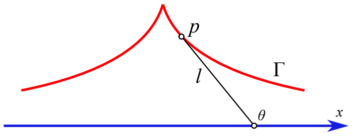
Notes no. 3

April 21, 2026

1 The tractrix

Consider an oriented curve Γ with the following property: at each pt $p \in \Gamma$, the distance to a fixed line along the positive direction of the tangent at p is a fixed number ℓ. Such a curve is called a *tractrix*.

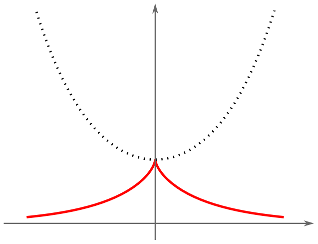


Exercise 3.1.

(a) If the fixed line is the x-axis, then the tractrix is a graph of solution of $y' = -y/\sqrt{\ell^2 - y^2}$. Solve this eqn.

(b) Let the intersection with the x-axis of the tangent at p be $(t, 0)$ and θ the angle it forms with the x-axis. Write an ODE for $\theta(t)$ and solve it. (*Ans.* $\ell\theta' = \cos\theta$.)

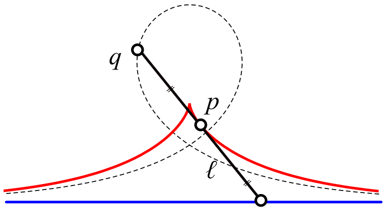
(c) Find the area under the tractrix. Can you find it without integrating one of the formulas you found in previous items? (*Ans.* $\pi\ell^2/2$).



(d) * The surface of revolution, obtained by rotating the tractrix about the fixed line, has constant negative curvature $-1/\ell^2$.

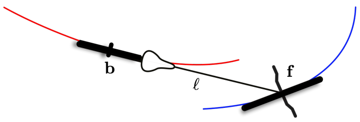
(e) * Find the evolute of the tractrix. (*Ans.* The *catenary*: $y = \ell\cosh(x/\ell)$.)

(f) ** Let q be the pt on the tangent to Γ at p, at a distance ℓ from p, in the *negative* direction. The curve traced by q, as p moves along the tractrix Γ, is called the Syntractrix, or Convict's Curve. Find an explicit formula for it and show that it is an *elastic curve*: the curvature at q is proportional to its y coordinate.



2 Bicycle tracks

A simple mathematical model for bicycle motion consists of a directed line segment in $\mathbb{R}^2$ of length ℓ (the 'bicycle frame'), with end pts $\mathbf{b}$ and $\mathbf{f}$ (the 'back' and 'front' wheels), moving so that

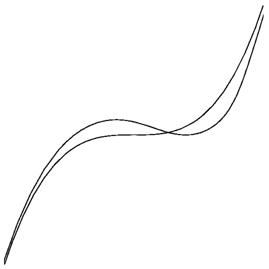


(1) its length is unchanged, $|\mathbf{f}(t) - \mathbf{b}(t)| = \ell$ for all t, and

(2) the back end moves in the direction of the frame, ie $\mathbf{b}'(t)$ is parallel $\mathbf{f}(t) - \mathbf{b}(t)$ for all t (possibly vanishes).

Condition (2) is called *the no-skid condition.* The pair of curves traced by the front and back ends of the line segment are the bicycle *front* and *back track* (resp.). The back track reduces to the tractrix when the front track is a straight line.

Exercise 3.2. Here is an example of such a pair of curves. Can you tell which is the front and which is the back track?



If the back track $\mathbf{b}(t)$ is given then clearly the front track is $\mathbf{f}(t) = \mathbf{b}(t) + \ell\mathbf{b}'(t)/|\mathbf{b}'(t)|$, provided $\mathbf{b}'(t)$ does not vanish. Suppose the front track $\mathbf{f}(t) = (x(t), y(t))$ is given (and is smooth; $\mathbf{f}'(t)$ is non-vanishing). Can we determine the back track $\mathbf{b}(t)$?

We will henceforth assume the front track is a regular curve, ie $\mathbf{f}'(t)$ does not vanish. From condition (1) we can write $\mathbf{b}(t) = \mathbf{f}(t) + \ell\mathbf{r}(t)$, for some $\mathbf{r}(t) \in S^1$, ie $|\mathbf{r}(t)| = 1$ for all t. This takes care of condition (1). How about the no-skid condition (condition (2))?

Proposition. Let $\mathbf{b}(t) = \mathbf{f}(t) + \ell\mathbf{r}(t)$, *where* $\mathbf{f}(t) = (x(t), y(t))$ *and* $\mathbf{r}(t) = (\cos\theta(t), \sin\theta(t))$. *Then the no skid condition is equivalent to*

$$\ell\theta' = x'\sin\theta - y'\cos\theta. \tag{1}$$

The last equation is the *bicycle equation.*

Let us fix two points along the front track, say $\mathbf{f}(t_0), \mathbf{f}(t_1)$. For each θ_0, solving the bicycle eqn (1) with the initial condition $\theta(0) = \theta_0$, defines a terminal condition $\theta_1 = \theta(t_1)$. This defines a map $M : S^1 \to S^1$, $e^{i\theta_0} \mapsto e^{i\theta_1}$.

Exercise 3.3. Show that M is a diffeomorphism (a smooth map with a smooth inverse). Suggestion: use the basic existence and uniqueness theorem for solutions of ODE.

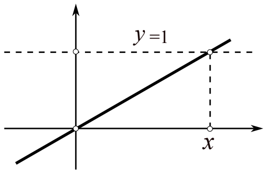
Proposition. *M is a Möbius transformation.*

A reminder on Möbius transformations. This is a class of diffeomorphisms of S^1, forming a group (ie the composition of two such maps is also a Möbius transformation, same for the inverse). They are also defined for S^n, $n > 1$, but here we are concerned only with S^1.

Consider a 2 by 2 invertible real matrix $g \in \mathrm{GL}_2(\mathbb{R})$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc \neq 0.$$

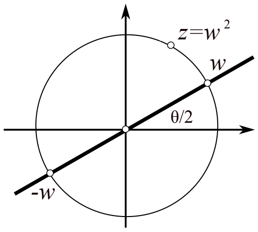
It maps each 1-dimensional subspace of $\mathbb{R}^2$ (a line through the origin) to the same kind of object, so g acts on $\mathbb{RP}^1$, the space of 1-dimensional subspaces of $\mathbb{R}^2$, the *projectivization* of the linear action on $\mathbb{R}^2$. We assign to a line through the origin in the xy plane, distinct from the x-axis, the x-coordinate of the intersection pt with the line $y = 1$.



Exercise 3.4. Show that with this coordinate on $\mathbb{RP}^1$, the projectivization of g acts by

$$x \mapsto \frac{ax+b}{cx+d}.$$

We can also identify $\mathbb{RP}^1$ with $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, by assigning to a line through the origin the point $z = w^2 \in S^1$, where $\pm w \in S^1$ are the intersection pts of the line with S^1.



Exercise 3.5. Let $z = e^{i\theta}$. Express the x coordinate on $\mathbb{RP}^1$ in terms of θ. (*Ans.* $x = \cot(\theta/2)$.)

Exercise 3.6. Write the projectivized action of $\mathrm{GL}_2(\mathbb{R})$ in the coordinate z.

Note that the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{RP}^1$ is not effective: the non-zero scalar multiples of the identity matrix act trivially on $\mathbb{RP}^1$. The quotient group is the *projective*, or *Möbius* group, $\mathrm{PGL}_2(\mathbb{R}) := \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^*\mathrm{Id}$.

Exercise 3.7.

(a) The action of $\mathrm{PGL}_2(\mathbb{R})$ on $\mathbb{RP}^1$ is triply transitive: for every two triples of *distinct* pts, $p_i, q_i \in \mathbb{RP}^1$, $i = 1, 2, 3$, there is a $g \in \mathrm{GL}_2(\mathbb{R})$ mapping the 1st triple to the 2nd.

(b) Furthermore, g is unique up to multiplication of a non-zero scalar; ie, $[g] \in \mathrm{PGL}_2(\mathbb{R})$ is unique.

(c) The $\mathrm{PGL}_2(\mathbb{R})$-action on $\mathbb{RP}^1$ preserves the *cross ratio*:

$$[x_1, x_2, x_3, x_4] := \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}.$$

(d)* A diffeomorphism of S^1 which preserves the cross ratio is a Möbius transformation.

(e) Every Mobius transformation is a composition of a *dilation*, $x \mapsto ax$, $a \neq 0$, a *translation*, $x \mapsto x + b$, and *inversion*, $x \mapsto -1/x$.

Proposition. *A non-trivial Mobius transformation* $[g] \in \mathrm{PGL}_2(\mathbb{R})$ *may have 0,1, or 2 fixed points in* S^1. *If* $\det(g) = 1$ *then there are* $0, 1,$ *or* 2 *fixed points according to* $|\mathrm{tr}(g)| < 2$, $= 2$ *or* > 2 *(resp.).*

Proof. The equation $g \cdot x = x$ is quadratic in x. Its discriminant is $\mathrm{tr}(g)^2 - 4$. $\square$

Definition. A Mobius tranformation is called elliptic, parabolic or hyperbolic according to having 0, 1 or 2 fixed points (resp.). The identity transformation is considered parabolic.

Exercise 3.8. A Mobius transformation is elliptic iff it is conjugate to a rotation, $\theta \mapsto \theta + \theta_0$, $\theta \notin 2\pi\mathbb{Z}$; it is parabolic iff it is conjugate to a translation,

$x \mapsto x + x_0$; it is hyperbolic if it is conjugate to a dilation, $x \mapsto \lambda x$, $\lambda \neq 0, 1$. The corresponding matrices are

$$\begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}.$$

Now given 4 functions $a(t), b(t), c(t), d(t)$, consider the linear system of ODE,

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \tag{2}$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \\ x_2(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a(t) & b(t) \\ \\ c(t) & d(t) \end{pmatrix}.$$

Let $g(t) \in \mathrm{GL}_2(\mathbb{R})$ be the fundamental system of solutions of (2), that is, the 1st and 2nd columns are the solutions satisfying $\mathbf{x}(0) = (1, 0)^t$ (1st column), $\mathbf{x}(0) = (0, 1)^t$ (2nd column). In other words, $g(0) = \mathrm{Id}$ and $\dot{g} = A(t)g$.

Exercise 3.9. If $\mathrm{tr}(A(t)) = a(t) + d(t) = 0$, then $\det(g(t)) = 1$.

Proposition. *Let $\mathbf{x}(t) = (x_1(t), x_2(t))^t$ be a solution of (2). Then $p(t) := x_2(t)/x_1(t)$ is a solution of*

$$\dot{p} = c + (d - a)p - bp^2. \tag{3}$$

The last type of eqn is called a *Ricatti equation.* It is the vector field induced on $\mathbb{RP}^1$ by the linear system (2).

Exercise 3.10. Write down the ODE on $\mathbb{RP}^1$ satisfied by $\theta(t)$, corresponding to the linear system (2). (Suggestion. Make the change of variable $p = \tan(\theta/2)$ in eqn (3).)

Now given a front track $\mathbf{f}(t) = (x(t), y(t))$, let

$$A(t) := -\frac{1}{2\ell} \begin{pmatrix} x(t) & y(t) \\ \\ y(t) & -x(t). \end{pmatrix}.$$

Exercise 3.11. Show that the Ricatti eqn on $\mathbb{RP}^1$ corresponding to this last $A(t)$, in the coordinate θ, is the bicycle eqn. This proves that the bicycle monodromy is a Mobius transformation.

2.1 The Menzin conjecture

We consider a closed front track, bicycle length ℓ and the resulting bicycle Monodromy $M = M(\ell)$ (a Möbius transformation). We are looking for a condition on the front track that implies hyperbolic monodromy. This is the subject of the Menzin conjecture (1906), proved in 2009 by Mark Levy and Sergei Tabachnikov.

Theorem. *If the front track is a smooth closed convex curve enclosing area* $A > \pi \ell^2$ *then the associated bicycle monodromy* M *is hyperbolic.*

The proof of this theorem can be found here.

There is a sort of converse to the Menzin conjecture. A necessary geometric condition for the monodromy to be hyperbolic.

Theorem. *If the front track is a smooth closed convex curve and the bicycle monodromy is hyperbolic then the length of the front track is at most* $2\pi\ell$.

The proof of this theorem can be found here.