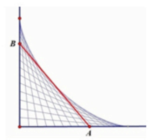
Notes no. 1

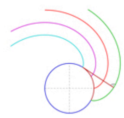
March 23, 2024

1 Curve gallery

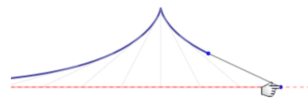
Catenary



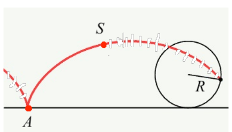
Astroid



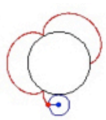
Circle involutes



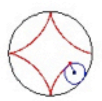
Tractrix



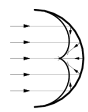
Cycloid



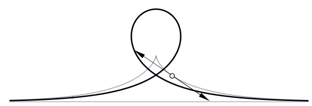
Epicycloid



Hypocycloid

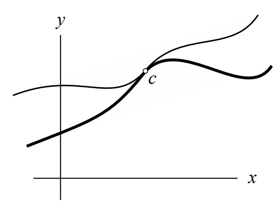


Cardioid



Syntractrix (convict's curve)

2 Order of contact, the tangent line, inflection points



Consider 2 plane curves C_1, C_2, given as the graphs of 2 functions, $y = f_1(x)$, $y = f_2(x)$ (resp.). Suppose that $c = (x_0, y_0) \in C_1 \cap C_2$, ie $f_1(x_0) = f_2(x_0)$. We say that c is a *point of contact of order* n if the 1st n derivatives of f_1 and f_2 at $x = x_0$ coincide:

$$f_1'(x_0) = f_1'(x_0), \ \ldots, \ f_1^{(n)}(x_0) = f_2^{(n)}(x_0).$$

If C is the graph of $y = f(x)$, with Taylor series at x_0 starting with

$$y = y_0 + f'(x_0)(x - x_0) + \ldots,$$

then the line $y = y_0 + f'(x_0)(x - x_0)$ has contact with C at c of order ≥ 1. This requirement determines this line, called the *tangent line* to C at c.

For example, the tangent line to the graph of $y = x^2$ at the origin $(0, 0)$ is the x-axis $(y = 0)$ and the tangent line at $(1, 1)$ is given by $y = 1 + 2(x - 1) = 2x - 1$.

Sometimes it happens that the tangent line at some point has order of contact with the curve greater then expected, ie ≥ 2. Such a point is called an *inflection point.*

Exercise 1.1. If the curve is the graph of $y = f(x)$ the point $(x_0, f(x_0))$ is an inflection point iff $f''(x_0) = 0$.

For example, the only inflection pt of the graph of $y = x^3$ is $(0, 0)$.

Exercise 1.2. Find the inflection points of $y = \sin(x)$. Make a drawing.

Exercise 1.3. You are driving a car with many curves. How do you know when you are at an inflection point? (Hint: think about the stirring wheel.)

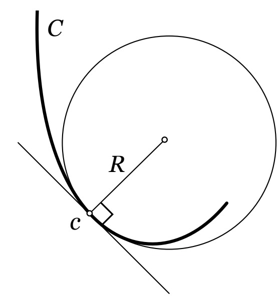
Exercise 1.4. Let $p(x) = (x-1)^{100}(x-2)^{200}$. Find the contact points of the graph of $y = p(x)$ with the x-axis and their order of contact.

Exercise 1.5.* Show that the notion of "order of contact" of an intersection point of two curves is invariant under diffeomorphism (change of coordinate system): if $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is diffeomorphism (a smooth bijection with a smooth inverse, so that $\det(D\Phi)$ does not vanish), then $c \in C_1 \cap C_2$ is a contact point of order n iff $\Phi(c)$ is a contact point of order n of $\Phi(C_1), \Phi(C_2)$.

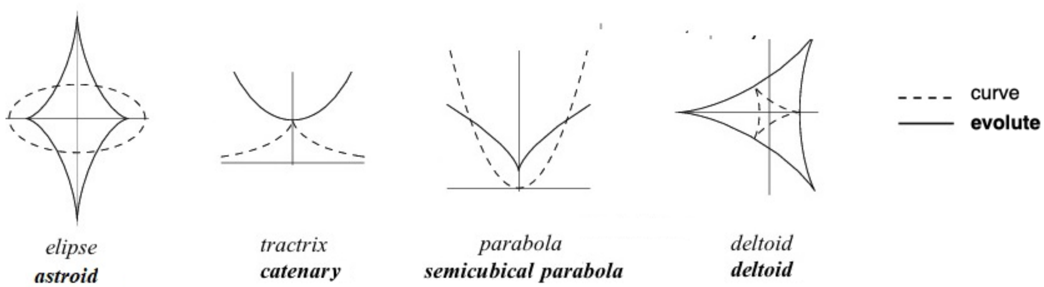
Note. Problems marked with * are less elementary (optional).

3 The osculating circle, evolute

Proposition/definition. If $c \in C$ is not an inflection point there is a unique circle with contact of order ≥ 2 with C at c, called the *osculating circle* of C at c. The radius R of the osculating circle at c is the *radius of curvature* at c, and its center is the *center of curvature.* The reciprocal $\kappa = 1/R$ is the *curvature* of C at c. At an inflection point we define $\kappa = 0$. The locus of centers of curvature is the *evolute* of C.

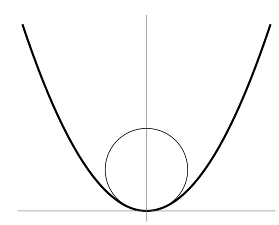


Exercise 1.6. Prove the above proposition.



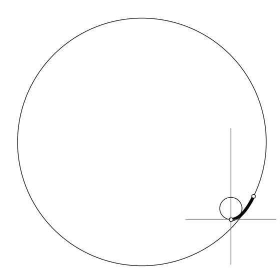
For example, the osculating circle to the parabola $y = x^2$ at $c = (0,0)$ is tangent to the x-axis at $(0,0)$, hence is given by an equation of the form

$$x^2 + (y-R)^2 = R^2.$$



To find R, we take twice the (implicit) derivative wrt x of the last equation, obtaining $1 + (y')^2 + (y-R)y'' = 0$. At $x = y = 0$ one has $y' = 0, y'' = 2$, obtaining $R = 1/2$, hence $\kappa = 2$.

Let us find the osculating circle to $y = x^2$ at $(1,1)$. This is a level curve of $F(x,y) = y - x^2$. The gradient at $(1,1)$ is $(-2,1)$. The center of the circle is thus at $(x_0, y_0) := (1,1) + R(-2,1)/\sqrt{5}$ and its equation is $(x-x_0)^2 + (y-y_0)^2 = R^2$. Differentiating twice (implicitly)



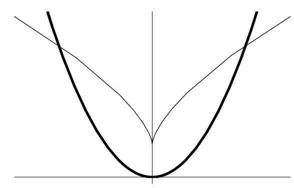
we get $1 + y'^2 + (y - y_0)y'' = 0$. Setting in this equation $y = 1, y' = y'' = 2, y_0 = 1 + R/\sqrt{5}$ and solving for R we obtain $R = 5^{3/2}/2 \approx 5.6$.

Exercise 1.7. (a) Draw the parabola $y = x^2$ and the two osculating circles at $(0,0)$ and $(1,1)$. Note that the 1st is contained in the 2nd. This is a special case of the *Tate-Kneser Theorem* (later). (b) The osculating circle to $y = x^2$ at $(0,0)$ has a contact of order 3 there (instead of the expected 2). (c) This is the only point on the curve where this happens.

Definition. A *vertex* of a curve is a point where the osculating circle has contact of order ≥ 3 with the curve.

Exercise 1.8.* A vertex is a critical point of the curvature, $\kappa' = 0$ (in the parabola example its a maximum). Note that the derivative of κ is wrt any regular parametrization of the curve, eg $x \mapsto (x, f(x))$.

Exercise 1.9. (a) If C is the graph of $y = f(x)$ and $f'(x_0) = 0$ then $\kappa = 1/R = f''(x_0)$. (b)* More generally, $\kappa = f''(x_0)/(1 + f'(x_0)^2)^{3/2}$.



Exercise 1.10. (a) Find a formula for the radius and center of the osculating circle of the parabola $y = x^2$ at the point (t, t^2). (b) Eliminating t from the formulas for the centre of the osculanting circle, find an equation for the evolute of the parabola. (*Ans.* $2(2y-1)^3 = 27x^2$.)

Proposition. *The evolute of a curve is the envelope of the family of lines perpendicular to the curve.*

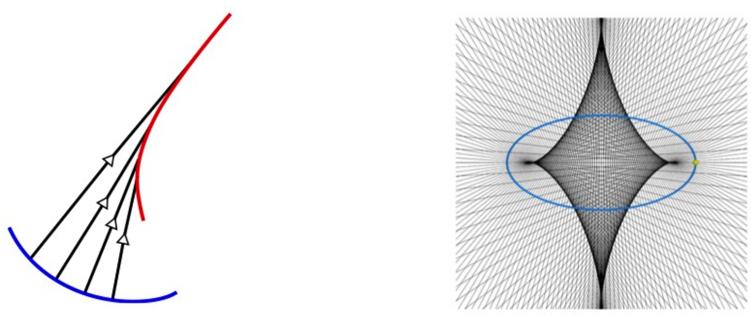


Figure 1: left: the red curve is the evolute of the blue and the blue curve is an involute of the red; the lines normal to the blue curve are tangent to the red. Right: the evolute of an ellipse (blue) as an envelope of its normal lines. (Source: *Osculating curves: around the Tait-Kneser Theorem*, de E. Ghys, S. Tabachnikov, V. Timorin.)

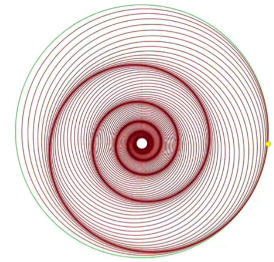
Exercise 1.11. Using the last proposition, determine again the equation of the evolute of the parabola. Determine also the equation of the evolute of the ellipse $(x/a)^2 + (y/b)^2 = 1$, where $a > b > 0$.

Suggestion. Suppose a family of curves (in our case lines) is given by $F(x, y, t) = 0$, where t is the parameter of the family. Then the envelope is given by eliminating t from the pair of equations $F(x, y, t) = F_t(x, y, t) = 0$ (the subindex t denotes partial derivative wrt t).

Exercise 1.12.* The vertices of a curve correspond to the *cusps* ('sharp points') of its evolute. That is, the points of the evolute which are not smooth.

4 The Tait-Kneser theorem, a paradox

Theorem (approx 1900). *The osculating circles along a vertex-free segment of a curve are pairwise disjoint and nested.*



That is, if the curve has a regular parametrization $C(t)$ with $R'(t) > 0$, then if $t_1 < t_2$ then the osculating circle at $C(t_1)$ is *contained and disjoint* from the osculating circle at $C(t_2)$.

In the article cited in the caption of the previous figure is an elementary proof. Another proof, new, is found in my article *Variations on the Tait-Kneser theorem* (with C. Jackman and S. Tabachnikov). Here is a sketch of this proof.

Proof (Sketch). A circle is given by an equation of the form $(x-a)^2 + (y-b)^2 = r^2$, where $(a, b) \in \mathbb{R}^2$ is the center and $r > 0$ the radius. Let $\mathbb{R}^{2,1}_+$ be the 3-dimensional vector space $\mathbb{R}^3$ with coordinates a, b, r such that $r > 0$, equipped with the *Minkowski norm* $|(a, b, r)|^2 := r^2 - a^2 - b^2$ (although this expression might be negative!).

Exercise 1.13. Show that two circles, with corresponding points $v_1, v_2 \in \mathbb{R}^{2,1}_+$, are nested iff $|v_1 - v_2|^2 \geq 0$, with equality if in addition they are tangent.

To a regularly parametrized curve $\gamma(t)$ in $\mathbb{R}^2$, without inflection points, we associate the curve $\Gamma(t)$ in $\mathbb{R}^{2,1}_+$ of its osculating circles.

Exercise 1.14. Γ is a null curve, that is $|\Gamma'(t)|^2 = 0$ for all t, and $\Gamma'(t_0) = 0$ for some t_0 iff $\gamma(t_0)$ is a vertex, ie $\kappa'(t_0) = 0$.

Exercise 1.15. Let $\Gamma(t)$, $t_1 \leq t \leq t_2$ be a segment of a regular null curve (Γ' does not vanish). Then $|\Gamma(t_2) - \Gamma(t_1)|^2 \geq 0$, with equality iff Γ is a segment of a null *line*.

To finish the proof, we note that a null line segment of Γ corresponds to a family of nested circles with variable radius tangent at a point. Such a family clearly cannot occur as a family of osculating circles of a curve in $\mathbb{R}^2$. $\square$

A paradox. Let $C(t)$ be curve with regular parametrization and without vertices. If $R' > 0$ then the osculating circles along $C(t)$, $t_1 \leq t \leq t_2$, form a "foliation" of the region bound by nested circles at $C(t_1), C(t_2)$ (an anulus). The curve traverses this region, starting at the interior boundary to the exterior one, is tangent at each of its points to the osculating circle through this point, and yet it does not coincide with any of these circles!

This is strange, for the following: let R be the function in the annulus which assigns to a point the radius of osculating circle passing through this point. Clearly, this function is constant along each of the circles. In particular its derivative at each point of the annulus in the direction tangent to the circle passing through this point is zero. But then, since C is tangent at each of its points to the osculating circle at this point, the derivative of R along C is zero as well, contradicting $R' > 0$.

The explanation of the paradox is that the foliation of the annulus by the osculating circles is not differentiable along the curve. In other words, the function R, defined in the annulus, is not differentiable (its partial derivative along the curve a direction transverse to the curve does not exist).

Exercise 1.16. (a) At each point of the graph of $y = x^3$, different from $(0, 0)$, find the *osculating vertical parabola*, that is, the parabola $y = ax^2 + bx + c$ with maximum order of contact with the curve at this point (in fact, 2). (b) Show that these parabolas are nested. (c)* Show a similar result for a general curve.