Chapter 3

Matrices

3.1 INTRODUCTION

Matrices were first introduced in Chapter 1 and their elements were related to the coefficients of systems of linear equations. Here we will reintroduce these matrices and we will study certain algebraic operations defined on them. The material here is mainly computational. However, as with linear equations, the abstract treatment presented later on will give us new insight into the structure of matrices.

The entries in our matrices shall come from some arbitrary, but fixed, field K. The elements of K are called scalars. Nothing essential is lost if the reader assumes that K is the real field $\mathbf{R}$ or the complex field $\mathbf{C}$.

Last, we remark that the elements of $\mathbf{R}^n$ or $\mathbf{C}^n$ are conveniently represented by "row vectors" or "column vectors," which are special cases of matrices.

3.2 MATRICES

A *matrix over a field* K (or simply a *matrix* if K is implicit) is a rectangular array of scalars a_{ij} of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The above matrix is also denoted by (a_{ij}), $i = 1, \ldots, m$, $j = 1, \ldots, n$, or simply by (a_{ij}). The m horizontal n-tuples

$$(a_{11}, a_{12}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots, (a_{m1}, a_{m2}, \ldots, a_{mn})$$

are the *rows* of the matrix, and the n vertical m-tuples

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{pmatrix}$$

are its *columns*. Note that the element a_{ij}, called the *ij-entry* or *ij-component*, appears in the ith row and the jth column. A matrix with m rows and n columns is called an m by n matrix, or $m \times n$ matrix; the pair of numbers (m, n) is called its *size* or *shape*.

Matrices will usually be denoted by capital letters $A, B, \ldots$, and the elements of the field K by lowercase letters $a, b, \ldots$. Two matrices A and B are *equal*, written $A = B$, if they have the same shape and if corresponding elements are equal. Thus the equality of two $m \times n$ matrices is equivalent to a system of mn equalities, one for each pair of elements.

Example 3.1

(*a*) The following is a 2×3 matrix: $\begin{pmatrix} 1 & -3 & 4 \\ 0 & 5 & -2 \end{pmatrix}$.

Its rows are $(1, -3, 4)$ and $(0, 5, -2)$; its columns are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$, and $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

(*b*) The statement $\begin{pmatrix} x+y & 2z+w \\ x-y & z-w \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$ is equivalent to the following system of equations:

$$\begin{cases} x + y = 3 \\ x - y = 1 \\ 2z + w = 5 \\ z - w = 4 \end{cases}$$

The solution of the system is $x = 2$, $y = 1$, $z = 3$, $w = -1$.

> **Remark:** A matrix with one row is also referred to as a *row vector*, and with one column as a *column vector*. In particular, an element in the field K can be viewed as a 1×1 matrix.

3.3 MATRIX ADDITION AND SCALAR MULTIPLICATION

Let A and B be two matrices with the same size, i.e., the same number of rows and of columns, say, $m \times n$ matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

The *sum* of A and B, written $A + B$, is the matrix obtained by adding corresponding entries:

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

The *product* of a matrix A by the scalar k, written $k \cdot A$ or simply kA, is the matrix obtained by multiplying each entry of A by k:

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix}$$

Observe that $A + B$ and kA are also $m \times n$ matrices. We also define

$$-A = -1 \cdot A \qquad \text{and} \qquad A - B = A + (-B)$$

The sum of matrices with different sizes is not defined.

Example 3.2. Let $A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{pmatrix}$. Then

$$A + B = \begin{pmatrix} 1+3 & -2+0 & 3+2 \\ 4-7 & 5+1 & -6+8 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 5 \\ -3 & 6 & 2 \end{pmatrix}$$

$$3A = \begin{pmatrix} 3 \cdot 1 & 3 \cdot (-2) & 3 \cdot 3 \\ 3 \cdot 4 & 3 \cdot 5 & 3 \cdot (-6) \end{pmatrix} = \begin{pmatrix} 3 & -6 & 9 \\ 12 & 15 & -18 \end{pmatrix}$$

$$2A - 3B = \begin{pmatrix} 2 & -4 & 6 \\ 8 & 10 & -12 \end{pmatrix} + \begin{pmatrix} -9 & 0 & -6 \\ 21 & -3 & -24 \end{pmatrix} = \begin{pmatrix} -7 & -4 & 0 \\ 29 & 7 & -36 \end{pmatrix}$$

The $m \times n$ matrix whose entries are all zero is called the zero matrix; for example, the 2×3 zero matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The zero matrix is similar to the scalar 0, and the same symbol will be used for both. For any matrix A,

$$A + 0 = 0 + A = A$$

Basic properties of matrices under the operations of matrix addition and scalar multiplication follow.

Theorem 3.1: Let V be the set of all $m \times n$ matrices over a field K. Then for any matrices $A, B, C \in V$ and any scalars $k_1, k_2 \in K$,

(i) $(A + B) + C = A + (B + C)$	(v) $k_1(A + B) = k_1 A + k_1 B$
(ii) $A + 0 = A$	(vi) $(k_1 + k_2)A = k_1 A + k_2 A$
(iii) $A + (-A) = 0$	(vii) $(k_1 k_2)A = k_1(k_2 A)$
(iv) $A + B = B + A$	(viii) $1 \cdot A = A$

Using (vi) and (viii) above, we also have that $A + A = 2A$, $A + A + A = 3A, \ldots$.

Remark: Suppose vectors in $\mathbf{R}^n$ are represented by row vectors (or by column vectors); say,

$$u = (a_1, a_2, \ldots, a_n) \quad \text{and} \quad v = (b_1, b_2, \ldots, b_n)$$

Then viewed as matrices, the sum $u + v$ and the scalar product ku are as follows:

$$u + v = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \quad \text{and} \quad ku = (ka_1, ka_2, \ldots, ka_n)$$

But this corresponds precisely to the sum and scalar product as defined in Chapter 2. In other words, the above operations on matrices may be viewed as a generalization of the corresponding operations defined in Chapter 2.

3.4 MATRIX MULTIPLICATION

The product of matrices A and B, written AB, is somewhat complicated. For this reason, we first begin with a special case.

The product $A \cdot B$ of a row matrix $A = (a_i)$ and a column matrix $B = (b_i)$ with the same number of elements is defined as follows:

$$(a_1, a_2, \ldots, a_n)\begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k$$

Note that $A \cdot B$ is a scalar (or a 1×1 matrix). The product $A \cdot B$ is not defined when A and B have different numbers of elements.

Example 3.3

$$(8, -4, 5)\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 8 \cdot 3 + (-4) \cdot 2 + 5 \cdot (-1) = 24 - 8 - 5 = 11$$

Using the above definition, we now define matrix multiplication in general.

Definition: Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are matrices such that the number of columns of A is equal to the number of rows of B; say, A is an $m \times p$ matrix and B is a $p \times n$ matrix. Then the *product* AB is the $m \times n$ matrix whose ij-entry is obtained by multiplying the ith row A_i of A by the jth column B^j of B:

$$AB = \begin{pmatrix} A_1 \cdot B^1 & A_1 \cdot B^2 & \dots & A_1 \cdot B^n \\ A_2 \cdot B^1 & A_2 \cdot B^2 & \dots & A_2 \cdot B^n \\ \dots & \dots & \dots & \dots \\ A_m \cdot B^1 & A_m \cdot B^2 & \dots & A_m \cdot B^n \end{pmatrix}$$

That is,

$$\begin{pmatrix} a_{11} & \dots & a_{1p} \\ \cdot & \dots & \cdot \\ a_{i1} & \dots & a_{ip} \\ \cdot & \dots & \cdot \\ a_{m1} & \dots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1n} \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ b_{p1} & \dots & b_{pj} & \dots & b_{pn} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \cdot & \dots & \cdot \\ \cdot & c_{ij} & \cdot \\ \cdot & \dots & \cdot \\ c_{m1} & \dots & c_{mn} \end{pmatrix}$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$.

We emphasize that the product AB is not defined if A is an $m \times p$ matrix and B is a $q \times n$ matrix, where $p \neq q$.

Example 3.4

(a) $\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} ra_1 + sb_1 & ra_2 + sb_2 & ra_3 + sb_3 \\ ta_1 + ub_1 & ta_2 + ub_2 & ta_3 + ub_3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 & 1 \cdot 1 + 2 \cdot 2 \\ 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 1 + 4 \cdot 2 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 3 & 11 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 1 \cdot 3 & 1 \cdot 2 + 1 \cdot 4 \\ 0 \cdot 1 + 2 \cdot 3 & 0 \cdot 2 + 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 8 \end{pmatrix}$

The above example shows that matrix multiplication is not commutative, i.e., the products AB and BA of matrices need not be equal.

Matrix multiplication does, however, satisfy the following properties:

Theorem 3.2:
(i) $(AB)C = A(BC)$ (associative law)
(ii) $A(B + C) = AB + AC$ (left distributive law)
(iii) $(B + C)A = BA + CA$ (right distributive law)
(iv) $k(AB) = (kA)B = A(kB)$, where k is a scalar

We assume that the sums and products in the above theorem are defined.

We remark that $0A = 0$ and $B0 = 0$ where 0 is the zero matrix.

3.5 TRANSPOSE OF A MATRIX

The *transpose* of a matrix A, denoted A^T, is the matrix obtained by writing the rows of A, in order, as columns:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

In other words, if $A = (a_{ij})$ is an $m \times n$ matrix, then $A^T = (a_{ij}^T)$ is the $n \times m$ matrix where $a_{ij}^T = a_{ji}$, for all i and j.

Note that the transpose of a row vector is a column vector and vice versa.

The transpose operation on matrices satisfies the following properties:

Theorem 3.3: (i) $(A + B)^T = A^T + B^T$ (iii) $(kA)^T = kA^T$ (k a scalar)
(ii) $(A^T)^T = A$ (iv) $(AB)^T = B^T A^T$

Observe in (iv) that the transpose of a product is the product of transposes, but in the reverse order.

3.6 MATRICES AND SYSTEMS OF LINEAR EQUATIONS

Consider again a system of m linear equations in n unknowns:

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \tag{3.1}$$

The above system is equivalent to the following matrix equation:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix} \qquad \text{or simply} \qquad AX = B$$

where $A = (a_{ij})$ is the matrix of coefficients, called the *coefficient matrix*, $X = (x_j)$ is the column of unknowns, and $B = (b_i)$ is the column of constants. The statement that they are equivalent means that every solution of the system (*3.1*) is a solution of the matrix equation, and vice versa.

The *augmented matrix* of the system (*3.1*) is the following matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

That is, the augmented matrix of the system $AX = B$ is the matrix which consists of the matrix A of coefficients followed by the column B of constants. Observe that the system (*3.1*) is completely determined by its augmented matrix.

Example 3.5. The following are, respectively, a system of linear equations and its equivalent matrix equation:

$$\begin{array}{r} 2x + 3y - 4z = 7 \\ x - 2y - 5z = 3 \end{array} \qquad \text{and} \qquad \begin{pmatrix} 2 & 3 & -4 \\ 1 & -2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

(Note that the size of the column of unknowns is not equal to the size of the column of constants.)

The augmented matrix of the system is

$$\begin{pmatrix} 2 & 3 & -4 & 7 \\ 1 & -2 & -5 & 3 \end{pmatrix}$$

In studying linear equations it is usually simpler to use the language and theory of matrices, as indicated by the following theorems.

Theorem 3.4: Suppose $u_1, u_2, \ldots, u_n$ are solutions of a homogeneous system of linear equations $AX = 0$. Then every linear combination of the u_i of the form $k_1u_1 + k_2u_2 + \cdots + k_nu_n$ where the k_i are scalars, is also a solution of $AX = 0$. Thus, in particular, every multiple ku of any solution u of $AX = 0$ is also a solution of $AX = 0$.

Proof. We are given that $Au_1 = 0, Au_2 = 0, \ldots, Au_n = 0$. Hence

$$\begin{aligned} A(ku_1 + ku_2 + \cdots + ku_n) &= k_1Au_1 + k_2Au_2 + \cdots + k_nAu_n \\ &= k_1 0 + k_2 0 + \cdots + k_n 0 = 0 \end{aligned}$$

Accordingly, $k_1u_1 + \cdots + k_nu_n$ is a solution of the homogeneous system $AX = 0$.

Theorem 3.5: The general solution of a nonhomogeneous system $AX = B$ may be obtained by adding the solution space W of the homogeneous system $AX = 0$ to a particular solution v_0 of the nonhomogeneous system $AX = B$. (That is, $v_0 + W$ is the general solution of $AX = B$.)

Proof. Let w be any solution of $AX = 0$. Then

$$A(v_0 + w) = A(v_0) + A(w) = B + 0 = B$$

That is, the sum $v_0 + w$ is a solution of $AX = B$.

On the other hand, suppose v is any solution of $AX = B$ (which may be distinct from v_0). Then

$$A(v - v_0) = Av - Av_0 = B - B = 0$$

That is, the difference $v - v_0$ is a solution of the homogeneous system $AX = 0$. But

$$v = v_0 + (v - v_0)$$

Thus any solution of $AX = B$ can be obtained by adding a solution of $AX = 0$ to the particular solution v_0 of $AX = B$.

Theorem 3.6: Suppose the field K is infinite (e.g., K is the real field $\mathbf{R}$ or the complex field $\mathbf{C}$). Then the system $AX = B$ has no solution, a unique solution, or an infinite number of solutions.

Proof. It suffices to show that if $AX = B$ has more than one solution, then it has infinitely many. Suppose u and v are distinct solutions of $AX = B$; that is, $Au = B$ and $Av = B$. Then, for any $k \in K$,

$$A[u + k(u - v)] = Au + k(Au - Av) = B + k(B - B) = B$$

In other words, for each $k \in K$, $u + k(u - v)$ is a solution of $AX = B$. Since all such solutions are distinct (Problem 3.21), $AX = B$ has an infinite number of solutions as claimed.

3.7 BLOCK MATRICES

Using a system of horizontal and vertical (dashed) lines, we can partition a matrix A into smaller matrices called *blocks* (or *cells*) of A. The matrix A is then called a *block matrix*. Clearly, a given matrix may be divided into blocks in different ways; for example,

$$\begin{pmatrix} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ 3 & 1 & 4 & 5 & 9 \end{pmatrix} = \left(\begin{array}{cc|cc|c} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hline 3 & 1 & 4 & 5 & 9 \end{array}\right) = \left(\begin{array}{ccc|cc} 1 & -2 & 0 & 1 & 3 \\ \hline 2 & 3 & 5 & 7 & -2 \\ \hline 3 & 1 & 4 & 5 & 9 \end{array}\right)$$

The convenience of the partition into blocks is that the result of operations on block matrices can be obtained by carrying out the computation with the blocks, just as if they were the actual elements of the matrices. This is illustrated below.

Suppose A is partitioned into blocks; say

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix}$$

Multiplying each block by a scalar k, multiplies each element of A by k; thus

$$kA = \begin{pmatrix} kA_{11} & kA_{12} & \dots & kA_{1n} \\ kA_{21} & kA_{22} & \dots & kA_{2n} \\ \dots & \dots & \dots & \dots \\ kA_{m1} & kA_{m2} & \dots & kA_{mn} \end{pmatrix}$$

Now suppose a matrix B is partitioned into the same number of blocks as A; say

$$B = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \dots & \dots & \dots & \dots \\ B_{m1} & B_{m2} & \dots & B_{mn} \end{pmatrix}$$

Furthermore, suppose the corresponding blocks of A and B have the same size. Adding these corresponding blocks adds the corresponding elements of A and B. Accordingly,

$$A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} & \dots & A_{1n} + B_{1n} \\ A_{21} + B_{21} & A_{22} + B_{22} & \dots & A_{2n} + B_{2n} \\ \dots & \dots & \dots & \dots \\ A_{m1} + B_{m1} & A_{m2} + B_{m2} & \dots & A_{mn} + B_{mn} \end{pmatrix}$$

The case of matrix multiplication is less obvious but still true. That is, suppose matrices U and V are partitioned into blocks as follows

$$U = \begin{pmatrix} U_{11} & U_{12} & \dots & U_{1p} \\ U_{21} & U_{22} & \dots & U_{2p} \\ \dots & \dots & \dots & \dots \\ U_{m1} & U_{m2} & \dots & U_{mp} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V_{11} & V_{12} & \dots & V_{1n} \\ V_{21} & V_{22} & \dots & V_{2n} \\ \dots & \dots & \dots & \dots \\ V_{p1} & V_{22} & \dots & V_{pn} \end{pmatrix}$$

such that the number of columns of each block U_{ik} is equal to the number of rows of each block V_{kj}. Then

$$UV = \begin{pmatrix} W_{11} & W_{12} & \dots & W_{1n} \\ W_{21} & W_{22} & \dots & W_{2n} \\ \dots & \dots & \dots & \dots \\ W_{m1} & W_{m2} & \dots & W_{mn} \end{pmatrix}$$

where

$$W_{ij} = U_{i1}V_{1j} + U_{i2}V_{2j} + \cdots + U_{ip}V_{pj}$$

The proof of the above formula for UV is straightforward, but detailed and lengthy. It is left for Problem 3.31.

Solved Problems

MATRIX ADDITION AND SCALAR MULTIPLICATION

3.1. Compute:

(*a*) $A + B$ for $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \end{pmatrix}$.

(*b*) $3A$ and $-5A$, where $A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix}$.

(*a*) Add corresponding elements:

$$A + B = \begin{pmatrix} 1+1 & 2+(-1) & 3+2 \\ 4+0 & 5+3 & 6+(-5) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 8 & 1 \end{pmatrix}$$

(*b*) Multiply each entry by the given scalar:

$$3A = \begin{pmatrix} 3 \cdot 1 & 3 \cdot (-2) & 3 \cdot 3 \\ 3 \cdot 4 & 3 \cdot 5 & 3 \cdot (-6) \end{pmatrix} = \begin{pmatrix} 3 & -6 & 9 \\ 12 & 15 & -18 \end{pmatrix}$$

$$-5A = \begin{pmatrix} -5 \cdot 1 & -5 \cdot (-2) & -5 \cdot 3 \\ -5 \cdot 4 & -5 \cdot 5 & -5 \cdot (-6) \end{pmatrix} = \begin{pmatrix} -5 & 10 & -15 \\ -20 & -25 & 30 \end{pmatrix}$$

3.2. Find $2A - 3B$, where $A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{pmatrix}$.

First perform the scalar multiplications, and then a matrix addition:

$$2A - 3B = \begin{pmatrix} 2 & -4 & 6 \\ 8 & 10 & -12 \end{pmatrix} + \begin{pmatrix} -9 & 0 & -6 \\ 21 & -3 & -24 \end{pmatrix} = \begin{pmatrix} -7 & -4 & 0 \\ 29 & 7 & -36 \end{pmatrix}$$

(Note that we multiply B by -3 and then add, rather than multiplying B by 3 and subtracting. This usually avoids errors.)

3.3. Find x, y, z, and w if $3\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 6 \\ -1 & 2w \end{pmatrix} + \begin{pmatrix} 4 & x+y \\ z+w & 3 \end{pmatrix}$.

First write each side as a single matrix:

$$\begin{pmatrix} 3x & 3y \\ 3z & 3w \end{pmatrix} = \begin{pmatrix} x+4 & x+y+6 \\ z+w-1 & 2w+3 \end{pmatrix}$$

Set corresponding entries equal to each other to obtain the system of four equations,

$$\begin{aligned} 3x &= x + 4 \\ 3y &= x + y + 6 \\ 3z &= z + w - 1 \\ 3w &= 2w + 3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2x &= 4 \\ 2y &= 6 + x \\ 2z &= w - 1 \\ w &= 3 \end{aligned}$$

The solution is: $x = 2$, $y = 4$, $z = 1$, $w = 3$.

3.4. Prove Theorem 3.1(v): Let A and B be $m \times n$ matrices and k a scalar. Then $k(A + B) = kA + kB$.

Suppose $A = (a_{ij})$ and $B = (b_{ij})$. Then $a_{ij} + b_{ij}$ is the ij-entry of $A + B$, and so $k(a_{ij} + b_{ij})$ is the ij-entry of $k(A + B)$. On the other hand, ka_{ij} and kb_{ij} are the ij-entries of kA and kB, respectively, and so $ka_{ij} + kb_{ij}$

is the ij-entry of $kA + kB$. But k, a_{ij} and b_{ij} are scalars in a field; hence

$$k(a_{ij} + b_{ij}) = ka_{ij} + kb_{ij} \qquad \text{for every } i, j$$

Thus $k(A + B) = kA + kB$, as corresponding entries are equal.

MATRIX MULTIPLICATION

3.5. Calculate: (*a*) $(3, 8, -2, 4)\begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix}$ (*b*) $(1, 8, 3, 4)(6, 1, -3, 5)$

(*a*) The product is not defined when the row matrix and column matrix have different numbers of elements.

(*b*) The product of a row matrix and a row matrix is not defined.

3.6. Let $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{pmatrix}$. Find (*a*) AB, (*b*) BA.

(*a*) Since A is 2×2 and B is 2×3, the product AB is defined and is a 2×3 matrix. To obtain the entries in the first row of AB, multiply the first row $(1, 3)$ of A by the columns $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$, and $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ of B, respectively:

$$\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + 3 \cdot 3 & 1 \cdot 0 + 3 \cdot (-2) & 1 \cdot (-4) + 3 \cdot 6 \\ & & \end{pmatrix}$$

$$= \begin{pmatrix} 2 + 9 & 0 - 6 & -4 + 18 \\ & & \end{pmatrix} = \begin{pmatrix} 11 & -6 & 14 \\ & & \end{pmatrix}$$

To obtain the entries in the second row of AB, multiply the second row $(2, -1)$ of A by the columns of B, respectively:

$$\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{pmatrix} = \begin{pmatrix} 11 & -6 & 14 \\ 4 - 3 & 0 + 2 & -8 - 6 \end{pmatrix}$$

Thus
$$AB = \begin{pmatrix} 11 & -6 & 14 \\ 1 & 2 & -14 \end{pmatrix}$$

(*b*) Note that B is 2×3 and A is 2×2. Since the inner numbers 3 and 2 are not equal, the product BA is not defined.

3.7. Given $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{pmatrix}$, find (*a*) AB, (*b*) BA.

(*a*) Since A is 3×2 and B is 2×3, the product AB is defined and is a 3×3 matrix. To obtain the first row of AB, multiply the first row of A by each column of B, respectively:

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 2 - 3 & -4 - 4 & -10 + 0 \\ & & \\ & & \end{pmatrix} = \begin{pmatrix} -1 & -8 & -10 \\ & & \\ & & \end{pmatrix}$$

To obtain the second row of AB, multiply the second row of A by each column of B, respectively:

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -8 & -10 \\ 1 + 0 & -2 + 0 & -5 + 0 \\ & & \end{pmatrix} = \begin{pmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ & & \end{pmatrix}$$

To obtain the third row of AB, multiply the third row of A by each column of B, respectively:

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ -3+12 & 6+16 & 15+0 \end{pmatrix} = \begin{pmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{pmatrix}$$

Thus
$$AB = \begin{pmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{pmatrix}$$

(b) Since B is 2×3 and A is 3×2, the product BA is defined and is a 2×2 matrix. To obtain the first row of BA, multiply the first row of B by each column of A, respectively:

$$\begin{pmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 2-2+15 & -1+0-20 \\ & \end{pmatrix} = \begin{pmatrix} 15 & -21 \\ & \end{pmatrix}$$

To obtain the second row of BA, multiply the second row of B by each column of A, respectively:

$$\begin{pmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 15 & -21 \\ 6+4+0 & -3+0+0 \end{pmatrix} = \begin{pmatrix} 15 & -21 \\ 10 & -3 \end{pmatrix}$$

Thus
$$BA = \begin{pmatrix} 15 & -21 \\ 10 & -3 \end{pmatrix}$$

Remark: Observe that in this problem both AB and BA are defined, but they are not equal; in fact they do not even have the same shape.

3.8. Find AB, where

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & 0 & 6 \\ 1 & 3 & -5 & 1 \\ 4 & 1 & -2 & 2 \end{pmatrix}$$

Since A is 2×3 and B is 3×4, the product is defined as a 2×4 matrix. Multiply the rows of A by the columns of B to obtain:

$$AB = \begin{pmatrix} 4+3-4 & -2+9-1 & 0-15+2 & 12+3-2 \\ 8-2+20 & -4-6+5 & 0+10-10 & 24-2+10 \end{pmatrix} = \begin{pmatrix} 3 & 6 & -13 & 13 \\ 26 & -5 & 0 & 32 \end{pmatrix}$$

3.9. Refer to Problem 3.8. Suppose that only the third column of the product AB were of interest. How could it be computed independently?

By the rule for matrix multiplication, the jth column of a product is equal to the first factor times the jth column vector of the second. Thus

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 0-15+2 \\ 0+10-10 \end{pmatrix} = \begin{pmatrix} -13 \\ 0 \end{pmatrix}$$

Similarly, the ith row of a product is equal to the ith row vector of the first factor times the second factor.

3.10. Let A be an $m \times n$ matrix, with $m > 1$ and $n > 1$. Assuming u and v are vectors, discuss the conditions under which (a) Au, (b) vA is defined.

(a) The product Au is defined only when u is a column vector with n components, i.e., an $n \times 1$ matrix. In such case, Au is a column vector with m components.

(*b*) The product vA is defined only when v is a row vector with m components, i.e., a $1 \times m$ matrix. In such case, vA is a row vector with n components.

3.11. Compute: (*a*) $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}(6, -4, 5)$ and (*b*) $(6, -4, 5)\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

(*a*) The first factor is 3×1 and the second factor is 1×3, so the product is defined as a 3×3 matrix:

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}(6, -4, 5) = \begin{pmatrix} (2)(6) & (2)(-4) & (2)(5) \\ (3)(6) & (3)(-4) & (3)(5) \\ (-1)(6) & (-1)(-4) & (-1)(5) \end{pmatrix} = \begin{pmatrix} 12 & -8 & 10 \\ 18 & -12 & 15 \\ -6 & 4 & -5 \end{pmatrix}$$

(*b*) The first factor is 1×3 and the second factor is 3×1, so the product is defined as a 1×1 matrix, which we frequently write as a scalar.

$$(6, -4, 5)\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = (12 - 12 - 5) = (-5) = -5$$

3.12. Prove Theorem 3.2(i): $(AB)C = A(BC)$.

Let $A = (a_{ij})$, $B = (b_{jk})$, and $C = (c_{kl})$. Furthermore, let $AB = S = (s_{ik})$ and $BC = T = (t_{jl})$. Then

$$s_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{im}b_{mk} = \sum_{j=1}^{m} a_{ij}b_{jk}$$

$$t_{jl} = b_{j1}c_{1l} + b_{j2}c_{2l} + \cdots + b_{jn}c_{nl} = \sum_{k=1}^{n} b_{jk}c_{kl}$$

Now multiplying S by C, i.e., (AB) by C, the element in the ith row and lth column of the matrix $(AB)C$ is

$$s_{i1}c_{1l} + s_{i2}c_{2l} + \cdots + s_{in}c_{nl} = \sum_{k=1}^{n} s_{ik}c_{kl} = \sum_{k=1}^{n}\sum_{j=1}^{m} (a_{ij}b_{jk})c_{kl}$$

On the other hand, multiplying A by T, i.e., A by BC, the element in the ith row and lth column of the matrix $A(BC)$ is

$$a_{i1}t_{1l} + a_{i2}t_{2l} + \cdots + a_{im}t_{ml} = \sum_{j=1}^{m} a_{ij}t_{jl} = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{ij}(b_{jk}c_{kl})$$

Since the above sums are equal, the theorem is proven.

3.13. Prove Theorem 3.2(ii): $A(B + C) = AB + AC$.

Let $A = (a_{ij})$, $B = (b_{jk})$, and $C = (c_{jk})$. Furthermore, let $D = B + C = (d_{jk})$, $E = AB = (e_{ik})$, and $F = AC = (f_{ik})$. Then

$$d_{jk} = b_{jk} + c_{jk}$$

$$e_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{im}b_{mk} = \sum_{j=1}^{m} a_{ij}b_{jk}$$

$$f_{ik} = a_{i1}c_{1k} + a_{i2}c_{2k} + \cdots + a_{im}c_{mk} = \sum_{j=1}^{m} a_{ij}c_{jk}$$

Hence the element in the ith row and kth column of the matrix $AB + AC$ is

$$e_{ik} + f_{ik} = \sum_{j=1}^{m} a_{ij}b_{jk} + \sum_{j=1}^{m} a_{ij}c_{jk} = \sum_{j=1}^{m} a_{ij}(b_{jk} + c_{jk})$$

On the other hand, the element in the ith row and kth column of the matrix $AD = A(B + C)$ is

$$a_{i1}d_{1k} + a_{i2}d_{2k} + \cdots + a_{im}d_{mk} = \sum_{j=1}^{m} a_{ij}d_{jk} = \sum_{j=1}^{m} a_{ij}(b_{jk} + c_{jk})$$

Thus $A(B + C) = AB + AC$ since the corresponding elements are equal.

TRANSPOSE

3.14. Given $A = \begin{pmatrix} 1 & 3 & 5 \\ 6 & -7 & -8 \end{pmatrix}$, find A^T and $(A^T)^T$.

Rewrite the rows of A as columns to obtain A^T, and then rewrite the rows of A^T as columns to obtain $(A^T)^T$:

$$A^T = \begin{pmatrix} 1 & 6 \\ 3 & -7 \\ 5 & -8 \end{pmatrix} \qquad (A^T)^T = \begin{pmatrix} 1 & 3 & 5 \\ 6 & -7 & -8 \end{pmatrix}$$

[As expected from Theorem 3.3(ii), $(A^T)^T = A$.]

3.15. Show that the matrices AA^T and A^TA are defined for any matrix A.

If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix. Hence AA^T is defined as an $m \times m$ matrix, and A^TA is defined as an $n \times n$ matrix.

3.16. Find AA^T and A^TA, where $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$.

Obtain A^T by rewriting the rows of A as columns:

$$A^T = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{pmatrix} \qquad \text{whence} \qquad AA^T = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 26 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 1+9 & 2-3 & 0+12 \\ 2-3 & 4+1 & 0-4 \\ 0+12 & 0-4 & 0+16 \end{pmatrix} = \begin{pmatrix} 10 & -1 & 12 \\ -1 & 5 & -4 \\ 12 & -4 & 16 \end{pmatrix}$$

3.17. Prove Theorem 3.3(iv): $(AB)^T = B^TA^T$.

If $A = (a_{ij})$ and $B = (b_{kj})$, the ij-entry of AB is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj} \tag{1}$$

Thus (1) is the ji-entry (reverse order) of $(AB)^T$.

On the other hand, column j of B becomes row j of B^T, and row i of A becomes column i of A^T. Consequently, the ji-entry of B^TA^T is

$$(b_{1j}, b_{2j}, \ldots, b_{mj})\begin{pmatrix} a_{i1} \\ a_{i2} \\ \cdots \\ a_{im} \end{pmatrix} = b_{1j}a_{i1} + b_{2j}a_{i2} + \cdots + b_{mj}a_{im}$$

Thus, $(AB)^T = B^TA^T$, since corresponding entries are equal.

BLOCK MATRICES

3.18. Compute AB using block multiplication, where

$$A = \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \\ \hline 0 & 0 & 2 \end{array}\right) \quad \text{and} \quad B = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Here $A = \begin{pmatrix} E & F \\ 0_{1\times 2} & G \end{pmatrix}$ and $B = \begin{pmatrix} R & S \\ 0_{1\times 3} & T \end{pmatrix}$, where E, F, G, R, S, and T are the given blocks. Hence

$$AB = \begin{pmatrix} ER & ES + FT \\ 0_{1\times 3} & GT \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \end{pmatrix} & \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ (0 \quad 0 \quad 0) & (2) \end{pmatrix} = \begin{pmatrix} 9 & 12 & 15 & 4 \\ 19 & 26 & 33 & 7 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

3.19. Compute CD by block multiplication, where

$$C = \left(\begin{array}{cc|ccc} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 5 & 1 & 2 \\ 0 & 0 & 3 & 4 & 1 \end{array}\right) \quad \text{and} \quad D = \left(\begin{array}{cc|cc} 3 & -2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ \hline 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & -4 & 1 \end{array}\right)$$

$$CD = \begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & 4 \end{pmatrix} & 0_{2\times 2} \\ 0_{2\times 2} & \begin{pmatrix} 5 & 1 & 2 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -4 & 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 3+4 & -2+8 \\ 9+8 & -6+16 \end{pmatrix} & 0_{2\times 2} \\ 0_{2\times 2} & \begin{pmatrix} 5+2-8 & 10-3+2 \\ 3+8-4 & 6-12+1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 7 & 6 & 0 & 0 \\ 17 & 10 & 0 & 0 \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 7 & -5 \end{pmatrix}$$

MISCELLANEOUS PROBLEMS

3.20. Show: (*a*) If A has a zero row, then AB has a zero row. (*b*) If B has a zero column, then AB has a zero column.

(*a*) Let R_i be the zero row of A, and $B^1, \ldots, B^n$ the columns of B. Then the ith row of AB is

$$(R_i \cdot B^1, R_i \cdot B^2, \ldots, R_i \cdot B^n) = (0, 0, \ldots, 0)$$

(*b*) Let C_j be the zero column of B, and $A_1, \ldots, A_m$ the rows of A. Then the jth column of AB is

$$\begin{pmatrix} A_1 \cdot C_j \\ A_2 \cdot C_j \\ \cdots \\ A_m \cdot C_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

3.21. Let u and v be distinct vectors. Show that, for distinct scalars $k \in K$, the vectors $u + k(u - v)$ are distinct.

It suffices to show that if

$$u + k_1(u - v) = u + k_2(u - v) \tag{1}$$

then $k_1 = k_2$. Suppose (1) holds. Then

$$k_1(u - v) = k_2(u - v) \qquad \text{or} \qquad (k_1 - k_2)(u - v) = 0$$

Since u and v are distinct, $u - v \neq 0$. Hence $k_1 - k_2 = 0$ and $k_1 = k_2$.

Supplementary Problems

MATRIX OPERATIONS

Problems 3.22–3.25 refer to the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 0 \\ -6 & 7 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -3 & 4 \\ 2 & 6 & -5 \end{pmatrix}$$

3.22. Find $5A - 2B$ and $2A + 3B$.

3.23. Find: (a) AB and $(AB)C$, (b) BC and $A(BC)$. [Note $(AB)C = A(BC)$.]

3.24. Find A^T, B^T, and A^TB^T. [Note $A^TB^T \neq (AB)^T$.]

3.25. Find $AA = A^2$ and AC.

3.26. Suppose $e_1 = (1, 0, 0)$, $e_2 = (0, 0, 1)$, $e_3 = (0, 0, 1)$, and $A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}$. Find e_1A, e_2A, and e_3A.

3.27. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is the ith component. Show the following:

(a) $e_iA = A_i$, the ith row of a matrix A.
(b) $Be_j^T = B^j$, the jth column of B.
(c) If $e_iA = e_iB$ for each i, then $A = B$.
(d) If $Ae_j^T = Be_j^T$ for each j, then $A = B$.

3.28. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. Find a 2×3 matrix B with distinct entries such that $AB = 0$.

3.29. Prove Theorem 3.2(iii): $(B + C)A = BA + CA$; (iv) $k(AB) = (kA)B = A(kB)$, where k is a scalar. [Parts (i) and (ii) were proven in Problems 3.12 and 3.13, respectively.]

3.30. Prove Theorem 3.3: (i) $(A + B)^T = A^T + B^T$; (ii) $(A^T)^T = A$; (iii) $(kA)^T = kA^T$, for k scalar. [Part (iv) was proven in Problem 3.17.]

3.31. Suppose $A = (A_{ik})$ and $B = (B_{kj})$ are block matrices for which AB is defined and the number of columns of each block A_{ik} is equal to the number of rows of each block B_{kj}. Show that

$$AB = (C_{ij}), \text{ where } C_{ij} = \sum_k A_{ik}B_{kj}.$$

Answers to Supplementary Problems

3.22. $\begin{pmatrix} -5 & 10 \\ 27 & -36 \end{pmatrix}, \begin{pmatrix} 17 & 4 \\ -12 & 13 \end{pmatrix}$

3.23. (*a*) $\begin{pmatrix} -7 & 14 \\ 39 & -28 \end{pmatrix}, \begin{pmatrix} 21 & 105 & -98 \\ -17 & -285 & 296 \end{pmatrix}$; (*b*) $\begin{pmatrix} 5 & -15 & 20 \\ 8 & 60 & -59 \end{pmatrix}, \begin{pmatrix} 21 & 105 & -98 \\ -17 & -285 & 296 \end{pmatrix}$

3.24. $\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}, \begin{pmatrix} 5 & -6 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 5 & 15 \\ 10 & -40 \end{pmatrix}$

3.25. $\begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}, \begin{pmatrix} 5 & 9 & -6 \\ -5 & -33 & 32 \end{pmatrix}$

3.26. (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4), the rows of A.

3.28. $\begin{pmatrix} 2 & 4 & 6 \\ -1 & -2 & -3 \end{pmatrix}$