well-adapted.

Given a well-adapted coframe (ϕ, ϕ_1), decomposing $\mathrm{d}\phi, \mathrm{d}\phi_1$ in the same coframe we get

$$\begin{aligned} \mathrm{d}\phi &= i\phi_1 \wedge \bar{\phi}_1 \\ \mathrm{d}\phi_1 &= a\,\phi_1 \wedge \bar{\phi}_1 + b\,\phi \wedge \phi_1 + c\,\phi \wedge \bar{\phi}_1, \end{aligned} \tag{1}$$

for some complex valued functions a, b, c on M. For a left-invariant CR structure on a 3-dimensional group G one can choose a (global) well-adapted coframe of left-invariant 1-forms, and then a, b, c are constants.

Proposition 3.2. *Consider a CR structure on a 3-manifold given by a well adapted coframe ϕ, ϕ_1, satisfying equations* (1) *for some constants $a, b, c \in \mathbb{C}$. The CR structure is spherical if and only if* $c\left(2|a|^2 + 9ib\right) = 0$.

This is a consequence of Cartan equivalence method. See Corollary A.2 in the Appendix.

3.3. **Realizability.** Let (M, D, J) be a CR 3-manifold and N a complex manifold. A smooth function $f : M \to N$ is a *CR map*, or simply *CR*, if $\tilde{J} \circ (\mathrm{d}f|_D) = (\mathrm{d}f|_D) \circ J$, where $\tilde{J} : TN \to TN$ is the almost complex structure on N. Equivalently, $(\mathrm{d}f)_{\mathbb{C}}V \subset T^{(0,1)}N$. A *realization* of (M, D, J) is a CR embedding of M in a (complex) 2-dimensional N. A *local realization* is a CR immersion in such N.

The following lemma is useful for finding CR immersions and embeddings of left-invariant CR structures on Lie groups.

Lemma 3.1. *Let G be a 3-dimensional Lie group with a left-invariant CR structure (D, J), with corresponding $[L] \in \mathrm{P}(\mathfrak{g}_{\mathbb{C}})_{\mathrm{reg}}$. Let $\rho : G \to \mathrm{GL}(U)$ be a finite dimensional complex representation, $u \in U$ and $\mu : G \to U$ the evaluation map $g \mapsto \rho(g)u$. Then μ is a CR map if and only if $\rho'(L)u = 0$, where $\rho' : \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(U)$ is the complex linear extension of $(\mathrm{d}\rho)_e : \mathfrak{g} \to \mathrm{End}(U)$ to $\mathfrak{g}_{\mathbb{C}}$.*

Proof. μ is clearly G-equivariant, hence μ is CR if and only if $\mathrm{d}\mu(JX) = i\,\mathrm{d}\mu(X)$ for some (and thus all) non-zero $X \in D_e$. Now $\mathrm{d}\mu(X) = \rho'(X)u$, hence the CR condition on μ is $\rho'(X + iJX)u = 0$, for all $X \in D_e$. Equivalently, $\rho'(L)u = 0$ for some (and thus all) non-zero $L \in \mathfrak{g}_{\mathbb{C}}$ of type $(0,1)$. □

Here is an application of the last lemma, often used by Cartan in Chapter II of [5].

Proposition 3.3. *Let G be a 3-dimensional Lie group with a left-invariant CR structure $[L] \in \mathrm{P}(\mathfrak{g}_{\mathbb{C}})_{\mathrm{reg}}$. Then the evaluation map $\mu : G \to \mathrm{P}(\mathfrak{g}_{\mathbb{C}})$, $g \mapsto [\mathrm{Ad}_g(L)]$, is a G-equivariant CR map, whose image $\mu(G) \subset \mathrm{P}(\mathfrak{g}_{\mathbb{C}})$, the Ad_G-orbit of $[L] \in \mathrm{P}(\mathfrak{g}_{\mathbb{C}})$, is of dimension 2 or 3. It follows that if L has a trivial centralizer in $\mathfrak{g}$ then $\mu(G)$ is 3-dimensional and hence μ is a local realization of the CR structure on G in $\mathrm{P}(\mathfrak{g}_{\mathbb{C}}) \simeq \mathbb{C}\mathrm{P}^2$.*

Proof. Let $\tilde{\mu} : G \to \mathfrak{g}_{\mathbb{C}} \setminus \{0\}$, $g \mapsto \mathrm{Ad}_g L$, and $\pi : \mathfrak{g}_{\mathbb{C}} \setminus \{0\} \to \mathrm{P}(\mathfrak{g}_{\mathbb{C}})$, $B \mapsto [B]$. Then $\mu = \pi \circ \tilde{\mu}$ and π is holomorphic, hence it is enough to show that $\tilde{\mu}$ is CR at $e \in G$. Applying Lemma 3.1 with $\rho = \mathrm{Ad}_G$, $u = L$, we have that $\rho'(L)L = [L, L] = 0$, hence $\tilde{\mu}$ is CR, and so is μ.

Let $\mathcal{O} = \mu(G)$. Since μ is CR, $\mathrm{d}\mu(D)$ is a $\tilde{J}$-invariant and G-invariant subbundle of $T\mathcal{O}$, where $\tilde{J}$ is the complex structure of $\mathrm{P}(\mathfrak{g}_{\mathbb{C}})$. Thus in order to show that $\dim(\mathcal{O}) \geq 2$ it is enough to show that $\mathrm{d}\mu(D_e) \neq 0$. Equivalently, $\mathrm{d}\tilde{\mu}(D_e) \not\subset \mathrm{Ker}((\mathrm{d}\pi)_L) = \mathbb{C}L$. Let $L = L_1 + iL_2$, with $L_1, L_2 \in \mathfrak{g}$. Then $L_2 = JL_1$ and so $\mathrm{d}\tilde{\mu}(L_2) = [L_2, L] = -[L_1, L_2]$. But $[L]$ is non-real, so $(\mathbb{C}L) \cap \mathfrak{g} = \{0\}$, hence $[L_1, L_2] \in \mathbb{C}L$ implies $[L_1, L_2] = 0$, so $D_e = \mathrm{Span}\{L_1, L_2\} \subset \mathfrak{g}$ is an (abelian) subalgebra, in contradiction to the non-degeneracy assumption on the CR structure. □

4. $\mathrm{SL}_2(\mathbb{R})$

We illustrate the results of the previous section first of all with a detailed description of left-invariant CR structures on the group $G = \mathrm{SL}_2(\mathbb{R})$, where $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, the set of 2×2 traceless real matrices and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$, the set of 2×2 traceless complex matrices.

Here is a summary of the results: for $G = \mathrm{SL}_2(\mathbb{R})$, the set of left-invariant CR structures $\mathrm{P}(\mathfrak{g}_{\mathbb{C}})_{\mathrm{reg}}$ is identified $\mathrm{Aut}(G)$-equivariantly with the set of unordered pairs of points $\zeta_1, \zeta_2 \in \mathbb{C} \setminus \mathbb{R}$, $\zeta_1 \neq \bar{\zeta}_2$, on which $\mathrm{Aut}(G)$ acts by orientation preserving isometries of the usual hyperbolic metric in each of the half planes. With this description, it is easy to determine the $\mathrm{Aut}(G)$-orbits. There are two families of orbits: the 'elliptic' family corresponds to pairs of points in the same half-plane, with the spherical structure corresponding to a 'double point', $\zeta_1 = \zeta_2$; the 'hyperbolic' family corresponds to non-conjugate pairs of points in opposite half planes. Each orbit is labeled uniquely by the hyperbolic distance $d(\zeta_1, \zeta_2)$ in the elliptic case, or $d(\zeta_1, \bar{\zeta}_2)$ in the hyperbolic case. All structures, except the spherical elliptic one, are locally realized as generic adjoint orbits in $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C})) = \mathbb{C}\mathrm{P}^2$, either inside S^3 (in the hyperbolic case) or in its exterior (in the elliptic case). The elliptic spherical structure embeds as any of the generic orbit of the standard action on $\mathbb{C}^2$.

We begin with the conjugation action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))$ (this will be useful also for the next example of $G = \mathrm{SU}_2$). With each $[L] \in \mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))$ we associate an unordered pair of points $\zeta_1, \zeta_2 \in \mathbb{C} \cup \infty$, possibly repeated, the roots of the quadratic polynomial

$$p_L(\zeta) := c\zeta^2 - 2a\zeta - b = c(\zeta - \zeta_1)(\zeta - \zeta_2), \qquad L = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \tag{2}$$

Clearly, multiplying L by a non-zero complex constant does not affect ζ_1, ζ_2.

Lemma 4.1. *Let $S^2(\mathbb{C}\mathrm{P}^1)$ be the set of unordered pairs of points $\zeta_1, \zeta_2 \in \mathbb{C} \cup \infty = \mathbb{C}\mathrm{P}^1$. Then:*

(a) *The map $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C})) \to S^2(\mathbb{C}\mathrm{P}^1)$, assigning to $[L] \in \mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))$ the roots of p_L, as in equation (2), is an $\mathrm{SL}_2(\mathbb{C})$-equivariant bijection, where $\mathrm{SL}_2(\mathbb{C})$ acts on $S^2(\mathbb{C}\mathrm{P}^1)$ via Möbius transformations on $\mathbb{C}\mathrm{P}^1$ (projectivization of the standard action on $\mathbb{C}^2$);*

(b) *Complex conjugation, $[L] \mapsto [\bar{L}]$, corresponds, under the above bijection, to complex conjugation of the roots of p_L, $\{\zeta_1, \zeta_2\} \mapsto \{\bar{\zeta}_1, \bar{\zeta}_2\}$.*

Proof. The map $[L] \mapsto \{\bar{\zeta}_1, \bar{\zeta}_2\}$ is clearly a bijection (a polynomial is determined, up to a scalar multiple, by its roots). The $\mathrm{SL}_2(\mathbb{C})$-equivariance, as well as item (b), can be easily checked by direct computation.

Here is a more illuminating argument, explaining also the origin of the formula for p_L in equation (2). We first show that the adjoint representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to H_2, the space of quadratic forms on $\mathbb{C}^2$, or complex homogeneous polynomials $q(z_1, z_2)$ of degree 2 in two variables, with $g \in \mathrm{SL}_2(\mathbb{C})$ acting by substitutions, $q \mapsto q \circ g^{-1}$. To derive an explicit isomorphism, let U be the standard representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ and U^* the dual representation, where $g \in \mathrm{SL}_2(\mathbb{C})$ acts on $\alpha \in \mathrm{U}^*$ by $\alpha \mapsto \alpha \circ g^{-1}$. The induced action on $\Lambda^2(\mathrm{U}^*)$ (skew symmetric bilinear forms on U) is trivial (this amounts to $\det(g) = 1$). Let us fix $\omega := z_1 \wedge z_2 \in \Lambda^2(\mathrm{U}^*)$. Since ω is $\mathrm{SL}_2(\mathbb{C})$-invariant, it defines an $\mathrm{SL}_2(\mathbb{C})$-equivariant isomorphism $\mathrm{U} \to \mathrm{U}^*$, $u \mapsto \omega(\cdot, u)$, mapping $\mathbf{e}_1 \mapsto -z_2$, $\mathbf{e}_2 \mapsto z_1$, where $\mathbf{e}_1, \mathbf{e}_2$ is the standard basis of U, dual to $z_1, z_2 \in \mathrm{U}^*$. We thus obtain an isomorphism of $\mathrm{SL}_2(\mathbb{C})$ representations, $\mathrm{End}(\mathrm{U}) \simeq \mathrm{U} \otimes \mathrm{U}^* \simeq \mathrm{U}^* \otimes \mathrm{U}^*$. Under this isomorphism, $\mathfrak{sl}_2(\mathbb{C}) \subset \mathrm{End}(\mathrm{U})$ is mapped unto $S^2(\mathrm{U}^*) \subset \mathrm{U}^* \otimes \mathrm{U}^*$ (symmetric bilinear forms on U), which in turn is identified with H_2, $\mathrm{SL}_2(\mathbb{C})$-equivariantly, via $B \mapsto q$, $q(u) = B(u,u)$. Following through these isomorphisms, we get the sought for $\mathrm{SL}_2(\mathbb{C})$-equivariant isomorphism $\mathfrak{sl}_2(\mathbb{C}) \xrightarrow{\sim} H_2$,

$$\begin{aligned} L = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} &\mapsto a\mathbf{e}_1 \otimes z_1 + b\mathbf{e}_1 \otimes z_2 + c\mathbf{e}_2 \otimes z_1 - a\mathbf{e}_2 \otimes z_2 \\ &\mapsto -az_2 \otimes z_1 - bz_2 \otimes z_2 + cz_1 \otimes z_1 - az_1 \otimes z_2 \\ &\mapsto q_L(z_1, z_2) = c(z_1)^2 - 2a\, z_1 z_2 - b(z_2)^2. \end{aligned}$$

Now every non-zero quadratic form $q \in H_2$ can be factored as the product of two non-zero linear forms, $q = \alpha_1\alpha_2$, where the kernel of each α_i determines a 'root' $\zeta_i \in \mathbb{C}\mathrm{P}^1$. Introducing the inhomogeneous coordinate $\zeta = z_1/z_2$ on $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \infty$, we get $c(z_1)^2 - 2a\, z_1 z_2 - b(z_2)^2 = (z_2)^2 p_L(\zeta)$, with p_L as in equation (2) with roots $\zeta_i \in \mathbb{C} \cup \infty$. □

Remark 4.1. There is a simple projective geometric interpretation of Lemma 4.1. See Figure 1(a). Consider in the projective plane $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C})) \simeq \mathbb{C}\mathrm{P}^2$ the conic $\mathcal{C} := \{[L] \mid \det(L) = 0\} \simeq \mathbb{C}\mathrm{P}^1$. Through a point $[L] \in \mathbb{C}\mathrm{P}^2 \setminus \mathcal{C}$ pass two (projective) lines tangent to $\mathcal{C}$, with tangency points $\zeta_1, \zeta_2 \in \mathcal{C}$ (if $[L] \in \mathcal{C}$ then $\zeta_1 = \zeta_2 = [L]$). Since $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{C}\mathrm{P}^2$ by projective transformations preserving $\mathcal{C}$, the map $[L] \mapsto \{\zeta_1, \zeta_2\}$ is $\mathrm{SL}_2(\mathbb{C})$-equivariant. The map $[L] \mapsto [\overline{L}]$ is the reflection about $\mathbb{R}\mathrm{P}^2 \subset \mathbb{C}\mathrm{P}^2$. Formula (2) is a coordinate expression of this geometric recipe.

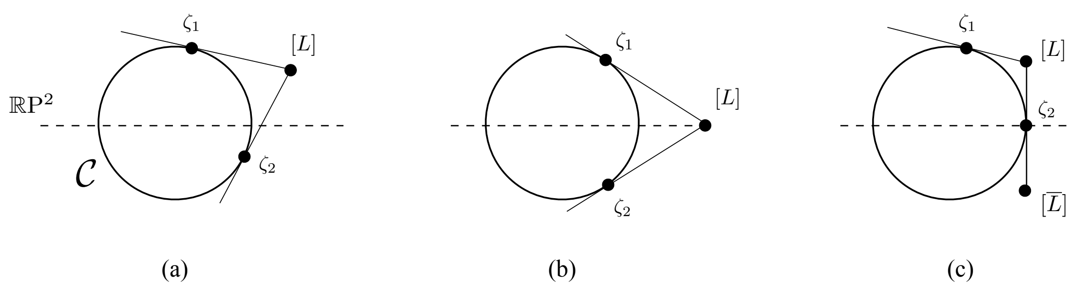

FIGURE 1. Distinct types of $[L] \in \mathrm{P}(\mathfrak{g}_\mathbb{C})$ for $G = \mathrm{SL}_2(\mathbb{R})$: (a) regular ; (b) real ; (c) non-real degenerate. See the proofs of Lemma 4.1, 4.2 and Remark 4.1.

Lemma 4.2. *Let $L \in \mathfrak{sl}_2(\mathbb{C})$, $L \neq 0$. Then $[L] \in \mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))_{\mathrm{reg}}$ if and only if both roots of p_L are non-real and are non-conjugate, i.e., $\zeta_1, \zeta_2 \in \mathbb{C} \setminus \mathbb{R}$ and $\zeta_1 \neq \bar{\zeta}_2$.*

Proof. Let ζ_1, ζ_2 be the roots of p_L. By Lemma 4.1 part (b), $[L]$ is real, $[L] = [\overline{L}]$, if and only if ζ_1, ζ_2 are both real or $\zeta_1 = \bar{\zeta}_2$. We claim that if $[L] \neq [\overline{L}]$ then $[L]$ is degenerate, i.e., $L, \overline{L}$ span a 2-dimensional subalgebra of $\mathfrak{sl}_2(\mathbb{C})$, exactly when one of the two roots ζ_1, ζ_2 is real and the other is non-real. This is perhaps best seen with Figure 1(c). A 2-dimensional subspace of $\mathfrak{sl}_2(\mathbb{C})$ corresponds to a projective line in $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))$. The 2-dimensional subalgebras of $\mathfrak{sl}_2(\mathbb{C})$ are all conjugate (by $\mathrm{SL}_2(\mathbb{C})$) to the subalgebra of upper triangular matrices and are represented in Figure 1 by lines tangent to $\mathcal{C}$. Now the line passing through $[L], [\overline{L}]$ is invariant under complex conjugation, hence if it is tangent to to $\mathcal{C}$ then the tangency point is real and is one of the roots of p_L. But $[L]$ is non-real, hence the other root is non-real. □

Next we describe $\mathrm{Aut}(\mathrm{SL}_2(\mathbb{R}))$. Clearly, $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathrm{SL}_2(\mathbb{R})$ by matrix conjugation as group automorphism. The ineffective kernel of this action is the center $\mathbb{R}^*\mathrm{I}$ of $\mathrm{GL}_2(\mathbb{R})$ (non-zero multiples of the identity matrix). The quotient group is denoted by $\mathrm{PGL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^*\mathrm{I}$. Thus there is a natural inclusion $\mathrm{PGL}_2(\mathbb{R}) \subset \mathrm{Aut}(\mathrm{SL}_2(\mathbb{R}))$.

Lemma 4.3. $\mathrm{PGL}_2(\mathbb{R}) = \mathrm{Aut}(\mathrm{SL}_2(\mathbb{R})) = \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{R}))$.

Proof. We have already seen the inclusions $\mathrm{PGL}_2(\mathbb{R}) \subset \mathrm{Aut}(\mathrm{SL}_2(\mathbb{R})) \subset \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{R}))$, so it is enough to show that $\mathrm{Aut}(\mathfrak{sl}_2(\mathbb{R})) \subset \mathrm{PGL}_2(\mathbb{R})$. Now the Killing form of a Lie algebra, $\langle X, Y\rangle = \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y)$, is defined in terms of the Lie bracket alone. For $\mathfrak{sl}_2(\mathbb{R})$, the associated quadratic form is $\det(X) = -a^2 - bc$ (up to a constant), a non-degenerate quadratic form of signature (2,1). Furthermore, the 'triple product' $(X, Y, Z) \mapsto \langle X, [Y, Z]\rangle$ defines a non vanishing volume form on $\mathfrak{sl}_2(\mathbb{R})$ in terms of the Lie bracket, hence $\mathrm{Aut}(\mathfrak{sl}_2(\mathbb{R})) \subset \mathrm{SO}_{2,1}$. Finally, $\mathrm{PGL}_2(\mathbb{R}) \subset \mathrm{SO}_{2,1}$ and both are 3-dimensional groups with two components, so they must coincide. □

Let us now examine the action of $\mathrm{Aut}(\mathrm{SL}_2(\mathbb{R}))$ on $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))$. It is convenient, instead of working with $\mathrm{Aut}(\mathrm{SL}_2(\mathbb{R})) = \mathrm{PGL}_2(\mathbb{R})$, to work with its double cover $\mathrm{SL}_2^{\pm}(\mathbb{R})$ (matrices with det $= \pm 1$.) The latter consists of two components, the identity component, $\mathrm{SL}_2(\mathbb{R})$, and $\sigma\mathrm{SL}_2(\mathbb{R})$, where σ is any matrix with det $= -1$; for example $\sigma = \mathrm{diag}(1, -1)$. According to Lemma 4.1, we need to consider first the action of $\mathrm{SL}_2^{\pm}(\mathbb{R})$ by Möbius transformations on $\mathbb{CP}^1$. The action of the identity component $\mathrm{SL}_2(\mathbb{R})$ has 3 orbits; in terms of the inhomogeneous coordinate ζ, these are

- the upper half-plane $\mathrm{Im}(\zeta) > 0$,
- the lower half-plane $\mathrm{Im}(\zeta) < 0$,
- their common boundary, the real projective line $\mathbb{RP}^1 = \mathbb{R} \cup \infty$.

The action on each half-plane is by orientation preserving hyperbolic isometries (isometries of the Poincaré metric $|d\zeta|/|\mathrm{Im}(\zeta)|$). The action of $\sigma = \mathrm{diag}(1, -1)$ is by reflection about the origin $\zeta = 0$, an orientation preserving hyperbolic isometry between the upper and lower half planes.

In summary, we get the following orbit structure:

Proposition 4.1. *Under the identification $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C})) \simeq S^2(\mathbb{CP}^1)$ of Lemma 4.1, the orbits of $\mathrm{Aut}(\mathrm{SL}_2(\mathbb{R}))$ in $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))_{\mathrm{reg}}$ correspond to the following two 1-parameter families of orbits in $S^2(\mathbb{CP}^1)$:*

I. *A 1-parameter family of orbits, corresponding to a pair of points $\zeta_1, \zeta_2 \in \mathbb{C} \setminus \mathbb{R}$ in the same half-plane (upper or lower). The parameter can be taken as the hyperbolic distance $d(\zeta_1, \zeta_2) \in [0, \infty)$. All these orbits are 3-dimensional, except the one corresponding to a double point $\zeta_1 = \zeta_2$, which is 2-dimensional.*

I. *A 1-parameter family of orbits, corresponding to pair of points $\zeta_1, \zeta_2 \in \mathbb{C}\setminus\mathbb{R}$ situated in opposite half planes and which are not complex conjugate, $\zeta_1 \neq \bar{\zeta}_2$. The parameter can be taken as the hyperbolic distance $d(\zeta_1, \bar{\zeta}_2) \in (0, \infty)$. All these orbits are 3-dimensional.*

The rest of the orbits are either real ($\zeta_1, \zeta_2 \in \mathbb{RP}^1 = \mathbb{R} \cup \infty$ or $\zeta_1 = \bar{\zeta}_2$) or degenerate (one of the points is real).

Proof. Most of the claims follow immediately from the previous lemmas so their proof is omitted. The claimed dimensions of the orbits follow from the dimension of the stabilizer in $\mathrm{Aut}(\mathrm{SL}_2(\mathbb{R}))$ of an unordered pair $\zeta_1, \zeta_2 \in \mathbb{C} \setminus \mathbb{R}$; for two distinct points in the same half-plane, or in opposite half-planes with $z_1 \neq \bar{z}_2$, the stabilizer is the two element subgroup interchanging the points. For a double point the stabilizer is a circle group of hyperbolic rotations about this point. □

Next, recall that the *Killing form* on $\mathfrak{sl}_2(\mathbb{R})$ is the bilinear form $\langle X, Y\rangle = (1/2)\mathrm{tr}(XY)$. The associated quadratic form $\langle X, X\rangle = -\det(X) = a^2 + bc$ is a non-degenerate indefinite form of signature $(2, 1)$, the unique Ad-invariant form on $\mathfrak{sl}_2(\mathbb{R})$, up to scalar multiple. The *null cone* $C \subset \mathfrak{sl}_2(\mathbb{R})$ is the subset of elements with $\langle X, X\rangle = 0$.

Definition 4.1. *A 2-dimensional subspace $\Pi \subset \mathfrak{sl}_2(\mathbb{R})$ is called* elliptic *(respectively,*hyperbolic*) if the Killing form restricts to a definite (respectively, indefinite, but non-degenerate) inner product* *on* Π*. Equivalently,* Π *is hyperbolic if its intersection with the null cone* C *consists of two of its generators and elliptic if it intersects it only at its vertex* $X = 0$*. A left-invariant CR structure* (D, J) *on* $\mathrm{SL}_2(\mathbb{R})$ *is elliptic (resp.* hyperbolic*) if* $D_e \subset \mathfrak{sl}_2(\mathbb{R})$ *is elliptic (resp.* hyperbolic*).*

Remark 4.2. There is a third type of a 2-dimensional subspace $\Pi \subset \mathfrak{sl}_2(\mathbb{R})$, called *parabolic*, consisting of 2-planes tangent to C, but these are subalgebras of $\mathfrak{sl}_2(\mathbb{R})$, hence are excluded by the non-degeneracy condition on the CR structure.

Remark 4.3. Our use of the terms elliptic and hyperbolic for the contact plane is natural from the point of view of Lie theory. However it conflicts with the terminology of analysis; CR vector fields are never elliptic or hyperbolic differential operators.

Lemma 4.4. *Let* $[L] \in \mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))_{\mathrm{reg}}$*, and* $D_e \subset \mathfrak{sl}_2(\mathbb{R})$ *the real part of the span of* $L, \overline{L}$*. Then* D_e *is elliptic if the roots of* p_L *lie in the same half plane (type I of Proposition 4.1), and is hyperbolic if they lie in opposite half planes (type II of proposition 4.1).*

Proof. Let ζ_1, ζ_2 be the roots of p_L. Acting by $\mathrm{Aut}(\mathrm{SL}_2(\mathbb{R}))$, we can assume, without loss of generality, that $\zeta_1 = i$ and $\zeta_2 = it$ for some $t \in \mathbb{R} \setminus \{-1, 0\}$. Thus, up to scalar multiple, $p_L = (\zeta - i)(\zeta - it) = \zeta^2 - i(1+t)\zeta - t$. A short calculation shows that D_e consists of matrices of the form $X = \begin{pmatrix} a(1+t) & tb \\ b & -a(1+t) \end{pmatrix}$, $a, b \in \mathbb{R}$, with $\det(X) = -a^2(1+t)^2 - tb^2$. This is negative definite for $t > 0$ and indefinite otherwise. □

Proposition 4.2. *Let* $V_t \subset T_{\mathbb{C}}\mathrm{SL}_2(\mathbb{R})$*,* $t \in \mathbb{R}$*, be the left-invariant complex line bundle spanned at* $e \in \mathrm{SL}_2(\mathbb{R})$ *by*

$$L_t = \begin{pmatrix} i\frac{1+t}{2} & t \\ 1 & -i\frac{1+t}{2} \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{C} = \mathfrak{sl}_2(\mathbb{C}). \tag{3}$$

Then

(a) V_t *is a left-invariant CR structure for all* $t \neq 0, -1$*, elliptic for* $t > 0$ *and hyperbolic for* $t < 0, t \neq -1$*.*
(b) V_t *is spherical if* $t = 1$ *or* $-3 \pm 2\sqrt{2}$ *and aspherical otherwise.*
(c) *Every left-invariant CR structure on* $\mathrm{SL}_2(\mathbb{R})$ *is CR equivalent to* V_t *for a unique* $t \in (-1, 0) \cup (0, 1]$*.*
(d) *The aspherical left-invariant CR structures* V_t*,* $t \in (-1, 1) \setminus \{0, -3 + 2\sqrt{2}\}$*, are pairwise non-equivalent, even locally.*

Proof. (a) The quadratic polynomial corresponding to L_t is

$$p(\zeta) = \zeta^2 - i(1+t)\zeta - t = (\zeta - i)(\zeta - it),$$

with roots i, it. For $t > 0$ the roots are in the upper half plane and thus, by Lemma 4.4, V_t is an elliptic CR structure. For $t < 0$ the roots are in opposite half planes and for $t \neq -1$ are not complex conjugate, hence V_t is an hyperbolic CR structure.

(b) Let

$$\Theta = g^{-1}\mathrm{d}g = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

be the left-invariant Maurer-Cartan $\mathfrak{sl}_2(\mathbb{R})$-valued 1-form on $\mathrm{SL}_2(\mathbb{R})$. A coframe adapted to V_t is

$$\theta = \beta - t\gamma, \quad \theta_1 = \alpha - i\frac{1+t}{2}\gamma, \tag{4}$$

i.e., $\theta(L_t) = \theta_1(L_t) = 0$, $\bar{\theta}_1(L_t) \neq 0$. The Maurer-Cartan equations, $\mathrm{d}\Theta = -\Theta \wedge \Theta$, are

$$\mathrm{d}\alpha = -\beta \wedge \gamma, \quad \mathrm{d}\beta = -2\alpha \wedge \beta, \quad \mathrm{d}\gamma = 2\alpha \wedge \gamma.$$

Using there equations, we calculate

$$\mathrm{d}\theta = i\frac{4t}{1+t}\theta_1 \wedge \bar{\theta}_1 + \theta \wedge \theta_1 + \theta \wedge \bar{\theta}_1.$$

Now

$$\phi := \mathrm{sign}(t)(\beta - t\gamma), \quad \phi_1 := \sqrt{\left|\frac{4t}{1+t}\right|}\left[\alpha - i\frac{1+t}{4}\left(\frac{\beta}{t} + \gamma\right)\right]$$

satisfy

$$\mathrm{d}\phi = i\phi_1 \wedge \bar{\phi}_1, \quad \mathrm{d}\phi_1 = b\phi \wedge \phi_1 + c\phi \wedge \bar{\phi}_1,$$

where

$$b = -i\frac{1+6t+t^2}{4|t|(1+t)}, \quad c = -i\frac{(1-t)^2}{4|t|(1+t)},$$

thus (ϕ, ϕ_1) is well-adapted to V_t. Applying Proposition 3.2, we conclude that V_t is spherical if and only if $(1+6t+t^2)(1-t) = 0$; that is, $t = 1$ or $-3 \pm 2\sqrt{2}$, as claimed.

(c) The hyperbolic distance $d(i, it)$ varies monotonically from 0 to ∞ as t varies from 1 to 0, hence every pair of points in the same half plane can be mapped by $\mathrm{Aut}(\mathrm{SL}_2(\mathbb{R}))$ to the pair (i, it) for a unique $t \in (0, 1]$. Consequently, every left-invariant elliptic CR structure is CR equivalent to V_t for a unique $t \in (0, 1]$.

Similarly, $d(i, -it)$ varies monotonically from 0 to ∞ as t varies from -1 to 0, hence every hyperbolic left-invariant CR structure is CR equivalent to V_t for a unique $t \in (-1, 0)$.

By Proposition 3.1, no pair of the aspherical V_t with $0 < |t| < 1$ are CR equivalent, even locally. It remains to show that the elliptic and hyperbolic spherical structures, namely, V_t for $t = 1$ and $-3 + 2\sqrt{2}$ (respectively), are not CR equivalent. In the next proposition, we find an embedding $\phi_1 : \mathrm{SL}_2(\mathbb{R}) \to S^3$ of the elliptic spherical structure in the standard spherical CR structure on S^3 and an immersion $\phi_2 : \mathrm{SL}_2(\mathbb{R}) \to S^3$ of the hyperbolic spherical structure which is not an embedding (it is a $2:1$ cover). It follows from Corollary 2.1 that these two spherical structures are not equivalent: if $f : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ were a diffeomorphism mapping the hyperbolic spherical structure to the elliptic one, then this would imply that the pull-backs to $\mathrm{SL}_2(\mathbb{R})$ of the spherical structure of S^3 by $\phi_1 \circ f$ and ϕ_2 coincide, and hence, by Corollary 2.1, there is an element $g \in \mathrm{PU}_{2,1}$ such that $\phi_2 = g \circ \phi_1 \circ f$. But this is impossible, since $g \circ \phi_1 \circ f$ is an embedding and ϕ_2 is not.

(d) As mentioned in the previous item, this is a consequence of Proposition 3.1. □

Remark 4.4. There is an alternative path, somewhat shorter (albeit less picturesque), to the classification of left-invariant CR structures on $\mathrm{SL}_2(\mathbb{R})$, leading to a family of 'normal forms' different then the V_t of Proposition 4.2. One shows first that, up to conjugation by $\mathrm{SL}_2(\mathbb{R})$, there are only two non-degenerate left-invariant contact structures $D \subset T\mathrm{SL}_2(\mathbb{R})$: an elliptic one, given by $D_e^+ = \{c = b\}$, and hyperbolic one, given by $D_e^- = \{c = -b\}$. The Killing form on $\mathfrak{sl}_2(\mathbb{R})$, $-\det(X) = a^2 + bc$, restricted to $D_e^\pm$, is given by $a^2 \pm b^2$, with orthonormal basis $A, B \pm C$, where A, B, C is the basis of $\mathfrak{sl}_2(\mathbb{R})$ dual to a, b, c. One then determines the stabilizer of $D_e^\pm$ in $\mathrm{Aut}(\mathrm{SL}_2(\mathbb{R}))$ (the subgroup that leaves $D_e^\pm$ invariant). In each case the stabilizer acts on $D_e^\pm$ as the full isometry group of $a^2 \pm b^2$, that is, O_2 in the elliptic case, and $\mathrm{O}_{1,1}$, in the hyperbolic case. Using this description one shows that, in the elliptic case, each almost complex structure on D_e^+ is conjugate to a unique one of the form $A \mapsto s(B + C)$, $s \geq 1$, with corresponding $(0,1)$ vector $A + is(B+C) = \begin{pmatrix} 1 & is \\ is & -1 \end{pmatrix}$, and in the hyperbolic case $A \mapsto s(B - C)$, $s > 0$, with corresponding $(0,1)$ vector $A + is(B - C) = \begin{pmatrix} 1 & is \\ -is & -1 \end{pmatrix}$. The spherical structures are given by $s = 1$ in both cases.

Regarding realizability of left-invariant CR structures on $\mathrm{SL}_2(\mathbb{R})$, we have the following.

Proposition 4.3. (a) *The elliptic left-invariant spherical CR structure on* $\mathrm{SL}_2(\mathbb{R})$ *(*$t = 1$ *in equation* (3)*) is realizable as any of the generic (3-dimensional)* $\mathrm{SL}_2(\mathbb{R})$*-orbits in* $\mathbb{C}^2$ *(complexification of the standard linear action on* $\mathbb{R}^2$*). This is also CR equivalent to the complement of a 'chain' in* $S^3 \subset \mathbb{C}^2$ *(a curve in* S^3 *given by the intersection of a complex affine line in* $\mathbb{C}^2$ *with* S^3*; e.g.,* $z_1 = 0$*)*

(b) *The rest of the left-invariant CR structures on* $\mathrm{SL}_2(\mathbb{R})$*, with* $0 < |t| < 1$ *in equation* (3)*, are either* $4:1$ *covers, in the aspherical elliptic case* $0 < t < 1$*, or* $2:1$ *covers, in the hyperbolic case* $-1 < t < 0$*, of the orbits of* $\mathrm{SL}_2(\mathbb{R})$ *in* $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))$*.*

(c) *The spherical hyperbolic orbit is also CR equivalent to the complement of* $S^3 \cap \mathbb{R}^2$ *in* $S^3 \subset \mathbb{C}^2$*.*

Proof. (a) Fix $v \in \mathbb{C}^2$ and define $\mu : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}^2$ by $\mu(g) = gv$. If the stabilizer of v in $\mathrm{SL}_2(\mathbb{R})$ is trivial and $L_1 v = 0$, then, by Lemma 3.1, μ is an $\mathrm{SL}_2(\mathbb{R})$-equivariant CR embedding. Both conditions are satisfied by $v = \binom{i}{1}$. In fact, all 3-dimensional $\mathrm{SL}_2(\mathbb{R})$-orbits in $\mathbb{C}^2$ are homothetic, hence are CR equivalent and any of them will do.

Now let $\mathcal{O} \subset \mathbb{C}^2$ be the $\mathrm{SL}_2(\mathbb{R})$-orbit of $v = \binom{i}{1}$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, with $\det(g) = ad - bc = 1$, $gv = \binom{b+ia}{d+ic}$, hence $\mathcal{O}$ is the quadric $\mathrm{Im}(z_1 \bar{z}_2) = 1$, where z_1, z_2 are the standard complex coordinates in $\mathbb{C}^2$. To map $\mathcal{O}$ onto the complement of $z_1 = 0$ in S^3 we first apply the complex linear transformation $\mathbb{C}^2 \to \mathbb{C}^2$, $(z_1, z_2) \mapsto (z_1 + iz_2, z_2 + iz_1)/2$, mapping $\mathcal{O}$ unto the hypersurface $|z_1|^2 - |z_2|^2 = 1$. Next let Z_1, Z_2, Z_3 be homogenous coordinates in $\mathbb{CP}^2$ and embed $\mathbb{C}^2$ as an affine chart, $(z_1, z_2) \mapsto [z_1 : z_2 : 1]$. The image of $|z_1|^2 - |z_2|^2 = 1$ is the complement of $Z_3 = 0$ in $|Z_1|^2 - |Z_2|^2 = |Z_3|^2$. This is mapped by $[Z_1 : Z_2 : Z_3] \mapsto [Z_3 : Z_2 : Z_1]$ to the complement of $Z_1 = 0$ in $|Z_1|^2 + |Z_2|^2 = |Z_3|^2$. In our affine chart this is the complement of $z_1 = 0$ in $|z_1|^2 + |z_2|^2 = 1$, as needed.

(b) By Proposition 3.3, to show that the map $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))$, $g \mapsto [\mathrm{Ad}_g L_t]$, is a CR immersion of V_t into $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))$, it is enough to to show that the stabilizer of $[L_t] \in \mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))$ in $\mathrm{SL}_2(\mathbb{R})$ is discrete. Using Lemma 4.1, we find that, in the aspherical elliptic case, where $t \in (0, 1)$, the roots are an unordered pair of distinct points in the upper half plane, so there is a single hyperbolic isometry in $\mathrm{PSL}_2(\mathbb{R})$ interchanging them, hence the stabilizer in $\mathrm{SL}_2(\mathbb{R})$ is a 4 element subgroup.

In the hyperbolic case, where $t \in (-1, 0)$, the roots ζ_1, ζ_2 are in opposite half-spaces and $\zeta_1 \neq \bar{\zeta}_2$. Hence an element $g \in \mathrm{SL}_2(\mathbb{R})$ that fixes the unordered pair ζ_1, ζ_2 has two distinct fixed points $\zeta_1, \bar{\zeta}_2$ in the same half plane. It follows that g acts trivially in this half plane and thus $g = \pm \mathrm{I}$.

(c) $\mathfrak{sl}_2(\mathbb{C})$ admits a pseudo-hermitian product of signature $(2,1)$, $\mathrm{tr}\left(X\overline{Y}\right)$, invariant under the conjugation action of $\mathrm{SL}_2(\mathbb{R})$. The associated projectivized null cone in $\mathbb{CP}^2$ is diffeomorphic to S^3, a model for the spherical CR structure on S^3. One can check that L_t is a null vector, i.e., $\mathrm{tr}(L_t \bar{L}_t) = 0$, for $t = -3 \pm \sqrt{2}$. Thus the hyperbolic spherical left-invariant structure on $\mathrm{SL}_2(\mathbb{R})$ is a $2:1$ cover of an $\mathrm{SL}_2(\mathbb{R})$-orbit in S^3, consisting of all regular elements $[L] \in S^3$, whose complement in S^3 is the set of elements which are either real or degenerate non-real (see Lemma 4.2 and its proof). One can check that the only degenerate element in S^3 satisfies $a = c = 0$, $b \neq 0$, which is real. Thus all irregular elements in S^3 are the real elements $\mathbb{RP}^2 \cap S^3 \subset \mathbb{CP}^2$, or $\mathbb{R}^2 \cap S^3 \subset \mathbb{C}^2$, as claimed. ☐

Remark 4.5. In Cartan's classification [5, p. 70], the left-invariant spherical elliptic CR structure on $\mathrm{SL}_2(\mathbb{R})$ appears in item 5°(B) of the first table, as a left-invariant CR structure on the group $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}/\mathbb{Z}$. This group is *not* isomorphic to $\mathrm{SL}_2(\mathbb{R})$, yet it admits a left-invariant spherical structure, CR equivalent to the spherical elliptic CR structure on $\mathrm{SL}_2(\mathbb{R})$. This shows that the asphericity condition in Proposition 3.1 is essential. Both groups are subgroups of the full 4-dimensional group of automorphism of this homogeneous spherical CR manifold (the stabilizer in $\mathrm{PU}_{2,1}$ of a chain in S^3). The hyperbolic spherical structure is item 8°(K′).

The elliptic and hyperbolic aspherical left-invariant structures on $\mathrm{SL}_2(\mathbb{R})$ appear in items 4°(K) and 5°(K′) (respectively) of the second table. In these items, Cartan gives explicit equations for the adjoint orbits in inhomogeneous coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{CP}^2$ (an affine chart). For the elliptic aspherical orbits he gives the equation $1+x\bar{x}-y\bar{y}=\mu|1+x^2-y^2|$, with $\mathrm{Im}(x(1+\bar{y}))>0$ and $\mu>1$; for the hyperbolic aspherical structures he gives the equation $x\bar{x}+y\bar{y}-1=\mu|x^2+y^2-1|$, with $(x,y)\in\mathbb{C}^2\setminus\mathbb{R}^2$ and $0<|\mu|<1$. Both equations are $\mathrm{tr}(L\bar{L})=\mu|\mathrm{tr}(L^2)|$, with $(x,y)=(b+c,b-c))/(2a)$ in the elliptic case, and $(x,y)=(2a,b-c)/(b+c)$ in he hyperbolic case. The elliptic orbits are the generic orbits in the exterior of S^3, given by $\mathrm{tr}(L\bar{L})>0$, while the hyperbolic orbits lie in its interior, given by $\mathrm{tr}(L\bar{L})<0$. The elliptic orbits come in complex-conjugate pairs; that is, for each orbit, given by the pairs of roots $\zeta_1,\zeta_2\in\mathbb{C}\setminus\mathbb{R}$ in the same (fixed) half-plane, with a fixed hyperbolic distance $d(\zeta_1,\zeta_2)$, there is a complex-conjugate orbit where the pair of roots lie in the opposite half plane. The condition $\mathrm{Im}(x(1+\bar{y}))>0$ constrain the roots to lie in one of the half planes, so picks up one of the orbits in each conjugate pair. The hyperbolic orbits are self conjugate.

5. SU_2

$\mathrm{SU}_2\simeq S^3$ is the group of 2×2 complex unitary matrices with det=1. Its Lie algebra $\mathfrak{su}_2$ consists of anti-hermitian 2×2 complex matrices with $\mathfrak{su}_2\otimes\mathbb{C}=\mathfrak{sl}_2(\mathbb{C})$. This case is easier then the previous case of $\mathrm{SL}_2(\mathbb{R})$, with no really new ideas, so we will be much briefer. The outcome is that there is a single 1-parameter family of left-invariant CR structures, exactly one of which is spherical, the standard spherical structure in S^3, realizable in $\mathbb{C}^2$. The rest of the structures are 4:1 covers of generic adjoint orbits in $\mathrm{P}(\mathfrak{g}_\mathbb{C})\simeq\mathbb{C}\mathrm{P}^2$.

Lemma 5.1. $\mathrm{Aut}(\mathrm{SU}_2)=\mathrm{Aut}(\mathfrak{su}_2)=\mathrm{Inn}(\mathrm{SU}_2)=\mathrm{SU}_2/\{\pm\mathrm{I}\}\simeq\mathrm{SO}_3$.

Proof. Similar to the $\mathrm{SL}_2(\mathbb{R})$ case, the Killing form and the triple product on $\mathfrak{su}_2$ are defined in terms of the Lie bracket alone. This gives a natural inclusion $\mathrm{Aut}(\mathrm{SU}_2)\subset\mathrm{SO}_3$. The conjugation action gives an embedding $\mathrm{Inn}(\mathrm{SU}_2)=\mathrm{SU}_2/\{\pm\mathrm{I}\}\subset\mathrm{SO}_3$. The last two groups are connected and 3-dimensional, hence coincide. □

Since $\mathrm{SU}_2\subset\mathrm{SL}_2(\mathbb{C})$, with $(\mathfrak{su}_2)_\mathbb{C}=\mathfrak{sl}_2(\mathbb{C})$, we can, like in the previous case of $G=\mathrm{SL}_2(\mathbb{R})$, identify $\mathrm{P}((\mathfrak{su}_2)_\mathbb{C})$, SU_2-equivariantly, with $S^2(\mathbb{C}\mathrm{P}^1)$, the set of unordered pairs of points on $\mathbb{C}\mathrm{P}^1=S^2$, with $\mathrm{Aut}(\mathrm{SU}_2)=\mathrm{SU}_2/\{\pm\mathrm{I}\}=\mathrm{SO}_3$ acting on $S^2(\mathbb{C}\mathrm{P}^1)$ by euclidean rotations of $\mathbb{C}\mathrm{P}^1=S^2$, and complex conjugation in $\mathrm{P}((\mathfrak{su}_2)_\mathbb{C})$ given by the antipodal map. Hence $\mathrm{P}((\mathfrak{su}_2)_\mathbb{C})$ consists of non-antipodal unordered pairs of points $\zeta_1,\zeta_2\in S^2$, each of which is given uniquely, up to $\mathrm{Aut}(\mathrm{SU}_2)=\mathrm{SO}_3$, by their spherical distance $d(\zeta_1,\zeta_2)\in[0,\pi)$.

Proposition 5.1. *Let $V_t\subset T_\mathbb{C}\mathrm{SU}_2$, $t\in\mathbb{R}$, be the left-invariant complex line bundle spanned at $e\in\mathrm{SU}_2$ by*

$$L_t=\begin{pmatrix}0 & t-1\\ t+1 & 0\end{pmatrix}\in\mathfrak{su}_2\otimes\mathbb{C}=\mathfrak{sl}_2(\mathbb{C}). \tag{5}$$

Then

(a) *V_t is a left-invariant CR structure on SU_2 for all $t\neq 0$.*
(b) *V_t is spherical if and only if $t=\pm1$.*
(c) *Every left-invariant CR structure on SU_2 is CR equivalent to V_t for a unique $t\geq 1$.*
(d) *The aspherical left-invariant CR structures V_t, $t>1$, are pairwise non-equivalent, even locally.*
(e) *V_1 is realized by any of the non-null orbits of the standard representation of SU_2 in $\mathbb{C}^2$. The aspherical structures are locally realized as $4:1$ covers of the adjoint orbits of SU_2 in $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))$.*

Proof. (a) Note that $L_t\in\mathfrak{su}_2$ only for $t=0$ and that $\mathfrak{su}_2$ does not have 2-dimensional subalgebras. It follows that $[L_t]$ is regular for all $t\neq 0$.

(b) We apply Proposition 3.2. The left-invariant $\mathfrak{su}_2$-valued Maurer Cartan form on SU_2 is

$$\Theta = g^{-1}\mathrm{d}g = \begin{pmatrix} i\alpha & \beta + i\gamma \\ -\beta + i\gamma & -i\alpha \end{pmatrix}. \tag{6}$$

The Maurer Cartan equation $\mathrm{d}\Theta = -\Theta \wedge \Theta$ gives

$$\mathrm{d}\alpha = -2\beta\wedge\gamma,\ \mathrm{d}\beta = -2\gamma\wedge\alpha,\ \mathrm{d}\gamma = -2\alpha\wedge\beta.$$

A coframe well adapted to V_t is

$$\phi = \alpha,\ \phi_1 = \sqrt{t}\beta + \frac{i}{\sqrt{t}}\gamma,$$

satisfying

$$\mathrm{d}\phi = i\phi_1\wedge\bar\phi_1, \quad \mathrm{d}\phi_1 = -i\left(\frac{1}{t}+t\right)\phi\wedge\phi_1 - i\left(\frac{1}{t}-t\right)\phi\wedge\bar\phi_1.$$

We conclude from Proposition 3.2 that V_t is spherical if and only if $\left(\frac{1}{t}+t\right)\left(\frac{1}{t}-t\right)=0$; that is, $t=\pm1$.

(c) The quadratic polynomial associated to L_t is $(t+1)\zeta^2-(t-1)$, with roots $\zeta_\pm = \pm\sqrt{(t-1)/(t+1)}$. For $t=1$ (the spherical structure) this is a double point at $\zeta=0$, and for $t>1$ these are a pair of points symmetrically situated on the real axis, in the interval $(-1,1)$. As t varies from 1 to ∞ the spherical distance $d(\zeta_+,\zeta_-)$ increases monotonically from 0 to π (see next paragraph). It follows that every pair of unordered non-antipodal pair of points on S^2 can be mapped by $\mathrm{Aut}(\mathrm{SU}_2)=\mathrm{SO}_3$ to a pair $\zeta_\pm$ for a unique $t\geq1$.

One way to see the claimed statement about $d(\zeta_+,\zeta_-)$ is to place the roots on the sphere S^2, using the inverse stereographic projection $\zeta\mapsto(2\zeta,1-|\zeta|^2)/(1+|\zeta|^2)\in\mathbb{C}\oplus\mathbb{R}$. Then $\zeta_\pm\mapsto(\pm\sin\theta,0,\cos\theta)\in\mathbb{R}^3$, where $\cos\theta=1/t$ and $\theta\in[0,\pi/2)$ for $t\in[1,\infty)$. Thus as t increases from $t=1$ to ∞ the pair of points on S^2 start from a double point at $(1,0,0)$, move in opposite directions along the meridian $y=0$ and tend towards the poles $(0,0,\pm1)$ as $t\to\infty$.

(e) Every non-null orbit of the standard action of SU_2 on $\mathbb{C}^2$ contains a point of the form $v=(\lambda,0)$, $\lambda\in\mathbb{C}^*$. Since the stabilizer of such a point is trivial and $L_1v=0$, it follows by Lemma 3.1 that $g\mapsto gv$ is a CR embedding of V_1 in $\mathbb{C}^2$. For $t>1$, we use Proposition 3.3 to realize the aspherical CR structure V_t as the SU_2-orbit of $[L_t]$ in $\mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))$. The stabilizer in SO_3 is the two element group interchanging the two roots in S^2, hence the stabilizer in SU_2 is a 4 element subgroup. □

Remark 5.1. As in the $\mathrm{SL}_2(\mathbb{R})$ case (see Remark 4.4), there is a somewhat quicker way to prove item (c). First note that $\mathrm{Aut}(\mathrm{SU}_2)=\mathrm{SO}_3$ acts transitively on the set of 2-dimensional subspaces of $\mathfrak{su}_2$, hence one can fix the contact plane D_e arbitrarily, say $D_e=\mathrm{Ker}(\alpha)=\mathrm{Span}\{B,C\}$, where A,B,C is the basis of $\mathfrak{su}_2$ dual to α,β,γ of equation (6). Then, using the subgroup $\mathrm{O}_2\subset\mathrm{SO}_3=\mathrm{Aut}(\mathrm{SU}_2)$ leaving invariant D_e, one can map any almost complex structure on D_e to $J_t: B\mapsto tC$, for a unique $t\geq1$, with associated $(0,1)$-vector $B+itC=-L_t$.

Remark 5.2. Proposition 5.1(e) gives a $4:1$ CR immersion $\mathrm{SU}_2\to\mathrm{P}(\mathfrak{sl}_2(\mathbb{C}))\simeq\mathbb{CP}^2$ of each of the aspherical left-invariant CR structures V_t, $t>1$. In fact, the proof shows that $\mathrm{SU}_2\to\mathfrak{sl}_2(\mathbb{C})\simeq\mathbb{C}^3$, $g\mapsto gL_tg^{-1}$, is a $2:1$ CR-immersion. It is still unknown, as far as we know, if one can find immersions into $\mathbb{C}^2$. However, it is known that one cannot find CR *embeddings* of the aspherical V_t into $\mathbb{C}^n$, $n\geq2$. This was first proved in [13], by showing that any function $f:\mathrm{SU}_2\to\mathbb{C}$ which is CR with respect to any of the aspherical V_t is necessarily *even*, i.e., $f(-g)=f(g)$. A simpler representation theoretic argument was later given in [2], which we proceed to sketch here (with minor notational modifications).

First, one embeds $\mu:\mathrm{SU}_2\to\mathbb{C}^2$, $g\mapsto g\binom{1}{0}$, with image $\mu(\mathrm{SU}_2)=S^3$, mapping the action of SU_2 on itself by left translations to the restriction to S^3 of the standard linear action of SU_2 on $\mathbb{C}^2$. Next, one uses the 'spherical harmonics' decomposition $L^2(S^3)=\bigoplus_{p,q\geq0}H^{p,q}$, where $H^{p,q}$ is the restriction to S^3 of the complex homogenous harmonic polynomials on $\mathbb{C}^2$ of bidegree (p,q); that is, complex polynomials $f(z_1, z_2, \bar z_1, \bar z_2)$ which are homogenous of degree p in z_1, z_2, homogenous of degree q in $\bar z_1, \bar z_2$, and satisfy $(\partial_{z_1}\partial_{\bar z_1} + \partial_{z_2}\partial_{\bar z_2})f = 0$. Each $H^{p,q}$ has dimension $p+q+1$, is SU_2-invariant and irreducible, with $-\mathrm{I} \in \mathrm{SU}_2$ acting by $(-1)^{p+q}$.

Next, one checks that $Z := \bar z_2 \partial_{z_1} - \bar z_1 \partial_{z_2}$ is an SU_2-invariant $(1,0)$-complex vector field on $\mathbb{C}^2$, tangent to S^3, mapping $H^{p,q} \to H^{p-1,q+1}$ for all $p > 0, q \geq 0$, SU_2-equivariantly. The latter is a non-zero map, hence, by Schur's Lemma, it is an *isomorphism*. Similarly, $\bar Z$ is a $(0,1)$-complex vector field on $\mathbb{C}^2$, tangent to S^3, defining an SU_2-isomorphism $H^{p,q} \to H^{p+1,q-1}$ for all $q > 0, p \geq 0$. It follows that each $H^k := \bigoplus_{p+q=k} H^{p,q}$, $k \geq 0$, is invariant under $Z, \bar Z$.

Next, one checks that $\bar Z_t := (1+t)\bar Z + (1-t)Z$, restricted to S^3, spans $\mathrm{d}\mu(V_t)$. That is, $f : S^3 \to \mathbb{C}$ is CR with respect to $\mathrm{d}\mu(V_t)$ if and only if $\bar Z_t f = 0$. By the previous paragraph, each H^k is $\bar Z_t$ invariant, hence $\bar Z_t f = 0$ implies $\bar Z_t f^k = 0$ for all $k \geq 0$, where $f^k \in H^k$ and $f = \sum f^k$. Now one uses the previous paragraph to show that for k odd and $t > 1$, $\bar Z_t$ restricted to H^k is *invertible*. It follows that $\bar Z_t f = 0$, for $t > 1$, implies that $f^k = 0$ for all k odd; that is, f is even, as claimed. $\square$

Remark 5.3. In Cartan's classification [5, p. 70], the spherical structure V_1 is item 1° of the first table. The aspherical structures appear in item 6°(L) of the second table. Note that Cartan has an error in this item (probably typographical): the equation for the SU_2-adjoint orbits, in homogenous coordinates in $\mathbb{C}\mathrm{P}^2$, should be $x_1\bar x_1 + x_2\bar x_2 + x_1\bar x_2 = \mu|x_1^2 + x_2^2 + x_3^2|$, $\mu > 1$ (as appears correctly on top of p. 67). This is a coordinate version of the equation $\mathrm{tr}(L\bar L^t) = \mu|\mathrm{tr}(L^2)|$.

6. The Heisenberg group

The Heisenberg group H is the group of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}.$$

Its Lie algebra $\mathfrak{h}$ consists of matrices of the form

$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad a, b, c \in \mathbb{R}.$$

Lemma 6.1. $\mathrm{Aut}(H) = \mathrm{Aut}(\mathfrak{h})$ *is the 6-dimensional Lie group, acting on* $\mathfrak{h}$ *by*

$$\begin{pmatrix} T & 0 \\ \mathbf{v} & \det(T) \end{pmatrix}, \quad T \in \mathrm{GL}_2(\mathbb{R}),\ \mathbf{v} \in \mathbb{R}^2 \tag{7}$$

(matrices with respect to the basis dual to a, b, c*).*

Proof. Let A, B, C be the basis of $\mathfrak{h}$ dual to a, b, c. Then

$$[A, B] = C,\ [A, C] = [B, C] = 0.$$

One can then verify by a direct calculation that the matrices in formula (7) are those preserving these commutation relations. $\square$

Remark 6.1. Here is a cleaner proof of the last Lemma (which works also for the higher dimensional Heisenberg group): the commutation relations imply that $\mathfrak{z} := \mathbb{R}C$ is the center of $\mathfrak{h}$, so any $\phi \in \mathrm{Aut}(H)$ leaves it invariant, acting on $\mathfrak{z}$ by some $\lambda \in \mathbb{R}^*$ and on $\mathfrak{h}/\mathfrak{z}$ by some $T \in \mathrm{Aut}(\mathfrak{h}/\mathfrak{z})$. The Lie bracket defines a non-zero element $\omega \in \Lambda^2((\mathfrak{h}/\mathfrak{z})^*) \otimes \mathfrak{z}$ fixed by ϕ. Now $\phi^*\omega = (\lambda/\det(T))\omega$, hence $\lambda = \det(T)$. This gives the desired form of ϕ, as in equation (7).

Proposition 6.1. *Let $V \subset T_{\mathbb{C}}H$ be the left-invariant complex line bundle spanned at $e \in H$ by*

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h} \otimes \mathbb{C}. \tag{8}$$

Then

(a) *V is the unique left-invariant CR structure on H, up to the action of* $\mathrm{Aut}(H)$.
(b) *V is spherical, CR equivalent to the complement of a point in S^3.*
(c) *V is also embeddable in $\mathbb{C}^2$ as the real quadric $\mathrm{Im}(z_1) = |z_2|^2$. In these coordinates, the group multiplication in H is given by*

$$(z_1, z_2) \cdot (w_1, w_2) = (z_1 + w_1, z_2 + w_2 + 2iz_1\bar{w_1}).$$

Proof. (a) The adjoint action is $(x, y, z)\cdot(a, b, c) = (a, b, c+bx-ay)$. This has 1-dimensional orbits, the affine lines parallel to the c axis, except the c axis itself (the center of $\mathfrak{h}$), which is pointwise fixed. The 'vertical' 2-dimensional subspaces in $\mathfrak{h}$, i.e., those containing the c axis, are subalgebras, so give degenerate CR structures. It is easy to see that any other 2-dimensional subspace can be mapped by the adjoint action to $D_e = \{c = 0\}$ and that the subgroup of $\mathrm{Aut}(H)$ preserving D_e consists of

$$\begin{pmatrix} T & 0 \\ 0 & \det(T) \end{pmatrix}, \quad T \in \mathrm{GL}_2(\mathbb{R}),$$

(written with respect to the basis of $\mathfrak{h}$ dual to a, b, c). These act transitively on the set of almost complex structures on D_e. One can thus take the almost complex structure on D_e mapping $A \mapsto B$, with associated $(0, 1)$ vector $L = A + iB$.

(b) Define a Lie algebra homomorphism $\rho' : \mathfrak{h} \to \mathrm{End}(\mathbb{C}^3)$

$$(a, b, c) \mapsto \begin{pmatrix} 0 & -b-ia & 2c \\ 0 & 0 & a+ib \\ 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

with associated complex linear representation $\rho : H \to \mathrm{GL}_3(\mathbb{C})$,

$$(x, y, z) \mapsto \begin{pmatrix} 1 & -y-ix & 2z-xy-\frac{i}{2}(x^2+y^2) \\ 0 & 1 & x+iy \\ 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

Then one can verify that ρ has the following properties:

- It preserves the pseudo-hermitian quadratic form $|Z_2|^2 - 2\mathrm{Im}(Z_1\bar{Z}_3)$ on $\mathbb{C}^3$, of signature $(2, 1)$.
- The induced H-action on $S^3 \subset \mathbb{CP}^2$ (the projectivized null cone of the pseudo-hermitian form) has 2 orbits: a fixed point $[\mathbf{e}_1] \in S^3$ and its complement.
- The H-action on $S^3 \setminus \{[\mathbf{e}_1]\}$ is free.
- $\rho'(L)\mathbf{e}_3 = 0$.

It follows, by Lemma 3.1, that $H \to S^3 \subset \mathbb{CP}^2$, $h \mapsto [\rho(h)\mathbf{e}_3]$, is a CR embedding of the CR structure V on H in S^3, whose image is the complement of $[\mathbf{e}_1]$.

(c) In the affine chart $\mathbb{C}^2 \subset \mathbb{CP}^2$, $(z_1, z_2) \mapsto [z_1 : z_2 : 1]$, the equation of $H = S^3 \setminus [\mathbf{e}_1]$ is $2\mathrm{Im}(z_1) = |z_2|^2$. After rescaling the z_1 coordinate one obtains $\mathrm{Im}(z_1) = |z_2|^2$. The claimed formula for the group product in these coordinates follows from the embedding $h \mapsto [\rho(h)\mathbf{e}_3]$ and formula (10). □

Remark 6.2. The origin of formula (9) is as follows. Consider the standard representation of $\mathrm{SU}_{2,1}$ on $\mathbb{C}^{2,1}$ and the resulting action on $S^3 \subset \mathbb{CP}^2 = \mathrm{P}(\mathbb{C}^{2,1})$. The stabilizer in $\mathrm{SU}_{2,1}$ of a point $\infty \in S^3$ is a 5-dimensional subgroup $P \subset \mathrm{SU}_{2,1}$, acting transitively on $S^3 \setminus \{\infty\}$. The stabilizer in P of a point $o \in S^3 \setminus \{\infty\}$ is a subgroup $\mathbb{C}^* \subset P$, whose conjugation action on P leaves invariant a

3-dimensional normal subgroup of P, isomorphic to our H, so that $P = H \rtimes \mathbb{C}^*$. To get formula (9), we consider the adjoint action of $\mathbb{C}^*$ on the Lie algebra $\mathfrak{p}$ of P, under which $\mathfrak{p}$ decomposes as $\mathfrak{p} = \mathfrak{h} \oplus \mathbb{C}$, as in (9). For more details, see [10, pp. 115-120].

Remark 6.3. In Cartan's classification [5, p. 70], the left-invariant spherical structure on H is item $2°$(A) of the first table.

7. The Euclidean Group

Let $\mathrm{E}_2 = \mathrm{SO}_2 \rtimes \mathbb{R}^2$ be the group of orientation preserving isometries of $\mathbb{R}^2$, equipped with the standard euclidean metric. Every element in E_2 is of the form $\mathbf{v} \mapsto R\mathbf{v} + \mathbf{w}$, for some $R \in \mathrm{SO}_2$, $\mathbf{w} \in \mathbb{R}^2$. If we embed $\mathbb{R}^2$ as the affine plane $z = 1$ in $\mathbb{R}^3$, $\mathbf{v} \mapsto (\mathbf{v}, 1)$, then E_2 is identified with the subgroup of $\mathrm{GL}_3(\mathbb{R})$ consisting of matrices in block form

$$\begin{pmatrix} R & \mathbf{w} \\ 0 & 1 \end{pmatrix}, \quad R \in \mathrm{SO}_2,\ \mathbf{w} \in \mathbb{R}^2. \tag{11}$$

Its Lie algebra $\mathfrak{e}_2$ consists of matrices of the form

$$\begin{pmatrix} 0 & -c & a \\ c & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad a, b, c \in \mathbb{R}. \tag{12}$$

Let CE_2 be the group of *similarity* transformations of $\mathbb{R}^2$ (not necessarily orientation preserving). That is, maps $\mathbb{R}^2 \to \mathbb{R}^2$ of the form $\mathbf{v} \mapsto T\mathbf{v} + \mathbf{w}$, where $\mathbf{w} \in \mathbb{R}^2$, $T \in \mathrm{CO}_2 = \mathbb{R}^* \times \mathrm{O}_2$. Then $\mathrm{E}_2 \subset \mathrm{CE}_2$ is a normal subgroup with trivial centralizer, hence there is a natural inclusion $\mathrm{CE}_2 \subset \mathrm{Aut}(\mathrm{E}_2)$.

Lemma 7.1. $\mathrm{CE}_2 = \mathrm{Aut}(\mathrm{E}_2) = \mathrm{Aut}(\mathfrak{e}_2)$.

Proof. One calculates that the inclusion $\mathrm{CE}_2 \subset \mathrm{Aut}(\mathfrak{e}_2)$ is given, with respect to the basis A, B, C of $\mathfrak{e}_2$ dual to a, b, c, by the matrices

$$(\mathbf{w}, T) \mapsto \begin{pmatrix} T & -\epsilon i \mathbf{w} \\ 0 & \epsilon \end{pmatrix}, \quad T \in \mathrm{CO}_2,\ \mathbf{w} \in \mathbb{R}^2, \tag{13}$$

where $\epsilon = \pm 1$ is the sign of $\det(T)$ and $i : (a, b) \mapsto (-b, a)$. To show that the map $\mathrm{CE}_2 \to \mathrm{Aut}(\mathfrak{e}_2)$ of equation (13) is surjective, let $\phi \in \mathrm{Aut}(\mathfrak{e}_2)$ and observe that ϕ must preserve the subspace $c = 0$, since it is the unique 2-dimensional ideal of $\mathfrak{e}_2$. Thus ϕ has the form

$$\phi = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

with respect to the basis A, B, C of $\mathfrak{e}_2$ dual to a, b, c. Next, using the commutation relations

$$[A, B] = 0,\ [A, C] = -B,\ [B, C] = A. \tag{14}$$

we get

$$a_{11} = a_{22}a_{33},\ a_{22} = a_{11}a_{33},\ a_{12} = -a_{21}a_{33},\ a_{21} = -a_{12}a_{33}.$$

From the first two equations we get $a_{11} = a_{11}(a_{33})^2$, and from the last two $a_{12} = a_{12}(a_{33})^2$. We cannot have $a_{11} = a_{12} = 0$, else $\det(\phi) = (a_{11}a_{22} - a_{12}a_{21})a_{33} = 0$. It follows that $a_{33} = \pm 1$. If $a_{33} = 1$ then we get from the above 4 equations $a_{22} = a_{11}, a_{12} = -a_{21}$, hence the top left 2×2 block of ϕ is in CO_2^+ (an orientation preserving linear similarity). If $a_{33} = -1$ then we get $a_{22} = -a_{11}, a_{12} = a_{21}$, hence the top left 2×2 block of ϕ is in CO_2^- (an orientation reversing linear similarity). These are exactly the matrices of equation (13). $\square$

Proposition 7.1. *Let $V \subset T_{\mathbb{C}}\mathrm{E}_2$ be the left-invariant line bundle whose value at $e \in \mathrm{E}_2$ is spanned by*

$$L = \begin{pmatrix} 0 & -i & 1 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in (\mathfrak{e}_2)_{\mathbb{C}}.$$

Then

(a) *Every left-invariant CR structure on* E_2 *is CR equivalent to* V *by* $\mathrm{Aut}(\mathrm{E}_2)$.
(b) V *is an aspherical left-invariant CR structure on* E_2.
(c) V *is realized in* $\mathrm{P}((\mathfrak{e}_2)_{\mathbb{C}}) = \mathbb{C}\mathrm{P}^2$ *by the adjoint orbit of* $[L]$. *This is CR equivalent to the real hypersurface* $[\mathrm{Re}(z_1)]^2 + [\mathrm{Re}(z_2)]^2 = 1$ *in* $\mathbb{C}^2$.

Proof. (a) Let A, B, C the basis of $\mathfrak{e}_2$ dual to a, b, c. Then $L = A + iC$, so $D_e = \mathrm{Span}\{A, C\} = \{b = 0\}$. The plane $c = 0$ is a subalgebra of $\mathfrak{e}_2$, so gives a degenerate CR structure. By equation (13), every other plane can be mapped by $\mathrm{Aut}(\mathrm{E}_2)$ to D_e. The subgroup of $\mathrm{Aut}(\mathrm{E}_2)$ preserving D_e acts on D_e, with respect to the basis A, C, by the matrices

$$\begin{pmatrix} r & s \\ 0 & \epsilon \end{pmatrix}, \quad r \in \mathbb{R}^*, \ s \in \mathbb{R}, \ \epsilon = \pm 1.$$

One can then show that this group acts transitively on the space of almost complex structures on D_e.

(b) Let α, β, γ be the left-invariant 1-forms on E whose value at e is a, b, c (respectively). Then

$$\Theta = \begin{pmatrix} 0 & -\gamma & \alpha \\ \gamma & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}$$

is the left-invariant Maurer-Cartan form on E, satisfying $\mathrm{d}\Theta = -\Theta \wedge \Theta$, from which we get

$$\mathrm{d}\alpha = -\beta \wedge \gamma, \ \mathrm{d}\beta = \alpha \wedge \gamma, \ \mathrm{d}\gamma = 0. \tag{15}$$

A coframe (ϕ, ϕ_1) adapted to V (i.e., $\phi(L) = \phi_1(L) = 0$, $\bar{\phi}_1(L) \neq 0$) is

$$\phi = \beta, \ \phi_1 = \frac{1}{\sqrt{2}}(\alpha + i\gamma).$$

Using equations (15), we find

$$\mathrm{d}\phi = i\phi_1 \wedge \bar{\phi}_1, \quad \mathrm{d}\phi_1 = \frac{i}{2}\phi \wedge \phi_1 - \frac{i}{2}\phi \wedge \bar{\phi}_1,$$

Thus (ϕ, ϕ_1) is well-adapted. By Proposition 3.2, the structure is aspherical.

(c) Using Proposition 3.3, this amount to showing that the stabilizer of $[L]$ in E_2 is trivial. This is a simple calculation using formula (13), with $L = A + iC$ and $T \in \mathrm{SO}_2$, $\epsilon = 1$. The E_2-orbit of $[L]$ in $\mathrm{P}((\mathfrak{e}_2)_{\mathbb{C}})$ is contained in the affine chart $c \neq 0$. Using the coordinates $z_1 = a/c, z_2 = b/c$ in this chart, the equation for the orbit is $[\mathrm{Re}(z_1)]^2 + [\mathrm{Re}(z_2)]^2 = 1$. □

Remark 7.1. In Cartan's classification [5, p. 70], the left-invariant aspherical structure on E_2 is item 3°(H) of the second table, with $m = 0$.

Appendix A. The Cartan equivalence method

We state the main result of É. Cartan's method of equivalence, as implemented for CR geometry in [5], and apply it to left-invariant CR structures on Lie groups. We follow mostly the notation and terminology of [11].

The equivalence method associates canonically to each CR 3-manifold M an H-principal bundle $B \to M$, where $H \subset \mathrm{PU}_{2,1} = \mathrm{SU}_{2,1}/\mathbb{Z}_3$ is the stabilizer of a point in $S^3 \subset \mathbb{C}\mathrm{P}^2 = \mathrm{P}(\mathbb{C}^{2,1})$ (a 5-dimensional parabolic subgroup). Furthermore, B is equipped with a certain 1-form $\Theta : TB \to \mathfrak{su}_{2,1}$, called the *Cartan connection form*, whose eight components are linearly independent at each point, defining a coframing on B (an 'e-structure'). In the special case of $M = S^3$, equipped with its standard spherical structure, B can be identified with $\mathrm{PU}_{2,1}$ and Θ with the left-invariant Maurer-Cartan form on this group. The *curvature* of Θ is the $\mathfrak{su}_{2,1}$-valued 2-form $\Omega := \mathrm{d}\Theta + \Theta\wedge\Theta$. It vanishes if and only if M is spherical and is the basic local invariant of CR geometry, much like the Riemann curvature tensor in Riemannian geometry. The construction is canonical in the sense that each CR equivalence $f : M \to M'$ lifts uniquely to a bundle map $\tilde{f} : B \to B'$, preserving the coframing, i.e., $\tilde{f}^*\Theta' = \Theta$. In fact, B is an H-reduction of the second order frame bundle of M (the 2-jets of germs of local diffeomorphisms $(\mathbb{R}^3, 0) \to M$), and $\tilde{f}$ is the restriction of the 2-jet of f to B.

More concretely, fix a pseudo-hermitian form on $\mathbb{C}^3$ of signature $(2, 1)$, $(z_1, z_2, z_3) \mapsto |z_2|^2 + i(z_3\bar{z}_1 - z_1\bar{z}_3)$, and let $\mathrm{SU}_{2,1} \subset \mathrm{SL}_3(\mathbb{C})$ be the subgroup preserving this hermitian form. A short calculation shows that its Lie algebra $\mathfrak{su}_{2,1}$ consists of matrices of the form

$$\begin{pmatrix} \frac{1}{3}(\bar{c}_2 + 2c_2) & i\bar{c}_3 & -c_4 \\ c_1 & \frac{1}{3}(\bar{c}_2 - c_2) & -c_3 \\ c & i\bar{c}_1 & -\frac{1}{3}(c_2 + 2\bar{c}_2) \end{pmatrix}, \tag{16}$$

where $c, c_4 \in \mathbb{R}$ and $c_1, c_2, c_3 \in \mathbb{C}$. Accordingly, Θ decomposes as

$$\Theta = \begin{pmatrix} \frac{1}{3}(\bar{\theta}_2 + 2\theta_2) & i\bar{\theta}_3 & -\theta_4 \\ \theta_1 & \frac{1}{3}(\bar{\theta}_2 - \theta_2) & -\theta_3 \\ \theta & i\bar{\theta}_1 & -\frac{1}{3}(\theta_2 + 2\bar{\theta}_2) \end{pmatrix}, \tag{17}$$

where θ, θ_4 are real-valued and $\theta_1, \theta_2, \theta_3$ are complex-valued 1-forms on B. Let $H \subset \mathrm{PU}_{2,1}$ be the stabilizer of $[1 : 0 : 0] \in S^3 \subset \mathbb{CP}^2$. Its Lie algebra $\mathfrak{h} \subset \mathfrak{su}_{2,1}$ is given by setting $c = c_1 = 0$ in formula (16). In the case of the spherical CR structure on S^3, where Θ is the left-invariant Maurer-Cartan form on $B = \mathrm{PU}_{2,1}$, the Maurer-Cartan equations give $\Omega = \mathrm{d}\Theta + \Theta \wedge \Theta = 0$. In general, Ω does not vanish but has a rather special form.

We summarize Cartan's main result of [5], as presented in [11]. We first give a global version, then a local one, using adapted coframes. Each has its advantage.

Theorem A.1 (Cartan's equivalence method, global version)**.** *With each CR 3-manifold M there is canonically associated an H-principal bundle $B \to M$ with Cartan connection $\Theta : TB \to \mathfrak{su}_{2,1}$, satisfying*

(a) *(H-equivariance) $R_h^*\Theta = \mathrm{Ad}_{h^{-1}}\Theta$ for all $h \in H$.*
(b) *The vertical distribution on B (the tangent spaces to the fibers of $B \to M$) is given by $\theta = \theta_1 = 0$.*
(c) *(e-structure) The eight components of Θ, namely θ, $\mathrm{Re}(\theta_1), \mathrm{Im}(\theta_1), \mathrm{Re}(\theta_2), \mathrm{Im}(\theta_2), \mathrm{Re}(\theta_3), \mathrm{Im}(\theta_3)$, θ_4, are pointwise linearly independent, defining a coframing on B.*
(d) *(The CR structure equations) There exist functions $R, S : B \to \mathbb{C}$ such that*

$$\Omega = \mathrm{d}\Theta + \Theta \wedge \Theta = \begin{pmatrix} 0 & -i\bar{R} & S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \theta \wedge \theta_1 + \begin{pmatrix} 0 & 0 & \bar{S} \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix} \theta \wedge \bar{\theta}_1.$$

Explicitly,

$$\begin{aligned} \mathrm{d}\theta &= i\theta_1 \wedge \bar{\theta}_1 - \theta \wedge (\theta_2 + \bar{\theta}_2), \\ \mathrm{d}\theta_1 &= -\theta_1 \wedge \theta_2 - \theta \wedge \theta_3, \\ \mathrm{d}\theta_2 &= 2i\,\theta_1 \wedge \bar{\theta}_3 + i\,\bar{\theta}_1 \wedge \theta_3 - \theta \wedge \theta_4, \\ \mathrm{d}\theta_3 &= -\theta_1 \wedge \theta_4 - \bar{\theta}_2 \wedge \theta_3 - R\,\theta \wedge \bar{\theta}_1, \\ \mathrm{d}\theta_4 &= i\,\theta_3 \wedge \bar{\theta}_3 - (\theta_2 + \bar{\theta}_2)\theta_4 + (S\,\theta_1 + \bar{S}\,\bar{\theta}_1) \wedge \theta. \end{aligned} \tag{18}$$

(e) *(Spherical structures) M is spherical if and only if $R \equiv 0$, in which case $S \equiv 0$ as well, hence $\Omega \equiv 0$.*

(f) *(Aspherical structures) If M is aspherical, i.e., R is non-vanishing, then $B_1 = \{R = 1\} \subset B$ is a $\mathbb{Z}_2$-principal subbundle of B. The restriction of (θ,θ_1) to B_1 defines a coframing on it.*

(g) *Any local CR diffeomorphism of CR manifolds $f : M \to M'$ lifts uniquely to an H-bundle map $\tilde{f} : B \to B'$ with $\tilde{f}^*\Theta' = \Theta$.*

Here is a reformulation of the last theorem using *adapted coframes.* Note that such coframes always exists, locally, for any CR manifold. See Definition 3.2 and the paragraph following it.

Theorem A.2 (Cartan's equivalence method, local version)**.** *Let M be a CR 3-manifold with an adapted coframe (ϕ, ϕ_1), satisfying $\mathrm{d}\phi = i\phi_1 \wedge \bar{\phi}_1 \pmod{\phi}$. Then*

(a) *There exist on M unique complex 1-forms ϕ_2, ϕ_3, a real 1-form ϕ_4 and complex functions r, s such that*

$$\begin{aligned} \mathrm{d}\phi &= i\phi_1 \wedge \bar{\phi}_1 - \phi \wedge (\phi_2 + \bar{\phi}_2), \\ \mathrm{d}\phi_1 &= -\phi_1 \wedge \phi_2 - \phi \wedge \phi_3, \\ \mathrm{d}\phi_2 &= 2i\,\phi_1 \wedge \bar{\phi}_3 + i\,\bar{\phi}_1 \wedge \phi_3 - \phi \wedge \phi_4, \\ \mathrm{d}\phi_3 &= -\phi_1 \wedge \phi_4 - \bar{\phi}_2 \wedge \phi_3 - r\,\phi \wedge \bar{\phi}_1, \\ \mathrm{d}\phi_4 &= i\,\phi_3 \wedge \bar{\phi}_3 + (s\,\phi_1 + \bar{s}\,\bar{\phi}_1) \wedge \phi. \end{aligned} \tag{19}$$

(b) *If (ϕ, ϕ_1) is well-adapted, i.e., $\mathrm{d}\phi = i\phi_1 \wedge \bar{\phi}_1$, then ϕ_2 is imaginary, $\phi_2 + \bar{\phi}_2 = 0$.*

(c) *M is spherical if and only if $r \equiv 0$, in which case $s \equiv 0$ as well.*

(d) *If M is aspherical, i.e., r is non-vanishing, then there exist on M exactly two well-adapted coframes $(\tilde{\phi}, \tilde{\phi}_1)$ for which $r = 1$ in equations (19), given by $\tilde{\phi} = |\lambda|^2\phi$, $\tilde{\phi}_1 = \lambda(\phi + \mu\phi_1)$, where λ, μ are complex functions given as follows: let L be the complex vector field of type $(0, 1)$ defined by $\theta(L) = \theta_1(L) = 0$, $\bar{\theta}_1(L) = 1$, then $\lambda = \pm(|r|^{-1/2}\bar{r})^{1/2}$, $\mu = i\,L(u)/u$ and $u = |\lambda|^2 = |r|^{1/2}$.*

(e) *The previous items are related to Theorem A.1 as follows: there exists a unique section $\sigma : M \to B$ such that $\phi = \sigma^*\theta$ and $\phi_1 = \sigma^*\theta_1$. Furthermore, $\phi_i = \sigma^*\theta_i$, $i = 2, 3, 4$, $r = R \circ \sigma$ and $s = S \circ \sigma$. If M is aspherical then B_1 is trivialized by the two sections corresponding to the two well-adapted coframes of the previous item.*

Proofs of these theorems are found in Chap. 6 and Chap. 7 of [11]. Note that the function r in equations (19), sometimes called 'the Cartan CR curvature', is a *relative invariant* of the CR structure: only its vanishing is independent of the coframe. Put differently, due to the H-equivariance of Θ, and hence of Ω, the function $R : B \to \mathbb{C}$ of Theorem A.1 varies non-trivially along any of the fibers of $B \to M$, unless it vanishes along it.

Corollary A.1. *For any connected CR 3-manifold,*

(a) *$\mathrm{Aut}_{\mathrm{CR}}(M)$ and $\mathfrak{aut}_{\mathrm{CR}}(M)$ are a Lie group and a Lie algebra (respectively) of dimension at most 8. The maximum dimension 8 is obtained if and only if M is spherical.*

(b) *If M is aspherical then $\mathrm{Aut}_{\mathrm{CR}}(M)$ and $\mathfrak{aut}_{\mathrm{CR}}(M)$ have dimension at most 3.*

(c) *$\mathrm{Aut}_{\mathrm{CR}}(S^3) = \mathrm{PU}_{2,1}$.*

(d) *If U and V are open connected subsets of S^3 and $f : U \to V$ is a CR diffeomorphism then f is the restriction to U of some element in $\mathrm{PU}_{2,1}$.*

Proof. (a) The essential observation is that any local diffeomorphism of coframed manifolds, preserving the coframing, is determined, in each connected component of its domain, by its value at a single point in it. This is a consequence of the uniqueness theorem of solutions to ODEs. It follows that the group of symmetries of a coframed connected manifold embeds in the manifold itself. This implies, by Theorem A.1 above, item (g), that $\mathrm{Aut}_{\mathrm{CR}}(M)$ embeds in B, which is 8-dimensional. The same argument applies to $\mathfrak{aut}_{\mathrm{CR}}(M)$, by restricting to open connected subsets of M. If $\dim \mathrm{Aut}_{\mathrm{CR}}(M) = 8$, then it acts with open orbits in B, hence R is locally constant. In particular, R must be constant along the fibers of $B \to M$. By the H-equivariance of Ω this can happen only if R vanish, which implies that M is spherical, by Theorem A.1, item (e).

(b) If M is aspherical then $\tilde{f}$ leaves B_1 invariant, preserving the coframing on it given by (θ, θ_1). Then, as in the previous item, $\mathrm{Aut}_{\mathrm{CR}}(M)$ embeds in B_1, hence it is of dimension at most $3 = \dim(B_1)$.

(c) As mentioned above, for $M = S^3$, $B = \mathrm{PU}_{2,1}$ and Θ is the left-invariant Maurer-Cartan form. For any $f \in \mathrm{Aut}_{\mathrm{CR}}(M)$, let $\tilde{f}(e) = g = ge \in B$. This coincides with the action of g on $\mathrm{PU}_{2,1}$ by left translations, hence $\tilde{f} = g$.

(d) This is the 'unique extension property' of Proposition 2.2. □

In general, given a well-adapted coframe ϕ, ϕ_1, it is not so simple to solve equations (19) to find the associated one-forms and the functions r, s. Fortunately, for a left-invariant CR structure on a Lie group, one can pick a left-invariant well-adapted coframe and then it is straightforward to write down explicitly the solutions in terms of ϕ, ϕ_1 and their structure constants.

Proposition A.1. *Let M be a manifold with a CR structure given by a well-adapted coframe ϕ, ϕ_1 satisfying*

$$\begin{aligned} \mathrm{d}\phi &= i\phi_1 \wedge \bar{\phi}_1, \\ \mathrm{d}\phi_1 &= a\,\phi_1 \wedge \bar{\phi}_1 + b\,\phi \wedge \phi_1 + c\,\phi \wedge \bar{\phi}_1, \end{aligned} \tag{20}$$

for some complex constants a, b, c. Then these constants satisfy

$$\bar{a}c = ab, \;\; b + \bar{b} = 0, \tag{21}$$

and equations (19) *are satisfied by $r, s, \phi_j = A_j\phi + B_j\phi_1 + C_j\bar{\phi}_1,\ j = 2, 3, 4,$ given by*

$$\begin{aligned} &A_2 = \frac{i|a|^2}{2} + \frac{3b}{4},\ B_2 = \bar{a},\ C_2 = -a, \\ &A_3 = \frac{4iab}{3},\ B_3 = \frac{i|a|^2}{2} - \frac{b}{4},\ C_3 = -c, \\ &A_4 = \frac{|a|^4}{4} + \frac{1}{16}|b|^2 + \frac{19}{12}ib|a|^2 - |c|^2,\ B_4 = \frac{2\bar{a}b}{3},\ C_4 = \frac{2a\bar{b}}{3} \\ &r = ic\left(\frac{|a|^2}{3} + \frac{3ib}{2}\right),\ s = \bar{a}\left(3|b|^2 + \frac{2i}{3}|a|^2 b\right). \end{aligned}$$

Proof. Taking exterior derivatives of equations (20) and substituting again equations (20) in the result, we obtain equations (21). The condition that ϕ_2 is imaginary and ϕ_4 is real is equivalent to $A_2 = -\overline{A}_2, C_2 = -\overline{B}_2, A_4 = \overline{A}_4, C_4 = \overline{B}_4$. Using this, substituting ϕ_2, ϕ_3, ϕ_4 into equations (19) and equating coefficients with respect to $\phi_1 \wedge \bar{\phi}_1,\ \phi \wedge \phi_1,\ \phi \wedge \bar{\phi}_1$ it is straightforward to obtain a system of algebraic equations whose solution is given by the stated formulas (we used Mathematica). □

Corollary A.2. *A locally homogeneous CR structure given by an adapted coframe satisfying equation* (20) *is spherical if and only if $c(2\,|a|^2 + 9ib) = 0$.*

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