Notes no. 2

April 16, 2026

1 Envelopes of curves

1.1 Fireworks

Consider the family of trajectories of objects (fireworks) which are being ejected simultaneously from a point, with fixed initial velocity, in all possible directions.

Q: What is the "envelope" of all these trajectories?

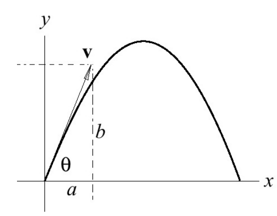
A: an inverted parabola, with a focus at the launching point.

Solution. Place the launching point of the objects at the origin of a vertical plane (the xy plane). If the initial velocity vector of a trajectory is $\mathbf{v} = (a, b)$, with $v = \|\mathbf{v}\| = \sqrt{a^2 + b^2} = const.$, then the trajectory, parametrized by time t, is given by

$$x = at, \quad y = -\frac{t^2}{2} + bt,$$

(supposing the acceleration of gravity is 1; which can be arranged by adjusting the unit of time).

Eliminating t from these two equations, and using $\tan\theta = b/a$, we get



$$y = -\frac{x^2}{2v^2}(1 + \tan^2\theta) + x(\tan\theta).$$

This is a parabola P_θ in the xy plane (its coefficients depend on the launching angle θ).

Denote by $\mathcal{E}$ the envelope of the family of parabolas $\{P_\theta\}$. Here is how one gets an equation for $\mathcal{E}$, somewhat informal (for more details one can consult for example the Vector Calculus book of Courant y John, vol. 2, Chap. 3.5).

We write 1st the equation of P_θ in the form

$$F(x, y, \theta) = y + \frac{x^2}{2v^2}(1 + \tan^2\theta) - x(\tan\theta) = 0. \tag{1}$$

Next, we parametrize the envelope $\mathcal{E}$ with the parameter θ of the curve P_θ that is tangent to $\mathcal{E}$; that is $F(x(\theta), y(\theta), \theta) = 0$. Taking derivative of the last equation of this equation wrt θ, one gets $F_x x' + F_y y' + F_\theta = 0$. Next, the velocity vector $(x'(\theta), y'(\theta))$ is *tangent* to P_θ at the point $(x(\theta), y(\theta))$, hence it is *perpendicular* to the gradient of the function defining P_θ in the point $(x(\theta), y(\theta))$; that is, $F_x x' + F_y y' = 0$. Substructing this equation from $F_x x' + F_y y' + F_\theta = 0$, we obtain $F_\theta = 0$.

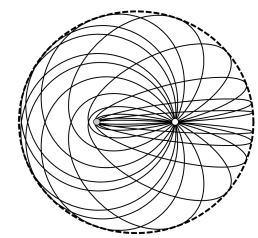
In summary, the equation for $\mathcal{E}$ is obtained by eliminating θ from the pair of equations in 3 variables

$$F(x, y, \theta) = 0, \quad F_\theta(x, y, \theta) = 0.$$

In our case, the equation $F_\theta = 0$ is $\tan(\theta) = v^2/x$. Sustituting this in $F = 0$ (equation (1)), the equation for $\mathcal{E}$ is

$$y = -\frac{x^2}{2v^2} + \frac{v^2}{2}. \tag{2}$$

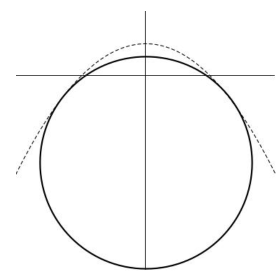
This is an equation of a parabola, with focus at the origen, whose "width" (at $y = 0$, the *latus rectum*) is $2v^2$ and whose "height" (the focal distance) is $v^2/2$ (this 1:4 ratio is for all parabolas, as they are all similar).



Note. In fact, the parabolic trajectories are an approximation to the true motion, where the acceleration due to gravity is not constant, but varies according to the distance to the center of the earth (the inverse square law). The trajectories are then *ellipses* instead of parabolas (for v not too big, $v < v_{esc} \approx 40{,}000$ kmh, the escape velocity), with one of their foci at the center of the earth (the other depend on the direction of launching). The envelope of the family of trajectories in this case turns out to be another elliptical trajectory, with one focus at the center of the earth and the other the launching point.

Excercise 2.1.[*] Prove the statement of the last sentence. (*Hint.* This family of elliptical trajectories all have the same size of major axis.)

1.2 An alternative derivation for the equation of the fireworks envelope $\mathcal{E}$ (without calculus!)



Let us imagine that all fireworks are launched at the same time $t = 0$ from the origin $(0, 0)$ in all directions, with the same initial velocity v. At some later time $t > 0$, they will all be on a curve given by

$$x = at, \quad y = -\frac{t^2}{2} + bt, \quad a^2 + b^2 = v^2.$$

Solving for a and b in the 1st two equations, then substituting in the 3rd, we obtain

$$x^2 + \left(y + \frac{t^2}{2}\right)^2 = (vt)^2. \tag{3}$$

An alternative derivation of eqn (3) is to consider a "free falling" observer. This observer senses no gravity, so at time t the particles are located on a concentric circle of radius vt centered at the origin. Going back to inertial frame, the particles at time t are located on a circle of radius vt with its canter "falling" at the same time that the radius increases. In the beginning, when $0 \leq t \leq 2v$, we have $vt - t^2/2 > 0$, so that part of the circle is above the x-axis. But later, when $t > 2v$, the circle is totally below the x-axis.

Now, the points (x, y) inside the envelope $\mathcal{E}$ are reached *twice* by the fireworks, those outside it (the "safe zone") are never reached, hence the points of $\mathcal{E}$ are the points eached exactly once. That is, those are the points (x, y) for which equation (3) has a unique solution $t \geq 0$. This is an equation quadratic in t^2, whose discriminant is $v^4 + 2v^2y - x^2$. This vanishes when $y = (v^2 - x^2/v^2)/2$, which is equation (2). □

1.3 More fun with envelopes

Excercise 2.2. Find the equation of the envelope of the family of line segments of length 1 connecting a point on the x-axis with a point of the y-axis. (Answer: the astroid $x^{2/3}+y^{2/3}=1$. See the Gallery of Curves in Notes no. 1).

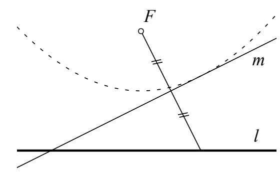
Excercise 2.3. Find the locus of midpoints of the segments of the previous exercise. But try 1st drawing this curve *before* making the calculation!

Excercise 2.4. Find the locus of the points that divide the segments of the last two exercises in some fixed proprotion.

Excercise 2.5. Find the envelope of family of ellipses with constant area, $(x/a)^2+(y/b)^2=1$, $ab=const.$

Excercise 2.6. Find the envelope of family of ellipses with constant sum of axes, $(x/a)^2+(y/b)^2=1$, $a+b=const.$

Excercise 2.7. Fix a point F and a line ℓ in the plane. Find the envelope of the perpendicular bisectors m of the line segments connecting F with ℓ.

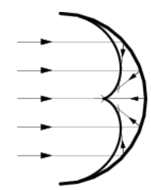


Answer. It is a parabola with focus at F and directrix ℓ. This provides a practical method for constructing a parabola with origami.

Excercise 2.8. Find a similar method to construct an ellipse.

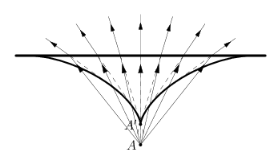
Excercise 2.9. Find the evolute of a cycloid. (*Answer.* A congruent cycloid.)

Excercise 2.10. Find the envelope of parallel rays, after reflection by a semi-circular mirror.



Answer. Half a *nephroid*, given by $64(x^2+y^2-1/4)^2=27y^2$.

Excercise 2.11.* Find the envelope of the refracted light rays ('cuastic') emanating from a point A below the x-axis, with indices of refraction $n_+ < n_-$ above and below the x-axis. In particular, find the location of the cusp A' of this envelope. This explains why pools seem shallower then they are. (Suggestion. Look up Snell's law).



2 The 4 vertex theorem

Theorem. *Every closed simple curve in the plane has at least 4 vertices (critical points of the curvature, where $\kappa'=0$). It follows that the evolute of the curve has at least 4 cusps.*

Excercise 2.12. Find an example showing that the theorem is not true without the simplicity assumption.

Here we show the theorem only for a *convex* curve (each segment connecting two points of the curve is contained in the interior of the curve).

Proof. A closed curve γ is compact, hence κ has at least two critical points, $c_{min}, c_{max} \in \gamma$. We show that in at least one of the two arcs of γ connecting c_{min} with c_{max} there is a local minimum, hence also a local maximum. If it is not true, then κ is a monotone function, along each of these two arcs, increasing or descending, depending on the sense in which the curve is parametrized. We now place the curve in the xy plane so that c_{min} and c_{max} are on the x-axis and we parametrize the curve by arc length s, $0 \leq s \leq L$. We thus have $\kappa' \geq 0$ for $y \geq 0$ and $\kappa' \leq 0$ for $y \leq 0$, so that $\kappa' y \geq 0$ along the whole curve. Integrating $\kappa' y$ around the curve we obtain

$$0 \leq \int_0^L \kappa' y = \kappa y \Big|_0^L - \int_0^L \kappa y'.$$

Now $\kappa y \Big|_0^L = 0$ (for being a closed curve) and $\kappa y' = -x''$, from the Frenet-Serret equations. Thus

$$0 \leq \int_0^L \kappa' y = \int_0^L x'' = x' \Big|_0^L = 0.$$

We conclude that $\kappa' y = 0$ identically, thus $\kappa' = 0$, since y vanishes only at c_{min} and c_{max}. $\square$

Note. This theorem has many other beautiful proofs. See for example page 81 of the book The geometry of billiards by S. Tabachnikov. Here is a sketch.

One chooses a point inside γ as the origin, then describe γ using its *support function* $p = f(\alpha)$, where α is the direction of a tangent line to the curve, and p is the distance of this line to the origin. We claim that (1) vertices of γ correspond to solutions α of the eqn $f'''(\alpha) + f'(\alpha) = 0$; (2) For any periodic smooth function f, $f''' + f'$ has at least 4 zeros in a period.

Claim (1) can be proved as follows: the support function of a circle centered at the origin is constant. When shifting the origin, the support function of a curve is modified by the addition of a 1st harmonic (linear combination of $\cos\alpha, \sin\alpha$). 1st harmonics are the solutions of $f'' + f = 0$, hence the support functions of circles (not necessarily centered at the origin) satisfy $f''' + f' = 0$. At a vertex of γ, it coincides with its osculating circle up to 3rd order, hence its support function satisfies $f''' + f' = 0$ at the contact pt as well.

Claim (2) is a special case of the *Sturm-Hurwitz theorem*: any periodic function has at least as many zeros as its 1st non trivial Fourier term (see this article for a nice elementary proof). Now writing the Fourier series of f, one sees that $f''' + f'$ starts with the 2nd harmonic (linear combination of $\cos(2\alpha), \sin(2\alpha)$). Such a function, by the Sturm-Hurwitz theorem, has at least 4 zeros, thus completing the proof of the 4v thm.

This proof generalized to other versions of the 4 vertex theorem, with 3 parameter family of curves different from circles. One such generalization is to *osculating Kepler conics*. See this article. The idea is to write the curve in "reciprocal polar coordinates" (ρ, θ), where $\rho = 1/r$. In this representation, a Kepler conic is written as $\rho = f(\theta)$,where $f(\theta) = a\cos\theta + b\sin\theta + c$, so again, "Kepler vertices" of a general curve $\rho = f(\theta)$ are given by the zeros of $f''' + f'$.

Another elegant proof of the 4 vertex theorem, not using the convexity of γ, was found by Robert Osserman in 1985. One can find it in the Wikipedia article on the 4 vertex theorem.