1. Expresar las siguientes raíces cuadradas en la forma $a + ib$:

a) $\sqrt{i}$ b) $\sqrt{-i}$ c) $\sqrt{1+i}$ d) $\sqrt{\dfrac{1-i\sqrt{3}}{2}}$.

a) $z^2 = i$, $z^2 - i$

↑ pol. cuad. (grado 2, con coef. compl.)

dos maneras: $<$ alg. / geom. ← ☺

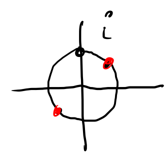
alg. $z = x+iy \Rightarrow z^2 = (x^2-y^2) + 2xy\,i = i$

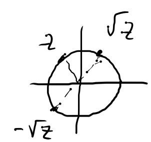
$$\Rightarrow \begin{cases} x^2 - y^2 = 0 \\ 2xy = 1 \end{cases} \qquad x, y \in \mathbb{R}.$$

$y^2 = x^2 \Rightarrow y = \pm x \Rightarrow \pm 2x^2 = 1 \Rightarrow x^2 = \pm 1/2$

$\Rightarrow x = \pm\sqrt{1/2} = \pm 1/\sqrt{2} \Rightarrow y = \pm 1/\sqrt{2}$

$\Rightarrow z = \pm(1+i)/\sqrt{2}$, $|z|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1.$





geom. $z = re^{i\theta}$, $z^2 = r^2 e^{2i\theta} = i = e^{i\pi/2} \Rightarrow r^2 = 1 \Rightarrow r = 1$

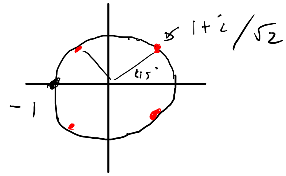
$2\theta = \pi/2 \mod 2\pi \Rightarrow \theta = \pi/4 \mod \pi$, $\theta = \pi/4, \frac{5\pi}{4}$.

$\Rightarrow z = \pm e^{i\pi/4} = \pm(1+i)/\sqrt{2}$.

2. Encontrar las raíces de los siguientes polinomios en la forma $a+ib$:

a) z^4+1 b) z^4-i c) z^3+2i d) * z^5-1.

a) $z^4=-1 \Rightarrow |z|=1$



$\Rightarrow z_1 = e^{i\pi/4} = (1+i)/\sqrt{2}$.

$z_2 = \bar{z}_1 = (1-i)/\sqrt{2}$

$z = e^{i\theta}$, $z^4 = e^{4i\theta} = e^{i\pi}$

$z^{10}=0$

$\Rightarrow 4\theta = \pi \mod 2\pi$

$\theta = \pi/4 \mod \pi/2$

$e^{i\theta_1} = e^{i\theta_2} \iff \theta_1 \equiv \theta_2 \mod 2\pi$

$= \pi/4,\ \pi/4+\pi/2,\ \pi/4+\pi,\ \pi/4+3\pi/2, \ldots$

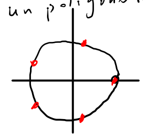
$= \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4,\ \cancel{9\pi/4 = \pi/4 \ (\mathrm{mod}\ 2\pi)}$

$w^n = 1$ (raíces n-ésimas de unidad)

$w = e^{i\theta}$, $w^n = e^{in\theta} = 1$ $\quad n\theta = 0 \pmod{2\pi}$

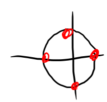
$\theta \equiv 0 \pmod{2\pi/n}$

vértices de un polígono regular

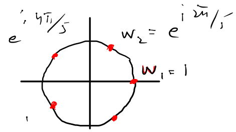


Obs: si $z^4=-1 \Rightarrow (zw)^4 = -1$ donde $w^4=-1$

O sea, la soluciones de $z^4=-1$ se obtiene de una de ellas, z_1, al multiplicarlo por la raíces $w_1, \dots, w_4$ de $w^4=1$.



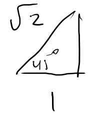
d)* $z^5-1=0$, $z^5=1$



$e^{i2\pi/5} = \cos 2\pi/5 + i \,\mathrm{sen}\, 2\pi/5$

#¿Se puede escribir $\cos 2\pi/5$ "explícitamente"?

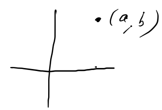
E.g. $e^{i\pi/4} = \underbrace{\cos\pi/4}_{1/\sqrt{2}} + i\,\underbrace{\mathrm{sen}\,\pi/4}_{1/\sqrt{2}}$,



Resp: Sí se puede, con raíces cuadradas,

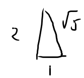
$\cos 72^\circ = \ldots + \ldots\sqrt{5}$

Pregunta relacionada: ¿cuáles polígonos regulares son construibles con regla y compás?



(a,b) construible $\iff$ a, b se puede expresar en términos $\mathbb{Q}$ y $\sqrt{\ }$.

$\sqrt{5}$ sí, se puede



$\sqrt[3]{5}$ no se puede.

$\cos \pi/7$ " " " .

$\to \sqrt{2}, \sqrt[3]{2}, \ldots$ no.

Resp parcial: pol. de n lados, $n = 2^k \cdot \underbrace{p_1 p_2 p_3 \ldots}_{\text{impares}}$

$p_1, p_2, \ldots$ son impares distintos

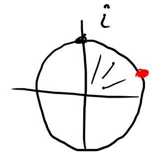
y de la forma $p = 2^{2^m}+1$ (primo de Mersenne)

Gauss.

3, 5, 17, 257, ?, no se sabe!

✓ ✓ !! !!!

2b) $z^4 = i$



$e^{i\pi/8} = a + ib$

$= \underbrace{\cos \pi/8}_{?} + i\, \text{sen}\, \pi/8$

$\cos\theta \Rightarrow \cos\theta/2$!

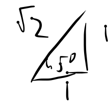
$\pi/8 = \frac{1}{2}(\pi/4)$

$\cos 2\alpha = \cos^2\alpha - \text{sen}^2\alpha = \underline{2\cos^2\alpha - 1}$

$= e^{i2\alpha} = (e^{i\alpha})^2 = (\cos\alpha + i\,\text{sen}\,\alpha)^2 = \cos^2\alpha - \text{sen}^2\alpha + 2i\cos\alpha\,\text{sen}\,\alpha$

$= \cos 2\alpha + i\,\text{sen}\,2\alpha$

$\cos\pi/4 = \cos 45^\circ = \frac{1}{\sqrt{2}}$



$\cos\theta = 2\cos^2\theta/2 - 1$

$1 + \cos\theta = 2\cos^2\theta/2$

$\cos\theta/2 = \sqrt{\frac{1}{2}(1+\cos\theta)}$, $0 < \theta < \pi/2$

$\Rightarrow \cos\pi/8 = \sqrt{\frac{1}{2}\left(1 + 1/\sqrt{2}\right)} = \ldots$ (simplificar)

$\Rightarrow \text{sen}\,\pi/8 = \sqrt{1 - \cos^2\pi/8} = \sqrt{1 - \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)} = \ldots$ (simplificar).