SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS

OF

LINEAR ALGEBRA

Second Edition

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SEYMOUR LIPSCHUTZ, Ph.D.

Professor of Mathematics
Temple University

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SEYMOUR LIPSCHUTZ, who is presently on the mathematics faculty of Temple University, formerly taught at the Polytechnic Institute of Brooklyn and was a visiting professor in the Computer Science Department of Brooklyn College. He received his Ph.D. in 1960 at the Courant Institute of Mathematical Sciences of New York University. Some of his other books in the Schaum's Outline Series are *Programming with Fortran* (with Arthur Poe), *Discrete Mathematics*, and *Probability*.

Schaum's Outline of Theory and Problems of
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Preface

Linear algebra has in recent years become an essential part of the mathematical background required of mathematicians, engineers, physicists and other scientists. This requirement reflects the importance and wide applications of the subject matter.

This book is designed for use as a textbook for a formal course in linear algebra or as a supplement to all current standard texts. It aims to present an introduction to linear algebra which will be found helpful to all readers regardless of their fields of specialization. More material has been included than can be covered in most first courses. This has been done to make the book more flexible, to provide a useful book of reference, and to stimulate further interest in the subject.

Each chapter begins with clear statements of pertinent definitions, principles and theorems together with illustrative and other descriptive material. This is followed by graded sets of solved and supplementary problems. The solved problems serve to illustrate and amplify the theory, bring into sharp focus those fine points without which the student continually feels himself on unsafe ground, and provide the repetition of basic principles so vital to effective learning. Numerous proofs of theorems are included among the solved problems. The supplementary problems serve as a complete review of the material of each chapter.

The first chapter treats systems of linear equations. This provides the motivation and basic computational tools for the subsequent material. After vectors and matrices are introduced, there are chapters on vector spaces and subspaces and on inner products. This is followed by chapters covering determinants, eigenvalues and eigenvectors, and diagonalizing matrices (under similarity) and quadratic forms (under congruence). The later chapters cover abstract linear maps and their canonical forms, specifically the triangular, Jordan and rational canonical forms. The last chapter treats abstract linear maps on inner product spaces.

The main changes in the second edition have been for pedagogical reasons (form) rather than in content. Here, the notion of a matrix mapping is introduced early in the text, and inner products are introduced right after the chapter on vector spaces and subspaces. Also, algorithms for row reduction, matrix inversion computing determinants, and diagonalizing matrices and quadratic forms are presented using algorithmic notation. Furthermore, such topics as elementary matrices, LU factorization, Fourier coefficients, and various norms in $\mathbf{R}^n$ are introduced directly in the text, rather than in the problem sections. Lastly, by treating the more advanced abstract topics in the latter part of the text, we make this edition more suitable for an elementary course or for a two-semester course in linear algebra.

I wish to thank the staff of the McGraw-Hill Schaum Series, especially John Aliano, David Beckwith and Margaret Tobin, for invaluable suggestions and for their very helpful cooperation. Lastly, I want to express my gratitude to Wilhelm Magnus, my teacher, advisor and friend, who introduced me to the beauty of mathematics.

Temple University
January, 1991

SEYMOUR LIPSCHUTZ

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Chapter 1

Systems of Linear Equations

1.1 INTRODUCTION

The theory of linear equations plays an important and motivating role in the subject of linear algebra. In fact, many problems in linear algebra are equivalent to studying a system of linear equations, e.g., finding the kernel of a linear mapping and characterizing the subspace spanned by a set of vectors. Thus the techniques introduced in this chapter will be applicable to the more abstract treatment given later. On the other hand, some of the results of the abstract treatment will give us new insights into the structure of "concrete" systems of linear equations.

This chapter investigates systems of linear equations and describes in detail the Gaussian elimination algorithm which is used to find their solution. Although matrices will be studied in detail in Chapter 3, matrices, together with certain operations on them, are also introduced here, since they are closely related to systems of linear equations and their solution.

All our equations will involve specific numbers called *constants* or *scalars*. For simplicity, we assume in this chapter that all our scalars belong to the real field **R**. The solutions of our equations will also involve n-tuples $u = (k_1, k_2, \ldots, k_n)$ of real numbers called *vectors*. The set of all such n-tuples is denoted by $\mathbf{R}^n$.

We note that the results in this chapter also hold for equations over the complex field **C** or over any arbitrary field K.

1.2 LINEAR EQUATIONS, SOLUTIONS

By a *linear equation* in unknowns $x_1, x_2, \ldots, x_n$, we mean an equation that can be put in the *standard form*:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1.1}$$

where $a_1, a_2, \ldots, a_n, b$ are constants. The constant a_k is called the *coefficient* of x_k and b is called the *constant* of the equation.

A *solution* of the above linear equation is a set of values for all the unknowns, say $x_1 = k_1$, $x_2 = k_2, \ldots, x_n = k_n$, or simply an n-tuple $u = (k_1, k_2, \ldots, k_n)$ of constants, with the property that the following statement (obtained by substituting each k_i for x_i in the equation) is true:

$$a_1k_1 + a_2k_2 + \cdots + a_nk_n = b$$

This set of values is then said to *satisfy* the equation.

The set of all such solutions is called the *solution set* or *general solution* or, simply, the *solution* of the equation.

> **Remark:** The above notions implicitly assume there is an ordering of the unknowns. In order to avoid subscripts, we will usually use variables x, y, z, as ordered, to denote three unknowns, x, y, z, t, as ordered, to denote four unknowns, and x, y, z, s, t, as ordered, to denote five unknowns.

Example 1.1

(*a*) The equation $2x - 5y + 3xz = 4$ is not linear since the product xz of two unknowns is of second degree.

(*b*) The equation $x + 2y - 4z + t = 3$ is linear in the four unknowns x, y, z, t.
The 4-tuple $u = (3, 2, 1, 0)$ is a solution of the equation since

$$3 + 2(2) - 4(1) + 0 = 3 \qquad \text{or} \qquad 3 = 3$$

is a true statement. However, the 4-tuple $v = (1, 2, 4, 5)$ is not a solution of the equation since

$$1 + 2(2) - 4(4) + 5 = 3 \qquad \text{or} \qquad -6 = 3$$

is not a true statement.

Linear Equations in One Unknown

The following basic result is proved in Problem 1.5.

Theorem 1.1: Consider the linear equation $ax = b$.

(i) If $a \neq 0$, then $x = b/a$ is a unique solution of $ax = b$.

(ii) If $a = 0$, but $b \neq 0$, then $ax = b$ has no solution.

(iii) If $a = 0$ and $b = 0$, then every scalar k is a solution of $ax = b$.

Example 1.2

(*a*) Solve $4x - 1 = x + 6$.
Transpose to obtain the equation in standard form: $4x - x = 6 + 1$ or $3x = 7$. Multiply by 1/3 to obtain the unique solution $x = \frac{7}{3}$ [Theorem 1.1(i)].

(*b*) Solve $2x - 5 - x = x + 3$.
Rewrite the equation in standard form: $x - 5 = x + 3$, or $x - x = 3 + 8$, or $0x = 8$. The equation has no solution [Theorem 1.1(ii)].

(*c*) Solve $4 + x - 3 = 2x + 1 - x$.
Rewrite the equation in standard form: $x + 1 = x + 1$, or $x - x = 1 - 1$, or $0x = 0$. Every scalar k is a solution [Theorem 1.1(iii)].

Degenerate Linear Equations

A linear equation is said to be *degenerate* if it has the form

$$0x_1 + 0x_2 + \cdots + 0x_n = b$$

that is, if every coefficient is equal to zero. The solution of such an equation is as follows:

Theorem 1.2: Consider the degenerate linear equation $0x_1 + 0x_2 + \cdots + 0x_n = b$.

(i) If the constant $b \neq 0$, then the equation has no solution.

(ii) If the constant $b = 0$, then every vector $u = (k_1, k_2, \ldots, k_n)$ is a solution.

Proof. (i) Let $u = (k_1, k_2, \ldots, k_n)$ be any vector. Suppose $b \neq 0$. Substituting u in the equation we obtain:

$$0k_1 + 0k_2 + \cdots + 0k_n = b \quad \text{or} \quad 0 + 0 + \cdots + 0 = b \quad \text{or} \quad 0 = b$$

This is not a true statement since $b \neq 0$. Hence no vector u is a solution.

(ii) Suppose $b = 0$. Substituting u in the equation we obtain:

$$0k_1 + 0k_2 + \cdots + 0k_n = 0 \quad \text{or} \quad 0 + 0 + \cdots + 0 = 0 \quad \text{or} \quad 0 = 0$$

which is a true statement. Thus every vector u in $\mathbf{R}^n$ is a solution, as claimed.

Example 1.3. Describe the solution of $4y - x - 3y + 3 = 2 + x - 2x + y + 1$.

Rewrite in standard form by collecting terms and transposing:

$$y - x + 3 = y - x + 3 \quad \text{or} \quad y - x - y + x = 3 - 3 \quad \text{or} \quad 0x + 0y = 0$$

The equation is degenerate with a zero constant; thus every vector $u = (a, b)$ in $\mathbf{R}^2$ is a solution.

Nondegenerate Linear Equations, Leading Unknown

This subsection covers the solution of a single nondegenerate linear equation in one or more unknowns, say

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

By the *leading unknown* in such an equation, we mean the first unknown with a nonzero coefficient. Its position p in the equation is therefore the smallest integral value of j for which $a_j \neq 0$. In other words, x_p is the leading unknown if $a_j = 0$ for $j < p$, but $a_p \neq 0$.

Example 1.4. Consider the linear equation $5y - 2z = 3$. Here y is the leading unknown. If the unknowns are x, y, and z, then $p = 2$ is its position; but if y and z are the only unknowns, then $p = 1$.

The following theorem, proved in Problem 1.9, applies.

Theorem 1.3: Consider a nondegenerate linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ with leading unknown x_p.

(i) Any set of values for the unknowns x_j with $j \neq p$ will yield a unique solution of the equation. (The unknowns x_j are called *free variables* since one can assign any values to them.)

(ii) Every solution of the equation is obtained in (i).

(The set of all solutions is called the *general solution* of the equation.)

Example 1.5

(*a*) Find three particular solutions to the equation $2x - 4y + z = 8$.

Here x is the leading unknown. Accordingly, assign any values to the free variables y and z, and then solve for x to obtain a solution. For example:

(1) Set $y = 1$ and $z = 1$. Substitution in the equation yields

$$2x - 4(1) + 1 = 8 \quad \text{or} \quad 2x - 4 + 1 = 8 \quad \text{or} \quad 2x = 11 \quad \text{or} \quad x = \tfrac{11}{2}$$

Thus $u_1 = (\frac{11}{2}, 1, 1)$ is a solution.

(2) Set $y = 1$, $z = 0$. Substitution yields $x = 6$. Hence $u_2 = (6, 1, 0)$ is a solution.
(3) Set $y = 0$, $z = 1$. Substitution yields $x = \frac{7}{2}$. Thus $u_3 = (\frac{7}{2}, 0, 1)$ is a solution.

(*b*) The general solution of the above equation $2x - 4y + z = 8$ is obtained as follows.

First, assign arbitrary values (called *parameters*) to the free variables, say, $y = a$ and $z = b$. Then substitute in the equation to obtain

$$2x - 4a + b = 8 \qquad \text{or} \qquad 2x = 8 + 4a - b \qquad \text{or} \qquad x = 4 + 2a - \tfrac{1}{2}b$$

Thus

$$x = 4 + 2a - \tfrac{1}{2}b,\ y = a,\ z = b \qquad \text{or} \qquad u = (4 + 2a - \tfrac{1}{2}b,\ a,\ b)$$

is the general solution.

1.3 LINEAR EQUATIONS IN TWO UNKNOWNS

This section considers the special case of linear equations in two unknowns, x and y, that is, equations that can be put in the standard form

$$ax + by = c$$

where a, b, c are real numbers. (We also assume that the equation is nondegenerate, i.e., that a and b are not both zero.) Each solution of the equation is a pair of real numbers, $u = (k_1, k_2)$, which can be found by assigning an arbitrary value to x and solving for y, or vice versa.

Every solution $u = (k_1, k_2)$ of the above equation determines a point in the cartesian plane $\mathbf{R}^2$. Since a and b are not both zero, all such solutions correspond precisely to the points on a straight line (whence the name "linear equation"). This line is called the *graph* of the equation.

Example 1.6. Consider the linear equation $2x + y = 4$. We find three solutions of the equation as follows. First choose any value for either unknown, say $x = -2$. Substitute $x = -2$ into the equation to obtain

$$2(-2) + y = 4 \qquad \text{or} \qquad -4 + y = 4 \qquad \text{or} \qquad y = 8$$

Thus $x = -2$, $y = -8$ or the point $(-2, 8)$ in $\mathbf{R}^2$ is a solution. Now find the y-intercept, that is, substitute $x = 0$ in the equation to get $y = 4$; hence $(0, 4)$ on the y axis is a solution. Next find the x-intercept, that is, substitute $y = 0$ in the equation to get $x = 2$; hence $(2, 0)$ on the x axis is a solution.

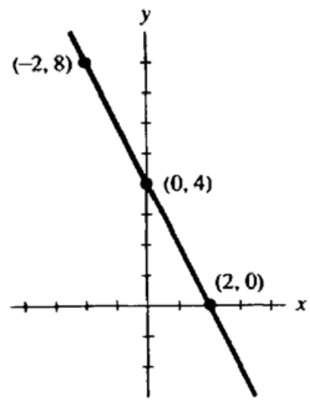
To plot the graph of the equation, first plot the three solutions, $(-2, 8)$, $(0, 4)$, and $(2, 0)$, in the plane $\mathbf{R}^2$ as pictured in Fig. 1-1. Then draw the line L determined by two of the solutions and note that the third solution also lies on L. (Indeed, L is the set of all solutions of the equation.) The line L is the graph of the equation.

System of Two Equations in Two Unknowns

This subsection considers a system of two (nondegenerate) linear equations in the two unknowns x and y:

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned} \tag{1.2}$$

(Thus a_1 and b_1 are not both zero, and a_2 and b_2 are not both zero.) This simple system is treated separately since it has a geometrical interpretation, and its properties motivate the general case.



Graph of $2x+y=4$

Fig. 1-1

A pair $u = (k_1, k_2)$ of real numbers which satisfies both equations is called a simultaneous solution of the given equations or a solution of the system of equations. There are three cases, which can be described geometrically.

(1) The system has exactly one solution. Here the graphs of the linear equations intersect in one point, as in Fig. 1-2(*a*).

(2) The system has no solutions. Here the graphs of the linear equations are parallel, as in Fig. 1-2(*b*).

(3) The system has an infinite number of solutions. Here the graphs of the linear equations coincide, as in Fig. 1-2(*c*).

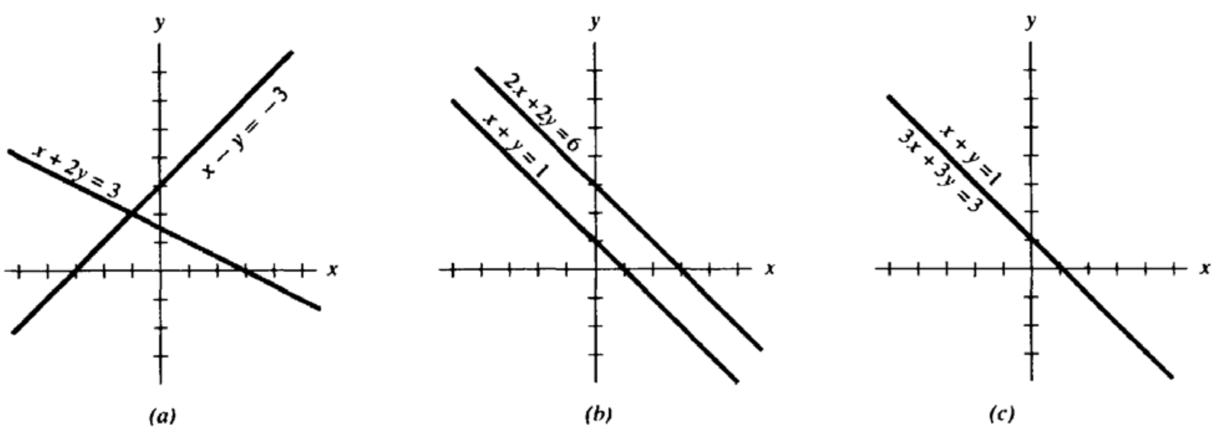


Fig. 1-2

The special cases (2) and (3) can only occur when the coefficients of x and y in the two linear equations are proportional; that is,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \quad \text{or} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = 0$$

Specifically, case (2) or (3) occurs if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \quad \text{or} \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

respectively. Unless otherwise stated or implied, we assume we are dealing with the general case (1).

Remark: The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, which has the value $a_1b_2 - a_2b_1$, is called a *determinant* of order two. Determinants will be studied in Chapter 7. Thus the system has a unique solution when the determinant of the coefficients is not zero.

Elimination Algorithm

The solution to system (*1.2*) can be obtained by the process known as elimination, whereby we reduce the system to a single equation in only one unknown. Assuming the system has a unique solution, this elimination algorithm consists of the following two steps:

Step 1. Add a multiple of one equation to the other equation (or to a nonzero multiple of the other equation) so that one of the unknowns is eliminated in the new equation.

Step 2. Solve the new equation for the given unknown, and substitute its value in one of the original equations to obtain the value of the other unknown.

Example 1.7

(*a*) Consider the system

$$\begin{aligned} L_1:&\quad 2x + 5y = 8 \\ L_2:&\quad 3x - 2y = -7 \end{aligned}$$

We eliminate x from the equations by forming the new equation $L = 3L_1 - 2L_2$; that is, by multiplying L_1 by 3 and multiplying L_2 by -2 and adding the resultant equations:

$$\begin{array}{lr} 3L_1: & 6x + 15y = 24 \\ -2L_2: & -6x + 4y = 14 \\ \hline \text{Addition:} & 19y = 38 \end{array}$$

Solving the new equation for y yields $y = 2$. Substituting $y = 2$ into one of the original equations, say L_1, yields

$$2x + 5(2) = 8 \qquad \text{or} \qquad 2x + 10 = 8 \qquad \text{or} \qquad 2x = -2 \qquad \text{or} \qquad x = -1$$

Thus $x = -1$ and $y = 2$, or the pair $(-1, 2)$, is the unique solution to the system.

(*b*) Consider the system

$$\begin{aligned} L_1:&\quad x - 3y = 4 \\ L_2:&\quad -2x + 6y = 5 \end{aligned}$$

Eliminate x from the equations by multiplying L_1 by 2 and adding it to L_2; that is, by forming the equation $L = 2L_1 + L_2$. This yields the new equation $0x + 0y = 13$. This is a degenerate equation which has a nonzero constant; therefore, the system has no solution. (Geometrically speaking, the lines are parallel.)

(*c*) Consider the system

$$\begin{aligned} L_1:&\quad x - 3y = 4 \\ L_2:&\quad -2x + 6y = -8 \end{aligned}$$

Eliminate x by multiplying L_1 by 2 and adding it to L_2. This yields the new equation $0x + 0y = 0$ which is a degenerate equation where the constant term is also zero. Hence the system has an infinite number of solutions, which correspond to the solutions of either equation. (Geometrically speaking, the lines coincide.) To find the general solution, let $y = a$ and substitute in L_1 to obtain $x - 3a = 4$ or $x = 3a + 4$. Accordingly, the general solution to the system is

$$(3a + 4, a)$$

where a is any real number.

1.4 SYSTEMS OF LINEAR EQUATIONS, EQUIVALENT SYSTEMS, ELEMENTARY OPERATIONS

This section considers a system of m linear equations, say $L_1, L_2, \ldots, L_m$, in n unknowns $x_1, x_2, \ldots, x_n$ which can be put in the *standard form*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1.3}$$

where the a_{ij}, b_i are constants.

A *solution* (or a *particular solution*) of the above system is a set of values for the unknowns, say $x_1 = k_1, x_2 = k_2, \ldots, x_n = k_n$, or an n-tuple $u = (k_1, k_2, \ldots, k_n)$ of constants, which is a solution of each of the equations of the system. The set of all such solutions is called the *solution set* or the *general solution* of the system.

Example 1.8. Consider the system

$$\begin{aligned} x_1 + 2x_2 - 5x_3 + 4x_4 &= 3 \\ 2x_1 + 3x_2 + x_3 - 2x_4 &= 1 \end{aligned}$$

Determine whether $x_1 = -8, x_2 = 4, x_3 = 1, x_4 = 2$ is a solution of the system.

Substitute in each equation to obtain

(1) $-8 + 2(4) - 5(1) + 4(2) = 3$ or $-8 + 8 - 5 + 8 = 3$ or $3 = 3$

(2) $2(-8) + 3(4) + 1 - 2(2) = 1$ or $-16 + 12 + 1 - 4 = 1$ or $-7 = 3$

No, it is not a solution since it is not a solution of the second equation.

Equivalent Systems, Elementary Operations

Systems of linear equations in the same unknowns are said to be *equivalent* if the systems have the same solution set. One way of producing a system which is equivalent to a given system, with linear equations $L_1, L_2, \ldots, L_m$, is by applying a sequence of the following operations called *elementary operations*:

[E_1] Interchange the ith equation and the jth equation: $L_i \leftrightarrow L_j$.

[E_2] Multiply the ith equation by a nonzero scalar k: $kL_i \to L_i$, $k \neq 0$.

[E_3] Replace the ith equation by k times the jth equation plus the ith equation: $(kL_j + L_i) \to L_i$.

In actual practice, we apply [E_2] and then [E_3] in one step, that is, the operation

[E] Replace the ith equation by k' times the jth equation plus k (nonzero) times the ith equation:

$$(k'L_j + kL_i) \to L_i, \; k \neq 0.$$

The above is formally stated in the following theorem proved in Problem 1.46.

Theorem 1.4: Suppose a system (#) of linear equations is obtained from a system (*) of linear equations by a finite sequence of elementary operations. Then (#) and (*) have the same solution set.

Our method for solving the system (*1.3*) of linear equations consists of two steps:

Step 1. Use the above elementary operations to reduce the system to an equivalent simpler system (in triangular or echelon form).

Step 2. Use back-substitution to find the solution of the simpler system.

The two steps are illustrated in Example 1.9. However, for pedagogical reasons, we first discuss Step 2 in detail in Section 1.5 and then we discuss Step 1 in detail in Section 1.6.

Example 1.9. The solution of the system

$$\begin{aligned} x + 2y - 4z &= -4 \\ 5x + 11y - 21z &= -22 \\ 3x - 2y + 3z &= 11 \end{aligned}$$

is obtained as follows:

Step 1. First we eliminate x from the second equation by the elementary operation $(-5L_1 + L_2) \to L_2$, that is, by multiplying L_1 by -5 and adding it to L_2; and then we eliminate x from the third equation by applying the elementary operation $(-3L_1 + L_3) \to L_3$, i.e., by multiplying L_1 by -3 and adding it to L_3:

$$\begin{array}{rr} -5 \times L_1: & -5x - 10y + 20z = 20 \\ L_2: & 5x + 11y - 21z = -22 \\ \hline \text{new } L_2: & y - z = -2 \end{array} \qquad \begin{array}{rr} -3 \times L_1: & -3x - 6y + 12z = 12 \\ L_3: & 3x - 2y + 3z = 11 \\ \hline \text{new } L_3: & -8y + 15z = 23 \end{array}$$

Thus the original system is equivalent to the system

$$\begin{aligned} x + 2y - 4z &= -4 \\ y - z &= -2 \\ -8y + 15z &= 23 \end{aligned}$$

Next we eliminate y from the third equation by applying $(8L_2 + L_3) \to L_3$ that is, by multiplying L_2 by 8 and adding it to L_3:

$$\begin{array}{rr} 8 \times L_2: & 8y - 8z = -16 \\ L_3: & -8y + 15z = 23 \\ \hline \text{new } L_3: & 7z = 7 \end{array}$$

Thus we obtain the following equivalent triangular system:

$$\begin{aligned} x + 2y - 4z &= -4 \\ y - z &= -2 \\ 7z &= 7 \end{aligned}$$

Step 2. Now we solve the simpler triangular system by back-substitution. The third equation gives $z = 1$. Substitute $z = 1$ into the second equation to obtain

$$y - 1 = -2 \quad \text{or} \quad y = -1$$

Now substitute $z = 1$ and $y = -1$ into the first equation to obtain

$$x + 2(-1) - 4(1) = -4 \quad \text{or} \quad x - 2 - 4 = -4 \quad \text{or} \quad x - 6 = -4 \quad \text{or} \quad x = 2$$

Thus $x = 2$, $y = -1$, $z = 1$, or, in other words, the ordered triple $(2, -1, 1)$, is the unique solution to the given system.

The above two-step algorithm for solving a system of linear equations is called *Gaussian elimination.* The following theorem will be used in Step 1 of the algorithm.

Theorem 1.5: Suppose a system of linear equations contains the degenerate equation

$$L: \quad 0x_1 + 0x_2 + \cdots + 0x_n = b$$

(*a*) If $b = 0$, then L may be deleted from the system without changing the solution set.

(*b*) If $b \neq 0$, then the system has no solution.

Proof. The proof follows directly from Theorem 1.2; that is, part (*a*) follows from the fact that every vector in $\mathbf{R}^n$ is a solution to L, and part (*b*) follows from the fact that L has no solution and hence the system has no solution.

1.5 SYSTEMS IN TRIANGULAR AND ECHELON FORM

This section considers two simple types of systems of linear equations: systems in triangular form and the more general systems in echelon form.

Triangular Form

A system of linear equations is in *triangular form* if the number of equations is equal to the number of unknowns and if x_k is the leading unknown of the kth equation. Thus a triangular system of linear equations has the following form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1,n-1}x_{n-1} + a_{1n}x_n &= b_1 \\ a_{22}x_2 + \cdots + a_{2,n-1}x_{n-1} + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n &= b_{n-1} \\ a_{nn}x_n &= b_n \end{aligned} \tag{1.4}$$

where $a_{11} \neq 0, a_{22} \neq 0, \ldots, a_{nn} \neq 0$.

The above triangular system of linear equations has a unique solution which may be obtained by the following process, known as *back-substitution*. First, we solve the last equation for the last unknown, x_n:

$$x_n = \frac{b_n}{a_{nn}}$$

Second, we substitute this value for x_n in the next-to-last equation and solve it for the next-to-last unknown, x_{n-1}:

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}(b_n/a_{nn})}{a_{n-1,n-1}}$$

Third, we substitute these values for x_n and x_{n-1} in the third-from-last equation and solve it for the third-from-last unknown, x_{n-2}:

$$x_{n-2} = \frac{b_{n-2} - (a_{n-2,n-1}/a_{n-1,n-1})[b_{n-1} - a_{n-1,n}(b_n/a_{nn})] - (a_{n-2,n}/a_{nn})b_n}{a_{n-2,n-2}}$$

In general, we determine x_k by substituting the previously obtained values of $x_n, x_{n-1}, \ldots, x_{k+1}$ in the kth equation:

$$x_k = \frac{b_k - \sum_{m=k+1}^{n} a_{km}x_m}{a_{kk}}$$

The process ceases when we have determined the first unknown, x_1. The solution is unique since, at each step of the algorithm, the value of x_k is, by Theorem 1.1(i), uniquely determined.

Example 1.10. Consider the system

$$\begin{aligned} 2x + 4y - z &= 11 \\ 5y + z &= 2 \\ 3z &= -9 \end{aligned}$$

Since the system is in triangular form it may be solved by back-substitution.

(i) The last equation yields $z = -3$.

(ii) Substitute in the second equation to obtain $5y - 3 = 2$ or $5y = 5$ or $y = 1$.

(iii) Substitute $z = -3$ and $y = 1$ in the first equation to obtain

$$2x + 4(1) - (-3) = 11 \quad \text{or} \quad 2x + 4 + 3 = 11 \quad \text{or} \quad 2x = 4 \quad \text{or} \quad x = 2$$

Thus the vector $u = (2, 1, -3)$ is the unique solution of the system.

Echelon Form, Free Variables

A system of linear equations is in *echelon form* if no equation is degenerate and if the leading unknown in each equation is to the right of the leading unknown of the preceding equation. The paradigm is:

$$\begin{array}{r} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \cdots\cdots\cdots + a_{1n}x_n = b_1 \\ a_{2j_2}x_{j_2} + a_{2,j_2+1}x_{j_2+1} + \cdots\cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{rj_r}x_{j_r} + a_{r,j_r+1}x_{j_r+1} + \cdots + a_{rn}x_n = b_r \end{array} \qquad (1.5)$$

where $1 < j_2 < \cdots < j_r$ and where $a_{11} \neq 0, a_{2j_2} \neq 0, \ldots, a_{rj_r} \neq 0$. Note that $r \leq n$.

An unknown x_k in the above echelon system (*1.5*) is called a *free variable* if x_k is not the leading unknown in any equation, that is, if $x_k \neq x_1, x_k \neq x_{j_2}, \ldots, x_k \neq x_{j_r}$.

The following theorem, proved in Problem 1.13, describes the solution set of an echelon system.

Theorem 1.6: Consider the system (*1.5*) of linear equations in echelon form. There are two cases.

(i) $r = n$. That is, there are as many equations as unknowns. Then the system has a unique solution.

(ii) $r < n$. That is, there are fewer equations than unknowns. Then we can arbitrarily assign values to the $n - r$ free variables and obtain a solution of the system.

Suppose the echelon system (*1.5*) does contain more unknowns than equations. Then the system has an infinite number of solutions since each of the $n - r$ free variables may be assigned any real number. The general solution of the system is obtained as follows. Arbitrary values, called *parameters*, say $t_1, t_2, \ldots, t_{n-r}$, are assigned to the free variables, and then back-substitution is used to obtain values of the nonfree variables in terms of the parameters. Alternatively, one may use back-substitution to solve for the nonfree variables $x_1, x_{j_2}, \ldots, x_{j_r}$ directly in terms of the free variables.

Example 1.11. Consider the system

$$\begin{array}{r} x + 4y - 3z + 2t = 5 \\ z - 4t = 2 \end{array}$$

The system is in echelon form. The leading unknowns are x and z; hence the free variables are the other unknowns y and t.

To find the general solution of the system, we assign arbitrary values to the free variables y and t, say $y = a$ and $t = b$, and then use back-substitution to solve for the nonfree variables x and z. Substituting in the last equation yields $z - 4b = 2$ or $z = 2 + 4b$. Substitute in the first equation to get

$$x + 4a - 3(2 + 4b) + 2b = 5 \quad \text{or} \quad x + 4a - 6 - 12b + 2b = 5 \quad \text{or} \quad x = 11 - 4a + 10b$$

Thus

$$x = 11 - 4a + 10b,\ y = a,\ z = 2 + 4b,\ t = b \quad \text{or} \quad (11 - 4a + 10b, a, 2 + 4b, b)$$

is the general solution in parametric form. Alternatively, we can use back-substitution to solve for the nonfree variables x and z directly in terms of the free variables y and t. The last equation gives $z = 2 + 4t$. Substitute in the first equation to obtain

$$x + 4y - 3(2 + 4t) + 2t = 5 \quad \text{or} \quad x + 4y - 6 - 12t + 2t = 5 \quad \text{or} \quad x = 11 - 4y + 10t$$

Accordingly,

$$\begin{aligned} x &= 11 - 4y + 10t \\ z &= 2 + 4t \end{aligned}$$

is another form for the general solution of the system.

1.6 REDUCTION ALGORITHM

The following algorithm (sometimes called row reduction) reduces the system (*1.3*) of m linear equations in n unknowns to echelon (possibly triangular) form, or determines that the system has no solution.

Reduction algorithm

Step 1. Interchange equations so that the first unknown, x_1, appears with a nonzero coefficient in the first equation; i.e., arrange that $a_{11} \neq 0$.

Step 2. Use a_{11} as a pivot to eliminate x_1 from all the equations except the first equation. That is, for each $i > 1$, apply the elementary operation (Section 1.4)

$$[E_3]: \; -(a_{i1}/a_{11})L_1 + L_i \to L_i \qquad \text{or} \qquad [E]: \; -a_{i1}L_1 + a_{11}L_i \to L_i$$

Step 3. Examine each new equation L:

(*a*) If L has the form $0x_1 + 0x_2 + \cdots + 0x_n = 0$ or if L is a multiple of another equation, then delete L from the system.

(*b*) If L has the form $0x_1 + 0x_2 + \cdots + 0x_n = b$ with $b \neq 0$, then exit from the algorithm. The system has no solution.

Step 4. Repeat Steps 1, 2, and 3 with the subsystem formed by all the equations, excluding the first equation.

Step 5. Continue the above process until the system is in echelon form or a degenerate equation is obtained in Step 3(*b*).

The justification of Step 3 is Theorem 1.5 and the fact that if $L = kL'$ for some other equation L' in the system, the operation $-kL' + L \to L$ replaces L by $0x_1 + 0x_2 + \cdots + 0x_n = 0$, which again may be deleted by Theorem 1.5.

Example 1.12

(*a*) The system

$$\begin{aligned} 2x + y - 2z &= 10 \\ 3x + 2y + 2z &= 1 \\ 5x + 4y + 3z &= 4 \end{aligned}$$

is solved by first reducing it to echelon form. To eliminate x from the second and third equations, apply the operations $-3L_1 + 2L_2 \to L_2$ and $-5L_1 + 2L_3 \to L_3$:

$$\begin{array}{rr} -3L_1: & -6x - 3y + 6z = -30 \\ 2L_2: & 6x + 4y + 4z = 2 \\ \hline -3L_1 + 2L_2: & y + 10z = -28 \end{array} \qquad \begin{array}{rr} -5L_1: & -10x - 5y + 10z = -50 \\ 2L_3: & 10x + 8y + 6z = 8 \\ \hline -5L_1 + 2L_3: & 3y + 16z = -42 \end{array}$$

This yields the following system, from which y is eliminated from the third equation by the operation $-3L_2 + L_3 \rightarrow L_3$:

$$\left.\begin{aligned} 2x + y - 2z &= 10 \\ y + 10z &= -28 \\ 3y + 16z &= -42 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} 2x + y - 2z &= 10 \\ y + 10z &= -28 \\ -14z &= 42 \end{aligned}\right.$$

The system is now in triangular form. Therefore, we can use back-substitution to obtain the unique solution $u = (1, 2, -3)$.

(*b*) The system

$$\begin{aligned} x + 2y - 3z &= 1 \\ 2x + 5y - 8z &= 4 \\ 3x + 8y - 13z &= 7 \end{aligned}$$

is solved by first reducing it to echelon form. To eliminate x from the second and third equations, apply $-2L_1 + L_2 \rightarrow L_2$ and $-3L_1 + L_3 \rightarrow L_3$ to obtain

$$\begin{aligned} x + 2y - 3z &= 1 \\ y - 2z &= 2 \\ 2y - 4z &= 4 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y - 3z &= 1 \\ y - 2z &= 2 \end{aligned}$$

(The third equation is deleted since it is a multiple of the second equation.) The system is now in echelon form, with free variable z.

To obtain the general solution, let $z = a$ and solve by back-substitution. Substitute $z = a$ into the second equation to obtain $y = 2 + 2a$. Then substitute $z = a$ and $y = 2 + 2a$ into the first equation to obtain $x + 2(2 + 2a) - 3a = 1$ or $x = -3 - a$. Thus the general solution is

$$x = -3 - a, \; y = 2 + 2a, \; z = a \qquad \text{or} \qquad (-3 - a, 2 + 2a, a)$$

where a is the parameter.

(*c*) The system

$$\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{aligned}$$

is solved by first reducing it to echelon form. To eliminate x from the second and third equations, apply the operations $-3L_1 + L_2 \rightarrow L_2$ and $-5L_1 + L_3 \rightarrow L_3$ to obtain the equivalent system

$$\begin{aligned} x + 2y - 3z &= -1 \\ -7y + 11z &= 10 \\ -7y + 11z &= 7 \end{aligned}$$

The operation $-L_2 + L_3 \rightarrow L_3$ yields the degenerate equation

$$0x + 0y + 0z = -3$$

Thus the system has no solution.

The following basic result was indicated previously.

Theorem 1.7: Any system of linear equations has either: (i) a unique solution, (ii) no solution, or (iii) an infinite number of solutions.

Proof. Applying the above algorithm to the system, we can either reduce it to echelon form or determine that it has no solution. If the echelon form has free variables, then the system has an infinite number of solutions.

Remark: A system is said to be *consistent* if it has one or more solutions [Case (i) or (iii) in Theorem 1.7], and is said to be *inconsistent* if it has no solutions [Case (ii) in Theorem 1.7]. Figure 1-3 illustrates this situation.

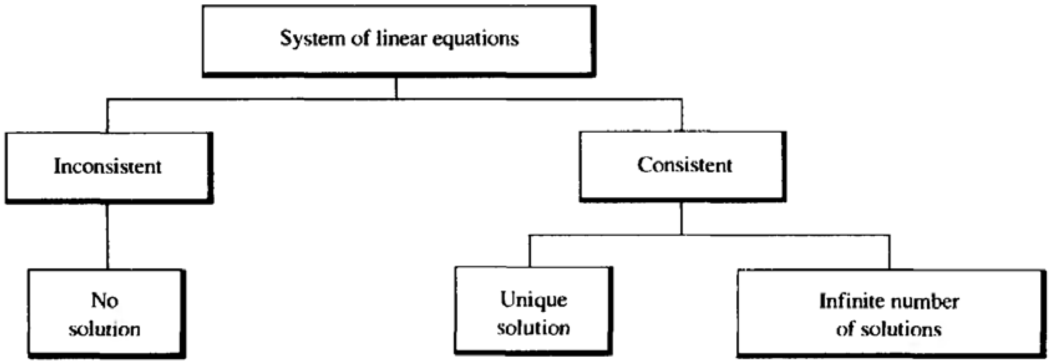


Fig. 1-3

1.7 MATRICES

Let A be a rectangular array of numbers as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The array A is called a *matrix*. Such a matrix may be denoted by writing $A = (a_{ij})$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, or simply $A = (a_{ij})$. The m horizontal n-tuples

$$(a_{11}, a_{12}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots, (a_{m1}, a_{m2}, \ldots, a_{mn})$$

are the *rows* of the matrix, and the n vertical m-tuples

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{pmatrix}$$

are its *columns*. Note that the element a_{ij}, called the *ij-entry* or *ij-component*, appears in the ith row and the jth column. A matrix with m rows and n columns is called an m by n matrix, or $m \times n$ matrix; the pair of numbers (m, n) is called its *size*.

Example 1.13. Let $A = \begin{pmatrix} 1 & -3 & 4 \\ 0 & 5 & -2 \end{pmatrix}$. Then A is a 2×3 matrix. Its rows are $(1, -3, 4)$ and $(0, 5, -2)$; its columns are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$, and $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

The first nonzero entry in a row R of a matrix A is called the *leading* nonzero entry of R. If R has no leading nonzero entry, i.e., if every entry in R is 0, then R is called a *zero row*. If all the rows of A are zero rows, i.e., if every entry of A is 0, then A is called the *zero matrix*, denoted by 0.

Echelon Matrices

A matrix A is called an *echelon matrix*, or is said to be in *echelon form* if the following two conditions hold:

(i) All zero rows, if any, are on the bottom of the matrix.

(ii) Each leading nonzero entry is to the right of the leading nonzero entry in the preceding row.

That is, $A = (a_{ij})$ is an echelon matrix if there exist nonzero entries

$$a_{1j_1}, a_{2j_2}, \ldots, a_{rj_r}, \qquad \text{where} \qquad j_1 < j_2 < \cdots < j_r$$

with the property that

$$a_{ij} = 0 \qquad \text{for (i) } i \le r,\ j < j_i, \text{ and (ii) } i > r$$

In this case, $a_{1j_1}, \ldots, a_{rj_r}$ are the leading nonzero entries of A.

Example 1.14. The following are echelon matrices whose leading nonzero entries have been circled:

$$\begin{pmatrix} \textcircled{2} & 3 & 2 & 0 & 4 & 5 & -6 \\ 0 & 0 & \textcircled{1} & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{6} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} \textcircled{1} & 2 & 3 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & \textcircled{1} & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & \textcircled{1} & 0 & -3 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 2 \end{pmatrix}$$

An echelon matrix A is said to be in *row canonical form* if it has the following two additional properties:

(iii) Each leading nonzero entry is 1.

(iv) Each leading nonzero entry is the only nonzero entry in its column.

The third matrix above is an example of a matrix in row canonical form. The second matrix is not in row canonical form since the leading nonzero entry in the second row is not the only nonzero entry in its column, there is a 3 above it. The first matrix is not in row canonical form since some leading nonzero entries are not 1.

The zero matrix 0, for any number of rows or columns, is also an example of a matrix in row canonical form.

1.8 ROW EQUIVALENCE AND ELEMENTARY ROW OPERATIONS

A matrix A is said to be *row equivalent* to a matrix B, written $A \sim B$, if B can be obtained from A by a finite sequence of the following *elementary row operations*:

[E_1] Interchange the ith row and the jth row: $R_i \leftrightarrow R_j$.

[E_2] Multiply the ith row by a nonzero scalar k: $kR_i \to R_i$, $k \neq 0$.

[E_3] Replace the ith row by k times the jth row plus the ith row: $kR_j + R_i \to R_i$.

In actual practice, we apply [E_2] and then [E_3] in one step, i.e., the operation

[E] Replace the ith row by k' times the jth row plus k (nonzero) times the ith row:

$$k'R_j + kR_i \to R_i, \ k \neq 0.$$

The reader no doubt recognizes the similarity of the above operations and those used in solving systems of linear equations.

The following algorithm row reduces a matrix A into echelon form. (The term "row reduce" or simply "reduce" shall mean to transform a matrix by row operations.)

Algorithm 1.8A

Here $A = (a_{ij})$ is an arbitrary matrix.

Step 1. Find the first column with a nonzero entry. Suppose it is the j_1 column.

Step 2. Interchange the rows so that a nonzero entry appears in the first row of the j_1 column, that is, so that $a_{1j_1} \neq 0$.

Step 3. Use a_{1j_1} as a pivot to obtain 0s below a_{1j_1}; that is, for each $i > 1$, apply the row operation $-a_{ij_1}R_1 + a_{1j_1}R_i \to R_i$ or $(-a_{ij_1}/a_{1j_1})R_1 + R_i \to R_i$.

Step 4. Repeat Steps 1, 2, and 3 with the submatrix formed by all the rows, excluding the first row.

Step 5. Continue the above process until the matrix is in echelon form.

Example 1.15. The matrix $A = \begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{pmatrix}$ is reduced to echelon form by Algorithm 1.8A as follows:

Use $a_{11} = 1$ as a pivot to obtain 0s below a_{11}, that is, apply the row operations $-2R_1 + R_2 \to R_2$ and $-3R_1 + R_3 \to R_3$ to obtain the matrix

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 3 \end{pmatrix}$$

Now use $a_{23} = 4$ as a pivot to obtain a 0 below a_{23}, that is, apply the row operation $-5R_2 + 4R_3 \to R_3$ to obtain the matrix

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

The matrix is now in echelon form.

The following algorithm row reduces an echelon matrix into its row canonical form.

Algorithm 1.8B

Here $A = (a_{ij})$ is in echelon form, say with leading nonzero entries

$$a_{1j_1}, a_{2j_2}, \ldots, a_{rj_r}$$

Step 1. Multiply the last nonzero row R_r by $1/a_{rj_r}$ so that the leading nonzero entry is 1.

Step 2. Use $a_{rj_r} = 1$ as a pivot to obtain 0s above the pivot; that is, for $i = r-1, r-2, \ldots, 1$, apply the operation

$$-a_{ir_1}R_r + R_i \to R_i$$

Step 3. Repeat Steps 1 and 2 for rows $R_{r-1}, R_{r-2}, \ldots, R_2$.

Step 4. Multiply R_1 by $1/a_{1j_1}$.

Example 1.16. Using Algorithm 1.8B, the echelon matrix

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 3 & 2 & 5 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

is reduced to row canonical form as follows:

Multiply R_3 by $\frac{1}{4}$ so that the leading nonzero entry equals 1; and then use $a_{35} = 1$ as a pivot to obtain 0s above it by applying the operations $-5R_3 + R_2 \to R_2$ and $-6R_3 + R_1 \to R_1$:

$$A \sim \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 3 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 4 & 5 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Multiply R_2 by $\frac{1}{3}$ so that the leading nonzero entry equals 1; and then use $a_{23} = 1$ as a pivot to obtain 0 above with the operation $-4R_2 + R_1 \to R_1$:

$$A \sim \begin{pmatrix} 2 & 3 & 4 & 5 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 0 & \frac{7}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Finally, multiply R_1 by $\frac{1}{2}$ to obtain

$$\begin{pmatrix} 1 & \frac{3}{2} & 0 & \frac{7}{6} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This matrix is the row canonical form of A.

Algorithms 1.8A and B show that any matrix is row equivalent to at least one matrix in row canonical form. In Chapter 5 we prove that such a matrix is unique, that is,

Theorem 1.8: Any matrix A is row equivalent to a unique matrix in row canonical form (called the *row canonical form* of A).

> **Remark:** If a matrix A is in echelon form, then its leading nonzero entries will be called *pivot entries*. The term comes from the above algorithm which row reduces a matrix to echelon form.

1.9 SYSTEMS OF LINEAR EQUATIONS AND MATRICES

The augmented matrix M of the system (*1.3*) of m linear equations in n unknowns is as follows:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

Observe that each row of M corresponds to an equation of the system, and each column of M corresponds to the coefficients of an unknown, except the last column which corresponds to the constants of the system.

The coefficient matrix A of the system (*1.3*) is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Note that the coefficient matrix A may be obtained from the augmented matrix M by omitting the last column of M.

One way to solve a system of linear equations is by working with its augmented matrix M, specifically, by reducing its augmented matrix to echelon form (which tells whether the system is consistent) and then reducing it to its row canonical form (which essentially gives the solution). The justification of this process comes from the following facts:

(1) Any elementary row operation on the augmented matrix M of the system is equivalent to applying the corresponding operation on the system itself.

(2) The system has a solution if and only if the echelon form of the augmented matrix M does not have a row of the form $(0, 0, \ldots, 0, b)$ with $b \neq 0$.

(3) In the row canonical form of the augmented matrix M (excluding zero rows) the coefficient of each nonfree variable is a leading nonzero entry which is equal to one and is the only nonzero entry in its respective column; hence the free-variable form of the solution is obtained by simply transferring the free variable terms to the other side.

This process is illustrated in the following example.

Example 1.17

(*a*) The system

$$\begin{aligned} x + y - 2z + 4t &= 5 \\ 2x + 2y - 3z + t &= 3 \\ 3x + 3y - 4z - 2t &= 1 \end{aligned}$$

is solved by reducing its augmented matrix M to echelon form and then to row canonical form as follows:

$$M = \begin{pmatrix} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & -10 & -9 \\ 0 & 0 & 1 & -7 & -7 \end{pmatrix}$$

[The third row (in the second matrix) is deleted since it is a multiple of the second row and will result in a zero row.] Thus the free variable form of the general solution of the system is as follows:

$$\begin{aligned} x + y - 10t &= -9 \\ z - 7t &= -7 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= -9 - y + 10t \\ z &= -7 + 7t \end{aligned}$$

Here the free variables are y and t, and the nonfree variables are x and z.

(*b*) The system

$$\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 &= 4 \\ 2x_1 + 3x_2 + 3x_3 - x_4 &= 3 \\ 5x_1 + 7x_2 + 4x_3 + x_4 &= 5 \end{aligned}$$

is solved as follows. First we reduce its augmented matrix to echelon form:

$$M = \begin{pmatrix} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & 15 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix}$$

There is no need to continue to find the row canonical form of the matrix since the echelon matrix already tells us that the system has no solution. Specifically, the third row of the echelon matrix corresponds to the degenerate equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = -5$$

which has no solution.

(*c*) The system

$$\begin{aligned} x + 2y + z &= 3 \\ 2x + 5y - z &= -4 \\ 3x - 2y - z &= 5 \end{aligned}$$

is solved by reducing its augmented matrix M to echelon form and then to row canonical form as follows:

$$M = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Thus the system has the unique solution $x = 2$, $y = -1$, $z = 3$ or $u = (2, -1, 3)$. (Note that the echelon form of M already indicated that the solution was unique since it corresponded to a triangular system.)

1.10 HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

The system (*1.3*) of linear equations is said to be *homogeneous* if all the constants are equal to zero, that is, if the system has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \tag{1.6}$$

In fact, the system (*1.6*) is called the homogeneous system associated with the system (*1.3*).

The homogeneous system (*1.6*) always has a solution, namely the zero n-tuple $0 = (0, 0, \ldots, 0)$ called the zero or trivial solution. (Any other solution, if it exists, is called a nonzero or nontrivial solution.) Thus it can always be reduced to an equivalent homogeneous system in echelon form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots\cdots + a_{1n}x_n &= 0 \\ a_{2j_2}x_{j_2} + a_{2,j_2+1}x_{j_2+1} + \cdots\cdots + a_{2n}x_n &= 0 \\ \cdots & \\ a_{rj_r}x_{j_r} + a_{r,j_r+1}x_{j_r+1} + \cdots + a_{rn}x_n &= 0 \end{aligned} \tag{1.7}$$

There are two possibilities:

(i) $r = n$. Then the system has only the zero solution.

(ii) $r < n$. Then the system has a nonzero solution.

Accordingly, if we begin with fewer equations than unknowns then, in echelon form, $r < n$ and hence the system has a nonzero solution. This proves the following important theorem.

Theorem 1.9: A homogeneous system of linear equations with more unknowns than equations has a nonzero solution.

Example 1.18

(*a*) The homogeneous system

$$\begin{aligned} x + 2y - 3z + w &= 0 \\ x - 3y + z - 2w &= 0 \\ 2x + y - 3z + 5w &= 0 \end{aligned}$$

has a nonzero solution since there are four unknowns but only three equations.

(*b*) We reduce the following system to echelon form:

$$\begin{aligned} x + y - z &= 0 \\ 2x - 3y + z &= 0 \\ x - 4y + 2z &= 0 \end{aligned} \qquad \begin{aligned} x + y - z &= 0 \\ -5y + 3z &= 0 \\ -5y + 3z &= 0 \end{aligned} \qquad \begin{aligned} x + y - z &= 0 \\ -5y + 3z &= 0 \end{aligned}$$

The system has a nonzero solution, since we obtained only two equations in the three unknowns in echelon form. For example, let $z = 5$; then $y = 3$ and $x = 2$. In other words, the 3-tuple (2, 3, 5) is a particular nonzero solution.

(*c*) We reduce the following system to echelon form:

$$\begin{aligned} x + y - z &= 0 \\ 2x + 4y - z &= 0 \\ 3x + 2y + 2z &= 0 \end{aligned} \qquad \begin{aligned} x + y - z &= 0 \\ 2y + z &= 0 \\ -y + 5z &= 0 \end{aligned} \qquad \begin{aligned} x + y - z &= 0 \\ 2y + z &= 0 \\ 11z &= 0 \end{aligned}$$

Since in echelon form there are three equations in three unknowns, the given system has only the zero solution (0, 0, 0).

Basis for the General Solution of a Homogeneous System

Let W denote the general solution of a homogeneous system. Nonzero solution vectors $u_1, u_2, \ldots, u_s$ are said to form a *basis* of W if every solution vector w in W can be expressed uniquely as a linear combination of $u_1, u_2, \ldots, u_s$. The number s of such basis vectors is called the *dimension* of W, written $\dim W = s$. (If $W = \{0\}$, we define $\dim W = 0$.)

The following theorem, proved in Chapter 5, tells us how to find such a basis.

Theorem 1.10: Let W be the general solution of a homogeneous system, and suppose an echelon form of the system has s free variables. Let $u_1, u_2, \ldots, u_s$ be the solutions obtained by setting one of the free variables equal to one (or any nonzero constant) and the remaining free variables equal to zero. Then $\dim W = s$ and $u_1, u_2, \ldots, u_s$ form a basis of W.

Remark: The above term *linear combination* refers to multiplying vectors by scalars and adding, where such operations are defined by

$$k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

These operations are studied in detail in Chapter 2.

Example 1.19 Suppose we want to find the dimension and a basis for the general solution W of the homogeneous system

$$\begin{aligned} x + 2y - 3z + 2s - 4t &= 0 \\ 2x + 4y - 5z + s - 6t &= 0 \\ 5x + 10y - 13z + 4s - 16t &= 0 \end{aligned}$$

First we reduce the system to echelon form. Applying the operations $-2L_1 + L_2 \to L_2$ and $-5L_2 + L_3 \to L_3$, and then $-2L_2 + L_3 \to L_3$, yields:

$$\begin{aligned} x + 2y - 3z + 2s - 4t &= 0 \\ z - 3s + 2t &= 0 \\ 2z - 6s + 4t &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x + 2y - 3z + 2s - 4t &= 0 \\ z - 3s + 2t &= 0 \end{aligned}$$

In echelon form, the system has three free variables, y, s and t; hence $\dim W = 3$. Three solution vectors which form a basis for W are obtained as follows:

(1) Set $y = 1, s = 0, t = 0$. Back-substitution yields the solution $u_1 = (-2, 1, 0, 0, 0)$.

(2) Set $y = 0, s = 1, t = 0$. Back-substitution yields the solution $u_2 = (7, 0, 3, 1, 0)$.

(3) Set $y = 0, s = 0, t = 1$. Back-substitution yields the solution $u_3 = (-2, 0, -2, 0, 1)$.

The set $\{u_1, u_2, u_3\}$ is a basis for W.

Now any solution of the system can be written in the form

$$\begin{aligned} au_1 + bu_2 + cu_3 &= a(-2, 1, 0, 0, 0) + b(7, 0, 3, 1, 0) + c(-2, 0, -2, 0, 1) \\ &= (-2a + 7b - 2c, a, 3b - 2c, b, c) \end{aligned}$$

where a, b, c are arbitrary constants. Observe that this is nothing other than the parametric form of the general solution under the choice of parameters $y = a, s = b, t = c$.

Nonhomogeneous and Associated Homogeneous Systems

The relationship between the nonhomogeneous system (*1.3*) and its associated homogeneous system (*1.6*) is contained in the following theorem whose proof is postponed until Chapter 3 (Theorem 3.5).

Theorem 1.11: Let v_0 be a particular solution and let U be the general solution of a nonhomogeneous system of linear equations. Then

$$U = v_0 + W = \{v_0 + w : w \in W\}$$

where W is the general solution of the associated homogeneous system.

That is, $U = v_0 + W$ may be obtained by adding v_0 to each element of W.

The above theorem has a geometrical interpretation in the space $\mathbf{R}^3$. Specifically, if W is a line through the origin, then, as pictured in Fig. 1-4, $U = v_0 + W$ is the line parallel to W which can be obtained by adding v_0 to each element in W. Similarly, whenever W is a plane through the origin, then $U = v_0 + W$ is a plane parallel to W.

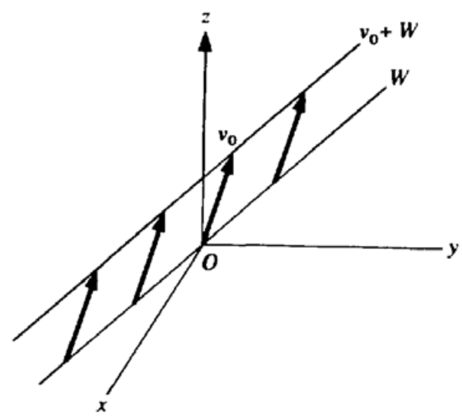


Fig. 1-4

Solved Problems

LINEAR EQUATIONS, SOLUTIONS

1.1. Determine whether each equation is linear:

(a) $5x + 7y - 8yz = 16$ (b) $x + \pi y + ez = \log 5$ (c) $3x + ky - 8z = 16$

(a) No, since the product yz of two unknowns is of second degree.

(b) Yes, since π, e, and $\log 5$ are constants.

(c) As it stands, there are four unknowns: x, y, z, k. Because of the term ky it is not a linear equation. However, assuming k is a constant, the equation is linear in the unknowns x, y, z.

1.2. Consider the linear equation $x + 2y - 3z = 4$. Determine whether $u = (8, 1, 2)$ is a solution.

Since x, y, z is the ordering of the unknowns, $u = (8, 1, 2)$ is short for $x = 8$, $y = 1$, $z = 2$. Substitute in the equation to obtain

$$8 + 2(1) - 3(2) = 4 \quad \text{or} \quad 8 + 2 - 6 = 4 \quad \text{or} \quad 4 = 4$$

Yes, it is a solution

1.3. Determine whether (a) $u = (3, 2, 1, 0)$ and (b) $v = (1, 2, 4, 5)$ are solutions of the equation $x_1 + 2x_2 - 4x_3 + x_4 = 3$.

(a) Substitute to obtain $3 + 2(2) - 4(1) + 0 = 3$, or $3 = 3$; yes, it is a solution.

(b) Substitute to obtain $1 + 2(2) - 4(4) + 5 = 3$, or $-6 = 3$; not a solution.

1.4. Is $u = (6, 4, -2)$ a solution of the equation $3x_2 + x_3 - x_1 = 4$?

By convention, the components of u are ordered according to the subscripts on the unknowns. That is, $u = (6, 4, -2)$ is short for $x_1 = 6$, $x_2 = 4$, $x_3 = -2$. Substitute in the equation to obtain $3(4) - 2 - 6 = 4$, or $4 = 4$. Yes, it is a solution.

1.5. Prove Theorem 1.1.

Suppose $a \neq 0$. Then the scalar b/a exists. Substituting b/a in $ax = b$ yields $a(b/a) = b$, or $b = b$; hence b/a is a solution. On the other hand, suppose x_0 is a solution to $ax = b$, so that $ax_0 = b$. Multiplying both sides by $1/a$ yields $x_0 = b/a$. Hence b/a is the unique solution of $ax = b$. Thus (i) is proved.

On the other hand, suppose $a = 0$. Then, for any scalar k, we have $ak = 0k = 0$. If $b \neq 0$, then $ak \neq b$. Accordingly, k is not a solution of $ax = b$ and so (ii) is proved. If $b = 0$, then $ak = b$. That is, any scalar k is a solution of $ax = b$ and so (iii) is proved.

1.6. Solve each equation:

$$(a)\ ex = \log 5 \qquad (c)\ 3x - 4 - x = 2x + 3$$
$$(b)\ cx = 0 \qquad (d)\ 7 + 2x - 4 = 3x + 3 - x$$

(*a*) Since $e \neq 0$, multiply by $1/e$ to obtain $x = (\log 5)/e$.

(*b*) If $c \neq 0$, then $0/c = 0$ is the unique solution. If $c = 0$, then every scalar k is a solution [Theorem 1.1(iii)].

(*c*) Rewrite in standard form, $2x - 4 = 2x + 3$ or $0x = 7$. The equation has no solution [Theorem 1.1(ii)].

(*d*) Rewrite in standard form, $3 + 2x = 2x + 3$ or $0x = 0$. Every scalar k is a solution [Theorem 1.1(iii)].

1.7. Describe the solutions of the equation $2x + y + x - 5 = 2y + 3x - y + 4$.

Rewrite in standard form by collecting terms and transposing:

$$3x + y - 5 = y + 3x + 4 \qquad \text{or} \qquad 0x + 0y = 9$$

The equation is degenerate with a nonzero constant; thus the equation has no solution.

1.8. Describe the solutions of the equation $2y + 3x - y + 4 = x + 3 + y + 1 + 2x$.

Rewrite in standard form by collecting terms and transposing:

$$y + 3x + 4 = 3x + 4 + y \qquad \text{or} \qquad 0x + 0y = 0$$

The equation is degenerate with a zero constant; thus every vector $u = (a, b)$ in $\mathbf{R}^2$ is a solution.

1.9. Prove Theorem 1.3.

First we prove (i). Set $x_j = k_j$ for $j \neq p$. Because $a_j = 0$ for $j < p$, substitution in the equation yields

$$a_p x_p + a_{p+1}k_{p+1} + \cdots + a_n k_n = b \qquad \text{or} \qquad a_p x_p = b - a_{p+1}k_{p+1} - \cdots - a_n k_n$$

with $a_p \neq 0$. By Theorem 1.1(i), x_p is uniquely determined as

$$x_p = \frac{1}{a_p}(b - a_{p+1}k_{p+1} - \cdots - a_n k_n)$$

Thus (i) is proved.

Now we prove (ii). Suppose $u = (k_1, k_2, \ldots, k_n)$ is a solution. Then

$$a_p k_p + a_{p+1}k_{p+1} + \cdots + a_n k_n = b \qquad \text{or} \qquad k_p = \frac{1}{a_p}(b - a_{p+1}k_{p+1} - \cdots - a_n k_n)$$

This, however, is precisely the solution

$$u = \left(k_1, \ldots, k_{p-1}, \frac{b - a_{p+1}k_{p+1} - \cdots - a_n k_n}{a_p}, k_{p+1}, \ldots, k_n\right)$$

obtained in (i). Thus (ii) is proved.

1.10. Consider the linear equation $x - 2y + 3z = 4$. Find (*a*) three particular solutions and (*b*) the general solution.

(*a*) Here x is the leading unknown. Accordingly, assign any values to the free variables y and z, and then solve for x to obtain a solution. For example:

(1) Set $y = 1$ and $z = 1$. Substitution in the equation yields

$$x - 2(1) + 3(1) = 4 \qquad \text{or} \qquad x - 2 + 3 = 4 \qquad \text{or} \qquad x = 3$$

Thus $u_1 = (3, 1, 1)$ is a solution.

(2) Set $y = 1, z = 0$. Substitution yields $x = 6$; hence $u_2 = (6, 1, 0)$ is a solution.

(3) Set $y = 0, z = 1$. Substitution yields $x = 1$; hence $u_3 = (1, 0, 1)$ is a solution.

(*b*) To find the general solution, assign arbitrary values to the free variables, say $y = a$ and $z = b$. (We call a and b parameters of the solution.) Then substitute in the equation to obtain

$$x - 2a + 3b = 4 \qquad \text{or} \qquad x = 4 + 2a - 3b$$

Thus $u = (4 + 2a - 3b, a, b)$ is the general solution.

SYSTEMS IN TRIANGULAR AND ECHELON FORM

1.11. Solve the system

$$\begin{aligned} 2x - 3y + 5z - 2t &= 9 \\ 5y - z + 3t &= 1 \\ 7z - t &= 3 \\ 2t &= 8 \end{aligned}$$

The system is in triangular form; hence we solve by back-substitution.

(i) The last equation gives $t = 4$.

(ii) Substituting in the third equation gives $7z - 4 = 3$, or $7z = 7$, or $z = 1$.

(iii) Substituting $z = 1$ and $t = 4$ in the second equation gives

$$5y - 1 + 3(4) = 1 \qquad \text{or} \qquad 5y - 1 + 12 = 1 \qquad \text{or} \qquad 5y = -10 \qquad \text{or} \qquad y = -2$$

(iv) Substituting $y = -2, z = 1, t = 4$ in the first equation gives

$$2x - 3(-2) + 5(1) - 2(4) = 9 \qquad \text{or} \qquad 2x + 6 + 5 - 8 = 9 \qquad \text{or} \qquad 2x = 6 \qquad \text{or} \qquad x = 3$$

Thus $x = 3, y = -2, z = 1, t = 4$ is the unique solution of the system.

1.12. Determine the free variables in each system:

$$\begin{aligned} 3x + 2y - 5z - 6s + 2t &= 4 \\ z + 8s - 3t &= 6 \\ s - 5t &= 5 \end{aligned} \qquad \begin{aligned} 5x - 3y + 7z &= 1 \\ 4y + 5z &= 6 \\ 4z &= 9 \end{aligned} \qquad \begin{aligned} x + 2y - 3z &= 2 \\ 2x - 3y + z &= 1 \\ 5x - 4y - z &= 4 \end{aligned}$$

(*a*) (*b*) (*c*)

(*a*) In the echelon form, any unknown that is not a leading unknown is termed a free variable. Here, y and t are the free variables.

(*b*) The leading unknowns are x, y, z. Hence there are no free variables (as in any triangular system).

(*c*) The notion of free variable applies only to a system in echelon form.

1.13. Prove Theorem 1.6.

There are two cases:

(i) $r = n$. That is, there are as many equations as unknowns. Then the system has a unique solution.

(ii) $r < n$. That is, there are fewer equations than unknowns. Then we can arbitrarily assign values to the $n - r$ free variables and obtain a solution of the system.

The proof is by induction on the number r of equations in the system. If $r = 1$, then we have a single, nondegenerate, linear equation, to which Theorem 1.3 applies when $n > r = 1$ and Theorem 1.1 applies when $n = r = 1$. Thus the theorem holds for $r = 1$.

Now assume that $r > 1$ and that the theorem is true for a system of $r - 1$ equations. We view the $r - 1$ equations

$$a_{2j_2}x_{j_2} + a_{2,j_2+1}x_{j_2+1} + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{rj_r}x_{j_r} + a_{r,j_r+1}x_{j_r+1} + \cdots + a_{rn}x_n = b_r$$

as a system in the unknowns $x_{j_2}, \ldots, x_n$. Note that the system is in echelon form. By the induction hypothesis, we can arbitrarily assign values to the $(n - j_2 + 1) - (r - 1)$ free variables in the reduced system to obtain a solution (say, $x_{j_2} = k_{j_2}, \ldots, x_n = k_n$). As in case $r = 1$, these values together with arbitrary values for the additional $j_2 - 2$ free variables (say $x_2 = k_2, \ldots, x_{j_2-1} = k_{j_2-1}$) yield a solution of the first equation with

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}k_2 - \cdots - a_{1n}k_n)$$

[Note that there are $(n - j_2 + 1) - (r - 1) + (j_2 - 2) = n - r$ free variables.] Furthermore, these values for $x_1, \ldots, x_n$ also satisfy the other equations since, in these equations, the coefficients of $x_1, \ldots, x_{j_2-1}$ are zero.

Now if $r = n$, then $j_2 = 2$. Thus by induction we obtain a unique solution of the subsystem and then a unique solution of the entire system. Accordingly, the theorem is proven.

1.14. Find the general solution of the echelon system

$$\begin{aligned} x - 2y - 3z + 5s - 2t &= 4 \\ 2z - 6s + 3t &= 2 \\ 5t &= 10 \end{aligned}$$

Since the equations begin with the unknowns x, z, and t, respectively, the other unknowns y and s are the free variables. To find the general solution, assign parameters to the free variables, say $y = a$ and $s = b$, and use back-substitution to solve for the nonfree variables x, z, and t.

(i) The last equation yields $t = 2$.

(ii) Substitute $t = 2$, $s = b$ in the second equation to obtain

$$2z - 6b + 3(2) = 2 \quad \text{or} \quad 2z - 6b + 6 = 2 \quad \text{or} \quad 2z = 6b - 4 \quad \text{or} \quad z = 3b - 2$$

(iii) Substitute $t = 2$, $s = b$, $z = 3b - 2$, $y = a$ in the first equation to obtain

$$x - 2a - 3(3b - 2) + 5b - 2(2) = 4 \quad \text{or} \quad x - 2a - 9b + 6 + 5b - 4 = 4$$
$$\text{or} \quad x = 2a + 4b + 2$$

Thus

$$x = 2a + 4b + 2 \qquad y = a \qquad z = 3b - 2 \qquad s = b \qquad t = 2$$

or, equivalently,

$$u = (2a + 4b + 2,\ a,\ 3b - 2,\ b,\ 2)$$

is the parametric form of the general solution.

Alternately, solving for x, z, and t in terms of the free variables y and s yields the following free-variable form of the general solution:

$$x = 2y + 4s + 2 \qquad z = 3s - 2 \qquad t = 2$$

SYSTEMS OF LINEAR EQUATIONS, GAUSSIAN ELIMINATION

1.15. Solve the system

$$\begin{aligned} x - 2y + \ z &= 7 \\ 2x - \ y + 4z &= 17 \\ 3x - 2y + 2z &= 14 \end{aligned}$$

Reduce to echelon form. Apply $-2L_1 + L_2 \to L_2$ and $-3L_1 + L_3 \to L_3$ to eliminate x from the second and third equations, and then apply $-4L_2 + 3L_3 \to L_3$ to eliminate y from the third equation. These operations yield

$$\begin{aligned} x - 2y + z &= 7 \\ 3y + 2z &= 3 \\ 4y - z &= -7 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x - 2y + z &= 7 \\ 3y + 2z &= 3 \\ -11z &= -33 \end{aligned}$$

The system is in triangular form, and hence, after back-substitution, has the unique solution $u = (2, -1, 3)$.

1.16. Solve the system

$$\begin{aligned} 2x - 5y + 3z - 4s + 2t &= 4 \\ 3x - 7y + 2z - 5s + 4t &= 9 \\ 5x - 10y - 5z - 4s + 7t &= 22 \end{aligned}$$

Reduce the system to echelon form. Apply the operations $-3L_1 + 2L_2 \to L_2$ and $-5L_1 + 2L_3 \to L_3$, and then $-5L_2 + L_3 \to L_3$ to obtain

$$\begin{aligned} 2x - 5y + 3z - 4s + 2t &= 4 \\ y - 5z + 2s + 2t &= 6 \\ 5y - 25z + 12s + 4t &= 24 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2x - 5y + 3z - 4s + 2t &= 4 \\ y - 5z + 2s + 2t &= 6 \\ 2s - 6t &= -6 \end{aligned}$$

The system is now in echelon form. Solving for the leading unknowns, x, y, and s, in terms of the free variables, z and t, we obtain the free-variable form of the general solution:

$$x = 26 + 11z - 15t \qquad y = 12 + 5z - 8t \qquad s = -3 + 3t$$

From this follows at once the parametric form of the general solution (where $z = a$, $t = b$):

$$x = 26 + 11a - 15b \qquad y = 12 + 5a - 8b \qquad z = a \qquad s = -3 + 3b \qquad t = b$$

1.17. Solve the system

$$\begin{aligned} x + 2y - 3z + 4t &= 2 \\ 2x + 5y - 2z + t &= 1 \\ 5x + 12y - 7z + 6t &= 7 \end{aligned}$$

Reduce the system to echelon form. Eliminate x from the second and third equations by the operations $-2L_1 + L_2 \to L_2$ and $-5L_1 + L_3 \to L_3$; this yields the system

$$\begin{aligned} x + 2y - 3z + 4t &= 2 \\ y + 4z - 7t &= -3 \\ 2y + 8z - 14t &= -3 \end{aligned}$$

The operation $-2L_2 + L_3 \to L_3$ yields the degenerate equation $0 = 3$. Thus the system has no solution (even though the system has more unknowns than equations).

1.18. Determine the values of k so that the following system in unknowns x, y, z has: (i) a unique solution, (ii) no solution, (iii) an infinite number of solutions.

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3y &= 2 \end{aligned}$$

Reduce the system to echelon form. Eliminate x from the second and third equations by the operations $-2L_1 + L_2 \to L_2$ and $-L_1 + L_3 \to L_3$ to obtain

$$\begin{aligned} x + \quad y - \quad z &= 1 \\ y + (k+2)z &= 1 \\ (k-1)y + \quad 4z &= 1 \end{aligned}$$

To eliminate y from the third equation, apply the operation $-(k-1)L_2 + L_3 \to L_3$ to obtain

$$\begin{aligned} x + y - \qquad\qquad z &= 1 \\ y + \qquad (k+2)z &= 1 \\ (3+k)(2-k)z &= 2-k \end{aligned}$$

The system has a unique solution if the coefficient of z in the third equation is not zero; that is, if $k \neq 2$ and $k \neq -3$. In case $k = 2$, the third equation reduces to $0 = 0$ and the system has an infinite number of solutions (one for each value of z). In case $k = -3$, the third equation reduces to $0 = 5$ and the system has no solution. Summarizing: (i) $k \neq 2$ and $k \neq 3$, (ii) $k = -3$, (iii) $k = 2$.

1.19. What condition must be placed on a, b, and c so that the following system in unknowns x, y, and z has a solution?

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 6y - 11z &= b \\ x - 2y + 7z &= c \end{aligned}$$

Reduce to echelon form. Eliminating x from the second and third equation by the operations $-2L_1 + L_2 \to L_2$ and $-L_1 + L_3 \to L_3$, we obtain the equivalent system

$$\begin{aligned} x + 2y - 3z &= a \\ 2y - 5z &= b - 2a \\ -4y + 10z &= c - a \end{aligned}$$

Eliminating y from the third equation by the operation $2L_2 + L_3 \to L_3$, we finally obtain the equivalent system

$$\begin{aligned} x + 2y - 3z &= a \\ 2y - 5z &= b - 2a \\ 0 &= c + 2b - 5a \end{aligned}$$

The system will have no solution if $c + 2b - 5a \neq 0$. Thus the system will have at least one solution if $c + 2b - 5a = 0$, or $5a = 2b + c$. Note, in this case, that the system will have infinitely many solutions. In other words, the system cannot have a unique solution.

MATRICES, ECHELON MATRICES, ROW REDUCTION

1.20. Interchange the rows in each of the following matrices to obtain an echelon matrix:

$$\begin{pmatrix} 0 & 1 & -3 & 4 & 6 \\ 4 & 0 & 2 & 5 & -3 \\ 0 & 0 & 7 & -2 & 8 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 5 & -4 & 7 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 2 & 2 & 2 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) (b) (c)

(a) Interchange the first and second rows, i.e., apply the elementary row operation $R_1 \leftrightarrow R_2$.

(b) Bring the zero row to the bottom of the matrix, i.e., apply $R_1 \leftrightarrow R_2$ and then $R_2 \leftrightarrow R_3$.

(c) No amount of row interchanges can produce an echelon matrix.

1.21. Row reduce the following matrix to echelon form:

$$A = \begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{pmatrix}$$

Use $a_{11} = 1$ as a pivot to obtain 0s below a_{11}; that is, apply the row operations $-2R_1 + R_2 \to R_2$ and $-3R_1 + R_3 \to R_3$ to obtain the matrix

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 3 \end{pmatrix}$$

Now use $a_{23} = 4$ as a pivot to obtain a 0 below a_{23}; that is, apply the row operation $-5R_2 + 4R_3 \to R_3$ to obtain the matrix

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

which is in echelon form.

1.22. Row reduce the following matrix to echelon form:

$$B = \begin{pmatrix} -4 & 1 & -6 \\ 1 & 2 & -5 \\ 6 & 3 & -4 \end{pmatrix}$$

Hand calculations are usually simpler if the pivot element equals 1. Therefore, first interchange R_1 and R_2; then apply $4R_1 + R_2 \to R_2$ and $-6R_1 + R_3 \to R_3$; and then apply $R_2 + R_3 \to R_3$:

$$B \sim \begin{pmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix is now in echelon form.

1.23. Describe the *pivoting* row reduction algorithm. Also, describe the advantages, if any, of using this pivoting algorithm.

The row reduction algorithm becomes a pivoting algorithm if the entry in column j of greatest absolute value is chosen as the pivot a_{1j_1} and if one uses the row operation

$$(-a_{ij_1}/a_{1j_1})R_1 + R_i \to R_i$$

The main advantage of the pivoting algorithm is that the above row operation involves division by the (current) pivot a_{1j_1} and, on the computer, roundoff errors may be substantially reduced when one divides by a number as large in absolute value as possible.

1.24. Use the pivoting algorithm to reduce the following matrix A to echelon form:

$$A = \begin{pmatrix} 2 & -2 & 2 & 1 \\ -3 & 6 & 0 & -1 \\ 1 & -7 & 10 & 2 \end{pmatrix}$$

First interchange R_1 and R_2 so that -3 can be used as the pivot, and then apply $(\frac{2}{3})R_1 + R_2 \to R_2$ and $(\frac{1}{3})R_1 + R_3 \to R_3$:

$$A \sim \begin{pmatrix} -3 & 6 & 0 & -1 \\ 2 & -2 & 2 & 1 \\ 1 & -7 & 10 & 2 \end{pmatrix} \sim \begin{pmatrix} -3 & 6 & 0 & -1 \\ 0 & 2 & 2 & \frac{1}{3} \\ 0 & -5 & 10 & \frac{5}{3} \end{pmatrix}$$

Now interchange R_2 and R_3 so that -5 may be used as the pivot, and apply $(\frac{2}{5})R_2 + R_3 \to R_3$:

$$A \sim \begin{pmatrix} -3 & 6 & 0 & -1 \\ 0 & -5 & 10 & \frac{5}{3} \\ 0 & 2 & 2 & \frac{1}{3} \end{pmatrix} \sim \begin{pmatrix} -3 & 6 & 0 & -1 \\ 0 & -5 & 10 & \frac{5}{3} \\ 0 & 0 & 6 & 1 \end{pmatrix}$$

The matrix has been brought to echelon form.

ROW CANONICAL FORM

1.25. Which of the following echelon matrices are in row canonical form?

$$\begin{pmatrix} 1 & 2 & -3 & 0 & 1 \\ 0 & 0 & 5 & 2 & -4 \\ 0 & 0 & 0 & 7 & 3 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 7 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 5 & 0 & 2 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

The first matrix is not in row canonical form since, for example, two leading nonzero entries are 5 and 7, not 1. Also, there are nonzero entries above the leading nonzero entries 5 and 7. The second and third matrices are in row canonical form.

1.26. Reduce the following matrix to row canonical form:

$$B = \begin{pmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 6 & 6 & 0 & 20 & 19 \end{pmatrix}$$

First, reduce B to an echelon form by applying $-2R_1 + R_2 \to R_2$ and $-3R_1 + R_3 \to R_3$, and then $-R_2 + R_3 \to R_3$:

$$B \sim \begin{pmatrix} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 3 & 2 & 7 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 0 & 4 & 2 \end{pmatrix}$$

Next reduce the echelon matrix to row canonical form. Specifically, first multiply R_3 by $\frac{1}{4}$, so the pivot $b_{34} = 1$, and then apply $2R_3 + R_2 \to R_2$ and $-6R_3 + R_1 \to R_1$:

$$B \sim \begin{pmatrix} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

Now multiply R_2 by $\frac{1}{3}$, making the pivot $b_{23} = 1$, and apply $R_2 + R_1 \to R_1$:

$$B \sim \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

Finally, multiply R_1 by $\frac{1}{2}$ to obtain the row canonical form

$$B \sim \begin{pmatrix} 1 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

1.27. Reduce the following matrix to row canonical form:

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 2 & 5 & 9 & -2 & 8 \end{pmatrix}$$

First reduce A to echelon form by applying $-R_1 + R_2 \to R_2$ and $-2R_1 + R_3 \to R_3$, and then applying $-3R_2 + R_3 \to R_3$:

$$A \sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 9 & 3 & -4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

Now use back-substitution. Multiply R_3 by $\frac{1}{2}$ to obtain the pivot $a_{34} = 1$, and then apply $2R_3 + R_2 \to R_2$ and $-R_3 + R_1 \to R_1$:

$$A \sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & 0 & \frac{3}{2} \\ 0 & 3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

Now multiply R_2 by $\frac{1}{3}$ to obtain the pivot $a_{22} = 1$, and then apply $2R_2 + R_1 \to R_1$:

$$A \sim \begin{pmatrix} 1 & -2 & 3 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{11}{3} & 0 & \frac{17}{6} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

Since $a_{11} = 1$, the last matrix is the desired row canonical form.

1.28. Describe the Gauss–Jordan elimination algorithm which reduces an arbitrary matrix A to its row canonical form.

The Gauss–Jordan algorithm is similar to the Gaussian elimination algorithm except that here the algorithm first normalizes a row to obtain a unit pivot and then uses the pivot to place 0s both below and above the pivot before obtaining the next pivot.

1.29. Use Gauss–Jordan elimination to obtain the row canonical form of the matrix of Problem 1.27.

Use the leading nonzero entry $a_{11} = 1$ as pivot to put 0s below it by applying $-R_1 + R_2 \to R_2$ and $-2R_1 + R_3 \to R_3$; this yields

$$A \sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 9 & 3 & -4 & 4 \end{pmatrix}$$

Multiply R_2 by $\frac{1}{3}$ to get the pivot $a_{22} = 1$ and produce 0s below and above a_{22} by applying $-9R_2 + R_3 \to R_3$ and $2R_2 + R_1 \to R_1$:

$$A \sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 9 & 3 & -4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{11}{3} & -\frac{1}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

Last, multiply R_3 by $\frac{1}{2}$ to get the pivot $a_{34} = 1$ and produce 0s above a_{34} by applying $\frac{2}{3}R_3 + R_2 \to R_2$ and $\frac{1}{3}R_3 + R_1 \to R_1$:

$$A \sim \begin{pmatrix} 1 & 0 & \frac{11}{3} & -\frac{1}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{11}{3} & 0 & \frac{17}{6} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

1.30. One speaks of "an" echelon form of a matrix A, "the" row canonical form of A. Why?

An arbitrary matrix A may be row equivalent to many echelon matrices. On the other hand, regardless of the algorithm that is used, a matrix A is row equivalent to a unique matrix in row canonical form. (The term "canonical" usually connotes uniqueness.) For example, the row canonical forms in Problems 1.27 and 1.29 are identical.

1.31. Given an $n \times n$ echelon matrix in triangular form,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2,n-1} & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3,n-1} & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix}$$

with all $a_{ii} \neq 0$. Find the row canonical form of A.

Multiplying R_n by $1/a_{nn}$ and using the new $a_{nn} = 1$ as pivot, we obtain the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} & 0 \\ 0 & a_{22} & a_{23} & \dots & a_{2,n-1} & 0 \\ 0 & 0 & a_{33} & \dots & a_{3,n-1} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Observe that the last column of A has been converted into a unit vector. Each succeeding back-substitution yields a new unit column vector, and the end result is

$$A \sim \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

i.e., A has the $n \times n$ *identity matrix* I as its row canonical form.

1.32. Reduce the following triangular matrix with nonzero diagonal elements to row canonical form:

$$C = \begin{pmatrix} 5 & -9 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 7 \end{pmatrix}$$

By Problem 1.31, C is row equivalent to the identity matrix. Alternatively, by back-substitution,

$$C \sim \begin{pmatrix} 5 & -9 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 5 & -9 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 5 & -9 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

SYSTEMS OF LINEAR EQUATIONS IN MATRIX FORM

1.33. Find the augmented matrix M and the coefficient matrix A of the following system:

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3y - 4z + 7x &= 5 \\ 6z + 8x - 9y &= 1 \end{aligned}$$

First align the unknowns in the system to obtain

$$\begin{aligned} x + 2y - 3z &= 4 \\ 7x + 3y - 4z &= 5 \\ 8x - 9y + 6z &= 1 \end{aligned}$$

Then

$$M = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 7 & 3 & -4 & 5 \\ 8 & -9 & 6 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 2 & -3 \\ 7 & 3 & -4 \\ 8 & -9 & 6 \end{pmatrix}$$

1.34. Solve, using the augmented matrix,

$$\begin{aligned} x - 2y + 4z &= 2 \\ 2x - 3y + 5z &= 3 \\ 3x - 4y + 6z &= 7 \end{aligned}$$

Reduce the augmented matrix to echelon form:

$$\begin{pmatrix} 1 & -2 & 4 & 2 \\ 2 & -3 & 5 & 3 \\ 3 & -4 & 6 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 2 & -6 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

The third row of the echelon matrix corresponds to the degenerate equation $0 = 3$; hence the system has no solution.

1.35. Solve, using the augmented matrix,

$$\begin{aligned} x + 2y - 3z - 2s + 4t &= 1 \\ 2x + 5y - 8z - s + 6t &= 4 \\ x + 4y - 7z + 5s + 2t &= 8 \end{aligned}$$

Reduce the augmented matrix to echelon form and then to row canonical form:

$$\begin{pmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ 1 & 4 & -7 & 5 & 2 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 0 & 1 & -2 & 3 & -2 & 2 \\ 0 & 2 & -4 & 7 & -2 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 0 & 1 & -2 & 3 & -2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & 0 & 8 & 7 \\ 0 & 1 & -2 & 0 & -8 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 24 & 21 \\ 0 & 1 & -2 & 0 & -8 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}$$

Thus the free-variable form of the solution is

$$\begin{aligned} x + z + 24t &= 21 \\ y - 2z - 8t &= -7 \\ s + 2t &= 3 \end{aligned} \quad \text{or} \quad \begin{aligned} x &= 21 - z - 24t \\ y &= -7 + 2z + 8t \\ s &= 3 - 2t \end{aligned}$$

where z and t are the free variables.

1.36. Solve, using the augmented matrix,

$$\begin{aligned} x + 2y - z &= 3 \\ x + 3y + z &= 5 \\ 3x + 8y + 4z &= 17 \end{aligned}$$

Reduce the augmented matrix to echelon form and then to row canonical form:

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 1 & 5 \\ 3 & 8 & 4 & 17 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 7 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & \frac{13}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix}$$

The system has the unique solution $x = \frac{17}{3}$, $y = -\frac{2}{3}$, $z = \frac{4}{3}$ or $u = (\frac{17}{3}, -\frac{2}{3}, \frac{4}{3})$.

HOMOGENEOUS SYSTEMS

1.37. Determine whether each system has a nonzero solution.

$$\begin{array}{ccc} \begin{aligned} x - 2y + 3z - 2w &= 0 \\ 3x - 7y - 2z + 4w &= 0 \\ 4x + 3y + 5z + 2w &= 0 \end{aligned} & \begin{aligned} x + 2y - 3z &= 0 \\ 2x + 5y + 2z &= 0 \\ 3x - y - 4z &= 0 \end{aligned} & \begin{aligned} x + 2y - z &= 0 \\ 2x + 5y + 2z &= 0 \\ x + 4y + 7z &= 0 \\ x + 3y + 3z &= 0 \end{aligned} \\ (a) & (b) & (c) \end{array}$$

(*a*) The system must have a nonzero solution since there are more unknowns than equations.

(*b*) Reduce to echelon form:

$$\begin{aligned} x + 2y - 3z &= 0 \\ 2x + 5y + 2z &= 0 \\ 3x - y - 4z &= 0 \end{aligned} \quad \text{to} \quad \begin{aligned} x + 2y - 3z &= 0 \\ y + 8z &= 0 \\ -7y + 5z &= 0 \end{aligned} \quad \text{to} \quad \begin{aligned} x + 2y - 3z &= 0 \\ y + 8z &= 0 \\ 61z &= 0 \end{aligned}$$

In echelon form there are exactly three equations in the three unknowns; hence the system has a unique solution, the zero solution.

(*c*) Reduce to echelon form:

$$\begin{aligned} x + 2y - z &= 0 \\ 2x + 5y + 2z &= 0 \\ x + 4y + 7z &= 0 \\ x + 3y + 3z &= 0 \end{aligned} \quad \text{to} \quad \begin{aligned} x + 2y - z &= 0 \\ y + 4z &= 0 \\ 2y + 8z &= 0 \\ y + 4z &= 0 \end{aligned} \quad \text{to} \quad \begin{aligned} x + 2y - z &= 0 \\ y + 4z &= 0 \end{aligned}$$

In echelon form there are only two equations in the three unknowns; hence the system has a nonzero solution.

1.38. Find the dimension and a basis for the general solution W of the homogeneous system

$$\begin{aligned} x + 3y - 2z + 5s - 3t &= 0 \\ 2x + 7y - 3z + 7s - 5t &= 0 \\ 3x + 11y - 4z + 10s - 9t &= 0 \end{aligned}$$

Show how the basis gives the parametric form of the general solution of the system.

Reduce the system to echelon form. Apply the operations $-2L_1 + L_2 \to L_2$ and $-3L_1 + L_3 \to L_3$, and then $-2L_2 + L_3 \to L_3$ to obtain

$$\begin{aligned} x + 3y - 2z + 5s - 3t &= 0 \\ y + z - 3s + t &= 0 \\ 2y + 2z - 5s \qquad &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x + 3y - 2z + 5s - 3t &= 0 \\ y + z - 3s + t &= 0 \\ s - 2t &= 0 \end{aligned}$$

In echelon form, the system has two free variables, z and t; hence $\dim W = 2$. A basis $[u_1, u_2]$ for W may be obtained as follows:

(1) Set $z = 1$, $t = 0$. Back-substitution yields $s = 0$, then $y = -1$, and then $x = 5$. Therefore, $u_1 = (5, -1, 1, 0, 0)$.

(2) Set $z = 0$, $t = 1$. Back-substitution yields $s = 2$, then $y = 5$, and then $x = -22$. Therefore, $u_2 = (-22, 5, 0, 2, 1)$.

Multiplying the basis vectors by the parameters a and b, respectively, yields

$$au_1 + bu_2 = a(5, -1, 1, 0, 0) + b(-22, 5, 0, 2, 1) = (5a - 22b, -a + 5b, a, 2b, b)$$

This is the parametric form of the general solution.

1.39. Find the dimension and a basis for the general solution W of the homogeneous system

$$\begin{aligned} x + 2y - 3z &= 0 \\ 2x + 5y + 2z &= 0 \\ 3x - y - 4z &= 0 \end{aligned}$$

Reduce the system to echelon form. From Problem 1.37(*b*) we have

$$\begin{aligned} x + 2y - 3z &= 0 \\ y + 8z &= 0 \\ 61z &= 0 \end{aligned}$$

There are no free variables (the system is in triangular form). Hence $\dim W = 0$ and W has no basis. Specifically, W consists only of the zero solution, $W = \{0\}$.

1.40. Find the dimension and a basis for the general solution W of the homogeneous system

$$\begin{aligned} 2x + 4y - 5z + 3t &= 0 \\ 3x + 6y - 7z + 4t &= 0 \\ 5x + 10y - 11z + 6t &= 0 \end{aligned}$$

Reduce the system to echelon form. Apply $-3L_1 + 2L_2 \to L_2$ and $-5L_1 + 2L_3 \to L_3$, and then $-3L_2 + L_3 \to L_3$ to obtain

$$\begin{aligned} 2x + 4y - 5z + 3t &= 0 \\ z - t &= 0 \\ 3z - 3t &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2x + 4y - 5z + 3t &= 0 \\ z - t &= 0 \end{aligned}$$

In echelon form, the system has two free variables, y and t; hence $\dim W = 2$. A basis $\{u_1, u_2\}$ for W may be obtained as follows:

(1) Set $y = 1, t = 0$. Back-substitution yields the solution $u_1 = (-2, 1, 0, 0)$.

(2) Set $y = 0, t = 1$. Back-substitution yields the solution $u_2 = (1, 0, 1, 1)$.

1.41. Consider the system

$$\begin{aligned} x - 3y - 2z + 4t &= 5 \\ 3x - 8y - 3z + 8t &= 18 \\ 2x - 3y + 5z - 4t &= 19 \end{aligned}$$

(*a*) Find the parametric form of the general solution of the system.

(*b*) Show that the result of (*a*) may be rewritten in the form given by Theorem 1.11.

(a) Reduce the system to echelon form. Apply $-3L_1 + L_2 \to L_2$ and $-2L_1 + L_3 \to L_3$, and then $-3L_2 + L_3 \to L_3$ to obtain

$$\begin{aligned} x - 3y - 2z + 4t &= 5 \\ y + 3z - 4t &= 3 \\ 3y + 9z - 12t &= 9 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x - 3y - 2z + 4t &= 5 \\ y + 3z - 4t &= 3 \end{aligned}$$

In echelon form, the free variables are z and t. Set $z = a$ and $t = b$, where a and b are parameters. Back-substitution yields $y = 3 - 3a + 4b$, and then $x = 14 - 7a + 8b$. Thus the parametric form of the solution is

$$x = 14 - 7a + 8b \qquad y = 3 - 3a + 4b \qquad z = a \qquad t = b \tag{$*$}$$

(b) Let $v_0 = (14, 3, 0, 0)$ be the vector of constant terms in $(*)$, let $u_1 = (-7, -3, 1, 0)$ be the vector of coefficients of a in $(*)$, and let $u_2 = (8, 4, 0, 1)$ be the vector of coefficients of b in $(*)$. Then the general solution $(*)$ may be rewritten in vector form as

$$(x, y, z, t) = v_0 + au_1 + bu_2 \tag{$**$}$$

We next show that $(**)$ is the general solution per Theorem 1.11. First note that v_0 is the solution of the inhomogeneous system obtained by setting $a = 0$ and $b = 0$. Consider the associated homogeneous system, in echelon form:

$$\begin{aligned} x - 3y - 2z + 4t &= 0 \\ y + 3z - 4t &= 0 \end{aligned}$$

The free variables are z and t. Set $z = 1$ and $t = 0$ to obtain the solution $u_1 = (-7, -3, 1, 0)$. Set $z = 0$ and $t = 1$ to obtain the solution $u_2 = (8, 4, 0, 1)$. By Theorem 1.10, $\{u_1, u_2\}$ is a basis for the solution space of the associated homogeneous system. Thus $(**)$ has the desired form.

MISCELLANEOUS PROBLEMS

1.42. Show that each of the elementary operations $[E_1]$, $[E_2]$, $[E_3]$ has an inverse operation of the same type.

$[E_1]$ Interchange the ith equation and the jth equation: $L_i \leftrightarrow L_j$.

$[E_2]$ Multiply the ith equation by a nonzero scalar k: $kL_i \to L_i$, $k \neq 0$.

$[E_3]$ Replace the ith equation by k times the jth equation plus the ith equation: $kL_j + L_i \to L_i$.

(a) Interchanging the same two equations twice, we obtain the original system; that is, $L_i \leftrightarrow L_j$ is its own inverse.

(b) Multiplying the ith equation by k and then by k^{-1}, or by k^{-1} and then k, we obtain the original system. In other words, the operations $kL_i \to L_i$ and $k^{-1}L_i \to L_i$ are inverses.

(c) Applying the operation $kL_j + L_i \to L_i$ and then the operation $-kL_j + L_i \to L_i$, or vice versa, we obtain the original system. In other words, the operations $kL_j + L_i \to L_i$ and $-kL_j + L_i \to L_i$ are inverses.

1.43. Show that the effect of applying the following operation $[E]$ can be obtained by applying $[E_2]$ and then $[E_3]$.

$[E]$ Replace the ith equation by k' times the jth equation plus k (nonzero) times the ith equation: $k'L_j + kL_i \to L_i$, $k \neq 0$.

Applying $kL_i \to L_i$ and then applying $k'L_j + L_i \to L_i$ has the same result as applying the operation $k'L_j + kL_i \to L_i$.

1.44. Suppose that each equation L_i in the system (*1.3*) is multiplied by a constant c_i, and that the resulting equations are added to yield

$$(c_1 a_{11} + \cdots + c_m a_{m1})x_1 + \cdots + (c_1 a_{1n} + \cdots + c_m a_{mn})x_n = c_1 b_1 + \cdots + c_m b_m \tag{1}$$

Such an equation is termed a linear combination of the equations L_i. Show that any solution of the system (*1.3*) is also a solution of the linear combination (*1*).

Suppose $u = (k_1, k_2, \ldots, k_n)$ is a solution of (*1.3*):

$$a_{i1}k_1 + a_{i2}k_2 + \cdots + a_{in}k_n = b_i \qquad (i = 1, \ldots, m) \tag{2}$$

To show that u is a solution of (*1*), we must verify the equation

$$(c_1a_{11} + \cdots + c_ma_{m1})k_1 + \cdots + (c_1a_{1n} + \cdots + c_ma_{mn})k_n = c_1b_1 + \cdots + c_mb_m$$

But this can be rearranged into

$$c_1(a_{11}k_1 + \cdots + a_{1n}k_n) + \cdots + c_m(a_{m1} + \cdots + a_{mn}k_n) = c_1b_1 + \cdots + c_mb_m$$

or, by (*2*)

$$c_1b_1 + \cdots + c_mb_m = c_1b_1 + \cdots + c_mb_m$$

which is clearly a true statement.

1.45. Suppose that a system (#) of linear equations is obtained from a system (*) of linear equations by applying a single elementary operation—$[E_1]$, $[E_2]$, or $[E_3]$. Show that (#) and (*) have all solutions in common (the two systems are equivalent).

Each equation in (#) is a linear combination of the equations in (*). Therefore, by Problem 1.44, any solution of (*) will be a solution of all the equations in (#). In other words, the solution set of (*) is contained in the solution set of (#). On the other hand, since the operations $[E_1]$, $[E_2]$, and $[E_3]$ have inverse elementary operations, the system (*) can be obtained from (#) by a single elementary operation. Accordingly, the solution set of (#) is contained in the solution set of (*). Thus (#) and (*) have the same solutions.

1.46. Prove Theorem 1.4.

By Problem 1.45, each step does not change the solution set. Hence the original system (*) and the final system (#) (and any system in between) have the same solution set.

1.47. Prove that the following three statements about a system of linear equations are equivalent: (i) The system is consistent (has a solution). (ii) No linear combination of the equations is the equation

$$0x_1 + 0x_2 + \cdots + 0x_n = b \neq 0 \tag{$*$}$$

(iii) The system is reducible to echelon form.

Suppose the system is reducible to echelon form. The echelon form has a solution, and hence the original system has a solution. Thus (iii) implies (i).

Suppose the system has a solution. By Problem 1.44, any linear combination of the equations also has a solution. But (*) has no solution; hence (*) is not a linear combination of the equations. Thus (i) implies (ii).

Finally, suppose the system is not reducible to echelon form. Then, in the Gaussian algorithm, it must yield an equation of the form (*). Hence (*) is a linear combination of the equations. Thus not-(iii) implies not-(ii), or, equivalently, (ii) implies (iii).

Supplementary Problems

SOLUTION OF LINEAR EQUATIONS

1.48. Solve:

$$(a)\ \begin{aligned} 2x + 3y &= 1 \\ 5x + 7y &= 3 \end{aligned} \qquad (b)\ \begin{aligned} 2x + 4y &= 10 \\ 3x + 6y &= 15 \end{aligned} \qquad (c)\ \begin{aligned} 4x - 2y &= 5 \\ -6x + 3y &= 1 \end{aligned}$$

1.49. Solve:

$$(a)\ \begin{aligned} 2x + y - 3z &= 5 \\ 3x - 2y + 2z &= 5 \\ 5x - 3y - z &= 16 \end{aligned} \qquad (b)\ \begin{aligned} 2x + 3y - 2z &= 5 \\ x - 2y + 3z &= 2 \\ 4x - y + 4z &= 1 \end{aligned} \qquad (c)\ \begin{aligned} x + 2y + 3z &= 3 \\ 2x + 3y + 8z &= 4 \\ 3x + 2y + 17z &= 1 \end{aligned}$$

1.50. Solve:

$$(a)\ \begin{aligned} 2x + 3y &= 3 \\ x - 2y &= 5 \\ 3x + 2y &= 7 \end{aligned} \qquad (b)\ \begin{aligned} x + 2y - 3z + 2t &= 2 \\ 2x + 5y - 8z + 6t &= 5 \\ 3x + 4y - 5z + 2t &= 4 \end{aligned} \qquad (c)\ \begin{aligned} x + 2y - z + 3t &= 3 \\ 2x + 4y + 4z + 3t &= 9 \\ 3x + 6y - z + 8t &= 10 \end{aligned}$$

1.51. Solve:

$$(a)\ \begin{aligned} x + 2y + 2z &= 2 \\ 3x - 2y - z &= 5 \\ 2x - 5y + 3z &= -4 \\ x + 4y + 6z &= 0 \end{aligned} \qquad (b)\ \begin{aligned} x + 5y + 4z - 13t &= 3 \\ 3x - y + 2z + 5t &= 2 \\ 2x + 2y + 3z - 4t &= 1 \end{aligned}$$

HOMOGENEOUS SYSTEMS

1.52. Determine whether each system has a nonzero solution:

$$(a)\ \begin{aligned} x + 3y - 2z &= 0 \\ x - 8y + 8z &= 0 \\ 3x - 2y + 4z &= 0 \end{aligned} \qquad (b)\ \begin{aligned} x + 3y - 2z &= 0 \\ 2x - 3y + z &= 0 \\ 3x - 2y + 2z &= 0 \end{aligned} \qquad (c)\ \begin{aligned} x + 2y - 5z + 4t &= 0 \\ 2x - 3y + 2z + 3t &= 0 \\ 4x - 7y + z - 6t &= 0 \end{aligned}$$

1.53. Find the dimension and a basis of the general solution W of each homogeneous system.

$$(a)\ \begin{aligned} x + 3y + 2z - s - t &= 0 \\ 2x + 6y + 5z + s - t &= 0 \\ 5x + 15y + 12z + s - 3t &= 0 \end{aligned} \qquad (b)\ \begin{aligned} 2x - 4y + 3z - s + 2t &= 0 \\ 3x - 6y + 5z - 2s + 4t &= 0 \\ 5x - 10y + 7z - 3s + t &= 0 \end{aligned}$$

ECHELON MATRICES AND ELEMENTARY ROW OPERATIONS

1.54. Reduce A to echelon form and then to its row canonical form, where

$$(a)\ A = \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{pmatrix} \qquad (b)\ A = \begin{pmatrix} 2 & 3 & -2 & 5 & 1 \\ 3 & -1 & 2 & 0 & 4 \\ 4 & -5 & 6 & -5 & 7 \end{pmatrix}$$

1.55. Reduce A to echelon form and then to its row canonical form, where

$$(a)\quad A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix} \qquad (b)\quad A = \begin{pmatrix} 0 & 1 & 3 & -2 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 5 & -3 & 4 \end{pmatrix}$$

1.56. Describe all the possible 2×2 matrices which are in row reduced echelon form.

1.57. Suppose A is a square row reduced echelon matrix. Show that if $A \neq I$, the identity matrix, then A has a zero row.

1.58. Show that each of the following elementary row operations has an inverse operation of the same type.

[E_1] Interchange the ith row and the jth row: $R_i \leftrightarrow R_j$.

[E_2] Multiply the ith row by a nonzero scalar k: $kR_i \to R_i$, $k \neq 0$.

[E_3] Replace the ith row by k times the jth row plus the ith row: $kR_j + R_i \to R_i$.

1.59. Show that row equivalence is an equivalence relation:

(i) A is row equivalent to A;

(ii) A row equivalent to B implies B row equivalent to A;

(iii) A row equivalent to B and B row equivalent to C implies A row equivalent to C.

MISCELLANEOUS PROBLEMS

1.60. Consider two general linear equations in two unknowns x and y over the real field $\mathbf{R}$:

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

Show that:

(i) If $\dfrac{a}{c} \neq \dfrac{b}{d}$, i.e., if $ad - bc \neq 0$, then the system has the unique solution $x = \dfrac{de - bf}{ad - bc}$, $y = \dfrac{af - ce}{ad - bc}$;

(ii) If $\dfrac{a}{c} = \dfrac{b}{d} \neq \dfrac{e}{f}$, then the system has no solution;

(iii) If $\dfrac{a}{c} = \dfrac{b}{d} = \dfrac{e}{f}$, then the system has more than one solution.

1.61. Consider the system

$$\begin{aligned} ax + by &= 1 \\ cx + dy &= 0 \end{aligned}$$

Show that if $ad - bc \neq 0$, then the system has the unique solution $x = d/(ad - bc)$, $y = -c/(ad - bc)$. Also show that if $ad - bc = 0$, $c \neq 0$ or $d \neq 0$, then the system has no solution.

1.62. Show that an equation of the form $0x_1 + 0x_2 + \cdots + 0x_n = 0$ may be added or deleted from a system without affecting the solution set.

1.63. Consider a system of linear equations with the same number of equations as unknowns:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array} \qquad (I)$$

(i) Suppose the associated homogeneous system has only the zero solution. Show that (I) has a unique solution for every choice of constants b_i.

(ii) Suppose the associated homogeneous system has a nonzero solution. Show that there are constants b_i for which (I) does not have a solution. Also show that if (I) has a solution, then it has more than one.

1.64. Suppose in a homogeneous system of linear equations the coefficients of one of the unknowns are all zero. Show that the system has a nonzero solution.

Answers to Supplementary Problems

1.48. (*a*) $x = 2, y = -1$; (*b*) $x = 5 - 2a, y = a$; (*c*) no solution

1.49. (*a*) $(1, -3, -2)$; (*b*) no solution; (*c*) $(-1 - 7a, 2 + 2a, a)$ or $\begin{cases} x = -1 - 7z \\ y = 2 + 2z \end{cases}$

1.50. (*a*) $x = 3, y = -1$; (*b*) $(-a + 2b, 1 + 2a - 2b, a, b)$ or $\begin{cases} x = -z + 2t \\ y = 1 + 2z - 2t \end{cases}$

(*c*) $(\frac{7}{2} - 5b/2 - 2a, a, \frac{1}{2} + b/2, b)$ or $\begin{cases} x = \frac{7}{2} - 5t/2 - 2y \\ z = \frac{1}{2} + t/2 \end{cases}$

1.51. (*a*) $(2, 1, -1)$; (*b*) no solution

1.52. (*a*) yes; (*b*) no; (*c*) yes, by Theorem 1.8

1.53. (*a*) Dim $W = 3$; $u_1 = (-3, 1, 0, 0, 0)$, $u_2 = (7, 0, -3, 1, 0)$, $u_3 = (3, 0, -1, 0, 1)$;
(*b*) Dim $W = 2$; $u_1 = (-2, 1, 0, 0, 0)$, $u_2 = (5, 0, -5, -3, 1)$

1.54. (*a*) $\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix}$;

(*b*) $\begin{pmatrix} 2 & 3 & -2 & 5 & 1 \\ 0 & -11 & 10 & -15 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & \frac{4}{11} & \frac{5}{11} & \frac{13}{11} \\ 0 & 1 & -\frac{10}{11} & \frac{15}{11} & -\frac{5}{11} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

1.55. (*a*) $\begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & \frac{4}{11} & \frac{13}{11} \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

(*b*) $\begin{pmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & -13 & 11 \\ 0 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$