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William Fulton Joe Harris

Representation Theory

A First Course

With 144 Illustrations

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Preface

The primary goal of these lectures is to introduce a beginner to the finite-dimensional representations of Lie groups and Lie algebras. Since this goal is shared by quite a few other books, we should explain in this Preface how our approach differs, although the potential reader can probably see this better by a quick browse through the book.

Representation theory is simple to define: it is the study of the ways in which a given group may act on vector spaces. It is almost certainly unique, however, among such clearly delineated subjects, in the breadth of its interest to mathematicians. This is not surprising: group actions are ubiquitous in 20th century mathematics, and where the object on which a group acts is not a vector space, we have learned to replace it by one that is (e.g., a cohomology group, tangent space, etc.). As a consequence, many mathematicians other than specialists in the field (or even those who think they might want to be) come in contact with the subject in various ways. It is for such people that this text is designed. To put it another way, we intend this as a book for beginners to learn from and not as a reference.

This idea essentially determines the choice of material covered here. As simple as is the definition of representation theory given above, it fragments considerably when we try to get more specific. For a start, what kind of group G are we dealing with—a finite group like the symmetric group $\mathfrak{S}_n$ or the general linear group over a finite field $GL_n(\mathbb{F}_q)$, an infinite discrete group like $SL_n(\mathbb{Z})$, a Lie group like $SL_n\mathbb{C}$, or possibly a Lie group over a local field? Needless to say, each of these settings requires a substantially different approach to its representation theory. Likewise, what sort of vector space is G acting on: is it over $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, or possibly a field of finite characteristic? Is it finite dimensional or infinite dimensional, and if the latter, what additional structure (such as norm, or inner product) does it carry? Various combinations of answers to these questions lead to areas of intense research activity in representation theory, and it is natural for a text intended to prepare students for a career in the subject to lead up to one or more of these areas. As a corollary, such a book tends to get through the elementary material as quickly as possible: if one has a semester to get up to and through Harish-Chandra modules, there is little time to dawdle over the representations of $\mathfrak{S}_4$ and $SL_3\mathbb{C}$.

By contrast, the present book focuses exactly on the simplest cases: representations of finite groups and Lie groups on finite-dimensional real and complex vector spaces. This is in some sense the common ground of the subject, the area that is the object of most of the interest in representation theory coming from outside.

The intent of this book to serve nonspecialists likewise dictates to some degree our approach to the material we do cover. Probably the main feature of our presentation is that we concentrate on examples, developing the general theory sparingly, and then mainly as a useful and unifying language to describe phenomena already encountered in concrete cases. By the same token, we for the most part introduce theoretical notions when and where they are useful for analyzing concrete situations, postponing as long as possible those notions that are used mainly for proving general theorems.

Finally, our goal of making the book accessible to outsiders accounts in part for the style of the writing. These lectures have grown from courses of the second author in 1984 and 1987, and we have attempted to keep the informal style of these lectures. Thus there is almost no attempt at efficiency: where it seems to make sense from a didactic point of view, we work out many special cases of an idea by hand before proving the general case; and we cheerfully give several proofs of one fact if we think they are illuminating. Similarly, while it is common to develop the whole semisimple story from one point of view, say that of compact groups, or Lie algebras, or algebraic groups, we have avoided this, as efficient as it may be.

It is of course not a strikingly original notion that beginners can best learn about a subject by working through examples, with general machinery only introduced slowly and as the need arises, but it seems particularly appropriate here. In most subjects such an approach means one has a few out of an unknown infinity of examples which are useful to illuminate the general situation. When the subject is the representation theory of complex semisimple Lie groups and algebras, however, something special happens: once one has worked through all the examples readily at hand—the "classical" cases of the special linear, orthogonal, and symplectic groups—one has not just a few useful examples, one has all but five "exceptional" cases.

This is essentially what we do here. We start with a quick tour through representation theory of finite groups, with emphasis determined by what is useful for Lie groups. In this regard, we include more on the symmetric groups than is usual. Then we turn to Lie groups and Lie algebras. After some preliminaries and a look at low-dimensional examples, and one lecture with some general notions about semisimplicity, we get to the heart of the course: working out the finite-dimensional representations of the classical groups.

For each series of classical Lie algebras we prove the fundamental existence theorem for representations of given highest weight by explicit construction. Our object, however, is not just existence, but to see the representations in action, to see geometric implications of decompositions of naturally occurring representations, and to see the relations among them caused by coincidences between the Lie algebras.

The goal of the last six lectures is to make a bridge between the example-oriented approach of the earlier parts and the general theory. Here we make an attempt to interpret what has gone before in abstract terms, trying to make connections with modern terminology. We develop the general theory enough to see that we have studied all the simple complex Lie algebras with five exceptions. Since these are encountered less frequently than the classical series, it is probably not reasonable in a first course to work out their representations as explicitly, although we do carry this out for one of them. We also prove the general Weyl character formula, which can be used to verify and extend many of the results we worked out by hand earlier in the book.

Of course, the point we reach hardly touches the current state of affairs in Lie theory, but we hope it is enough to keep the reader's eyes from glazing over when confronted with a lecture that begins: "Let G be a semisimple Lie group, P a parabolic subgroup, ..." We might also hope that working through this book would prepare some readers to appreciate the elegance (and efficiency) of the abstract approach.

In spirit this book is probably closer to Weyl's classic [Wel] than to others written today. Indeed, a secondary goal of our book is to present many of the results of Weyl and his predecessors in a form more accessible to modern readers. In particular, we include Weyl's constructions of the representations of the general and special linear groups by using Young's symmetrizers; and we invoke a little invariant theory to do the corresponding result for the orthogonal and symplectic groups. We also include Weyl's formulas for the characters of these representations in terms of the elementary characters of symmetric powers of the standard representations. (Interestingly, Weyl only gave the corresponding formulas in terms of the exterior powers for the general linear group. The corresponding formulas for the orthogonal and symplectic groups were only given recently by Koike and Terada. We include a simple new proof of these determinantal formulas.)

More about individual sections can be found in the introductions to other parts of the book.

Needless to say, a price is paid for the inefficiency and restricted focus of these notes. The most obvious is a lot of omitted material: for example, we include little on the basic topological, differentiable, or analytic properties of Lie groups, as this plays a small role in our story and is well covered in dozens of other sources, including many graduate texts on manifolds. Moreover, there are no infinite-dimensional representations, no Harish-Chandra or Verma modules, no Steifel diagrams, no Lie algebra cohomology, no analysis on symmetric spaces or groups, no arithmetic groups or automorphic forms, and nothing about representations in characteristic $p > 0$. There is no consistent attempt to indicate which of our results on Lie groups apply more generally to algebraic groups over fields other than $\mathbb{R}$ or $\mathbb{C}$ (e.g., local fields). And there is only passing mention of other standard topics, such as universal enveloping algebras or Bruhat decompositions, which have become standard tools of representation theory. (Experts who saw drafts of this book agreed that some topic we omitted must not be left out of a modern book on representation theory—but no two experts suggested the same topic.)

We have not tried to trace the history of the subjects treated, or assign credit, or to attribute ideas to original sources—this is far beyond our knowledge. When we give references, we have simply tried to send the reader to sources that are as readable as possible for one knowing what is written here. A good systematic reference for the finite-group material, including proofs of the results we leave out, is Serre [Se2]. For Lie groups and Lie algebras, Serre [Se3], Adams [Ad], Humphreys [Hu1], and Bourbaki [Bour] are recommended references, as are the classics Weyl [We1] and Littlewood [Lit1].

We would like to thank the many people who have contributed ideas and suggestions for this manuscript, among them J-F. Burnol, R. Bryant, J. Carrell, B. Conrad, P. Diaconis, D. Eisenbud, D. Goldstein, M. Green, P. Griffiths, B. Gross, M. Hildebrand, R. Howe, H. Kraft, A. Landman, B. Mazur, N. Chriss, D. Petersen, G. Schwartz, J. Towber, and L. Tu. In particular, we would like to thank David Mumford, from whom we learned much of what we know about the subject, and whose ideas are very much in evidence in this book.

Had this book been written 10 years ago, we would at this point thank the people who typed it. That being no longer applicable, perhaps we should thank instead the National Science Foundation, the University of Chicago, and Harvard University for generously providing the various Macintoshes on which this manuscript was produced. Finally, we thank Chan Fulton for making the drawings.

Bill Fulton and Joe Harris

Using This Book

A few words are in order about the practical use of this book. To begin with, prerequisites are minimal: we assume only a basic knowledge of standard first-year graduate material in algebra and topology, including basic notions about manifolds. A good undergraduate background should be more than enough for most of the text; some examples and exercises, and some of the discussion in Part IV may refer to more advanced topics, but these can readily be skipped. Probably the main practical requirement is a good working knowledge of multilinear algebra, including tensor, exterior, and symmetric products of finite dimensional vector spaces, for which Appendix B may help. We have indicated, in introductory remarks to each lecture, when any background beyond this is assumed and how essential it is.

For a course, this book could be used in two ways. First, there are a number of topics that are not logically essential to the rest of the book and that can be skimmed or skipped entirely. For example, in a minimal reading one could skip §§4, 5, 6, 11.3, 13.4, 15.3–15.5, 17.3, 19.5, 20, 22.1, 22.3, 23.3–23.4, 25.3, and 26.2; this might be suitable for a basic one-semester course. On the other hand, in a year-long course it should be possible to work through as much of the material as background and/or interest suggested. Most of the material in the Appendices is relevant only to such a long course. Again, we have tried to indicate, in the introductory remarks in each lecture, which topics are inessential and may be omitted.

Another aspect of the book that readers may want to approach in different ways is the profusion of examples. These are put in largely for didactic reasons: we feel that this is the sort of material that can best be understood by gaining some direct hands-on experience with the objects involved. For the most part, however, they do not actually develop new ideas; the reader whose tastes run more to the abstract and general than the concrete and special may skip many of them without logical consequence. (Of course, such a reader will probably wind up burning this book anyway.)

We include hundreds of exercises, of wildly different purposes and difficulties. Some are the usual sorts of variations of the examples in the text or are straightforward verifications of facts needed; a student will probably want to attempt most of these. Sometimes an exercise is inserted whose solution is a special case of something we do in the text later, if we think working on it will be useful motivation (again, there is no attempt at "efficiency," and readers are encouraged to go back to old exercises from time to time). Many exercises are included that indicate some further directions or new topics (or standard topics we have omitted); a beginner may best be advised to skim these for general information, perhaps working out a few simple cases. In exercises, we tried to include topics that may be hard for nonexperts to extract from the literature, especially the older literature. In general, much of the theory is in the exercises—and most of the examples in the text.

We have resisted the idea of grading the exercises by (expected) difficulty, although a "problem" is probably harder than an "exercise." Many exercises are starred: the * is not an indication of difficulty, but means that the reader can find some information about it in the section "Hints, Answers, and References" at the back of the book. This may be a hint, a statement of the answer, a complete solution, a reference to where more can be found, or a combination of any of these. We hope these miscellaneous remarks, as haphazard and uneven as they are, will be of some use.

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№2 Lectures 1÷3

PART I

FINITE GROUPS

Given that over three-quarters of this book is devoted to the representation theory of Lie groups and Lie algebras, why have a discussion of the representations of finite groups at all? There are certainly valid reasons from a logical point of view: many of the ideas, concepts, and constructions we will introduce here will be applied in the study of Lie groups and algebras. The real reason for us, however, is didactic, as we will now try to explain.

Representation theory is very much a 20th-century subject, in the following sense. In the 19th century, when groups were dealt with they were generally understood to be subsets of the permutations of a set, or of the automorphisms $GL(V)$ of a vector space V, closed under composition and inverse. Only in the 20th century was the notion of an abstract group given, making it possible to make a distinction between properties of the abstract group and properties of the particular realization as a subgroup of a permutation group or $GL(V)$. To give an analogy, in the 19th century a manifold was always a subset of $\mathbb{R}^n$; only in the 20th century did the notion of an abstract Riemannian manifold become common.

In both cases, the introduction of the abstract object made a fundamental difference to the subject. In differential geometry, one could make a crucial distinction between the intrinsic and extrinsic geometry of the manifold: which properties were invariants of the metric on the manifold and which were properties of the particular embedding in $\mathbb{R}^n$. Questions of existence or non-existence, for example, could be broken up into two parts: did the abstract manifold exist, and could it be embedded. Similarly, what would have been called in the 19th century simply "group theory" is now factored into two parts. First, there is the study of the structure of abstract groups (e.g., the classification of simple groups). Second is the companion question: given a group G, how can we describe all the ways in which G may be embedded in (or mapped to) a linear group $GL(V)$? This, of course, is the subject matter of representation theory.

Given this point of view, it makes sense when first introducing representation theory to do so in a context where the nature of the groups G in question is itself simple, and relatively well understood. It is largely for this reason that we are starting off with the representation theory of finite groups: for those readers who are not already familiar with the motivations and goals of representation theory, it seemed better to establish those first in a setting where the structure of the groups was not itself an issue. When we analyze, for example, the representations of the symmetric and alternating groups on 3, 4, and 5 letters, it can be expected that the reader is already familiar with the groups and can focus on the basic concepts of representation theory being introduced.

We will spend the first six lectures on the case of finite groups. Many of the techniques developed for finite groups will carry over to Lie groups; indeed, our choice of topics is in part guided by this. For example, we spend quite a bit of time on the symmetric group; this is partly for its own interest, but also partly because what we learn here gives one way to study representations of the general linear group and its subgroups. There are other topics, such as the alternating group $\mathfrak{A}_4$, and the groups $SL_2(\mathbb{F}_q)$ and $GL_2(\mathbb{F}_q)$ that are studied purely for their own interest and do not appear later. (In general, for those readers primarily concerned with Lie theory, we have tried to indicate in the introductory notes to each lecture which ideas will be useful in the succeeding parts of this book.) Nonetheless, this is by no means a comprehensive treatment of the representation theory of finite groups; many important topics, such as the Artin and Brauer theorems and the whole subject of modular representations, are omitted.

LECTURE 1

Representations of Finite Groups

In this lecture we give the basic definitions of representation theory, and prove two of the basic results, showing that every representation is a (unique) direct sum of irreducible ones. We work out as examples the case of abelian groups, and the simplest nonabelian group, the symmetric group on 3 letters. In the latter case we give an analysis that will turn out not to be useful for the study of finite groups, but whose main idea is central to the study of the representations of Lie groups.

§1.1: Definitions
§1.2: Complete reducibility; Schur's lemma
§1.3: Examples: Abelian groups; $\mathfrak{S}_3$

§1.1. Definitions

A *representation* of a finite group G on a finite-dimensional complex vector space V is a homomorphism $\rho: G \to GL(V)$ of G to the group of automorphisms of V; we say that such a map *gives V the structure of a G-module*. When there is little ambiguity about the map ρ (and, we're afraid, even sometimes when there is) we sometimes call V itself a representation of G; in this vein we will often suppress the symbol ρ and write $g \cdot v$ or gv for $\rho(g)(v)$. The dimension of V is sometimes called the *degree* of ρ.

A *map* φ between two representations V and W of G is a vector space map $\varphi: V \to W$ such that

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{\varphi} & W \end{array}$$

commutes for every $g \in G$. (We will call this a *G-linear map* when we want to distinguish it from an arbitrary linear map between the vector spaces V and W.) We can then define Ker φ, Im φ, and Coker φ, which are also G-modules.

A *subrepresentation* of a representation V is a vector subspace W of V which is invariant under G. A representation V is called *irreducible* if there is no proper nonzero invariant subspace W of V.

If V and W are representations, the *direct sum* $V \oplus W$ and the *tensor product* $V \otimes W$ are also representations, the latter via

$$g(v \otimes w) = gv \otimes gw.$$

For a representation V, the nth tensor power $V^{\otimes n}$ is again a representation of G by this rule, and the *exterior powers* $\Lambda^n(V)$ and *symmetric powers* $\mathrm{Sym}^n(V)$ are subrepresentations[1] of it. The *dual* $V^* = \mathrm{Hom}(V, \mathbb{C})$ of V is also a representation, though not in the most obvious way: we want the two representations of G to respect the natural pairing (denoted $\langle\ ,\ \rangle$) between V^* and V, so that if $\rho: G \to \mathrm{GL}(V)$ is a representation and $\rho^*: G \to \mathrm{GL}(V^*)$ is the dual, we should have

$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle$$

for all $g \in G$, $v \in V$, and $v^* \in V^*$. This in turn forces us to define the dual representation by

$$\rho^*(g) = {}^t\rho(g^{-1}): V^* \to V^*$$

for all $g \in G$.

Exercise 1.1. Verify that with this definition of ρ^*, the relation above is satisfied.

Having defined the dual of a representation and the tensor product of two representations, it is likewise the case that if V and W are representations, then $\mathrm{Hom}(V, W)$ is also a representation, via the identification $\mathrm{Hom}(V, W) = V^* \otimes W$. Unraveling this, if we view an element of $\mathrm{Hom}(V, W)$ as a linear map φ from V to W, we have

$$(g\varphi)(v) = g\varphi(g^{-1}v)$$

for all $v \in V$. In other words, the definition is such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \varphi\ } & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{\ g\varphi\ } & W \end{array}$$

commutes. Note that the dual representation is, in turn, a special case of this:

[1] For more on exterior and symmetric powers, including descriptions as quotient spaces of tensor powers, see Appendix B.

when $W = \mathbb{C}$ is the *trivial* representation, i.e., $gw = w$ for all $w \in \mathbb{C}$, this makes V^* into a G-module, with $g\varphi(v) = \varphi(g^{-1}v)$, i.e., $g\varphi = {}^t(g^{-1})\varphi$.

Exercise 1.2. Verify that in general the vector space of G-linear maps between two representations V and W of G is just the subspace $\mathrm{Hom}(V, W)^G$ of elements of $\mathrm{Hom}(V, W)$ fixed under the action of G. This subspace is often denoted $\mathrm{Hom}_G(V, W)$.

We have, in effect, taken the identification $\mathrm{Hom}(V, W) = V^* \otimes W$ as the definition of the representation $\mathrm{Hom}(V, W)$. More generally, the usual identities for vector spaces are also true for representations, e.g.,

$$V \otimes (U \oplus W) = (V \otimes U) \oplus (V \otimes W),$$

$$\Lambda^k(V \oplus W) = \bigoplus_{a+b=k} \Lambda^a V \otimes \Lambda^b W,$$

$$\Lambda^k(V^*) = \Lambda^k(V)^*,$$

and so on.

Exercise 1.3*. Let $\rho: G \to \mathrm{GL}(V)$ be any representation of the finite group G on an n-dimensional vector space V and suppose that for any $g \in G$, the determinant of $\rho(g)$ is 1. Show that the spaces $\Lambda^k V$ and $\Lambda^{n-k} V^*$ are isomorphic as representations of G.

If X is any finite set and G acts on the left on X, i.e., $G \to \mathrm{Aut}(X)$ is a homomorphism to the permutation group of X, there is an associated *permutation representation*: let V be the vector space with basis $\{e_x : x \in X\}$, and let G act on V by

$$g \cdot \sum a_x e_x = \sum a_x e_{gx}.$$

The *regular representation*, denoted R_G or R, corresponds to the left action of G on itself. Alternatively, R is the space of complex-valued functions on G, where an element $g \in G$ acts on a function α by $(g\alpha)(h) = \alpha(g^{-1}h)$.

Exercise 1.4*. (a) Verify that these two descriptions of R agree, by identifying the element e_x with the characteristic function which takes the value 1 on x, 0 on other elements of G.

(b) The space of functions on G can also be made into a G-module by the rule $(g\alpha)(h) = \alpha(hg)$. Show that this is an isomorphic representation.

§1.2. Complete Reducibility; Schur's Lemma

As in any study, before we begin our attempt to classify the representations of a finite group G in earnest we should try to simplify life by restricting our search somewhat. Specifically, we have seen that representations of G can be built up out of other representations by linear algebraic operations, most simply by taking the direct sum. We should focus, then, on representations that are "atomic" with respect to this operation, i.e., that cannot be expressed as a direct sum of others; the usual term for such a representation is *indecomposable*. Happily, the situation is as nice as it could possibly be: a representation is atomic in this sense if and only if it is irreducible (i.e., contains no proper subrepresentations); and every representation is the direct sum of irreducibles, in a suitable sense uniquely so. The key to all this is

Proposition 1.5. *If W is a subrepresentation of a representation V of a finite group G, then there is a complementary invariant subspace W' of V, so that $V = W \oplus W'$.*

PROOF. There are two ways of doing this. One can introduce a (positive definite) Hermitian inner product H on V which is preserved by each $g \in G$ (i.e., such that $H(gv, gw) = H(v, w)$ for all $v, w \in V$ and $g \in G$). Indeed, if H_0 is any Hermitian product on V, one gets such an H by averaging over G:

$$H(v, w) = \sum_{g \in G} H_0(gv, gw).$$

Then the perpendicular subspace $W^{\perp}$ is complementary to W in V. Alternatively (but similarly), we can simply choose an arbitrary subspace U complementary to W, let $\pi_0 \colon V \to W$ be the projection given by the direct sum decomposition $V = W \oplus U$, and average the map π_0 over G: that is, take

$$\pi(v) = \sum_{g \in G} g(\pi_0(g^{-1}v)).$$

This will then be a G-linear map from V onto W, which is multiplication by $|G|$ on W; its kernel will, therefore, be a subspace of V invariant under G and complementary to W. □

Corollary 1.6. *Any representation is a direct sum of irreducible representations.*

This property is called *complete reducibility*, or *semisimplicity*. We will see that, for continuous representations, the circle S^1, or any compact group, has this property; integration over the group (with respect to an invariant measure on the group) plays the role of averaging in the above proof. The (additive) group $\mathbb{R}$ does not have this property: the representation

$$a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

leaves the x axis fixed, but there is no complementary subspace. We will see other Lie groups such as $\mathrm{SL}_n(\mathbb{C})$ that are semisimple in this sense. Note also that this argument would fail if the vector space V was over a field of finite characteristic since it might then be the case that $\pi(v) = 0$ for $v \in W$. The failure

of complete reducibility is one of the things that makes the subject of *modular representations*, or representations on vector spaces over finite fields, so tricky.

The extent to which the decomposition of an arbitrary representation into a direct sum of irreducible ones is unique is one of the consequences of the following:

Schur's Lemma 1.7. *If V and W are irreducible representations of G and $\varphi \colon V \to W$ is a G-module homomorphism, then*

(1) *Either φ is an isomorphism, or $\varphi = 0$.*
(2) *If $V = W$, then $\varphi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$, I the identity.*

PROOF. The first claim follows from the fact that Ker φ and Im φ are invariant subspaces. For the second, since $\mathbb{C}$ is algebraically closed, φ must have an eigenvalue λ, i.e., for some $\lambda \in \mathbb{C}$, $\varphi - \lambda I$ has a nonzero kernel. By (1), then, we must have $\varphi - \lambda I = 0$, so $\varphi = \lambda I$. □

We can summarize what we have shown so far in

Proposition 1.8. *For any representation V of a finite group G, there is a decomposition*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations. The decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i.

PROOF. It follows from Schur's lemma that if W is another representation of G, with a decomposition $W = \bigoplus W_j^{\oplus b_j}$, and $\varphi \colon V \to W$ is a map of representations, then φ must map the factor $V_i^{\oplus a_i}$ into that factor $W_j^{\oplus b_j}$ for which $W_j \cong V_i$; when applied to the identity map of V to V, the stated uniqueness follows. □

In the next lecture we will give a formula for the projection of V onto $V_i^{\oplus a_i}$. The decomposition of the ith summand into a direct sum of a_i copies of V_i is not unique if $a_i > 1$, however.

Occasionally the decomposition is written

$$V = a_1 V_1 \oplus \cdots \oplus a_k V_k = a_1 V_1 + \cdots + a_k V_k, \tag{1.9}$$

especially when one is concerned only about the isomorphism classes and multiplicities of the V_i.

One more fact that will be established in the following lecture is that a finite group G admits only finitely many irreducible representations V_i up to isomorphism (in fact, we will say how many). This, then, is the framework of the classification of all representations of G: by the above, once we have described the irreducible representations of G, we will be able to describe an arbitrary representation as a linear combination of these. Our first goal, in analyzing the representations of any group, will therefore be:

(i) *Describe all the irreducible representations of G.*

Once we have done this, there remains the problem of carrying out in practice the description of a given representation in these terms. Thus, our second goal will be:

(ii) *Find techniques for giving the direct sum decomposition* (1.9), *and in particular determining the multiplicities a_i of an arbitrary representation V.*

Finally, it is the case that the representations we will most often be concerned with are those arising from simpler ones by the sort of linear- or multilinear-algebraic operations described above. We would like, therefore, to be able to describe, in the terms above, the representation we get when we perform these operations on a known representation. This is known generally as

(iii) *Plethysm: Describe the decompositions, with multiplicities, of representations derived from a given representation V,* such as $V \otimes V$, V^*, $\Lambda^k(V)$, $\mathrm{Sym}^k(V)$, and $\Lambda^k(\Lambda^l V)$. Note that if V decomposes into a sum of two representations, these representations decompose accordingly; e.g., if $V = U \oplus W$, then

$$\Lambda^k V = \bigoplus_{i+j=k} \Lambda^i U \otimes \Lambda^j W,$$

so it is enough to work out this plethysm for irreducible representations. Similarly, if V and W are two irreducible representations, we want to decompose $V \otimes W$; this is usually known as the *Clebsch–Gordon* problem.

§1.3. Examples: Abelian Groups; $\mathfrak{S}_3$

One obvious place to look for examples is with abelian groups. It does not take long, however, to deal with this case. Basically, we may observe in general that if V is a representation of the finite group G, abelian or not, each $g \in G$ gives a map $\rho(g)\colon V \to V$; but *this map is not generally a G-module homomorphism*: for general $h \in G$ we will have

$$g(h(v)) \neq h(g(v)).$$

Indeed, $\rho(g)\colon V \to V$ *will be G-linear for every ρ if (and only if) g is in the center* $Z(G)$ of G. In particular if G is abelian, and V is an irreducible representation, then by Schur's lemma every element $g \in G$ acts on V by a scalar multiple of the identity. Every subspace of V is thus invariant; so that V must be one dimensional. The irreducible representations of an abelian group G are thus simply elements of the dual group, that is, homomorphisms

$$\rho\colon G \to \mathbb{C}^*.$$

We consider next the simplest nonabelian group, $G = \mathfrak{S}_3$. To begin with, we have (as with any symmetric group) two one-dimensional representations: we have the trivial representation, which we will denote U, and the *alternating representation* U', defined by setting

$$gv = \mathrm{sgn}(g)v$$

for $g \in G$, $v \in \mathbb{C}$. Next, since G comes to us as a permutation group, we have a natural permutation representation, in which G acts on $\mathbb{C}^3$ by permuting the coordinates. Explicitly, if $\{e_1, e_2, e_3\}$ is the standard basis, then $g \cdot e_i = e_{g(i)}$, or, equivalently,

$$g \cdot (z_1, z_2, z_3) = (z_{g^{-1}(1)}, z_{g^{-1}(2)}, z_{g^{-1}(3)}).$$

This representation, like any permutation representation, is not irreducible: the line spanned by the sum $(1, 1, 1)$ of the basis vectors is invariant, with complementary subspace

$$V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + z_2 + z_3 = 0\}.$$

This two-dimensional representation V is easily seen to be irreducible; we call it the *standard representation* of $\mathfrak{S}_3$.

Let us now turn to the problem of describing an arbitrary representation of $\mathfrak{S}_3$. We will see in the next lecture a wonderful tool for doing this, called *character theory*; but, as inefficient as this may be, we would like here to adopt a more ad hoc approach. This has some virtues as a didactic technique in the present context (admittedly dubious ones, consisting mainly of making the point that there are other and far worse ways of doing things than character theory). The real reason we are doing it is that it will serve to introduce an idea that, while superfluous for analyzing the representations of finite groups in general, will prove to be the key to understanding representations of Lie groups.

The idea is a very simple one: since we have just seen that the representation theory of a finite abelian group is virtually trivial, we will start our analysis of an arbitrary representation W of $\mathfrak{S}_3$ by looking just at the action of the abelian subgroup $\mathfrak{A}_3 = \mathbb{Z}/3 \subset \mathfrak{S}_3$ on W. This yields a very simple decomposition: if we take τ to be any generator of $\mathfrak{A}_3$ (that is, any three-cycle), the space W is spanned by eigenvectors v_i for the action of τ, whose eigenvalues are of course all powers of a cube root of unity $\omega = e^{2\pi i/3}$. Thus,

$$W = \bigoplus V_i,$$

where

$$V_i = \mathbb{C}v_i \quad \text{and} \quad \tau v_i = \omega^{\alpha_i} v_i.$$

Next, we ask how the remaining elements of $\mathfrak{S}_3$ act on W in terms of this decomposition. To see how this goes, let σ be any transposition, so that τ and σ together generate $\mathfrak{S}_3$, with the relation $\sigma\tau\sigma = \tau^2$. We want to know where σ sends an eigenvector v for the action of τ, say with eigenvalue ω^i; to answer this, we look at how τ acts on $\sigma(v)$. We use the basic relation above to write

$$\begin{aligned}\tau(\sigma(v)) &= \sigma(\tau^2(v))\\ &= \sigma(\omega^{2i}\cdot v)\\ &= \omega^{2i}\cdot\sigma(v).\end{aligned}$$

The conclusion, then, is that *if v is an eigenvector for τ with eigenvalue ω^i, then $\sigma(v)$ is again an eigenvector for τ, with eigenvalue ω^{2i}.*

Exercise 1.10. Verify that with $\sigma = (12)$, $\tau = (123)$, the standard representation has a basis $\alpha = (\omega, 1, \omega^2)$, $\beta = (1, \omega, \omega^2)$, with

$$\tau\alpha = \omega\alpha, \qquad \tau\beta = \omega^2\beta, \qquad \sigma\alpha = \beta, \qquad \sigma\beta = \alpha.$$

Suppose now that we start with such an eigenvector v for τ. If the eigenvalue of v is $\omega^i \neq 1$, then $\sigma(v)$ is an eigenvector with eigenvalue $\omega^{2i} \neq \omega^i$, and so is independent of v; and v and $\sigma(v)$ together span a two-dimensional subspace V' of W invariant under $\mathfrak{S}_3$. In fact, V' is isomorphic to the standard representation, which follows from Exercise 1.10. If, on the other hand, the eigenvalue of v is 1, then $\sigma(v)$ may or may not be independent of v. If it is not, then v spans a one-dimensional subrepresentation of W, isomorphic to the trivial representation if $\sigma(v) = v$ and to the alternating representation if $\sigma(v) = -v$. If $\sigma(v)$ and v are independent, then $v + \sigma(v)$ and $v - \sigma(v)$ span one-dimensional representations of W isomorphic to the trivial and alternating representations, respectively.

We have thus accomplished the first two of the goals we have set for ourselves above in the case of the group $G = \mathfrak{S}_3$. First, we see from the above that *the only three irreducible representations of $\mathfrak{S}_3$ are the trivial, alternating, and standard representations U, U' and V.* Moreover, for an arbitrary representation W of $\mathfrak{S}_3$ we can write

$$W = U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c};$$

and we have a way to determine the multiplicities a, b, and c: c, for example, is the number of independent eigenvectors for τ with eigenvalue ω, whereas $a + c$ is the multiplicity of 1 as an eigenvalue of σ, and $b + c$ is the multiplicity of -1 as an eigenvalue of σ.

In fact, this approach gives us as well the answer to our third problem, finding the decomposition of the symmetric, alternating, or tensor powers of a given representation W, since if we know the eigenvalues of τ on such a representation, we know the eigenvalues of τ on the various tensor powers of W. For example, we can use this method to decompose $V \otimes V$, where V is the standard two-dimensional representation. For $V \otimes V$ is spanned by the vectors $\alpha\otimes\alpha$, $\alpha\otimes\beta$, $\beta\otimes\alpha$, and $\beta\otimes\beta$; these are eigenvectors for τ with eigenvalues ω^2, 1, 1, and ω, respectively, and σ interchanges $\alpha\otimes\alpha$ with $\beta\otimes\beta$, and $\alpha\otimes\beta$ with $\beta\otimes\alpha$. Thus $\alpha\otimes\alpha$ and $\beta\otimes\beta$ span a subrepresentation isomorphic to V, $\alpha\otimes\beta + \beta\otimes\alpha$ spans a trivial representation U, and $\alpha\otimes\beta - \beta\otimes\alpha$ spans U', so

$$V\otimes V \cong U \oplus U' \oplus V.$$

Exercise 1.11. Use this approach to find the decomposition of the representations $\mathrm{Sym}^2 V$ and $\mathrm{Sym}^3 V$.

Exercise 1.12. (a) Decompose the regular representation R of $\mathfrak{S}_3$.

(b) Show that $\mathrm{Sym}^{k+6}V$ is isomorphic to $\mathrm{Sym}^k V \oplus R$, and compute $\mathrm{Sym}^k V$ for all k.

Exercise 1.13*. Show that $\mathrm{Sym}^2(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^3(\mathrm{Sym}^2 V)$. Is $\mathrm{Sym}^m(\mathrm{Sym}^n V)$ isomorphic to $\mathrm{Sym}^n(\mathrm{Sym}^m V)$?

As we have indicated, the idea of studying a representation V of a group G by first restricting the action to an abelian subgroup, getting a decomposition of V into one-dimensional invariant subspaces, and then asking how the remaining generators of the group act on these subspaces, does not work well for finite G in general; for one thing, there will not in general be a convenient abelian subgroup to use. This idea will turn out, however, to be the key to understanding the representations of Lie groups, with a torus subgroup playing the role of the cyclic subgroup in this example.

Exercise 1.14*. Let V be an irreducible representation of the finite group G. Show that, up to scalars, there is a *unique* Hermitian inner product on V preserved by G.

LECTURE 2

Characters

This lecture contains the heart of our treatment of the representation theory of finite groups: the definition in §2.1 of the character of a representation, and the main theorem (proved in two steps in §2.2 and §2.4) that the characters of the irreducible representations form an orthonormal basis for the space of class functions on G. There will be more examples and more constructions in the following lectures, but this is what you need to know.

§2.1: Characters
§2.2: The first projection formula and its consequences
§2.3: Examples: $\mathfrak{S}_4$ and $\mathfrak{A}_4$
§2.4: More projection formulas; more consequences

§2.1. Characters

As we indicated in the preceding section, there is a remarkably effective tool for understanding the representations of a finite group G, called *character theory*. This is in some ways motivated by the example worked out in the last section where we saw that a representation of $\mathfrak{S}_3$ was determined by knowing the eigenvalues of the action of the elements τ and $\sigma \in \mathfrak{S}_3$. For a general group G, it is not clear what subgroups and/or elements should play the role of $\mathfrak{A}_3$, τ, and σ; but the example certainly suggests that knowing all the eigenvalues of each element of G should suffice to describe the representation.

Of course, specifying all the eigenvalues of the action of each element of G is somewhat unwieldy; but fortunately it is redundant as well. For example, if we know the eigenvalues $\{\lambda_i\}$ of an element $g \in G$, then of course we know the eigenvalues $\{\lambda_i^k\}$ of g^k for each k as well. We can thus use this redundancy to simplify the data we have to specify. The key observation here is it is enough to give, for example, just the *sum* of the eigenvalues of each element of G, since knowing the sums $\sum \lambda_i^k$ of the kth powers of the eigenvalues of a given element $g \in G$ is equivalent to knowing the eigenvalues $\{\lambda_i\}$ of g themselves. This then suggests the following:

Definition. If V is a representation of G, its *character* χ_V is the complex-valued function on the group defined by

$$\chi_V(g) = \mathrm{Tr}(g|_V),$$

the trace of g on V.

In particular, we have

$$\chi_V(hgh^{-1}) = \chi_V(g),$$

so that χ_V is constant on the conjugacy classes of G; such a function is called a *class function*. Note that $\chi_V(1) = \dim V$.

Proposition 2.1. *Let V and W be representations of G. Then*

$$\chi_{V \oplus W} = \chi_V + \chi_W, \qquad \chi_{V \otimes W} = \chi_V \cdot \chi_W,$$
$$\chi_{V^*} = \overline{\chi}_V \quad \textit{and} \quad \chi_{\wedge^2 V}(g) = \tfrac{1}{2}[\chi_V(g)^2 - \chi_V(g^2)].$$

PROOF. We compute the values of these characters on a fixed element $g \in G$. For the action of g, V has eigenvalues $\{\lambda_i\}$ and W has eigenvalues $\{\mu_i\}$. Then $\{\lambda_i + \mu_j\}$ and $\{\lambda_i \cdot \mu_j\}$ are eigenvalues for $V \oplus W$ and $V \otimes W$, from which the first two formulas follow. Similarly $\{\lambda_i^{-1} = \bar{\lambda}_i\}$ are the eigenvalues for g on V^*, since all eigenvalues are nth roots of unity, with n the order of g. Finally, $\{\lambda_i\lambda_j | i < j\}$ are the eigenvalues for g on $\wedge^2 V$, and

$$\sum_{i<j} \lambda_i\lambda_j = \frac{(\sum \lambda_i)^2 - \sum \lambda_i^2}{2};$$

and since g^2 has eigenvalues $\{\lambda_i^2\}$, the last formula follows. □

Exercise 2.2. For $\mathrm{Sym}^2 V$, verify that

$$\chi_{\mathrm{Sym}^2 V}(g) = \tfrac{1}{2}[\chi_V(g)^2 + \chi_V(g^2)].$$

Note that this is compatible with the decomposition

$$V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V.$$

Exercise 2.3*. Compute the characters of $\mathrm{Sym}^k V$ and $\wedge^k V$.

Exercise 2.4*. Show that if we know the character χ_V of a representation V, then we know the eigenvalues of each element g of G, in the sense that we know the coefficients of the characteristic polynomial of $g: V \to V$. Carry this out explicitly for elements $g \in G$ of orders 2, 3, and 4, and for a representation of G on a vector space of dimension 2, 3, or 4.

Exercise 2.5. (*The original fixed-point formula*). If V is the permutation representation associated to the action of a group G on a finite set X, show that $\chi_V(g)$ is the number of elements of X fixed by g.

As we have said, the character of a representation of a group G is really a function on the set of conjugacy classes in G. This suggests expressing the basic information about the irreducible representations of a group G in the form of a *character table*. This is a table with the conjugacy classes $[g]$ of G listed across the top, usually given by a representative g, with (for reasons that will become apparent later) the number of elements in each conjugacy class over it; the irreducible representations V of G listed on the left; and, in the appropriate box, the value of the character χ_V on the conjugacy class $[g]$.

Example 2.6. We compute the character table of $\mathfrak{S}_3$. This is easy: to begin with, the trivial representation takes the values (1, 1, 1) on the three conjugacy classes [1], [(12)], and [(123)], whereas the alternating representation has values (1, −1, 1). To see the character of the standard representation, note that the permutation representation decomposes: $\mathbb{C}^3 = U \oplus V$; since the character of the permutation representation has, by Exercise 2.5, the values (3, 1, 0), we have $\chi_V = \chi_{\mathbb{C}^3} - \chi_U = (3, 1, 0) - (1, 1, 1) = (2, 0, -1)$. In sum, then, the character table of $\mathfrak{S}_3$ is

	1	3	2
$\mathfrak{S}_3$	1	(12)	(123)
trivial U	1	1	1
alternating U'	1	−1	1
standard V	2	0	−1

This gives us another solution of the basic problem posed in Lecture 1: if W is any representation of $\mathfrak{S}_3$ and we decompose W into irreducible representations $W \cong U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$, then $\chi_W = a\chi_U + b\chi_{U'} + c\chi_V$. In particular, since the functions χ_U, $\chi_{U'}$ and χ_V are independent, we see that *W is determined up to isomorphism by its character χ_W.*

Consider, for example, $V \otimes V$. Its character is $(\chi_V)^2$, which has values 4, 0, and 1 on the three conjugacy classes. Since $V \oplus U \oplus U'$ has the same character, this implies that $V \otimes V$ decomposes into $V \oplus U \oplus U'$, as we have seen directly. Similarly, $V \otimes U'$ has values 2, 0, and −1, so $V \otimes U' \cong V$.

Exercise 2.7*. Find the decomposition of the representation $V^{\otimes n}$ using character theory.

Characters will be similarly useful for larger groups, although it is rare to find simple closed formulas for decomposing tensor products.

§2.2. The First Projection Formula and Its Consequences

In the last lecture, we asked (among other things) for a way of locating explicitly the direct sum factors in the decomposition of a representation into irreducible ones. In this section we will start by giving an explicit formula for the projection of an irreducible representation onto the direct sum of the trivial factors in this decomposition; as it will turn out, this formula alone has tremendous consequences.

To start, for any representation V of a group G, we set

$$V^G = \{v \in V: gv = v \quad \forall g \in G\}.$$

We ask for a way of finding V^G explicitly. The idea behind our solution to this is already implicit in the previous lecture. We observed there that for any representation V of G and any $g \in G$, the endomorphism $g: V \to V$ is, in general, not a G-module homomorphism. On the other hand, if we take the *average* of all these endomorphisms, that is, we set

$$\varphi = \frac{1}{|G|}\sum_{g \in G} g \in \mathrm{End}(V),$$

then the endomorphism φ will be G-linear since $\sum g = \sum hgh^{-1}$. In fact, we have

Proposition 2.8. *The map φ is a projection of V onto V^G.*

PROOF. First, suppose $v = \varphi(w) = (1/|G|)\sum gw$. Then, for any $h \in G$,

$$hv = \frac{1}{|G|}\sum hgw = \frac{1}{|G|}\sum gw,$$

so the image of φ is contained in V^G. Conversely, if $v \in V^G$, then $\varphi(v) = (1/|G|)\sum v = v$, so $V^G \subset \mathrm{Im}(\varphi)$; and $\varphi \circ \varphi = \varphi$. □

We thus have a way of finding explicitly the direct sum of the trivial subrepresentations of a given representation, although the formula can be hard to use if it does not simplify. If we just want to know the number m of copies of the trivial representation appearing in the decomposition of V, we can do this numerically, since this number will be just the trace of the projection φ. We have

$$\begin{aligned} m = \dim V^G &= \operatorname{Trace}(\varphi) \\ &= \frac{1}{|G|}\sum_{g\in G}\operatorname{Trace}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g). \end{aligned} \tag{2.9}$$

In particular, we observe that for an irreducible representation V other than the trivial one, the sum over all $g \in G$ of the values of the character χ_V is zero.

We can do much more with this idea, however. The key is to use Exercise 1.2: if V and W are representations of G, then with $\operatorname{Hom}(V, W)$, the representation defined in Lecture 1, we have

$$\operatorname{Hom}(V, W)^G = \{G\text{-module homomorphisms from } V \text{ to } W\}.$$

If V is irreducible then by Schur's lemma $\dim \operatorname{Hom}(V, W)^G$ is the multiplicity of V in W; similarly, if W is irreducible, $\dim \operatorname{Hom}(V, W)^G$ is the multiplicity of W in V, and in the case where both V and W are irreducible, we have

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

But now the character $\chi_{\operatorname{Hom}(V,W)}$ of the representation $\operatorname{Hom}(V, W) = V^* \otimes W$ is given by

$$\chi_{\operatorname{Hom}(V,W)}(g) = \overline{\chi_V(g)}\cdot\chi_W(g).$$

We can now apply formula (2.9) in this case to obtain the striking

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_W(g) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases} \tag{2.10}$$

To express this, let

$$\mathbb{C}_{\text{class}}(G) = \{\text{class functions on } G\}$$

and define an Hermitian inner product on $\mathbb{C}_{\text{class}}(G)$ by

$$(\alpha, \beta) = \frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)}\beta(g). \tag{2.11}$$

Formula (2.10) then amounts to

Theorem 2.12. *In terms of this inner product, the characters of the irreducible representations of G are orthonormal.*

For example, the orthonormality of the three irreducible representations of $\mathfrak{S}_3$ can be read from its character table in Example 2.6. The numbers over each conjugacy class tell how many times to count entries in that column.

Corollary 2.13. *The number of irreducible representations of G is less than or equal to the number of conjugacy classes.*

We will soon show that there are no nonzero class functions orthogonal to the characters, so that equality holds in Corollary 2.13.

Corollary 2.14. *Any representation is determined by its character.*

Indeed if $V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$, with the V_i distinct irreducible characters, then $\chi_V = \sum a_i\chi_{V_i}$, and the χ_{V_i} are linearly independent.

Corollary 2.15. *A representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$.*

In fact, if $V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$ as above, then $(\chi_V, \chi_V) = \sum a_i^2$.

The multiplicities a_i can be calculated via

Corollary 2.16. *The multiplicity a_i of V_i in V is the inner product of χ_V with χ_{V_i}, i.e., $a_i = (\chi_V, \chi_{V_i})$.*

We obtain some further corollaries by applying all this to the regular representation R of G. First, by Exercise 2.5 we know the character of R; it is simply

$$\chi_R(g) = \begin{cases} 0 & \text{if } g \neq e \\ |G| & \text{if } g = e. \end{cases}$$

Thus, we see first of all that R is not irreducible if $G \neq \{e\}$. In fact, if we set $R = \bigoplus V_i^{\oplus a_i}$, with V_i distinct irreducibles, then

$$a_i = (\chi_{V_i}, \chi_R) = \frac{1}{|G|}\chi_{V_i}(e)\cdot|G| = \dim V_i. \tag{2.17}$$

Corollary 2.18. *Any irreducible representation V of G appears in the regular representation $\dim V$ times.*

In particular, this proves again that there are only finitely many irreducible representations. As a numerical consequence of this we have the formula

$$|G| = \dim(R) = \sum_i \dim(V_i)^2. \tag{2.19}$$

Also, applying this to the value of the character of the regular representation on an element $g \in G$ other than the identity, we have

$$0 = \sum (\dim V_i)\cdot\chi_{V_i}(g) \quad \text{if } g \neq e. \tag{2.20}$$

These two formulas amount to the Fourier inversion formula for finite groups, cf. Example 3.32. For example, if all but one of the characters is known, they give a formula for the unknown character.

Exercise 2.21. The orthogonality of the rows of the character table is equivalent to an orthogonality for the columns (assuming the fact that there are as many rows as columns). Written out, this says:

(i) For $g \in G$,

$$\sum_\chi \overline{\chi(g)}\chi(g) = \frac{|G|}{c(g)},$$

where the sum is over all irreducible characters, and $c(g)$ is the number of elements in the conjugacy class of g.

(ii) If g and h are elements of G that are not conjugate, then

$$\sum_\chi \overline{\chi(g)}\chi(h) = 0.$$

Note that for $g = e$ these reduce to (2.19) and (2.20).

§2.3. Examples: $\mathfrak{S}_4$ and $\mathfrak{A}_4$

To see how the analysis of the characters of a group actually goes in practice, we now work out the character table of $\mathfrak{S}_4$. To start, we list the conjugacy classes in $\mathfrak{S}_4$ and the number of elements of $\mathfrak{S}_4$ in each. As with any symmetric group $\mathfrak{S}_d$, the conjugacy classes correspond naturally to the *partitions* of d, that is, expressions of d as a sum of positive integers $a_1, a_2, \ldots, a_k$, where the correspondence associates to such a partition the conjugacy class of a permutation consisting of disjoint cycles of length $a_1, a_2, \ldots, a_k$. Thus, in $\mathfrak{S}_4$ we have the classes of the identity element 1 ($4 = 1 + 1 + 1 + 1$), a transposition such as (12), corresponding to the partition $4 = 2 + 1 + 1$; a three-cycle (123) corresponding to $4 = 3 + 1$; a four-cycle (1234) ($4 = 4$); and the product of two disjoint transpositions (12)(34) ($4 = 2 + 2$).

Exercise 2.22. Show that the number of elements in each of these conjugacy classes is, respectively, 1, 6, 8, 6, and 3.

As for the irreducible representations of $\mathfrak{S}_4$, we start with the same ones that we had in the case of $\mathfrak{S}_3$: the trivial U, the alternating U', and the standard representation V, i.e., the quotient of the permutation representation associated to the standard action of $\mathfrak{S}_4$ on a set of four elements by the trivial subrepresentation. The character of the trivial representation on the five conjugacy classes is of course (1, 1, 1, 1, 1), and that of the alternating representation is (1, −1, 1, −1, 1). To find the character of the standard representation, we observe that by Exercise 2.5 the character of the permutation representation on $\mathbb{C}^4$ is $\chi_{\mathbb{C}^4} = (4, 2, 1, 0, 0)$ and, correspondingly,

$$\chi_V = \chi_{\mathbb{C}^4} - \chi_U = (3, 1, 0, -1, -1).$$

Note that $|\chi_V| = 1$, so V is irreducible. The character table so far looks like

	1	6	8	6	3
$\mathfrak{S}_4$	1	(12)	(123)	(1234)	(12)(34)
trivial U	1	1	1	1	1
alternating U'	1	−1	1	−1	1
standard V	3	1	0	−1	−1

Clearly, we are not done yet: since the sum of the squares of the dimensions of these three representations is $1 + 1 + 9 = 11$, by (2.19) there must be additional irreducible representations of $\mathfrak{S}_4$, the squares of whose dimensions add up to $24 - 11 = 13$. Since there are by Corollary 2.13 at most two of them, there must be exactly two, of dimensions 2 and 3. The latter of these is easy to locate: if we just tensor the standard representation V with the alternating one U', we arrive at a representation V' with character $\chi_{V'} = \chi_V \cdot \chi_{U'} = (3, -1, 0, 1, -1)$. We can see that this is irreducible either from its character (since $|\chi_{V'}| = 1$) or from the fact that it is the tensor product of an irreducible representation with a one-dimensional one; since its character is not equal to that of any of the first three, this must be one of the two missing ones. As for the remaining representation of degree two, we will for now simply call it W; we can determine its character from the orthogonality relations (2.10). We obtain then the complete character table for $\mathfrak{S}_4$:

	1	6	8	6	3
$\mathfrak{S}_4$	1	(12)	(123)	(1234)	(12)(34)
trivial U	1	1	1	1	1
alternating U'	1	−1	1	−1	1
standard V	3	1	0	−1	−1
$V' = V \otimes U'$	3	−1	0	1	−1
Another W	2	0	−1	0	2

Exercise 2.23. Verify the last row of this table from (2.10) or (2.20).

We now get a dividend: we can take the character of the mystery representation W, which we have obtained from general character theory alone, and use it to describe the representation W explicitly! The key is the 2 in the last column for χ_W: this says that the action of (12)(34) on the two-dimensional vector space W is an involution of trace 2, and so must be the identity. Thus, W is really a representation of the quotient group[1]

[1] If N is a normal subgroup of a group G, a representation $\rho\colon G \to GL(V)$ is trivial on N if and only if it factors through the quotient

$$G \to G/N \to GL(V).$$

Representations of G/N can be identified with representations of G that are trivial on N.

$$\mathfrak{S}_4/\{1, (12)(34), (13)(24), (14)(23)\} \cong \mathfrak{S}_3.$$

[One may see this isomorphism by letting $\mathfrak{S}_4$ act on the elements of the conjugacy class of (12)(34); equivalently, if we realize $\mathfrak{S}_4$ as the group of rigid motions of a cube (see below), by looking at the action of $\mathfrak{S}_4$ on pairs of opposite faces.] W must then be just the standard representation of $\mathfrak{S}_3$ pulled back to $\mathfrak{S}_4$ via this quotient.

Example 2.24. As we said above, the group of rigid motions of a cube is the symmetric group on four letters; $\mathfrak{S}_4$ acts on the cube via its action on the four long diagonals. It follows, of course, that $\mathfrak{S}_4$ acts as well on the set of faces, of edges, of vertices, etc.; and to each of these is associated a permutation representation of $\mathfrak{S}_4$. We may thus ask how these representations decompose; we will do here the case of the faces and leave the others as exercises.

We start, of course, by describing the character χ of the permutation representation associated to the faces of the cube. Rotation by 180° about a line joining the midpoints of two opposite edges is a transposition in $\mathfrak{S}_4$ and fixes no faces, so $\chi(12) = 0$. Rotation by 120° about a long diagonal shows $\chi(123) = 0$. Rotation by 90° about a line joining the midpoints of two opposite faces shows $\chi(1234) = 2$, and rotation by 180° gives $\chi((12)(34)) = 2$. Now $(\chi, \chi) = 3$, so χ is the sum of three distinct irreducible representations. From the table, $(\chi, \chi_U) = (\chi, \chi_{V'}) = (\chi, \chi_W) = 1$, and the inner products with the others are zero, so this representation is $U \oplus V' \oplus W$. In fact, the sums of opposite faces span a three-dimensional subrepresentation which contains U (spanned by the sum of all faces), so this representation is $U \oplus W$. The differences of opposite faces therefore span V'.

Exercise 2.25*. Decompose the permutation representation of $\mathfrak{S}_4$ on (i) the vertices and (ii) the edges of the cube.

Exercise 2.26. The alternating group $\mathfrak{A}_4$ has four conjugacy classes. Three representations U, U', and U'' come from the representations of

$$\mathfrak{A}_4/\{1, (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}/3,$$

so there is one more irreducible representation V of dimension 3. Compute the character table, with $\omega = e^{2\pi i/3}$:

	1	4	4	3
$\mathfrak{A}_4$	1	(123)	(132)	(12)(34)
U	1	1	1	1
U'	1	ω	ω^2	1
U''	1	ω^2	ω	1
V	3	0	0	-1

Exercise 2.27. Consider the representations of $\mathfrak{S}_4$ and their restrictions to $\mathfrak{A}_4$. Which are still irreducible when restricted, and which decompose? Which pairs of nonisomorphic representations of $\mathfrak{S}_4$ become isomorphic when restricted? Which representations of $\mathfrak{A}_4$ arise as restrictions from $\mathfrak{S}_4$?

§2.4. More Projection Formulas; More Consequences

In this section, we complete the analysis of the characters of the irreducible representations of a general finite group begun in §2.2 and give a more general formula for the projection of a general representation V onto the direct sum of the factors in V isomorphic to a given irreducible representation W. The main idea for both is a generalization of the "averaging" of the endomorphisms $g: V \to V$ used in §2.2, the point being that instead of simply averaging all the g we can ask the question: what linear combinations of the endomorphisms $g: V \to V$ are G-linear endomorphisms? The answer is given by

Proposition 2.28. *Let $\alpha: G \to \mathbb{C}$ be any function on the group G, and for any representation V of G set*

$$\varphi_{\alpha,V} = \sum \alpha(g) \cdot g: V \to V.$$

Then $\varphi_{\alpha,V}$ is a homomorphism of G-modules for all V if and only if α is a class function.

Proof. We simply write out the condition that $\varphi_{\alpha,V}$ be G-linear, and the result falls out: we have

$$\varphi_{\alpha,V}(hv) = \sum \alpha(g) \cdot g(hv)$$
$$= \sum \alpha(hgh^{-1}) \cdot hgh^{-1}(hv)$$

(substituting hgh^{-1} for g)

$$= h(\sum \alpha(hgh^{-1}) \cdot g(v))$$
$$= h(\sum \alpha(g) \cdot g(v))$$

(if α is a class function)

$$= h(\varphi_{\alpha,V}(v)).$$

Exercise 2.29*. Complete this proof by showing that conversely if α is not a class function, then there exists a representation V of G for which $\varphi_{\alpha,V}$ fails to be G-linear. □

As an immediate consequence of this proposition, we have

Proposition 2.30. *The number of irreducible representations of G is equal to the number of conjugacy classes of G. Equivalently, their characters $\{\chi_V\}$ form an orthonormal basis for $\mathbb{C}_{\text{class}}(G)$.*

PROOF. Suppose $\alpha\colon G \to \mathbb{C}$ is a class function and $(\alpha, \chi_V) = 0$ for all irreducible representations V; we must show that $\alpha = 0$. Consider the endomorphism

$$\varphi_{\alpha,V} = \sum \alpha(g)\cdot g\colon V \to V$$

as defined above. By Schur's lemma, $\varphi_{\alpha,V} = \lambda \cdot \mathrm{Id}$; and if $n = \dim V$, then

$$\begin{aligned} \lambda &= \frac{1}{n}\cdot \operatorname{trace}(\varphi_{\alpha,V}) \\ &= \frac{1}{n}\cdot \sum \alpha(g)\chi_V(g) \\ &= \frac{|G|}{n}\overline{(\alpha, \chi_{V^*})} \\ &= 0. \end{aligned}$$

Thus, $\varphi_{\alpha,V} = 0$, or $\sum \alpha(g)\cdot g = 0$ on any representation V of G; in particular, this will be true for the regular representation $V = R$. But in R the elements $\{g \in G\}$, thought of as elements of $\operatorname{End}(R)$, are linearly independent. For example, the elements $\{g(e)\}$ are all independent. Thus $\alpha(g) = 0$ for all g, as required. □

This proposition completes the description of the characters of a finite group in general. We will see in more examples below how we can use this information to build up the character table of a given group. For now, we mention another way of expressing this proposition, via the *representation ring* of the group G.

The representation ring $R(G)$ of a group G is easy to define. First, as a group we just take $R(G)$ to be the free abelian group generated by all (isomorphism classes of) representations of G, and mod out by the subgroup generated by elements of the form $V + W - (V \oplus W)$. Equivalently, given the statement of complete reducibility, we can just take all integral linear combinations $\sum a_i \cdot V_i$ of the irreducible representations V_i of G; elements of $R(G)$ are correspondingly called *virtual representations*. The ring structure is then given simply by tensor product, defined on the generators of $R(G)$ and extended by linearity.

We can express most of what we have learned so far about representations of a finite group G in these terms. To begin, the character defines a map

$$\chi\colon R(G) \to \mathbb{C}_{\text{class}}(G)$$

from $R(G)$ to the ring of complex-valued functions on G; by the basic formulas of Proposition 2.1, this map is in fact a ring homomorphism. The statement that a representation in determined by its character then says that χ is injective; the images of χ are called *virtual characters* and correspond thereby to virtual representations. Finally, our last proposition amounts to the statement that *χ induces an isomorphism*

$$\chi_{\mathbb{C}}\colon R(G) \otimes \mathbb{C} \to \mathbb{C}_{\text{class}}(G).$$

The virtual characters of G form a lattice $\Lambda \cong \mathbb{Z}^c$ in $\mathbb{C}_{\text{class}}(G)$, in which the actual characters sit as a cone $\Lambda_0 \cong \mathbb{N}^c \subset \mathbb{Z}^c$. We can thus think of the problem of describing the characters of G as having two parts: first, we have to find Λ, and then the cone $\Lambda_0 \subset \Lambda$ (once we know Λ_0, the characters of the irreducible representations will be determined). In the following lecture we will state theorems of Artin and Brauer characterizing $\Lambda \otimes \mathbb{Q}$ and Λ.

The argument for Proposition 2.30 also suggests how to obtain a more general projection formula. Explicitly, if W is a fixed irreducible representation, then for any representation V, look at the weighted sum

$$\psi = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_W(g)}\cdot g \in \operatorname{End}(V).$$

By Proposition 2.28, ψ is a G-module homomorphism. Hence, if V is irreducible, we have $\psi = \lambda \cdot \mathrm{Id}$, and

$$\begin{aligned} \lambda &= \frac{1}{\dim V}\operatorname{Trace}\psi \\ &= \frac{1}{\dim V}\cdot\frac{1}{|G|}\sum \overline{\chi_W(g)}\cdot \chi_V(g) \\ &= \begin{cases} \dfrac{1}{\dim V} & \text{if } V = W \\ 0 & \text{if } V \neq W. \end{cases} \end{aligned}$$

For arbitrary V,

$$\psi_V = \dim W \cdot \frac{1}{|G|}\sum_{g\in G}\overline{\chi_W(g)}\cdot g : V \to V \tag{2.31}$$

is the projection of V onto the factor consisting of the sum of all copies of W appearing in V. In other words, if $V = \bigoplus V_i^{\oplus a_i}$, then

$$\pi_i = \dim V_i \cdot \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{V_i}(g)}\cdot g \tag{2.32}$$

is the projection of V onto $V_i^{\oplus a_i}$.

Exercise 2.33*. (a) In terms of representations V and W in $R(G)$, the inner product on $\mathbb{C}_{\text{class}}(G)$ takes the simple form

$$(V, W) = \dim \operatorname{Hom}_G(V, W).$$

(b) If $\chi \in \mathbb{C}_{\text{class}}(G)$ is a virtual character, and $(\chi, \chi) = 1$, then either χ or $-\chi$ is the character of an irreducible representation, the plus sign occurring when $\chi(1) > 0$. If $(\chi, \chi) = 2$, and $\chi(1) > 0$, then χ is either the sum or the difference of two irreducible characters.

(c) If U, V, and W are irreducible representations, show that U appears in $V \otimes W$ if and only if W occurs in $V^* \otimes U$. Deduce that this cannot occur unless $\dim U \geq \dim W/\dim V$.

We conclude this lecture with some exercises that use characters to work out some standard facts about representations.

Exercise 2.34*. Let V and W be irreducible representations of G, and L_0: $V \to W$ any linear mapping. Define L: $V \to W$ by

$$L(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot L_0(g \cdot v).$$

Show that $L = 0$ if V and W are not isomorphic, and that L is multiplication by $\operatorname{trace}(L_0)/\dim(V)$ if $V = W$.

Exercise 2.35*. Show that, if the irreducible representations of G are represented by unitary matrices [cf. Exercise 1.14], the matrix entries of these representations form an orthogonal basis for the space of *all* functions on G [with inner product given by (2.11)].

Exercise 2.36*. If G_1 and G_2 are groups, and V_1 and V_2 are representations of G_1 and G_2, then the tensor product $V_1 \otimes V_2$ is a representation of $G_1 \times G_2$, by $(g_1 \times g_2) \cdot (v_1 \otimes v_2) = g_1 \cdot v_1 \otimes g_2 \cdot v_2$. To distinguish this "external" tensor product from the internal tensor product—when $G_1 = G_2$—this *external tensor product* is sometimes denoted $V_1 \boxtimes V_2$. If χ_i is the character of V_i, then the value of the character χ of $V_1 \boxtimes V_2$ is given by the product:

$$\chi(g_1 \times g_2) = \chi_1(g_1)\chi_2(g_2).$$

If V_1 and V_2 are irreducible, show that $V_1 \boxtimes V_2$ is also irreducible and show that every irreducible representation of $G_1 \times G_2$ arises this way. In terms of representation rings,

$$R(G_1 \times G_2) = R(G_1) \otimes R(G_2).$$

In these lectures we will often be given a subgroup G of a general linear group $\mathrm{GL}(V)$, and we will look for other representations inside tensor powers of V. The following problem, which is a theorem of Burnside and Molien, shows that for a finite group G, all irreducible representations can be found this way.

Problem 2.37*. Show that if V is a faithful representation of G, i.e., ρ: $G \to \mathrm{GL}(V)$ is injective, then any irreducible representation of G is contained in some tensor power $V^{\otimes n}$ of V.

Problem 2.38*. Show that the dimension of an irreducible representation of G divides the order of G.

Another challenge:

Problem 2.39*. Show that the character of any irreducible representation of dimension greater than 1 assumes the value 0 on some conjugacy class of the group.

LECTURE 3

Examples; Induced Representations; Group Algebras; Real Representations

This lecture is something of a grabbag. We start in §3.1 with examples illustrating the use of the techniques of the preceding lecture. Section 3.2 is also by way of an example. We will see quite a bit more about the representations of the symmetric groups in general later; §4 is devoted to this and will certainly subsume this discussion, but this should provide at least a sense of how we can go about analyzing representations of a class of groups, as opposed to individual groups. In §§3.3 and 3.4 we introduce two basic notions in representation theory, induced representations and the group algebra. Finally, in §3.5 we show how to classify representations of a finite group on a real vector space, given the answer to the corresponding question over $\mathbb{C}$, and say a few words about the analogous question for subfields of $\mathbb{C}$ other than $\mathbb{R}$. Everything in this lecture is elementary except Exercises 3.9 and 3.32, which involve the notions of Clifford algebras and the Fourier transform, respectively (both exercises, of course, can be skipped).

§3.1: Examples: $\mathfrak{S}_5$ and $\mathfrak{A}_5$
§3.2: Exterior powers of the standard representation of $\mathfrak{S}_d$
§3.3: Induced representations
§3.4: The group algebra
§3.5: Real representations and representations over subfields of $\mathbb{C}$

§3.1. Examples: $\mathfrak{S}_5$ and $\mathfrak{A}_5$

We have found the representations of the symmetric and alternating groups for $n \leq 4$. Before turning to a more systematic study of symmetric and alternating groups, we will work out the next couple of cases.

Representations of the Symmetric Group $\mathfrak{S}_5$

As before, we start by listing the conjugacy classes of $\mathfrak{S}_5$ and giving the number of elements of each: we have 10 transpositions, 20 three-cycles, 30 four-cycles and 24 five-cycles; in addition, we have 15 elements conjugate to (12)(34) and 10 elements conjugate to (12)(345). As for the irreducible representations, we have, of course, the trivial representation U, the alternating representation U', and the standard representation V; also, as in the case of $\mathfrak{S}_4$ we can tensor the standard representation V with the alternating one to obtain another irreducible representation V' with character $\chi_{V'} = \chi_V \cdot \chi_{U'}$.

Exercise 3.1. Find the characters of the representations V and V'; deduce in particular that V and V' are distinct irreducible representations.

The first four rows of the character table are thus

	1	10	20	30	24	15	20
$\mathfrak{S}_5$	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
V'	4	-2	1	0	-1	0	1

Clearly, we need three more irreducible representations. Where should we look for these? On the basis of our previous experience (and Problem 2.37), a natural place would be in the tensor products/powers of the irreducible representations we have found so far, in particular in $V \otimes V$ (the other two possible products will yield nothing new: we have $V' \otimes V = V \otimes V \otimes U'$ and $V' \otimes V' = V \otimes V$). Of course, $V \otimes V$ breaks up into $\wedge^2 V$ and $\mathrm{Sym}^2 V$, so we look at these separately. To start with, by the formula

$$\chi_{\wedge^2 V}(g) = \tfrac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$$

we calculate the character of $\wedge^2 V$:

$$\chi_{\wedge^2 V} = (6, 0, 0, 0, 1, -2, 0);$$

we see from this that it is indeed a fifth irreducible representation (and that $\wedge^2 V \otimes U' = \wedge^2 V$, so we get nothing new that way).

We can now find the remaining two representations in either of two ways. First, if n_1 and n_2 are their dimensions, we have

$$5! = 120 = 1^2 + 1^2 + 4^2 + 4^2 + 6^2 + n_1^2 + n_2^2,$$

so $n_1^2 + n_2^2 = 50$. There are no more one-dimensional representations, since these are trivial on normal subgroups whose quotient group is cyclic, and $\mathfrak{A}_5$

is the only such subgroup. So the only possibility is $n_1 = n_2 = 5$. Let W denote one of these five-dimensional representations, and set $W' = W \otimes U'$. In the table, if the row giving the character of W is

$$(5 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6),$$

that of W' is $(5 \quad -\alpha_1 \quad \alpha_2 \quad -\alpha_3 \quad \alpha_4 \quad \alpha_5 \quad -\alpha_6)$. Using the orthogonality relations or (2.20), one sees that $W' \not\cong W$; and with a little calculation, up to interchanging W and W', the last two rows are as given:

	1	10	20	30	24	15	20
$\mathfrak{S}_5$	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
V'	4	-2	1	0	-1	0	1
$\Lambda^2 V$	6	0	0	0	1	-2	0
W	5	1	-1	-1	0	1	1
W'	5	-1	-1	1	0	1	-1

From the decomposition $V \oplus U = \mathbb{C}^5$, we have also $\Lambda^4 V = \Lambda^5 \mathbb{C}^5 = U'$, and $V^* = V$. The perfect pairing[1]

$$V \times \Lambda^3 V \to \Lambda^4 V = U',$$

taking $v \times (v_1 \wedge v_2 \wedge v_3)$ to $v \wedge v_1 \wedge v_2 \wedge v_3$ shows that $\Lambda^3 V$ is isomorphic to $V^* \otimes U' = V'$.

Another way to find the representations W and W' would be to proceed with our original plan, and look at the representation $\mathrm{Sym}^2 V$. We will leave this in the form of an exercise:

Exercise 3.2. (i) Find the character of the representation $\mathrm{Sym}^2 V$.

(ii) Without using any knowledge of the character table of $\mathfrak{S}_5$, use this to show that $\mathrm{Sym}^2 V$ is the direct sum of three distinct irreducible representations.

(iii) Using our knowledge of the first five rows of the character table, show that $\mathrm{Sym}^2 V$ is the direct sum of the representations U, V, and a third irreducible representation W. Complete the character table for $\mathfrak{S}_5$.

Exercise 3.3. Find the decomposition into irreducibles of the representations $\Lambda^2 W$, $\mathrm{Sym}^2 W$, and $V \otimes W$.

[1] If V and W are n-dimensional vector spaces, and U is one dimensional, a *perfect pairing* is a bilinear map $\beta: V \times W \to U$ such that no nonzero vector v in V has $\beta(v, W) = 0$. Equivalently, the map $V \to \mathrm{Hom}(W, U) = W^* \otimes U$, $v \mapsto (w \mapsto \beta(v, w))$, is an isomorphism.

Representations of the Alternating Group $\mathfrak{A}_5$

What happens to the conjugacy classes above if we replace $\mathfrak{S}_d$ by $\mathfrak{A}_d$? Obviously, all the odd conjugacy classes disappear; but at the same time, since conjugation by a transposition is now an outer, rather than inner, automorphism, some conjugacy classes may break into two.

Exercise 3.4. Show that the conjugacy class in $\mathfrak{S}_d$ of permutations consisting of products of disjoint cycles of lengths $b_1, b_2, \ldots$ will break up into the union of two conjugacy classes in $\mathfrak{A}_d$ if all the b_k are odd and distinct; if any b_k are even or repeated, it remains a single conjugacy class in $\mathfrak{A}_d$. (We consider a fixed point as a cycle of length 1.)

In the case of $\mathfrak{A}_5$, this means we have the conjugacy class of three-cycles (as before, 20 elements), and of products of two disjoint transpositions (15 elements); the conjugacy class of five-cycles, however, breaks up into the conjugacy classes of (12345) and (21345), each having 12 elements.

As for the representations, the obvious first place to look is at restrictions to $\mathfrak{A}_5$ of the irreducible representations of $\mathfrak{S}_5$ found above. An irreducible representation of $\mathfrak{S}_5$ may become reducible when restricted to $\mathfrak{A}_5$; or two distinct representations may become isomorphic, as will be the case with U and U', V and V', or W and W'. In fact, U, V, and W stay irreducible since their characters satisfy $(\chi, \chi) = 1$. But the character of $\Lambda^2 V$ has values $(6, 0, -2, 1, 1)$ on the conjugacy classes listed above, so $(\chi, \chi) = 2$, and $\Lambda^2 V$ is the sum of two irreducible representations, which we denote by Y and Z. Since the sums of the squares of all the dimensions is 60, $(\dim Y)^2 + (\dim Z)^2 = 18$, so each must be three dimensional.

Exercise 3.5. Use the orthogonality relations to complete the character table of $\mathfrak{A}_5$:

	1	20	15	12	12
$\mathfrak{A}_5$	1	(123)	(12)(34)	(12345)	(21345)
U	1	1	1	1	1
V	4	1	0	-1	-1
W	5	-1	1	0	0
Y	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
Z	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

The representations Y and Z may in fact be familiar: $\mathfrak{A}_5$ can be realized as the group of motions of an icosahedron (or, equivalently, of a dodecahedron) and Y is the corresponding representation. Note that the two representations $\mathfrak{A}_5 \to GL_3(\mathbb{R})$ corresponding to Y and Z have the same image, but (as you can see from the fact that their characters differ only on the conjugacy classes of (12345) and (21345)) differ by an *outer* automorphism of $\mathfrak{A}_5$.

Note also that $\Lambda^2 V$ does not decompose over $\mathbb{Q}$; we could see this directly from the fact that the vertices of a dodecahedron cannot all have rational coordinates, which follows from the analogous fact for a regular pentagon in the plane.

Exercise 3.6. Find the decomposition of the permutation representation of $\mathfrak{A}_5$ corresponding to the (i) vertices, (ii) faces, and (iii) edges of the icosahedron.

Exercise 3.7. Consider the dihedral group D_{2n}, defined to be the group of isometries of a regular n-gon in the plane. Let $\Gamma \cong \mathbb{Z}/n \subset D_{2n}$ be the subgroup of rotations. Use the methods of Lecture 1 (applied there to the case $\mathfrak{S}_3 \cong D_6$) to analyze the representations of D_{2n}: that is, restrict an arbitrary representation of D_{2n} to Γ, break it up into eigenspaces for the action of Γ, and ask how the remaining generator of D_{2n} acts of these eigenspaces.

Exercise 3.8. Analyze the representations of the dihedral group D_{2n} using the character theory developed in Lecture 2.

Exercise 3.9. (a) Find the character table of the group of order 8 consisting of the quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$ under multiplication. This is the case $m = 3$ of a collection of groups of order 2^m, which we denote H_m. To describe them, let C_m denote the complex Clifford algebra generated by $v_1, \ldots, v_m$ with relations $v_i^2 = -1$ and $v_i \cdot v_j = -v_j \cdot v_i$, so C_m has a basis $v_I = v_{i_1} \cdot \ldots \cdot v_{i_r}$, as $I = \{i_1 < \cdots < i_r\}$ varies over subsets of $\{1, \ldots, m\}$. (See §20.1 for notation and basic facts about Clifford algebras). Set

$$H_m = \{\pm v_I \colon |I| \text{ is even}\} \subset (C_m^{\text{even}})^*.$$

This group is a 2-to-1 covering of the abelian 2-group of $m \times m$ diagonal matrices with ± 1 diagonal entries and determinant 1. The center of H_m is $\{\pm 1\}$ if m is odd and is $\{\pm 1, \pm v_{\{1,\ldots,m\}}\}$ if m is even. The other conjugacy classes consist of pairs of elements $\{\pm v_I\}$. The isomorphisms of C_m^{even} with a matrix algebra or a product of two matrix algebras give a 2^n-dimensional "spin" representation S of H_{2n+1}, and two 2^{n-1}-dimensional "spin" or "half-spin" representations S^+ and S^- of H_{2n}.

(b) Compute the characters of these spin representations and verify that they are irreducible.

(c) Deduce that the spin representations, together with the 2^{m-1} one-dimensional representations coming from the abelian group $H_m/\{\pm 1\}$ give a complete set of irreducible representations, and compute the character table for H_m.

For odd m the groups H_m are examples of *extra-special 2-groups*, cf. [Grie], [Qu].

Exercise 3.10. Find the character table of the group $SL_2(\mathbb{Z}/3)$.

Exercise 3.11. Let $H(\mathbb{Z}/3)$ be the *Heisenberg group* of order 27:

$$H(\mathbb{Z}/3) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, a, b, c \in \mathbb{Z}/3 \right\} \subset SL_3(\mathbb{Z}/3).$$

Analyze the representations of $H(\mathbb{Z}/3)$, first by the methods of Lecture 1 (restricting in this case to the center

$$Z = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b \in \mathbb{Z}/3 \right\} \cong \mathbb{Z}/3$$

of $H(\mathbb{Z}/3)$), and then by character theory.

§3.2. Exterior Powers of the Standard Representation of $\mathfrak{S}_d$

How should we go about constructing representations of the symmetric groups in general? The answer to this is not immediate; it is a subject that will occupy most of the next lecture (where we will produce all the irreducible representations of $\mathfrak{S}_d$). For now, as an example of the elementary techniques developed so far we will analyze directly one of the obvious candidates:

Proposition 3.12. *Each exterior power $\Lambda^k V$ of the standard representation V of $\mathfrak{S}_d$ is irreducible, $0 \le k \le d - 1$.*

PROOF. From the decomposition $\mathbb{C}^d = V \oplus U$, we see that V is irreducible if and only if $(\chi_{\mathbb{C}^d}, \chi_{\mathbb{C}^d}) = 2$. Similarly, since

$$\Lambda^k \mathbb{C}^d = (\Lambda^k V \otimes \Lambda^0 U) \oplus (\Lambda^{k-1} V \otimes \Lambda^1 U) = \Lambda^k V \oplus \Lambda^{k-1} V,$$

it suffices to show that $(\chi, \chi) = 2$, where χ is the character of the representation $\Lambda^k \mathbb{C}^d$. Let $A = \{1, 2, \ldots, d\}$. For a subset B of A with k elements, and $g \in G = \mathfrak{S}_d$, let

$$\{g\}_B = \begin{cases} 0 & \text{if } g(B) \ne B \\ 1 & \text{if } g(B) = B \text{ and } g|_B \text{ is an even permutation} \\ -1 & \text{if } g(B) = B \text{ and } g|_B \text{ is odd.} \end{cases}$$

Here, if $g(B) = B$, $g|_B$ denotes the permutation of the set B determined by g. Then $\chi(g) = \sum \{g\}_B$, and

$$(\chi, \chi) = \frac{1}{d!} \sum_{g \in G} \left(\sum_B \{g\}_B \right)^2$$

$$= \frac{1}{d!} \sum_{g \in G} \sum_B \sum_C \{g\}_B \{g\}_C$$

$$= \frac{1}{d!} \sum_B \sum_C \sum_g (\operatorname{sgn} g|_B) \cdot (\operatorname{sgn} g|_C),$$

where the sums are over subsets B and C of A with k elements, and in the last equation, the sum is over those g with $g(B) = B$ and $g(C) = C$. Such g is given by four permutations: one of $B \cap C$, one of $B \setminus B \cap C$, one of $C \setminus B \cap C$, and one of $A \setminus B \cup C$. Letting l be the cardinality of $B \cap C$, this last sum can be written

$$\frac{1}{d!} \sum_B \sum_C \sum_{a \in \mathfrak{S}_l} \sum_{b \in \mathfrak{S}_{k-l}} \sum_{c \in \mathfrak{S}_{k-l}} \sum_{h \in \mathfrak{S}_{d-2k+l}} (\operatorname{sgn} a)^2 (\operatorname{sgn} b)(\operatorname{sgn} c)$$

$$= \frac{1}{d!} \sum_B \sum_C l!(d - 2k + l)! \left(\sum_{b \in \mathfrak{S}_{k-l}} \operatorname{sgn} b \right) \left(\sum_{c \in \mathfrak{S}_{k-l}} \operatorname{sgn} c \right).$$

These last sums are zero unless $k - l = 0$ or 1. The case $k = l$ gives

$$\frac{1}{d!} \sum_B k!(d - k)! = \frac{1}{d!} \binom{d}{k} k!(n - k)! = 1.$$

Similarly, the terms with $k - l = 1$ also add up to 1, so $(\chi, \chi) = 2$, as required. □

Note by way of contrast that the symmetric powers of the standard representation of $\mathfrak{S}_d$ are almost never irreducible. For example, we already know that the representation $\operatorname{Sym}^2 V$ contains one copy of the trivial representation: this is just the statement that every irreducible real representation (such as V) admits an inner product (unique, up to scalars) invariant under the group action; nor is the quotient of $\operatorname{Sym}^2 V$ by this trivial subrepresentation necessarily irreducible, as witness the case of $\mathfrak{S}_5$.

§3.3. Induced Representations

If $H \subset G$ is a subgroup, any representation V of G restricts to a representation of H, denoted $\operatorname{Res}_H^G V$ or simple $\operatorname{Res} V$. In this section, we describe an important construction which produces representations of G from representations of H. Suppose V is a representation of G, and $W \subset V$ is a subspace which is H-invariant. For any g in G, the subspace $g \cdot W = \{g \cdot w : w \in W\}$ depends only on the left coset gH of g modulo H, since $gh \cdot W = g \cdot (h \cdot W) = g \cdot W$; for a coset

σ in G/H, we write $\sigma \cdot W$ for this subspace of V. We say that V is *induced* by W if every element in V can be written uniquely as a sum of elements in such translates of W, i.e.,

$$V = \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$

In this case we write $V = \operatorname{Ind}_H^G W = \operatorname{Ind} W$.

Example 3.13. The permutation representation associated to the left action of G on G/H is induced from the trivial one-dimensional representation W of H. Here V has basis $\{e_\sigma : \sigma \in G/H\}$, and $W = \mathbb{C} \cdot e_H$, with H the trivial coset.

Example 3.14. The regular representation of G is induced from the regular representation of H. Here V has basis $\{e_g : g \in G\}$, whereas W has basis $\{e_h : h \in H\}$.

We claim that, given a representation W of H, such V exists and is unique up to isomorphism. Although we will later give several fancier ways to see this, it is not hard to do it by hand. Choose a representative $g_\sigma \in G$ for each coset $\sigma \in G/H$, with e representing the trivial coset H. To see the uniqueness, note that each element of V has a unique expression $v = \sum g_\sigma w_\sigma$, for elements w_σ in W. Given g in G, write $g \cdot g_\sigma = g_\tau \cdot h$ for some $\tau \in G/H$ and $h \in H$. Then we must have

$$g \cdot (g_\sigma w_\sigma) = (g \cdot g_\sigma) w_\sigma = (g_\tau \cdot h) w_\sigma = g_\tau (h w_\sigma).$$

This proves the uniqueness and tells us how to construct $V = \operatorname{Ind}(W)$ from W. Take a copy W^σ of W for each left coset $\sigma \in G/H$; for $w \in W$, let $g_\sigma w$ denote the element of W^σ corresponding to w in W. Let $V = \bigoplus_{\sigma \in G/H} W^\sigma$, so every element of V has a unique expression $v = \sum g_\sigma w_\sigma$ for elements w_σ in W. Given $g \in G$, define

$$g \cdot (g_\sigma w_\sigma) = g_\tau (h w_\sigma) \quad \text{if } g \cdot g_\sigma = g_\tau \cdot h.$$

To show that this defines as action of G on V, we must verify that $g' \cdot (g \cdot (g_\sigma w_\sigma)) = (g' \cdot g) \cdot (g_\sigma w_\sigma)$ for another element g' in G. Now if $g' \cdot g_\tau = g_\rho \cdot h'$, then

$$g' \cdot (g \cdot (g_\sigma w_\sigma)) = g' \cdot (g_\tau (h w_\sigma)) = g_\rho (h' (h w_\sigma)).$$

Since $(g' \cdot g) \cdot g_\sigma = g' \cdot (g \cdot g_\sigma) = g' \cdot g_\tau \cdot h = g_\rho \cdot h' \cdot h$, we have

$$(g' \cdot g) \cdot (g_\sigma w_\sigma) = g_\rho ((h' \cdot h) w_\sigma) = g_\rho (h' \cdot (h w_\sigma)),$$

as required.

Example 3.15. If $W = \bigoplus W_i$, then $\operatorname{Ind} W = \bigoplus \operatorname{Ind} W_i$.

The existence of the induced representation follows from Examples 3.14 and 3.15 since any W is a direct sum of summands of the regular representation.

Exercise 3.16. (a) If U is a representation of G and W a representation of H, show that (with all tensor products over $\mathbb{C}$)

$$U \otimes \operatorname{Ind} W = \operatorname{Ind}(\operatorname{Res}(U) \otimes W).$$

In particular, $\operatorname{Ind}(\operatorname{Res}(U)) = U \otimes P$, where P is the permutation representation of G on G/H. For a formula for $\operatorname{Res}(\operatorname{Ind}(W))$, for W a representation of H, see [Se2, p. 58].

(b) Like restriction, induction is transitive: if $H \subset K \subset G$ are subgroups, show that

$$\operatorname{Ind}_H^G(W) = \operatorname{Ind}_K^G(\operatorname{Ind}_H^K W).$$

Note that Example 3.15 says that the map Ind gives a group homomorphism between the representation rings $R(H)$ and $R(G)$, in the opposite direction from the ring homomorphism Res: $R(G) \to R(H)$ given by restriction; Exercise 3.16(a) says that this map satisfies a "push–pull" formula $\alpha \cdot \operatorname{Ind}(\beta) = \operatorname{Ind}(\operatorname{Res}(\alpha) \cdot \beta)$ with respect to the restriction map.

Proposition 3.17. *Let W be a representation of H, U a representation of G, and suppose $V = \operatorname{Ind} W$. Then any H-module homomorphism $\varphi\colon W \to U$ extends uniquely to a G-module homomorphism $\tilde{\varphi}\colon V \to U$.* i.e.,

$$\operatorname{Hom}_H(W, \operatorname{Res} U) = \operatorname{Hom}_G(\operatorname{Ind} W, U).$$

In particular, this universal property determines Ind W *up to canonical isomorphism.*

PROOF. With $V = \bigoplus_{\sigma \in G/H} \sigma \cdot W$ as before, define $\tilde{\varphi}$ on $\sigma \cdot W$ by

$$\sigma \cdot W \xrightarrow{g_\sigma^{-1}} W \xrightarrow{\varphi} U \xrightarrow{g_\sigma} U,$$

which is independent of the representative g_σ for σ since φ is H-linear. □

To compute the character of $V = \operatorname{Ind} W$, note that $g \in G$ maps σW to $g\sigma W$, so the trace is calculated from those cosets σ with $g\sigma = \sigma$, i.e., $s^{-1}gs \in H$ for $s \in \sigma$. Therefore,

$$\chi_{\operatorname{Ind} W}(g) = \sum_{g\sigma=\sigma} \chi_W(s^{-1}gs) \qquad (s \in \sigma \text{ arbitrary}). \tag{3.18}$$

Exercise 3.19. (a) If C is a conjugacy class of G, and $C \cap H$ decomposes into conjugacy classes $D_1, \ldots, D_r$ of H, (3.18) can be rewritten as: the value of the character of Ind W on C is

$$\chi_{\operatorname{Ind} W}(C) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi_W(D_i).$$

(b) If W is the trivial representation of H, then

$$\chi_{\operatorname{Ind} W}(C) = \frac{[G:H]}{|C|} \cdot |C \cap H|.$$

Corollary 3.20 (Frobenius Reciprocity). *If W is a representation of H, and U a representation of G, then*

$$(\chi_{\operatorname{Ind} W}, \chi_U)_G = (\chi_W, \chi_{\operatorname{Res} U})_H.$$

PROOF. It suffices by linearity to prove this when W and U are irreducible. The left-hand side is the number of times U appears in Ind W, which is the dimension of $\operatorname{Hom}_G(\operatorname{Ind} W, U)$. The right-hand side is the dimension of $\operatorname{Hom}_H(W, \operatorname{Res} U)$. These dimensions are equal by the proposition. □

If W and U are irreducible, Frobenius reciprocity says: *the number of times U appears in* Ind W *is the same as the number of times W appears in* Res U.

Frobenius reciprocity can be used to find characters of G if characters of H are known.

Example 3.21. We compute $\operatorname{Ind}_H^G W$, when $H = \mathfrak{S}_2 \subset G = \mathfrak{S}_3$, $W = V_2$ (the standard representation) $= U_2'$ (the alternating representation). We know the irreducible represenatations of $\mathfrak{S}_3$: U_3, U_3', V_3, which restrict to U_2, $U_2' = V_2$, $U_2 \oplus U_2'$, respectively. Thus, by Frobenius, Ind $V_2 = U_3' \oplus V_3$.

Example 3.22. Consider next $H = \mathfrak{S}_3 \subset G = \mathfrak{S}_4$, $W = V_3$. Again we know the irreducible representations, and Res $U_4 = U_3$, Res $U_4' = U_3'$, Res $V_4 = U_3 \oplus V_3$ [the vector

$$(1, 1, 1, -3) \in V_4 = \{(x_1, x_2, x_3, x_4)\colon \textstyle\sum x_i = 0\}$$

is fixed by H], Res $V_4' = U_3' \oplus V_3'$, with $V_3' = V_3$, and Res $W_4 = V_3$ (as one may see directly). Hence, Ind $V_3 = V_4 \oplus V_4' \oplus W_4$. (Note that the isomorphism Res $W_4 = V_3$ actually follows, since one W_4 is all that could be added to $V_4 \oplus V_4'$ to get Ind V_3.)

Exercise 3.23. Determine the isomorphism classes of the representations of $\mathfrak{S}_4$ induced by (i) the one-dimensional representation of the group generated by (1234) in which $(1234)\cdot v = iv$, $i = \sqrt{-1}$; (ii) the one-dimensional representation of the group generated by (123) in which $(123)\cdot v = e^{2\pi i/3}v$.

Exercise 3.24. Let $H = \mathfrak{A}_5 \subset G = \mathfrak{S}_5$. Show that Ind $U = U \oplus U'$, Ind $V = V \oplus V'$, and Ind $W = W \oplus W'$, whereas Ind Y = Ind $Z = \Lambda^2 V$.

Exercise 3.25*. Which irreducible representations of $\mathfrak{S}_d$ remain irreducible when restricted to $\mathfrak{A}_d$? Which are induced from $\mathfrak{A}_d$? How much does this tell you about the irreducible representations of $\mathfrak{A}_d$?

Exercise 3.26*. There is a unique nonabelian group of order 21, which can be realized as the group of affine transformations $x \mapsto \alpha x + \beta$ of the line over the field with seven elements, with α a cube root of unity in that field. Find the irreducible representations and character table for this group.

Now that we have introduced the notion of induced representation, we can state two important theorems describing the characters of representations of a finite group. In the preceding lecture we mentioned the notion of *virtual character*; this is just an element of the image Λ of the character map

$$\chi\colon R(G) \to \mathbb{C}_{\text{class}}(G)$$

from the representation ring $R(G)$ of virtual representations. The following two theorems both state that in order to generate $\Lambda \otimes \mathbb{Q}$ (resp. Λ) it is enough to consider the simplest kind of induced representations, namely, those induced from cyclic (respective elementary) subgroups of G. For the proofs of these theorems we refer to [Se2, §9, 10]. We will not need them in these lectures.

Artin's Theorem 3.27. *The characters of induced representations from cyclic subgroups of G generate a lattice of finite index in Λ.*

A subgroup H of G is *p-elementary* if $H = A \times B$, with A cyclic of order prime to p and B a p-group.

Brauer's Theorem 3.28. *The characters of induced representations from elementary subgroups of G generate the lattice Λ.*

§3.4. The Group Algebra

There is an important notion that we have already dealt with implicitly but not explicitly; this is the group algebra $\mathbb{C}G$ associated to a finite group G. This is an object that for all intents and purposes can completely replace the group G itself; any statement about the representations of G has an exact equivalent statement about the group algebra. Indeed, to a large extent the choice of language is a matter of taste.

The underlying vector space of the group algebra of G is the vector space with basis $\{e_g\}$ corresponding to elements of the group G, that is, the underlying vector space of the regular representation. We define the algebra structure on this vector space simply by

$$e_g \cdot e_h = e_{gh}.$$

By a representation of the algebra $\mathbb{C}G$ on a vector space V we mean simply an algebra homomorphism

$$\mathbb{C}G \to \operatorname{End}(V),$$

so that a representation V of $\mathbb{C}G$ is the same thing as a left $\mathbb{C}G$-module. Note that a representation $\rho\colon G \to \operatorname{Aut}(V)$ will extend by linearity to a map $\tilde{\rho}\colon \mathbb{C}G \to \operatorname{End}(V)$, so that representations of $\mathbb{C}G$ correspond exactly to representations of G; the left $\mathbb{C}G$-module given by $\mathbb{C}G$ itself corresponds to the regular representation.

If $\{W_i\}$ are the irreducible representations of G, then we have seen that the regular representation R decomposes

$$R = \bigoplus (W_i)^{\oplus \dim(W_i)}.$$

We can now refine this statement in terms of the group algebra: we have

Proposition 3.29. *As algebras,*

$$\mathbb{C}G \cong \bigoplus \operatorname{End}(W_i).$$

PROOF. As we have said, for any representation W of G, the map $G \to \operatorname{Aut}(W)$ extends by linearity to a map $\mathbb{C}G \to \operatorname{End}(W)$; applying this to each of the irreducible representations W_i gives us a canonical map

$$\varphi\colon \mathbb{C}G \to \bigoplus \operatorname{End}(W_i).$$

This is injective since the representation on the regular representation is faithful. Since both have dimension $\sum(\dim W_i)^2$, the map is an isomorphism. □

A few remarks are in order about the isomorphism φ of the proposition. First, φ can be interpreted as the Fourier transform, cf. Exercise 3.32. Note also that Proposition 2.28 has a natural interpretation in terms of the group algebra: it says that the center of $\mathbb{C}G$ consists of those $\sum \alpha(g)e_g$ for which α is a class function.

Next, we can think of φ as the decomposition of the semisimple algebra $\mathbb{C}G$ into a product of matrix algebras. It implies that the matrix entries of the irreducible representations give a basis for the space of all functions on G, cf. Exercise 2.35.

Note in particular that any irreducible representation is isomorphic to a (minimal) left ideal in $\mathbb{C}G$. These left ideals are generated by idempotents. In fact, we can interpret the projection formulas of the last lecture in the language of the group algebra: the formulas say simply that the elements

$$\dim W \cdot \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot e_g \in \mathbb{C}G$$

are the idempotents in the group algebra corresponding to the direct sum factors in the decomposition of Proposition 3.29. To locate the irreducible representations W_i of a group G [not just a direct sum of $\dim(W_i)$ copies], we want to find other idempotents of $\mathbb{C}G$. We will see this carried out for the symmetric groups in the following lecture.

The group algebra also gives us another description of induced representations: if W is a representation of a subgroup H of G, then the induced representation may be constructed simply by

$$\operatorname{Ind} W = \mathbb{C}G \otimes_{\mathbb{C}H} W,$$

where G acts on the first factor: $g \cdot (e_{g'} \otimes w) = e_{gg'} \otimes w$. The isomorphism of the reciprocity theorem is then a special case of a general formula for a change of rings $\mathbb{C}H \to \mathbb{C}G$:

$$\mathrm{Hom}_{\mathbb{C}H}(W, U) = \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{C}H} W, U).$$

Exercise 3.30*. The induced representation $\mathrm{Ind}(W)$ can also be realized concretely as a space of W-valued functions on G, which can be useful to produce matrix realizations, or when trying to decompose $\mathrm{Ind}(W)$ into irreducible pieces. Show that $\mathrm{Ind}(W)$ is isomorphic to

$$\mathrm{Hom}_H(\mathbb{C}G, W) \cong \{f: G \to W: f(hg) = hf(g), \quad \forall h \in H, g \in G\},$$

where G acts by $(g' \cdot f)(g) = f(gg')$.

Exercise 3.31. If $\mathbb{C}G$ is identified with the space of functions on G, the function φ corresponding to $\sum_{g \in G} \varphi(g) e_g$, show that the product in $\mathbb{C}G$ corresponds to the convolution $*$ of functions:

$$(\varphi * \psi)(g) = \sum_{h \in G} \varphi(h)\psi(h^{-1}g).$$

(With integration replacing summation, this indicates how one may extend the notion of regular representation to compact groups.)

Exercise 3.32*. If $\rho: G \to \mathrm{GL}(V_\rho)$ is a representation, and φ is a function on G, define the *Fourier transform* $\hat{\varphi}(\rho)$ in $\mathrm{End}(V_\rho)$ by the formula

$$\hat{\varphi}(\rho) = \sum_{g \in G} \varphi(g) \cdot \rho(g).$$

(a) Show that $\widehat{\varphi * \psi}(\rho) = \widehat{\varphi}(\rho) \cdot \widehat{\psi}(\rho)$.

(b) Prove the *Fourier inversion formula*

$$\varphi(g) = \frac{1}{|G|} \sum \dim(V_\rho) \cdot \mathrm{Trace}(\rho(g^{-1}) \cdot \hat{\varphi}(\rho)),$$

the sum over the irreducible representations ρ of G. This formula is equivalent to formulas (2.19) and (2.20).

(c) Prove the *Plancherel formula* for functions φ and ψ on G:

$$\sum_{g \in G} \varphi(g^{-1})\psi(g) = \frac{1}{|G|} \sum_{\rho} \dim(V_\rho) \cdot \mathrm{Trace}(\hat{\varphi}(\rho)\hat{\psi}(\rho)).$$

Our choice of left action of a group on a space has been perfectly arbitrary, and the entire story is the same if G acts on the *right* instead. Moreover, there is a standard way to change a right action into a left action, and vice versa: Given a right action of G on V, define the left action by

$$g \cdot v = v \cdot (g^{-1}), \qquad g \in G, v \in V.$$

If $A = \mathbb{C}G$ is the group algebra, a right action of G on V makes V a right A-module. To turn right modules into left modules, we can use the anti-involution $a \mapsto \hat{a}$ of A defined by $(\sum a_g e_g)^\wedge = \sum a_g e_{g^{-1}}$. A right A-module is then turned into a left A-module by setting $a \cdot v = v \cdot \hat{a}$.

The following exercise will take you back to the origins of representation theory in the 19th century, when Frobenius found the characters by factoring this determinant.

Exercise 3.33*. Given a finite group G of order n, take a variable x_g for each element g in G, and order the elements of G arbitrarily. Let F be the determinant of the $n \times n$ matrix whose entry in the row labeled by g and column labeled by h is $x_{g \cdot h^{-1}}$. This is a form of degree n in the n variables x_g, which is independent of the ordering. Normalize the factors of F to take the value 1 when $x_e = 1$ and $x_g = 0$ for $g \neq e$. Show that the irreducible factors of F correspond to the irreducible representations of G. Moreover, if F_ρ is the factor corresponding to the representation ρ, show that the degree of F_ρ is the degree $d(\rho)$ of the representation ρ, and that each F_ρ occurs in F $d(\rho)$ times. If χ_ρ is the character of ρ, show that $\chi_\rho(g)$ is the coefficient of $x_g \cdot x_e^{d(\rho)-1}$ in F_ρ.

§3.5. Real Representations and Representations over Subfields of $\mathbb{C}$

If a group G acts on a real vector space V_0, then we say the corresponding complex representation of $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ is *real*. To the extent that we are interested in the action of a group G on real rather than complex vector spaces, the problem we face is to say which of the complex representations of G we have studied are in fact real.

Our first guess might be that a representation is real if and only if its character is real-valued. This turns out not to be the case: the character of a real representation is certainly real-valued, but the converse need not be true. To find an example, suppose $G \subset \mathrm{SU}(2)$ is a finite, nonabelian subgroup. Then G acts on $\mathbb{C}^2 = V$ with a real-valued character since the trace of any matrix in $\mathrm{SU}(2)$ is real. If V were a real representation, however, then G would be a subgroup of $\mathrm{SO}(2) = S^1$, which is abelian. To produce such a group, note that $\mathrm{SU}(2)$ can be identified with the unit quaternions. Set $G = \{\pm 1, \pm i, \pm j, \pm k\}$. Then $G/\{\pm 1\}$ is abelian, so has four one-dimensional representations, which give four one-dimensional representations of G. Thus, G has one irreducible two-dimensional representation, whose character is real, but which is not real.

Exercise 3.34*. Compute the character table for this quaternion group G, and compare it with the character table of the dihedral group of order 8.

A more successful approach is to note that if V is a real representation of G, coming from V_0 as above, then one can find a positive definite symmetric bilinear form on V_0 which is preserved by G. This gives a symmetric bilinear form on V which is preserved by G. Not every representation will have such a form since degeneracies may arise when one tries to construct one following the construction of Proposition 1.5. In fact,

Lemma 3.35. *An irreducible representation V of G is real if and only if there is a nondegenerate symmetric bilinear form B on V preserved by G.*

PROOF. If we have such B, and an arbitrary nondegenerate Hermitian form H, also G-invariant, then

$$V \xrightarrow{B} V^* \xrightarrow{H} V$$

gives a conjugate linear isomorphism φ from V to V: given $x \in V$, there is a unique $\varphi(x) \in V$ with $B(x, y) = H(\varphi(x), y)$, and φ commutes with the action of G. Then $\varphi^2 = \varphi \circ \varphi$ is a complex linear G-module homomorphism, so $\varphi^2 = \lambda \cdot \mathrm{Id}$. Moreover,

$$H(\varphi(x), y) = B(x, y) = B(y, x) = H(\varphi(y), x) = \overline{H(x, \varphi(y))},$$

from which it follows that $H(\varphi^2(x), y) = H(x, \varphi^2(y))$, and therefore λ is a positive real number. Changing H by a scalar, we may assume $\lambda = 1$, so $\varphi^2 = \mathrm{Id}$. Thus, V is a sum of real eigenspaces V_+ and V_- for φ corresponding to eigenvalues 1 and -1. Since φ commutes with G, V_+ and V_- are G-invariant subspaces. Finally, $\varphi(ix) = -i\varphi(x)$, so $iV_+ = V_-$, and $V = V_+ \otimes \mathbb{C}$. □

Note from the proof that a real representation is also characterized by the existence of a conjugate linear endomorphism of V whose square is the identity; if $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$, it is given by conjugation: $v_0 \otimes \lambda \mapsto v_0 \otimes \bar{\lambda}$.

A warning is in order here: an irreducible representation of G on a vector space over $\mathbb{R}$ may become reducible when we extend the group field to $\mathbb{C}$. To give the simplest example, the representation of $\mathbb{Z}/n$ on $\mathbb{R}^2$ given by

$$\rho\colon k \mapsto \begin{pmatrix} \cos\dfrac{2\pi k}{n} & -\sin\dfrac{2\pi k}{n} \\ \sin\dfrac{2\pi k}{n} & \cos\dfrac{2\pi k}{n} \end{pmatrix}$$

is irreducible over $\mathbb{R}$ for $n > 2$ (no line in $\mathbb{R}^2$ is fixed by the action of $\mathbb{Z}/n$), but will be reducible over $\mathbb{C}$. Thus, classifying the irreducible representations of G over $\mathbb{C}$ that are real does not mean that we have classified all the irreducible real representations. However, we will see in Exercise 3.39 below how to finish the story once we have found the real representations of G that are irreducible over $\mathbb{C}$.

Suppose V is an irreducible representation of G with χ_V real. Then there is a G-equivariant isomorphism $V \cong V^*$, i.e., there is a G-equivariant (nondegenerate) bilinear form B on V; but, in general, B need not be symmetric. Regarding B in

$$V^* \otimes V^* = \mathrm{Sym}^2 V^* \oplus \wedge^2 V^*,$$

and noting the uniqueness of B up to multiplication by scalars, we see that B is either symmetric or skew-symmetric. If B is skew-symmetric, proceeding as above one can scale so $\varphi^2 = -\mathrm{Id}$. This makes V "quaternionic," with φ becoming multiplication[2] by j:

Definition 3.36. A *quaternionic* representation is a (complex) representation V which has a G-invariant homomorphism $J\colon V \to V$ that is conjugate linear, and satisfies $J^2 = -\mathrm{Id}$. Thus, a skew-symmetric nondegenerate G-invariant B determines a quaternionic structure on V.

Summarizing the preceding discussion we have the

Theorem 3.37. *An irreducible representation V is one and only one of the following:*

(1) Complex: *χ_V is not real-valued; V does not have a G-invariant nondegenerate bilinear form.*

(2) Real: *$V = V_0 \otimes \mathbb{C}$, a real representation; V has a G-invariant symmetric nondegenerate bilinear form.*

(3) Quaternionic: *χ_V is real, but V is not real; V has a G-invariant skew-symmetric nondegeneate bilinear form.*

Exercise 3.38. Show that for V irreducible,

$$\frac{1}{|G|}\sum_{g \in G} \chi_V(g^2) = \begin{cases} 0 & \text{if } V \text{ is complex} \\ 1 & \text{if } V \text{ is real} \\ -1 & \text{if } V \text{ is quaternionic.} \end{cases}$$

This verifies that the three cases in the theorem are mutually exclusive. It also implies that if the order of G is odd, all nontrivial representations must be complex.

Exercise 3.39. Let V_0 be a real vector space on which G acts irreducibly, $V = V_0 \otimes \mathbb{C}$ the corresponding real representation of G. Show that if V is not irreducible, then it has exactly two irreducible factors, and they are conjugate complex representations of G.

[2] See §7.2 for more on quaternions and quaternionic representations.

Exercise 3.40. Classify the real representations of $\mathfrak{A}_4$.

Exercise 3.41*. The group algebra $\mathbb{R}G$ is a product of simple $\mathbb{R}$-algebras corresponding to the irreducible representations over $\mathbb{R}$. These simple algebras are matrix algebras over $\mathbb{C}$, $\mathbb{R}$, or the quaternions $\mathbb{H}$ according as the representation is complex, real, or quaternionic.

Exercise 3.42*. (a) Show that all characters of a group are real if and only if every element is conjugate to its inverse.

(b) Show that an element σ in a split conjugacy class of $\mathfrak{A}_d$ is conjugate to its inverse if and only if the number of cycles in σ whose length is congruent to 3 modulo 4 is even.

(c) Show that the only d's for which every character of $\mathfrak{A}_d$ is real-valued are $d = 1, 2, 5, 6, 10$, and 14.

Exercise 3.43*. Show that: (i) the tensor product of two real or two quaternionic representations is real; (ii) for any V, $V^* \otimes V$ is real; (iii) if V is real, so are all $\Lambda^k V$; (iv) if V is quaternionic, $\Lambda^k V$ is real for k even, quaternionic for k odd.

Representations over Subfields of $\mathbb{C}$ in General

We consider next the generalization of the preceding problem to more general subfields of $\mathbb{C}$. Unfortunately, our results will not be nearly as strong in general, but we can at least express the problem neatly in terms of the representation ring of G.

To begin with, our terminology in this general setting is a little different. Let $K \subset \mathbb{C}$ be any subfield. We define a K-representation of G to be a vector space V_0 over K on which G acts; in this case we say that the complex representation $V = V_0 \otimes \mathbb{C}$ is *defined over* K.

One way to measure how many of the representations of G are defined over a field K is to introduce the *representation ring* $R_K(G)$ *of* G *over* K. This is defined just like the ordinary representation ring; that is, it is just the group of formal linear combinations of K-representations of G modulo relations of the form $V + W - (V \oplus W)$, with multiplication given by tensor product.

Exercise 3.44*. Describe the representation ring of G over $\mathbb{R}$ for some of the groups G whose complex representation we have analyzed above. In particular, is the rank of $R_{\mathbb{R}}(G)$ always the same as the rank of $R(G)$?

Exercise 3.45*. (a) Show that $R_K(G)$ is the subring of the ring of class functions on G generated (as an additive group) by characters of representations defined over K.

(b) Show that the characters of irreducible representations over K form an orthogonal basis for $R_K(G)$.

(c) Show that a complex representation of G can be defined over K if and only if its character belongs to $R_K(G)$.

For more on the relation between $R_K(G)$ and $R(G)$, see [Se2].

(N°3) Lectures 4÷6

LECTURE 4

Representations of $\mathfrak{S}_d$: Young Diagrams and Frobenius's Character Formula

In this lecture we get to work. Specifically, we give in §4.1 a complete description of the irreducible representations of the symmetric group, that is, a construction of the representations (via Young symmetrizers) and a formula (Frobenius' formula) for their characters. The proof that the representations constructed in §4.1 are indeed the irreducible representations of the symmetric group is given in §4.2; the proof of Frobenius' formula, as well as a number of others, in §4.3. Apart from their intrinsic interest (and undeniable beauty), these results turn out to be of substantial interest in Lie theory: analogs of the Young symmetrizers will give a construction of the irreducible representations of $SL_n\mathbb{C}$. At the same time, while the techniques of this lecture are completely elementary (we use only a few identities about symmetric polynomials, proved in Appendix A), the level of difficulty is clearly higher than in preceding lectures. The results in the latter half of §4.3 (from Corollary 4.39 on) in particular are quite difficult, and inasmuch as they are not used later in the text may be skipped by readers who are not symmetric group enthusiasts.

§4.1: Statements of the results
§4.2: Irreducible representations of $\mathfrak{S}_d$
§4.3: Proof of Frobenius's formula

§4.1. Statements of the Results

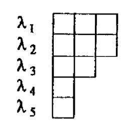
The number of irreducible representaton of $\mathfrak{S}_d$ is the number of conjugacy classes, which is the number $p(d)$ of partitions[1] of d: $d = \lambda_1 + \cdots + \lambda_k$, $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$. We have

[1] It is sometimes convenient, and sometimes a nuisance, to have partitions that end in one or more zeros; if convenient, we allow some of the λ_i on the end to be zero. Two sequences define the same partition, of course, if they differ only by zeros at the end.

$$\sum_{d=0}^{\infty} p(d)t^d = \prod_{n=1}^{\infty}\left(\frac{1}{1-t^n}\right)$$
$$= (1 + t + t^2 + \cdots)(1 + t^2 + t^4 + \cdots)(1 + t^3 + \cdots)\cdots$$

which converges exactly in $|t| < 1$. This partition number is an interesting arithmetic function, whose congruences and growth behavior as a function of d have been much studied (cf. [Har], [And]). For example, $p(d)$ is asymptotically equal to $(1/\alpha d)e^{\beta\sqrt{d}}$, with $\alpha = 4\sqrt{3}$ and $\beta = \pi\sqrt{2/3}$.

To a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is associated a *Young diagram* (sometimes called a Young frame or Ferrers diagram)



with λ_i boxes in the ith row, the rows of boxes lined up on the left. The *conjugate partition* $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ to the partition λ is defined by interchanging rows and columns in the Young diagram, i.e., reflecting the diagram in the 45° line. For example, the diagram above is that of the partition $(3, 3, 2, 1, 1)$, whose conjugate is $(5, 3, 2)$. (Without reference to the diagram, the conjugate partition to λ can be defined by saying λ'_i is the number of terms in the partition λ that are greater than or equal to i.)

Young diagrams can be used to describe projection operators for the regular representation, which will then give the irreducible representations of $\mathfrak{S}_d$. For a given Young diagram, number the boxes, say consecutively as shown:

1	2	3
4	5	
6	7	
8		

More generally, define a *tableau* on a given Young diagram to be a numbering of the boxes by the integers $1, \ldots, d$. Given a tableau, say the canonical one shown, define two subgroups[2] of the symmetric group

[2] If a tableau other than the canonical one were chosen, one would get different groups in place of P and Q, and different elements in the group ring, but the representations constructed this way will be isomorphic.

$$P = P_\lambda = \{g \in \mathfrak{S}_d: g \text{ preserves each row}\}$$

and

$$Q = Q_\lambda = \{g \in \mathfrak{S}_d: g \text{ preserves each column}\}.$$

In the group algebra $\mathbb{C}\mathfrak{S}_d$, we introduce two elements corresponding to these subgroups: we set

$$a_\lambda = \sum_{g \in P} e_g \quad \text{and} \quad b_\lambda = \sum_{g \in Q} \operatorname{sgn}(g) \cdot e_g. \tag{4.1}$$

To see what a_λ and b_λ do, observe that if V is any vector space and $\mathfrak{S}_d$ acts on the dth tensor power $V^{\otimes d}$ by permuting factors, the image of the element $a_\lambda \in \mathbb{C}\mathfrak{S}_d \to \operatorname{End}(V^{\otimes d})$ is just the subspace

$$\operatorname{Im}(a_\lambda) = \operatorname{Sym}^{\lambda_1} V \otimes \operatorname{Sym}^{\lambda_2} V \otimes \cdots \otimes \operatorname{Sym}^{\lambda_k} V \subset V^{\otimes d},$$

where the inclusion on the right is obtained by grouping the factors of $V^{\otimes d}$ according to the rows of the Young tableaux. Similarly, the image of b_λ on this tensor power is

$$\operatorname{Im}(b_\lambda) = \Lambda^{\mu_1} V \otimes \Lambda^{\mu_2} V \otimes \cdots \otimes \Lambda^{\mu_l} V \subset V^{\otimes d},$$

where μ is the conjugate partition to λ.

Finally, we set

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}\mathfrak{S}_d; \tag{4.2}$$

this is called a *Young symmetrizer*. For example, when $\lambda = (d)$, $c_{(d)} = a_{(d)} = \sum_{g \in \mathfrak{S}_d} e_g$, and the image of $c_{(d)}$ on $V^{\otimes d}$ is $\operatorname{Sym}^d V$. When $\lambda = (1, \ldots, 1)$, $c_{(1,\ldots,1)} = b_{(1,\ldots,1)} = \sum_{g \in \mathfrak{S}_d} \operatorname{sgn}(g) e_g$, and the image of $c_{(1,\ldots,1)}$ on $V^{\otimes d}$ is $\Lambda^d V$. We will eventually see that the image of the symmetrizers c_λ in $V^{\otimes d}$ provide essentially all the finite-dimensional irreducible representations of $GL(V)$. Here we state the corresponding fact for representations of $\mathfrak{S}_d$:

Theorem 4.3. *Some scalar multiple of c_λ is idempotent, i.e., $c_\lambda^2 = n_\lambda c_\lambda$, and the image of c_λ (by right multiplication on $\mathbb{C}\mathfrak{S}_d$) is an irreducible representation V_λ of $\mathfrak{S}_d$. Every irreducible representation of $\mathfrak{S}_d$ can be obtained in this way for a unique partition.*

We will prove this theorem in the next section. Note that, as a corollary, each irreducible representation of $\mathfrak{S}_d$ can be defined over the rational numbers since c_λ is in the rational group algebra $\mathbb{Q}\mathfrak{S}_d$. Note also that the theorem gives a direct correspondence between conjugacy classes in $\mathfrak{S}_d$ and irreducible representations of $\mathfrak{S}_d$, something which has never been achieved for general groups.

For example, for $\lambda = (d)$,

$$V_{(d)} = \mathbb{C}\mathfrak{S}_d \cdot \sum_{g \in \mathfrak{S}_d} e_g = \mathbb{C} \cdot \sum_{g \in \mathfrak{S}_d} e_g$$

is the trivial representation U, and when $\lambda = (1, \ldots, 1)$,

$$V_{(1,\ldots,1)} = \mathbb{C}\mathfrak{S}_d \cdot \sum_{g \in \mathfrak{S}_d} \operatorname{sgn}(g) e_g = \mathbb{C} \cdot \sum_{g \in \mathfrak{S}_d} \operatorname{sgn}(g) e_g$$

is the alternating representation U'. For $\lambda = (2, 1)$,

$$c_{(2,1)} = (e_1 + e_{(12)}) \cdot (e_1 - e_{(13)}) = 1 + e_{(12)} - e_{(13)} - e_{(132)}$$

in $\mathbb{C}\mathfrak{S}_3$, and $V_{(2,1)}$ is spanned by $c_{(2,1)}$ and $(13) \cdot c_{(2,1)}$, so $V_{(2,1)}$ is the standard representation of $\mathfrak{S}_3$.

Exercise 4.4*. Set $A = \mathbb{C}\mathfrak{S}_d$, so $V_\lambda = Ac_\lambda = Aa_\lambda b_\lambda$.

(a) Show that $V_\lambda \cong Ab_\lambda a_\lambda$.

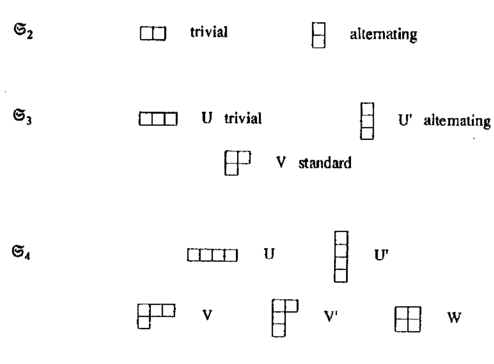
(b) Show that V_λ is the image of the map from Aa_λ to Ab_λ given by right multiplication by b_λ. By (a), this is isomorphic to the image of $Ab_\lambda \to Aa_\lambda$ given by right multiplication by a_λ.

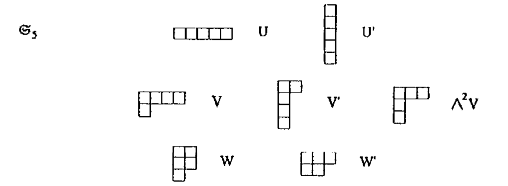
(c) Using (a) and the description of V_λ in the theorem show that

$$V_{\lambda'} = V_\lambda \otimes U',$$

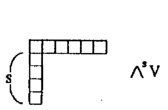
where λ' is the conjugate partition to λ and U' is the alternating representation.

Examples 4.5. In earlier lectures we described the irreducible representations of $\mathfrak{S}_d$ for $d \le 5$. From the construction of the representation corresponding to a Young diagram it is not hard to work out which representations come from which diagrams:





Exercise 4.6*. Show that for general d, the standard representation V corresponds to the partition $d = (d-1) + 1$. As a challenge, you can try to prove that the exterior powers of the standard representation V are represented by a "hook":



Note that this recovers our theorem that the $\Lambda^s V$ are irreducible.

Next we turn to Frobenius's formula for the character χ_λ of V_λ, which includes a formula for its dimension. Let $C_\mathbf{i}$ denote the conjugacy class in $\mathfrak{S}_d$ determined by a sequence

$$\mathbf{i} = (i_1, i_2, \ldots, i_d) \quad \text{with } \sum \alpha i_\alpha = d:$$

$C_\mathbf{i}$ consists of those permutations that have i_1 1-cycles, i_2 2-cycles, ..., and i_d d-cycles.

Introduce independent variables $x_1, \ldots, x_k$, with k at least as large as the number of rows in the Young diagram of λ. Define the *power sums* $P_j(x)$, $1 \le j \le d$, and the *discriminant* $\Delta(x)$ by

$$\begin{aligned} P_j(x) &= x_1^j + x_2^j + \cdots + x_k^j, \\ \Delta(x) &= \prod_{i<j} (x_i - x_j). \end{aligned} \tag{4.7}$$

If $f(x) = f(x_1, \ldots, x_k)$ is a formal power series, and $(l_1, \ldots, l_k)$ is a k-tuple of non-negative integers, let

$$[f(x)]_{(l_1,\ldots,l_k)} = \text{coefficient of } x_1^{l_1} \cdot \cdots \cdot x_k^{l_k} \text{ in } f. \tag{4.8}$$

Given a partition λ: $\lambda_1 \ge \cdots \ge \lambda_k \ge 0$ of d, set

$$l_1 = \lambda_1 + k - 1, \qquad l_2 = \lambda_2 + k - 2, \ldots, l_k = \lambda_k, \tag{4.9}$$

a strictly decreasing sequence of k non-negative integers. The character of V_λ evaluated on $g \in C_\mathbf{i}$ is given by the remarkable

Frobenius Formula 4.10

$$\chi_\lambda(C_\mathbf{i}) = \left[\Delta(x) \cdot \prod_j P_j(x)^{i_j}\right]_{(l_1,\ldots,l_k)}.$$

For example, if $d = 5$, $\lambda = (3, 2)$, and $C_\mathbf{i}$ is the conjugacy class of $(12)(345)$, i.e., $i_1 = 0$, $i_2 = 1$, $i_3 = 1$, then

$$\chi_{(3,2)}(C_\mathbf{i}) = [(x_1 - x_2)\cdot(x_1^2 + x_2^2)(x_1^3 + x_2^3)]_{(4,2)} = 1.$$

Other entries in our character tables for $\mathfrak{S}_3$, $\mathfrak{S}_4$, and $\mathfrak{S}_5$ can be verified as easily, verifying the assertions of Examples 4.5.

In terms of certain symmetric functions S_λ called *Schur polynomials*, Frobenius's formula can be expressed by

$$\prod_j P_j(x)^{i_j} = \sum \chi_\lambda(C_\mathbf{i}) S_\lambda,$$

the sum over all partitions λ of d in at most k parts (cf. Proposition 4.37 and (A.27)). Although we do not use Schur polynomials explicitly in this lecture, they play the central role in the algebraic background developed in Appendix A.

Let us use the Frobenius formula to compute the dimension of V_λ. The conjugacy class of the identity corresponds to $\mathbf{i} = (d)$, so

$$\dim V_\lambda = \chi_\lambda(C_{(d)}) = [\Delta(x)\cdot(x_1 + \cdots + x_k)^d]_{(l_1,\ldots,l_k)}.$$

Now $\Delta(x)$ is the Vandermonde determinant:

$$\begin{vmatrix} 1 & x_k & \cdots & x_k^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_1 & \cdots & x_1^{k-1} \end{vmatrix} = \sum_{\sigma \in \mathfrak{S}_k} (\operatorname{sgn} \sigma) x_k^{\sigma(1)-1} \cdot \cdots \cdot x_1^{\sigma(k)-1}.$$

The other term is

$$(x_1 + \cdots + x_k)^d = \sum \frac{d!}{r_1! \cdot \ldots \cdot r_k!} x_1^{r_1} x_2^{r_2} \cdot \ldots \cdot x_k^{r_k},$$

the sum over k-tuples $(r_1, \ldots, r_k)$ that sum to d. To find the coefficient of $x_1^{l_1} \cdot \ldots \cdot x_k^{l_k}$ in the product, we pair off corresponding terms in these two sums, getting

$$\sum \operatorname{sgn}(\sigma) \cdot \frac{d!}{(l_1 - \sigma(k) + 1)! \cdots (l_k - \sigma(1) + 1)!},$$

the sum over those σ in $\mathfrak{S}_k$ such that $l_{k-i+1} - \sigma(i) + 1 \ge 0$ for all $1 \le i \le k$. This sum can be written as

$$\frac{d!}{l_1!\cdots l_k!}\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)\prod_{j=1}^{k} l_j(l_j-1)\cdot\ldots\cdot(l_j-\sigma(k-j+1)+2)$$

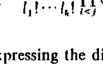
$$=\frac{d!}{l_1!\cdots l_k!}\begin{vmatrix} 1 & l_k & l_k(l_k-1) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & l_1 & l_1(l_1-1) & \cdots \end{vmatrix}.$$

By column reduction this determinant reduces to the van der Monde determinant, so

$$\dim V_\lambda = \frac{d!}{l_1!\cdots l_k!}\prod_{i<j}(l_i - l_j), \tag{4.11}$$

with $l_i = \lambda_i + k - i$.

There is another way of expressing the dimensions of the V_λ. The *hook length* of a box in a Young diagram is the number of squares directly below or directly to the right of the box, including the box once.



In the following diagram, each box is labeled by its hook length:

6	4	3	1
4	2	1	
1			

Hook Length Formula 4.12.

$$\dim V_\lambda = \frac{d!}{\prod(\text{Hook lengths})}.$$

For the above partition $4+3+1$ of 8, the dimension of the corresponding representation of $\mathfrak{S}_8$ is therefore $8!/6\cdot 4\cdot 4\cdot 2\cdot 3 = 70$.

Exercise 4.13*. Deduce the hook length formula from the Frobenius formula (4.11).

Exercise 4.14*. Use the hook length formula to show that the only irreducible representations of $\mathfrak{S}_d$ of dimension less than d are the trivial and alternating representations U and U' of dimension 1, the standard representation V and $V' = V\otimes U'$ of dimension $d-1$, and three other examples: the two-dimensional representation of $\mathfrak{S}_4$ corresponding to the partition $4 = 2+2$, and the two five-dimensional representations of $\mathfrak{S}_6$ corresponding to the partitions $6 = 3+3$ and $6 = 2+2+2$.

Exercise 4.15*. Using Frobenius's formula or otherwise, show that:

$$\chi_{(d-1,1)}(C_i) = i_1 - 1;$$
$$\chi_{(d-2,1,1)}(C_i) = \tfrac{1}{2}(i_1-1)(i_1-2) - i_2;$$
$$\chi_{(d-2,2)}(C_i) = \tfrac{1}{2}(i_1-1)(i_1-2) + i_2 - 1.$$

Can you continue this list?

Exercise 4.16*. If g is a cycle of length d in $\mathfrak{S}_d$, show that $\chi_\lambda(g)$ is ± 1 if λ is a hook, and zero if λ is not a hook:

$$\chi_\lambda(g) = \begin{cases}(-1)^s & \text{if } \lambda = (d-s, 1, \ldots, 1),\ 0\le s\le d-1\\ 0 & \text{otherwise.}\end{cases}$$

Exercise 4.17. Frobenius [Fro1] used his formula to compute the value of χ_λ on a cycle of length $m \le d$.

(a) Following the procedure that led to (4.11)—which was the case $m = 1$—show that

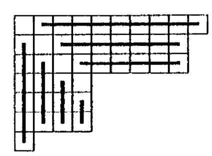
$$\chi_\lambda((12\ldots m)) = \frac{\dim V_\lambda}{-m^2 h_m}\sum_{p=1}^{k}\frac{\psi(l_p)}{\varphi'(l_p)}, \tag{4.18}$$

where $h_m = d!/(d-m)!m$ is the number of cycles of length m (if $m > 1$), and

$$\varphi(x) = \prod_{i=1}^{k}(x - l_i), \qquad \psi(x) = \varphi(x-m)\prod_{j=1}^{m}(x-j+1).$$

The sum in (4.18) can be realized as the coefficient of x^{-1} in the Laurent expansion of $\psi(x)/\varphi(x)$ at $x = \infty$.

Define the *rank* r of a partition to be the length of the diagonal of its Young diagram, and let a_i and b_i be the number of boxes below and to the right of the ith box of the diagonal, reading from lower right to upper left. Frobenius called $\begin{pmatrix} a_1 a_2 \ldots a_r \\ b_1 b_2 \ldots b_r\end{pmatrix}$ the *characteristics* of the partition. (Many writers now use a reverse notation for the characteristics, writing $(b_r, \ldots, b_1 | a_r, \ldots, a_1)$ instead.) For the partition (10, 9, 9, 4, 4, 4, 1):



$r = 4$

$$\text{characteristics} = \begin{pmatrix} 2 & 3 & 4 & 6 \\ 0 & 6 & 7 & 9\end{pmatrix}$$

Algebraically, r and the characteristics $a_1 < \cdots < a_r$ and $b_1 < \cdots < b_r$ are determined by requiring the equality of the two sets

$$\{l_1, \ldots, l_k, k-1-a_1, \ldots, k-1-a_r\} \quad \text{and}$$
$$\{0, 1, \ldots, k-1, k+b_1, \ldots, k+b_r\}.$$

(b) Show that $\psi(x)/\varphi(x) = g(y)/f(y)$, where $y = x - d$ and

$$f(y) = \frac{\prod_{i=1}^{r}(y - b_i)}{\prod_{i=1}^{r}(y + a_i + 1)}, \qquad g(y) = f(y-m)\prod_{j=1}^{m}(y - j + 1).$$

Deduce that the sum in (4.18) is the coefficient of x^{-1} in $g(x)/f(x)$.

(c) When $m = 2$, use this to prove the formula

$$\chi_\lambda((12)) = \frac{\dim V_\lambda}{d(d-1)}\sum_{i=1}^{r}(b_i(b_i+1) - a_i(a_i+1)).$$

Hurwitz [Hur] used this formula of Frobenius to calculate the number of ways to write a given permutation as a product of transpositions. From this he gave a formula for the number of branched coverings of the Riemann sphere with a given number of sheets and given simple branch points. Ingram [In] has given other formulas for $\chi_\lambda(g)$, when g is a somewhat more complicated conjugacy class.

Exercise 4.19*. If V is the standard representation of $\mathfrak{S}_d$, prove the decompositions into irreducible representations:

$$\mathrm{Sym}^2 V \cong U \oplus V \oplus V_{(d-2,2)},$$
$$V \otimes V = \mathrm{Sym}^2 V \oplus \Lambda^2 V \cong U \oplus V \oplus V_{(d-2,2)} \oplus V_{(d-2,1,1)}.$$

Exercise 4.20*. Suppose λ is symmetric, i.e., $\lambda = \lambda'$, and let $q_1 > q_2 > \cdots > q_r > 0$ be the lengths of the symmetric hooks that form the diagram of λ; thus, $q_1 = 2\lambda_1 - 1$, $q_2 = 2\lambda_2 - 3, \ldots$. Show that if g is a product of disjoint cycles of lengths $q_1, q_2, \ldots, q_r$, then

$$\chi_\lambda(g) = (-1)^{(d-r)/2}.$$

§4.2. Irreducible Representations of $\mathfrak{S}_d$

We show next that the representations V_λ constructed in the first section are exactly the irreducible representations of $\mathfrak{S}_d$. This proof appears in many standard texts (e.g. [C-R], [Ja-Ke], [N-S], [Wel]), so we will be a little concise.

Let $A = \mathbb{C}\mathfrak{S}_d$ be the group ring of $\mathfrak{S}_d$. For a partition λ of d, let P and Q be the corresponding subgroups preserving the rows and columns of a Young tableau T corresponding to λ, let $a = a_\lambda$, $b = b_\lambda$, and let $c = c_\lambda = ab$ be

the corresponding Young symmetrizer, so $V_\lambda = Ac_\lambda$ is the corresponding representation. (These groups and elements should really be subscripted by T to denote dependence on the tableau chosen, but the assertions made depend only on the partition, so we usually omit reference to T.)

Note that $P \cap Q = \{1\}$, so an element of $\mathfrak{S}_d$ can be written in at most one way as a product $p \cdot q$, $p \in P$, $q \in Q$. Thus, c is the sum $\sum \pm e_g$, the sum over all g that can be written as $p \cdot q$, with coefficient ± 1 being $\mathrm{sgn}(q)$; in particular, the coefficient of e_1 in c is 1.

Lemma 4.21. (1) *For $p \in P$, $p \cdot a = a \cdot p = a$.*

(2) *For $q \in Q$, $(\mathrm{sgn}(q)q) \cdot b = b \cdot (\mathrm{sgn}(q)q) = b$.*

(3) *For all $p \in P$, $q \in Q$, $p \cdot c \cdot (\mathrm{sgn}(q)q) = c$, and, up to multiplication by a scalar, c is the only such element in A.*

PROOF. Only the last assertion is not obvious. If $\sum n_g e_g$ satisfies the condition in (3), then $n_{pgq} = \mathrm{sgn}(q)n_g$ for all g, p, q; in particular, $n_{pq} = \mathrm{sgn}(q)n_1$. Thus, it suffices to verify that $n_g = 0$ if $g \notin PQ$. For such g it suffices to find a transposition t such that $p = t \in P$ and $q = g^{-1}tg \in Q$; for then $g = pgq$, so $n_g = -n_g$. If $T' = gT$ is the tableau obtained by replacing each entry i of T by $g(i)$, the claim is that there is are two distinct integers that appear in the same row of T and in the same column of T'; t is then the transposition of these two integers. We must verify that if there were no such pair of integers, then one could write $g = p \cdot q$ for some $p \in P$, $q \in Q$. To do this, first take $p_1 \in P$ and $q_1' \in Q' = gQg^{-1}$ so that $p_1 T$ and $q_1' T'$ have the same first row; repeating on the rest of the tableau, one gets $p \in P$ and $q' \in Q'$ so that $pT = q'T'$. Then $pT = q'gT$, so $p = q'g$, and therefore $g = pq$, where $q = g^{-1}(q')^{-1}g \in Q$, as required. □

We order partitions *lexicographically*:

$$\lambda > \mu \quad \text{if the first nonvanishing } \lambda_i - \mu_i \text{ is positive.} \tag{4.22}$$

Lemma 4.23. (1) *If $\lambda > \mu$, then for all $x \in A$, $a_\lambda \cdot x \cdot b_\mu = 0$. In particular, if $\lambda > \mu$, then $c_\lambda \cdot c_\mu = 0$.*

(2) *For all $x \in A$, $c_\lambda \cdot x \cdot c_\lambda$ = is a scalar multiple of c_λ. In particular, $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{C}$.*

PROOF. For (1), we may take $x = g \in \mathfrak{S}_d$. Since $g \cdot b_\mu \cdot g^{-1}$ is the element constructed from gT', where T' is the tableau used to construct b_μ, it suffices to show that $a_\lambda \cdot b_\mu = 0$. One verifies that $\lambda > \mu$ implies that there are two integers in the same row of T and the same column of T'. If t is the transposition of these integers, then $a_\lambda \cdot t = a_\lambda$, $t \cdot b_\mu = -b_\mu$, so $a_\lambda \cdot b_\mu = a_\lambda \cdot t \cdot t \cdot b_\mu = -a_\lambda \cdot b_\mu$, as required. Part (2) follows from Lemma 4.21 (3). □

Exercise 4.24*. Show that if $\lambda \neq \mu$, then $c_\lambda \cdot A \cdot c_\mu = 0$; in particular, $c_\lambda \cdot c_\mu = 0$.

Lemma 4.25. (1) *Each V_λ is an irreducible representation of $\mathfrak{S}_d$.*
(2) *If $\lambda \neq \mu$, then V_λ and V_μ are not isomorphic.*

Proof. For (1) note that $c_\lambda V_\lambda \subset \mathbb{C}c_\lambda$ by Lemma 4.23. If $W \subset V_\lambda$ is a subrepresentation, then $c_\lambda W$ is either $\mathbb{C}c_\lambda$ or 0. If the first is true, then $V_\lambda = A \cdot c_\lambda \subset W$. Otherwise $W \cdot W \subset A \cdot c_\lambda W = 0$, but this implies $W = 0$. Indeed, a projection from A onto W is given by right multiplication by an element $\varphi \in A$ with $\varphi = \varphi^2 \in W \cdot W = 0$. This argument also shows that $c_\lambda V_\lambda \neq 0$, i.e., that the number n_λ of the previous lemma is nonzero.

For (2), we may assume $\lambda > \mu$. Then $c_\lambda V_\lambda = \mathbb{C}c_\lambda \neq 0$, but $c_\lambda V_\mu = c_\lambda \cdot Ac_\mu = 0$, so they cannot be isomorphic A-modules. □

Lemma 4.26. *For any λ, $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$, with $n_\lambda = d!/\dim V_\lambda$.*

Proof. Let F be right multiplication by c_λ on A. Since F is multiplication by n_λ on V_λ, and zero on $\operatorname{Ker}(c_\lambda)$, the trace of F is n_λ times the dimension of V_λ. But the coefficient of e_g in $e_g \cdot c_\lambda$ is 1, so $\operatorname{trace}(F) = |\mathfrak{S}_d| = d!$. □

Since there are as many irreducible representations V_λ as conjugacy classes of $\mathfrak{S}_d$, these must form a complete set of isomorphism classes of irreducible representations, which completes the proof of Theorem 4.3. In the next section we will prove Frobenius's formula for the character of V_λ, and, in a series of exercises, discuss a little of what else is known about them: how to decompose tensor products or induced or restricted representations, how to find a basis for V_λ, etc.

§4.3. Proof of Frobenius's Formula

For any partition λ of d, we have a subgroup, often called a *Young subgroup*,

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k} \hookrightarrow \mathfrak{S}_d. \tag{4.27}$$

Let U_λ be the representation of $\mathfrak{S}_d$ induced from the trivial representation of $\mathfrak{S}_\lambda$. Equivalently, $U_\lambda = A \cdot a_\lambda$, with a_λ as in the preceding section. Let

$$\psi_\lambda = \chi_{U_\lambda} = \text{character of } U_\lambda. \tag{4.28}$$

Key to this investigation is the relation between U_λ and V_λ, i.e., between ψ_λ and the character χ_λ of V_λ. Note first that V_λ appears in U_λ, since there is a surjection

$$U_\lambda = Aa_\lambda \twoheadrightarrow V_\lambda = Aa_\lambda b_\lambda, \quad x \mapsto x \cdot b_\lambda. \tag{4.29}$$

Alternatively,

$$V_\lambda = Aa_\lambda b_\lambda \cong Ab_\lambda a_\lambda \subset Aa_\lambda = U_\lambda,$$

by Exercise 4.4. For example, we have

$$U_{(d-1,1)} \cong V_{(d-1,1)} \oplus V_{(d)}$$

which expresses the fact that the permutation representation $\mathbb{C}^d$ of $\mathfrak{S}_d$ is the sum of the standard representation and the trivial representation. Eventually we will see that every U_λ contains V_λ with multiplicity one, and contains only other V_μ for $\mu > \lambda$.

The character of U_λ is easy to compute directly since U_λ is an induced representation, and we do this next.

For $\mathbf{i} = (i_1, \ldots, i_d)$ a d-tuple of non-negative integers with $\sum \alpha i_\alpha = d$, denote by

$$C_\mathbf{i} \subset \mathfrak{S}_d$$

the conjugacy class consisting of elements made up of i_1 1-cycles, i_2 2-cycles, $\ldots$, i_d d-cycles. The number of elements in $C_\mathbf{i}$ is easily counted to be

$$|C_\mathbf{i}| = \frac{d!}{1^{i_1} i_1! \, 2^{i_2} i_2! \cdot \ldots \cdot d^{i_d} i_d!}. \tag{4.30}$$

By the formula for characters of induced representations (Exercise 3.19),

$$\begin{aligned}\psi_\lambda(C_\mathbf{i}) &= \frac{1}{|C_\mathbf{i}|}[\mathfrak{S}_d : \mathfrak{S}_\lambda] \cdot |C_\mathbf{i} \cap \mathfrak{S}_\lambda| \\ &= \frac{1^{i_1} i_1! \cdot \ldots \cdot d^{i_d} i_d!}{d!} \cdot \frac{d!}{\lambda_1! \cdot \ldots \cdot \lambda_k!} \cdot \sum \prod_{p=1}^{k} \frac{\lambda_p!}{1^{r_{p1}} r_{p1}! \cdot \ldots d^{r_{pd}} r_{pd}!},\end{aligned}$$

where the sum is over all collections $\{r_{pq}: 1 \leq p \leq k, 1 \leq q \leq d\}$ of non-negative integers satisfying

$$\begin{aligned} i_q &= r_{1q} + r_{2q} + \cdots + r_{kq}, \\ \lambda_p &= r_{p1} + 2r_{p2} + \cdots + dr_{pd}. \end{aligned}$$

(To count $C_\mathbf{i} \cap \mathfrak{S}_\lambda$, write the pth component of an element of $\mathfrak{S}_\lambda$ as a product of r_{p1} 1-cycles, r_{p2} 2-cycles, ….) Simplifying,

$$\psi_\lambda(C_\mathbf{i}) = \sum \prod_{q=1}^{d} \frac{i_q!}{r_{1q}! r_{2q}! \cdot \ldots \cdot r_{kq}!}, \tag{4.31}$$

the sum over the same collections of integers $\{r_{pq}\}$.

This sum is exactly the coefficient of the monomial $X^\lambda = x_1^{\lambda_1} \cdot \ldots \cdot x_k^{\lambda_k}$ in the power sum symmetric polynomial

$$P^{(\mathbf{i})} = (x_1 + \cdots + x_k)^{i_1} \cdot (x_1^2 + \cdots + x_k^2)^{i_2} \cdot \ldots \cdot (x_1^d + \cdots + x_k^d)^{i_d}. \tag{4.32}$$

So we have the formula

$$\psi_\lambda(C_\mathbf{i}) = [P^{(\mathbf{i})}]_\lambda = \text{coefficient of } X^\lambda \text{ in } P^{(\mathbf{i})}. \tag{4.33}$$

To prove Frobenius's formula, we need to compare these coefficients with the coefficients $\omega_\lambda(\mathbf{i})$ defined by

$$\omega_\lambda(\mathbf{i}) = [\Delta \cdot P^{(\mathbf{i})}]_l, \quad l = (\lambda_1 + k - 1, \lambda_2 + k - 2, \ldots, \lambda_k). \tag{4.34}$$

Our goal, Frobenius's formula, is the assertion that $\chi_\lambda(C_\mathbf{i}) = \omega_\lambda(\mathbf{i})$.

There is a general identity, valid for any symmetric polynomial P, relating such coefficients:

$$[P]_\lambda = \sum_\mu K_{\mu\lambda}[\Delta \cdot P]_{(\mu_1+k-1, \mu_2+k-2, \ldots, \mu_k)},$$

where the coefficients $K_{\mu\lambda}$ are certain universally defined integers, called *Kostka numbers.* For any partitions λ and μ of d, the integer $K_{\mu\lambda}$ may be defined combinatorially as the number of ways to fill the boxes of the Young diagram for μ with λ_1 1's, λ_2 2's, up to λ_k k's, in such a way that the entries in each row are nondecreasing, and those in each column are strictly increasing; such are called *semistandard tableaux on μ of type λ.* In particular,

$$K_{\lambda\lambda} = 1, \quad \text{and } K_{\mu\lambda} = 0 \text{ for } \mu < \lambda.$$

The integer $K_{\mu\lambda}$ may be also be defined to be the coefficient of the monomial $X^\lambda = x_1^{\lambda_1} \cdot \ldots \cdot x_k^{\lambda_k}$ in the Schur polynomial S_μ corresponding to μ. For the proof that these are equivalent definitions, see (A.9) and (A.19) of Appendix A. In the present case, applying Lemma A.26 to the polynomial $P = P^{(\mathbf{i})}$, we deduce

$$\psi_\lambda(C_\mathbf{i}) = \sum_\mu K_{\mu\lambda}\omega_\mu(\mathbf{i}) = \omega_\lambda(\mathbf{i}) + \sum_{\mu > \lambda} K_{\mu\lambda}\omega_\mu(\mathbf{i}). \tag{4.35}$$

The result of Lemma A.28 can be written, using (4.30), in the form

$$\frac{1}{d!}\sum_\mathbf{i} |C_\mathbf{i}|\omega_\lambda(\mathbf{i})\omega_\mu(\mathbf{i}) = \delta_{\lambda\mu}. \tag{4.36}$$

This indicates that the functions ω_λ, regarded as functions on the conjugacy classes of $\mathfrak{S}_d$, satisfy the same orthogonality relations as the irreducible characters of $\mathfrak{S}_d$. In fact, one can deduce formally from these equations that the ω_λ must be the irreducible characters of $\mathfrak{S}_d$, which is what Frobenius proved. A little more work is needed to see that ω_λ is actually the character of the representation V_λ, that is, to prove

Proposition 4.37. *Let $\chi_\lambda = \chi_{V_\lambda}$ be the character of V_λ. Then for any conjugacy class $C_\mathbf{i}$ of $\mathfrak{S}_d$,*

$$\chi_\lambda(C_\mathbf{i}) = \omega_\lambda(\mathbf{i}).$$

PROOF. We have seen in (4.29) that the representation U_λ, whose character is ψ_λ, contains the irreducible representation V_λ. In fact, this is all that we need to know about the relation between U_λ and V_λ. It implies that we have

$$\psi_\lambda = \sum_\mu n_{\lambda\mu}\chi_\mu, \quad n_{\lambda\lambda} \geq 1, \text{ all } n_{\lambda\mu} \geq 0. \tag{4.38}$$

Consider this equation together with (4.35). We deduce first that each ω_λ is a

virtual character: we can write

$$\omega_\lambda = \sum m_{\lambda\mu}\chi_\mu, \quad m_{\lambda\mu} \in \mathbb{Z}.$$

But the ω_λ, like the χ_λ, are orthonormal by (4.36), so

$$1 = (\omega_\lambda, \omega_\lambda) = \sum_\mu m_{\lambda\mu}^2,$$

and hence ω_λ is $\pm\chi$ for some irreducible character χ. (It follows from the hook length formula that the plus sign holds here, but we do not need to assume this.)

Fix λ, and assume inductively that $\chi_\mu = \omega_\mu$ for all $\mu > \lambda$, so by (4.35)

$$\psi_\lambda = \omega_\lambda + \sum_{\mu > \lambda} K_{\mu\lambda}\chi_\mu.$$

Comparing this with (4.38), and using the linear independence of characters, the only possibility is that $\omega_\lambda = \chi_\lambda$. □

Corollary 4.39 (Young's rule). *The integer $K_{\mu\lambda}$ is the multiplicity of the irreducible representation V_μ in the induced representation U_λ:*

$$U_\lambda \cong V_\lambda \oplus \bigoplus_{\mu > \lambda} K_{\mu\lambda}V_\mu, \qquad \psi_\lambda = \chi_\lambda + \sum_{\mu > \lambda} K_{\mu\lambda}\chi_\mu.$$

Note that when $\lambda = (1, \ldots, 1)$, U_λ is just the regular representation, so $K_{\mu(1,\ldots,1)} = \dim V_\mu$. This shows that *the dimension of V_λ is the number of standard tableaux on λ*, i.e., the number of ways to fill the Young diagram of λ with the numbers from 1 to d, such that all rows and columns are increasing. The hook length formula gives another combinatorial formula for this dimension. Frame, Robinson, and Thrall proved that these two numbers are equal. For a short and purely combinatorial proof, see [G-N-W]. For another proof that the dimension of V_λ is the number of standard tableaux, see [Jam]. The latter leads to a canonical decomposition of the group ring $A = \mathbb{C}\mathfrak{S}_d$ as the direct sum of left ideals Ae_T, summing over all standard tableaux, with $e_T = (\dim V_\lambda/d!) \cdot c_T$, and c_T the Young symmetrizer corresponding to T, cf. Exercises 4.47 and 4.50. This, in turn, leads to explicit calculation of matrices of the representations V_λ with integer coefficients.

For another example of Young's rule, we have a decomposition

$$U_{(d-a,a)} = \bigoplus_{i=0}^{a} V_{(d-i,i)}.$$

In fact, the only μ whose diagrams can be filled with $d - a$ 1's and a 2's, nondecreasing in rows and strictly increasing in columns, are those with at most two rows, with the second row no longer than a; and such a diagram has only one such tableau, so there are no multiplicities.

Exercise 4.40*. The characters ψ_λ of $\mathfrak{S}_d$ have been defined only when λ is a partition of d. Extend the definition to any k-tuple $a = (a_1, \ldots, a_k)$ of integers that add up to d by setting $\psi_a = 0$ if any of the a_i are negative, and otherwise $\psi_a = \psi_\lambda$, where λ is the reordering of $a_1, \ldots, a_k$ in descending order. In this case ψ_a is the character of the representation induced from the trivial representation by the inclusion of $\mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_k}$ in $\mathfrak{S}_d$. Use (A.5) and (A.9) of Appendix A to prove the *determinantal formula* for the irreducible characters χ_λ in terms of the induced characters ψ_μ:

$$\chi_\lambda = \sum_{\tau \in \mathfrak{S}_k} \operatorname{sgn}(\tau) \psi_{(\lambda_1 + \tau(1) - 1, \lambda_2 + \tau(2) - 2, \ldots, \lambda_k + \tau(k) - k)}.$$

If one writes ψ_a as a formal product $\psi_{a_1} \cdot \psi_{a_2} \cdot \ldots \cdot \psi_{a_k}$, the preceding formula can be written

$$\chi_\lambda = |\psi_{\lambda_i + j - i}| = \begin{vmatrix} \psi_{\lambda_1} & \psi_{\lambda_1 + 1} & \psi_{\lambda_1 + k - 1} \\ \psi_{\lambda_2 - 1} & \psi_{\lambda_2} \cdots & \\ \vdots & & \vdots \\ \psi_{\lambda_k - k + 1} \cdots & & \psi_{\lambda_k} \end{vmatrix}.$$

The formal product of the preceding exercise is the character version of an "outer product" of representations. Given any non-negative integers $d_1, \ldots, d_k$, and representations V_i of $\mathfrak{S}_{d_i}$, denote by $V_1 \circ \cdots \circ V_k$ the (isomorphism class of the) representation of $\mathfrak{S}_d$, $d = \sum d_i$, induced from the tensor product representation $V_1 \boxtimes \cdots \boxtimes V_k$ of $\mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_k}$ by the inclusion of $\mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_k}$ in $\mathfrak{S}_d$ (see Exercise 2.36). This product is commutative and associative. It will turn out to be useful to have a procedure for decomposing such a representation into its irreducible pieces. For this it is enough to do the case of two factors, and with the individual representations V_i irreducible. In this case, one has, for V_λ the representation of $\mathfrak{S}_d$ corresponding to the partition λ of d and V_μ the representation of $\mathfrak{S}_m$ corresponding to the partition μ of m,

$$V_\lambda \circ V_\mu = \sum N_{\lambda\mu\nu} V_\nu, \tag{4.41}$$

the sum over all partitions ν of $d + m$, with $N_{\lambda\mu\nu}$ the coefficients given by the *Littlewood–Richardson rule* (A.8) of Appendix A. Indeed, by the exercise, the character of $V_\lambda \circ V_\mu$ is the product of the corresponding determinants, and, by (A.8), that is the sum of the characters $N_{\lambda\mu\nu}\chi_\nu$.

When $m = 1$ and $\mu = (m)$, V_μ is trivial; this gives

$$\operatorname{Ind}_{\mathfrak{S}_d}^{\mathfrak{S}_{d+1}} V_\lambda = \sum V_\nu, \tag{4.42}$$

the sum over all ν whose Young diagram is obtained from that of λ by adding one box. This formula uses only a simpler form of the Littlewood–Richardson rule known as Pieri's formula, which is proved in (A.7).

Exercise 4.43*. Show that the Littlewood–Richardson number $N_{\lambda\mu\nu}$ is the multiplicity of the irreducible representation $V_\lambda \boxtimes V_\mu$ in the restriction of V_ν from $\mathfrak{S}_{d+m}$ to $\mathfrak{S}_d \times \mathfrak{S}_m$. In particular, taking $m = 1$, $\mu = (1)$, Pieri's formula (A.7) gives

$$\operatorname{Res}_{\mathfrak{S}_d}^{\mathfrak{S}_{d+1}} V_\nu = \sum V_\lambda,$$

the sum over all λ obtained from ν by removing one box. This is known as the "branching theorem," and is useful for inductive proofs and constructions, particularly because the decomposition is multiplicity free. For example, you can use it to reprove the fact that the multiplicity of V_λ in U_μ is the number of semistandard tableaux on μ of type λ. It can also be used to prove the assertion made in Exercise 4.6 that the representations corresponding to hooks are exterior powers of the standard representation.

Exercise 4.44* (Pieri's rule). Regard $\mathfrak{S}_d$ as a subgroup of $\mathfrak{S}_{d+m}$ as usual. Let λ be a partition of d and ν a partition of $d + m$. Use Exercise 4.40 to show that the multiplicity of V_ν in the induced representation $\operatorname{Ind}(V_\lambda)$ is zero unless the Young diagram of λ is contained in that of ν, and then it is the number of ways to number the skew diagram lying between them with the numbers from 1 to m, increasing in both row and column. By Frobenius reciprocity, this is the same as the multiplicity of V_λ in $\operatorname{Res}(V_\nu)$.

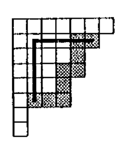
When applied to $d = 0$ (or 1), this implies again that the dimension of V_ν is the number of standard tableaux on the Young diagram of ν.

For a sampling of the many applications of these rules, see [Dia §7, §8].

Problem 4.45*. The *Murnaghan–Nakayama rule* gives an efficient inductive method for computing character values: If λ is a partition of d, and $g \in \mathfrak{S}_d$ is written as a product of an m-cycle and a disjoint permutation $h \in \mathfrak{S}_{d-m}$, then

$$\chi_\lambda(g) = \sum (-1)^{r(\mu)} \chi_\mu(h),$$

where the sum is over all partitions μ of $d - m$ that are obtained from λ by removing a skew hook of length m, and $r(\mu)$ is the number of vertical steps in the skew hook, i.e., one less than the number of rows in the hook. A *skew hook* for λ is a connected region of boundary boxes for its Young diagram such that removing them leaves a smaller Young diagram; there is a one-to-one correspondence between skew hooks and ordinary hooks of the same size, as indicated:



$\lambda = (7, 6, 5, 5, 4, 4, 1, 1)$

$\mu = (7, 4, 4, 3, 3, 1, 1, 1)$

hook length $= 9$, $r = 4$

For example, if λ has no hooks of length m, then $\chi_\lambda(g) = 0$.

The Murnaghan–Nakayama rule may be written inductively as follows: If g is a written as a product of disjoint cycles of lengths $m_1, m_2, \ldots, m_p$, with the lengths m_i taken in any order, then $\chi_\lambda(g)$ is the sum $\sum (-1)^{r(s)}$, where the sum is over all ways s to decompose the Young diagram of λ by successively removing p skew hooks of lengths $m_1, \ldots, m_p$, and $r(s)$ is the total number of vertical steps in the hooks of s.

(a) Deduce the Murnaghan–Nakayama rule from (4.41) and Exercise 4.16, using the Littlewood–Richardson rule. Or:

(b) With the notation of Exercise 4.40, show that

$$\psi_{a_1}\psi_{a_2}\cdot\ldots\cdot\psi_{a_k}(g) = \sum_{i=1}^{k} \psi_{a_1}\psi_{a_2}\cdot\ldots\cdot\psi_{a_i-m}\psi_{a_{i+1}}\cdot\ldots\cdot\psi_{a_k}(h).$$

Exercise 4.46*. Show that Corollary 4.39 implies the "Snapper conjecture": the irreducible representation V_μ occurs in the induced representation U_λ if and only if

$$\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i \quad \text{for all } j \geq 1.$$

Problem 4.47*. There is a more intrinsic construction of the irreducible representation V_λ, called a *Specht module*, which does not involve of the choice of a tableau; it is also useful for studying representations of $\mathfrak{S}_d$ in positive characteristic. Define a *tabloid* $\{T\}$ to be an equivalence class of tableaux (numberings by the integers 1 to d) on λ, two being equivalent if the rows are the same up to order. Then $\mathfrak{S}_d$ acts by permutations on the tabloids, and the corresponding representation, with basis the tabloids, is isomorphic to U_λ. For each tableau T, define an element E_T in this representation space, by

$$E_T = b_T\{T\} = \sum \operatorname{sgn}(q)\{qT\},$$

the sum over the q that preserve the columns of T. The span of all E_T's is isomorphic to V_λ, and the E_T's, where T varies over the standard tableaux, form a basis.

Another construction of V_λ is to take the subspace of the polynomial ring $\mathbb{C}[x_1, \ldots, x_d]$ spanned by all polynomials F_T, where $F_T = \prod (x_i - x_j)$, the product over all pairs $i < j$ which occur in the same column in the tableau T.

Exercise 4.48*. Let U'_λ be the representation $A \cdot b_\lambda$, which is the representation of $\mathfrak{S}_d$ induced from the tensor product of the alternating representations on the subgroup $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_r}$, where $\mu = \lambda'$ is the conjugate partition. Show that the decomposition of U'_λ is

$$U'_\lambda = \sum_\mu K_{\mu'\lambda'} V_\mu.$$

Deduce that V_λ is the only irreducible representation that occurs in both U_λ and U'_λ, and it occurs in each with multiplicity one.

Note, however, that in general $A \cdot c_\lambda \neq A \cdot a_\lambda \cap A \cdot b_\lambda$ since $A \cdot c_\lambda$ may not be contained in $A \cdot a_\lambda$.

Exercise 4.49*. With notation as in (4.41), if $U' = V_{(1,\ldots,1)}$ is the alternating representation of $\mathfrak{S}_m$, show that $V_\lambda \circ V_{(1,\ldots,1)}$ decomposes into a direct sum $\bigoplus V_\pi$, the sum over all π whose Young diagram can be obtained from that of λ by adding m boxes, with no two in the same row.

Exercise 4.50. We have seen that $A = \mathbb{C}\mathfrak{S}_d$ is isomorphic to a direct sum of m_λ copies of $V_\lambda = Ac_\lambda$, where $m_\lambda = \dim V_\lambda$ is the number of standard tableaux on λ. This can be seen explicitly as follows. For each standard tableau T on each λ, let c_T be the element of $\mathbb{C}\mathfrak{S}_d$ constructed from T. Then $A = \bigoplus A \cdot c_T$. Indeed, an argument like that in Lemma 4.23 shows that $c_T \cdot c_{T'} = 0$ whenever T and T' are tableaux on the same diagram and $T > T'$, i.e., the first entry (reading from left to right, then top to bottom) where the tableaux differ has the entry of T larger than that of T'. From this it follows that the sum $\Sigma A \cdot c_T$ is direct. A dimension count concludes the proof. (This also gives another proof that the dimension of V_λ is the number of standard tableaux on λ, provided one verifies that the sum of the squares of the latter numbers is $d!$, cf. [Boe] or [Ke].)

Exercise 4.51*. There are several methods for decomposing a tensor product of two representations of $\mathfrak{S}_d$, which amounts to finding the coefficients $C_{\lambda\mu\nu}$ in the decomposition

$$V_\lambda \otimes V_\mu \cong \Sigma_\nu C_{\lambda\mu\nu} V_\nu,$$

for λ, μ, and ν partitions of d. Since one knows how to express V_μ in terms of the induced representations U_ν, it suffices to compute $V_\lambda \otimes U_\nu$, which is isomorphic to $\operatorname{Ind}(\operatorname{Res}(V_\lambda))$, restricting and inducing from the subgroup $\mathfrak{S}_\nu = \mathfrak{S}_{\nu_1} \times \mathfrak{S}_{\nu_2} \times \cdots$; this restriction and induction can be computed by the Littlewood–Richardson rule. For $d \leq 5$, you can work out these coefficients using only restriction to $\mathfrak{S}_{d-1}$ and Pieri's formula.

(a) Prove the following closed-form formula for the coefficients, which shows in particular that they are independent of the ordering of the subscripts λ, μ, and ν:

$$C_{\lambda\mu\nu} = \sum_{\mathbf{i}} \frac{1}{z(\mathbf{i})} \omega_\lambda(\mathbf{i})\omega_\mu(\mathbf{i})\omega_\nu(\mathbf{i}),$$

the sum over all $\mathbf{i} = (i_1, \ldots, i_d)$ with $\Sigma \alpha i_\alpha = d$, and with $\omega_\lambda(\mathbf{i}) = \chi_\lambda(C_{\mathbf{i}})$ and $z(\mathbf{i}) = i_1! 1^{i_1} \cdot i_2! 2^{i_2} \cdot \ldots \cdot i_d! d^{i_d}$.

(b) Show that

$$C_{\lambda\mu(d)} = \begin{cases} 1 & \text{if } \mu = \lambda \\ 0 & \text{otherwise,} \end{cases} \qquad C_{\lambda\mu(1,\ldots,1)} = \begin{cases} 1 & \text{if } \mu = \lambda' \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 4.52*. Let $R_d = R(\mathfrak{S}_d)$ denote the representation ring, and set $R = \bigoplus_{d=0}^{\infty} R_d$. The outer product of (4.41) determines maps

$$R_n \otimes R_m \to R_{n+m},$$

which makes R into a commutative, graded $\mathbb{Z}$-algebra. Restriction determines maps

$$R_{n+m} = R(\mathfrak{S}_{n+m}) \to R(\mathfrak{S}_n \times \mathfrak{S}_m) = R_n \otimes R_m,$$

which defines a *co-product* $\delta: R \to R \otimes R$. Together, these make R into a (graded) Hopf algebra. (This assertion implies many of the formulas we have proved in this lecture, as well as some we have not.)

(a) Show that, as an algebra,

$$R \cong \mathbb{Z}[H_1, \ldots, H_d, \ldots],$$

where H_d is an indeterminate of degree d; H_d corresponds to the trivial representation of $\mathfrak{S}_d$. Show that the co-product δ is determined by

$$\delta(H_n) = H_n \otimes 1 + H_{n-1} \otimes H_1 + \cdots + 1 \otimes H_n.$$

If we set $\Lambda = \mathbb{Z}[H_1, \ldots, H_d, \ldots] = \bigoplus \Lambda_d$, we can identify Λ_d with the symmetric polynomials of degree d in $k \geq d$ variables. The basic symmetric polynomials in Λ_d defined in Appendix A therefore correspond to virtual representations of $\mathfrak{S}_d$.

(b) Show that E_d corresponds to the alternating representation U', and

$$H_\lambda \leftrightarrow U_\lambda, \qquad S_\lambda \leftrightarrow V_\lambda, \qquad E_\lambda \leftrightarrow U'_{\lambda'}.$$

(c) Show that the scalar product $\langle\ ,\ \rangle$ defined on Λ_d in (A.16) corresponds to the scalar product defined on class functions in (2.11).

(d) Show that the involution ϑ of Exercise A.32 corresponds to tensoring a representation with the alternating representation U'.

(e) Show that the inverse map from R_d to Λ_d takes a representation W to

$$\sum_{\mathbf{i}} \frac{1}{z(\mathbf{i})} \chi_W(C_{(\mathbf{i})}) P^{(\mathbf{i})},$$

where $z(\mathbf{i}) = i_1! 1^{i_1} \cdot i_2! 2^{i_2} \cdot \ldots \cdot i_d! d^{i_d}$.

The (inner) tensor product of representations of $\mathfrak{S}_d$ gives a map $R_d \otimes R_d \to R_d$ which corresponds to an "inner product" on symmetric functions, sometimes denoted $*$.

(f) Show that

$$P^{(\mathbf{i})} * P^{(\mathbf{j})} = \begin{cases} 0 & \text{for } \mathbf{j} \neq \mathbf{i} \\ z(\mathbf{i}) P^{(\mathbf{i})} & \text{if } \mathbf{j} = \mathbf{i}. \end{cases}$$

Since these $P^{(\mathbf{i})}$ form a basis for $\Lambda_d \otimes \mathbb{Q}$, this formula determines the inner product.

LECTURE 5

Representations of $\mathfrak{A}_d$ and $\mathrm{GL}_2(\mathbb{F}_q)$

In this lecture we analyze the representation of two more types of groups: the alternating groups $\mathfrak{A}_d$ and the linear groups $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{SL}_2(\mathbb{F}_q)$ over finite fields. In the former case, we prove some general results relating the representations of a group to the representations of a subgroup of index two, and use what we know about the symmetric group; this should be completely straightforward given just the basic ideas of the preceding lecture. In the latter case we start essentially from scratch. The two sections can be read (or not) independently; neither is logically necessary for the remainder of the book.

§5.1: Representations of $\mathfrak{A}_d$
§5.2: Representations of $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{SL}_2(\mathbb{F}_q)$

§5.1. Representations of $\mathfrak{A}_d$

The alternating groups $\mathfrak{A}_d$, $d \geq 5$, form one of the infinite families of simple groups. In this section, continuing the discussion of §3.1, we describe their irreducible representations. The basic method for analyzing representations of $\mathfrak{A}_d$ is by restricting the representations we know from $\mathfrak{S}_d$.

In general when H is a subgroup of index two in a group G, there is a close relationship between their representations. We will see this phenomenon again in Lie theory for the subgroups SO_n of the orthogonal groups O_n.

Let U and U' denote the trivial and nontrivial representation of G obtained from the two representations of G/H. For any representation V of G, let $V' = V \otimes U'$; the character of V' is the same as the character of V on elements of H, but takes opposite values on elements not in H. In particular, $\mathrm{Res}_H^G V' = \mathrm{Res}_H^G V$.

If W is any representation of H, there is a *conjugate* representation defined by conjugating by any element t of G that is not in H; if ψ is the character of W, the character of the conjugate is $h \mapsto \psi(tht^{-1})$. Since t is unique up to multiplication by an element of H, the conjugate representation is unique up to isomorphism.

Proposition 5.1. *Let V be an irreducible representation of G, and let $W = \mathrm{Res}_H^G V$ be the restriction of V to H. Then exactly one of the following holds:*

(1) V is not isomorphic to V'; W is irreducible and isomorphic to its conjugate; $\mathrm{Ind}_H^G W \cong V \oplus V'$.

(2) $V \cong V'$; $W = W' \oplus W''$, where W' and W'' are irreducible and conjugate but not isomorphic; $\mathrm{Ind}_H^G W' \cong \mathrm{Ind}_H^G W'' \cong V$.

Each irreducible representation of H arises uniquely in this way, noting that in case (1) V' and V determine the same representation.

PROOF. Let χ be the character of V. We have

$$|G| = 2|H| = \sum_{h \in H} |\chi(h)|^2 + \sum_{t \notin H} |\chi(t)|^2.$$

Since the first sum is an integral multiple of $|H|$, this multiple must be 1 or 2, which are the two cases of the proposition. This shows that W is either irreducible or the sum of two distinct irreducible representations W' and W''. Note that the second case happens when $\chi(t) = 0$ for all $t \notin H$, which is the case when V' is isomorphic to V. In the second case, W' and W'' must be conjugate since W is self-conjugate, and if W' and W'' were self-conjugate V would not be irreducible. The other assertions in (1) and (2) follow from the isomorphism $\mathrm{Ind}(\mathrm{Res}\, V) = V \otimes (U \oplus U')$ of Exercise 3.16. Similarly, for any representation W of H, $\mathrm{Res}(\mathrm{Ind}\, W)$ is the direct sum of W and its conjugate—as follows say from Exercise 3.19—from which the last statement follows readily. □

Most of this discussion extends with little change to the case where H is a normal subgroup of arbitrary prime index in G, cf. [B-tD, pp. 293–296]. Clifford has extended much of this proposition to arbitrary normal subgroups of finite index, cf. [Dor, §14].

There are two types of conjugacy classes c in H: those that are also conjugacy classes in G, and those such that $c \cup c'$ is a conjugacy class in G, where $c' = tct^{-1}$, $t \notin H$; the latter are called *split*. When W is irreducible, its character assumes the same values—those of the character of the representation V of G that restricts to W—on pairs of split conjugacy classes, whereas in the other case the characters of W' and W'' agree on nonsplit classes, but they must disagree on some split classes. If $\chi_{W'}(c) = \chi_{W''}(c') = x$, and $\chi_{W'}(c') = \chi_{W''}(c) = y$, we know the sum $x + y$, since it is the value of the character of the representation V that gives rise to W' and W'' on $c \cup c'$. Often the exact values of x and y can be determined from orthogonality considerations.

Exercise 5.2*. Show that the number of split conjugacy classes is equal to the number of irreducible representations V of G that are isomorphic to V', or to the number of irreducible representations of H that are not isomorphic to their conjugates. Equivalently, the number of nonsplit classes in H is same as the number of conjugacy classes of G that are not in H.

We apply these considerations to the alternating subgroup of the symmetric group. Consider restrictions of the representations V_λ from $\mathfrak{S}_d$ to $\mathfrak{A}_d$. Recall that if λ' is the conjugate partition to λ, then

$$V_{\lambda'} = V_\lambda \otimes U',$$

with U' the alternating representation. The two cases of the proposition correspond to the cases (1) $\lambda' \neq \lambda$ and (2) $\lambda' = \lambda$. If $\lambda' \neq \lambda$, let W_λ be the restriction of V_λ to $\mathfrak{A}_d$. If $\lambda' = \lambda$, let W'_λ and W''_λ be the two representations whose sum is the restriction of V_λ. We have

$$\mathrm{Ind}\, W_\lambda = V_\lambda \oplus V_{\lambda'}, \qquad \mathrm{Res}\, V_\lambda = \mathrm{Res}\, V_{\lambda'} = W_\lambda \quad \text{when } \lambda' \neq \lambda,$$

$$\mathrm{Ind}\, W'_\lambda = \mathrm{Ind}\, W''_\lambda = V_\lambda, \qquad \mathrm{Res}\, V_\lambda = W'_\lambda \oplus W''_\lambda \quad \text{when } \lambda' = \lambda.$$

Note that

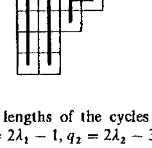
{self-conjugate representations of $\mathfrak{S}_d$}

= # {symmetric Young diagrams}

= # {split pairs of conjugacy classes in $\mathfrak{A}_d$}

= # {conjugacy classes in $\mathfrak{S}_d$ breaking into two classes in $\mathfrak{A}_d$}.

Now a conjugacy class of an element written as a product of disjoint cycles is split if and only if there is no odd permutation commuting with it, which is equivalent to all the cycles having odd length, and no two cycles having the same length. So the number of self-conjugate representations is the number of partitions of d as a sum of distinct odd numbers. In fact, there is a natural correspondence between these two sets: any such partition corresponds to a symmetric Young diagram, assembling hooks as indicated:



If λ is the partition, the lengths of the cycles in the corresponding split conjugacy classes are $q_1 = 2\lambda_1 - 1$, $q_2 = 2\lambda_2 - 3$, $q_3 = 2\lambda_3 - 5, \ldots$.

For a self-conjugate partition λ, let χ'_λ and χ''_λ denote the characters of W'_λ and W''_λ, and let c and c' be a pair of split conjugacy classes, consisting of cycles of odd lengths $q_1 > q_2 > \cdots > q_r$. The following proposition of Frobenius completes the description of the character table of $\mathfrak{A}_d$.

Proposition 5.3. (1) *If c and c' do not correspond to the partition λ, then*

$$\chi'_\lambda(c) = \chi'_\lambda(c') = \chi''_\lambda(c) = \chi''_\lambda(c') = \tfrac{1}{2}\chi_\lambda(c \cup c').$$

(2) *If c and c' correspond to λ, then*

$$\chi'_\lambda(c) = \chi''_\lambda(c') = x, \qquad \chi'_\lambda(c') = \chi''_\lambda(c) = y,$$

with x and y the two numbers

$$\tfrac{1}{2}\left((-1)^m \pm \sqrt{(-1)^m q_1 \cdot \ldots \cdot q_r}\right),$$

and $m = \frac{1}{2}(\prod q_i - 1) = \frac{1}{2}(d - r)$.

For example, if $d = 4$ and $\lambda = (2, 2)$, we have $r = 2$, $q_1 = 3$, $q_2 = 1$, and x and y are the cube roots of unity; the representations W'_λ and W''_λ are the representations labeled U' and U'' in the table in §2.3. For $d = 5$, $\lambda = (3, 1, 1)$, $r = 1$, $q_1 = 5$, and we find the representations called Y and Z in §3.1. For $d \le 7$, there is at most one split pair, so the character table can be derived from orthogonality alone.

Note that since only one pair of character values is not taken care of by the first case of Frobenius's formula, the choice of which representation is W'_λ and which W''_λ is equivalent to choosing the plus and minus sign in (2). Note also that the integer m occurring in (2) is the number of squares above the diagonal in the Young diagram of λ.

We outline a proof of the proposition as an exercise:

Exercise 5.4*. *Step 1.* Let $q = (q_1 > \cdots > q_r)$ be a sequence of positive odd integers adding to d, and let $c' = c'(q)$ and $c'' = c''(q)$ be the corresponding conjugacy classes in $\mathfrak{A}_d$. Let λ be a self-conjugate partition of d, and let χ'_λ and χ''_λ be the corresponding characters of $\mathfrak{A}_d$. Assume that χ'_λ and χ''_λ take on the same values on each element of $\mathfrak{A}_d$ that is not in c' or c''. Let $u = \chi'_\lambda(c') = \chi''_\lambda(c'')$ and $v = \chi'_\lambda(c'') = \chi''_\lambda(c')$.

(i) Show that u and v are real when $m = \frac{1}{2}\Sigma(q_i - 1)$ is even, and $\bar{u} = v$ when m is odd.

(ii) Let $\vartheta = \chi'_\lambda - \chi''_\lambda$. Deduce from the equation $(\vartheta, \vartheta) = 2$ that $|u - v|^2 = q_1 \cdot \ldots \cdot q_r$.

(iii) Show that λ is the partition that corresponds to q and that $u + v = (-1)^m$, and deduce that u and v are the numbers specified in (2) of the proposition.

Step 2. Prove the proposition by induction on d, and for fixed d, look at that q which has smallest q_1, and for which some character has values on the classes $c'(q)$ and $c''(q)$ other than those prescribed by the proposition.

(i) If $r = 1$, so $q_1 = d = 2m + 1$, the corresponding self-conjugate partition is $\lambda = (m + 1, 1, \ldots, 1)$. By induction, Step 1 applies to χ'_λ and χ''_λ.

(ii) If $r > 1$, consider the imbedding $H = \mathfrak{A}_{q_1} \times \mathfrak{A}_{d-q_1} \subset G = \mathfrak{A}_d$, and let X' and X'' be the representations of G induced from the representations $W'_1 \boxtimes W'_2$ and $W''_1 \boxtimes W'_2$, where W'_1 and W''_1 are the representations of $\mathfrak{A}_{q_1}$ corresponding to q_1, i.e., to the self-conjugate partition $(\frac{1}{2}(q_1 - 1), 1, \ldots, 1)$ of q_1; W'_2 is one of the representations of $\mathfrak{A}_{d-q_1}$ corresponding to $(q_2, \ldots, q_r)$; and $\boxtimes$ denotes the external tensor product (see Exercise 2.36). Show that X' and X'' are conjugate representations of $\mathfrak{A}_d$, and their characters χ' and χ'' take equal values on each pair of split conjugacy classes, with the exception of $c'(q)$ and $c''(q)$, and compute the values of these characters on $c'(q)$ and $c''(q)$.

(iii) Let $\vartheta = \chi' - \chi''$, and show that $(\vartheta, \vartheta) = 2$. Decomposing X' and X'' into their irreducible pieces, deduce that $X' = Y \oplus W'_\lambda$ and $X'' = Y \oplus W''_\lambda$ for some self-conjugate representation Y and some self-conjugate partition λ of d.

(iv) Apply Step 1 to the characters χ'_λ and χ''_λ, and conclude the proof.

Exercise 5.5*. Show that if $d > 6$, the only irreducible representations of $\mathfrak{A}_d$ of dimension less than d are the trivial representation and the $(n - 1)$-dimensional restriction of the standard representation of $\mathfrak{S}_d$. Find the exceptions for $d \le 6$.

We have worked out the character tables for all $\mathfrak{S}_d$ and $\mathfrak{A}_d$ for $d \le 5$. With the formulas of Frobenius, an interested reader can construct the tables for a few more d—until the number of partitions of d becomes large.

§5.2. Representations of $GL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q)$

The groups $GL_2(\mathbb{F}_q)$ of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_q$ with q elements, where q is a prime power, form another important series of finite groups, as do their subgroups $SL_2(\mathbb{F}_q)$ consisting of matrices of determinant one. The quotient $PGL_2(\mathbb{F}_q) = GL_2(\mathbb{F}_q)/\mathbb{F}_q^*$ is the automorphism group of the finite projective line $\mathbb{P}^1(\mathbb{F}_q)$. The quotients $PSL_2(\mathbb{F}_q) = SL_2(\mathbb{F}_q)/\{\pm 1\}$ are simple groups if $q \ne 2, 3$ (Exercise 5.9). In this section we sketch the character theory of these groups.

We begin with $G = GL_2(\mathbb{F}_q)$. There are several key subgroups:

$$G \supset B = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right\} \supset N = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right\}.$$

(This "Borel subgroup" B and the group of upper triangular unipotent matrices N will reappear when we look at Lie groups.) Since G acts transitively on the projective line $\mathbb{P}^1(\mathbb{F}_q)$, with B the isotropy group of the point $(1{:}0)$, we have

$$|G| = |B| \cdot |\mathbb{P}^1(\mathbb{F}_q)| = (q - 1)^2 q(q + 1).$$

We will also need the diagonal subgroup

$$D = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right\} = \mathbb{F}^* \times \mathbb{F}^*,$$

where we write $\mathbb{F}$ for $\mathbb{F}_q$. Let $\mathbb{F}' = \mathbb{F}_{q^2}$ be the extension of $\mathbb{F}$ of degree two, unique up to isomorphism. We can identify $\mathrm{GL}_2(\mathbb{F}_q)$ as the group of all $\mathbb{F}$-linear invertible endomorphisms of $\mathbb{F}'$. This makes evident a large cyclic subgroup $K = (\mathbb{F}')^*$ of G. At least if q is odd, we may make this isomorphism explicit by choosing a generator ε for the cyclic group $\mathbb{F}^*$ and choosing a square root $\sqrt{\varepsilon}$ in $\mathbb{F}'$. Then 1 and $\sqrt{\varepsilon}$ form a basis for $\mathbb{F}'$ as a vector space over $\mathbb{F}$, so we can make the identification:

$$K = \left\{\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}\right\} \cong (\mathbb{F}')^*, \qquad \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} \leftrightarrow \zeta = x + y\sqrt{\varepsilon};$$

K is a cyclic subgroup of G of order $q^2 - 1$. We often make this identification, leaving it as an exercise to make the necessary modifications in case q is even.

The conjugacy classes in G are easily found:

Representative	No. Elements in Class	No. Classes
$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	1	$q - 1$
$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	$q^2 - 1$	$q - 1$
$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, x \neq y$	$q^2 + q$	$\dfrac{(q-1)(q-2)}{2}$
$d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}, y \neq 0$	$q^2 - q$	$\dfrac{q(q-1)}{2}$

Here $c_{x,y}$ and $c_{y,x}$ are conjugate by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $d_{x,y}$ and $d_{x,-y}$ are conjugate by any $\begin{pmatrix} a & -\varepsilon c \\ c & -a \end{pmatrix}$. To count the number of elements in the conjugacy class of b_x, look at the action of G on this class by conjugation; the isotropy group is $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right\}$, so the number of elements in the class is the index of this group in G, which is $q^2 - 1$. Similarly the isotropy group for $c_{x,y}$ is D, and the isotropy group for $d_{x,y}$ is K. To see that the classes are disjoint, consider the eigenvalues and the Jordan canonical forms. Since they account for $|G|$ elements, the list is complete.

There are $q^2 - 1$ conjugacy classes, so we must find the same number of irreducible representations. Consider first the permutation representation of G on $\mathbb{P}^1(\mathbb{F})$, which has dimension $q + 1$. It contains the trivial representation; let V be the complementary q-dimensional representation. The values of the character χ of V on the four types of conjugacy classes are $\chi(a_x) = q$, $\chi(b_x) = 0$, $\chi(c_{x,y}) = 1$, $\chi(d_{x,y}) = -1$, which we display as the table:

$$V: \quad q \quad 0 \quad 1 \quad -1$$

Since $(\chi, \chi) = 1$, V is irreducible.

For each of the $q - 1$ characters $\alpha\colon \mathbb{F}^* \to \mathbb{C}^*$ of $\mathbb{F}^*$, we have a one-dimensional representation U_α of G defined by $U_\alpha(g) = \alpha(\det(g))$. We also have the representations $V_\alpha = V \otimes U_\alpha$. The values of the characters of these representations are

U_α:	$\alpha(x)^2$	$\alpha(x)^2$	$\alpha(x)\alpha(y)$	$\alpha(x^2 - \varepsilon y^2)$
V_α:	$q\alpha(x)^2$	0	$\alpha(x)\alpha(y)$	$-\alpha(x^2 - \varepsilon y^2)$

Note that if we identify $\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$ with $\zeta = x + y\sqrt{\varepsilon}$ in $\mathbb{F}'$, then

$$x^2 - \varepsilon y^2 = \det\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = \mathrm{Norm}_{\mathbb{F}'/\mathbb{F}}(\zeta) = \zeta \cdot \zeta^q = \zeta^{q+1}.$$

The next place to look for representations is at those that are induced from large subgroups. For each pair α, β of characters of $\mathbb{F}^*$, there is a character of the subgroup B:

$$B \to B/N = D = \mathbb{F}^* \times \mathbb{F}^* \to \mathbb{C}^* \times \mathbb{C}^* \xrightarrow{\cdot} \mathbb{C}^*,$$

which takes $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ to $\alpha(a)\beta(d)$. Let $W_{\alpha,\beta}$ be the representation induced from B to G by this representation; this is a representation of dimension $[G : B] = q + 1$. By Exercise 3.19 its character values are found to be:

$$W_{\alpha,\beta}: \quad (q+1)\alpha(x)\beta(x) \quad \alpha(x)\beta(x) \quad \alpha(x)\beta(y) + \alpha(y)\beta(x) \quad 0$$

We see from this that $W_{\alpha,\beta} \cong W_{\beta,\alpha}$, that $W_{\alpha,\alpha} \cong U_\alpha \oplus V_\alpha$, and that for $\alpha \neq \beta$ the representation is irreducible. This gives $\frac{1}{2}(q-1)(q-2)$ more irreducible representations, of dimension $q + 1$.

Comparing with the list of conjugacy classes, we see that there are $\frac{1}{2}q(q-1)$ irreducible characters left to be found. A natural way to find new characters is to induce characters from the cyclic subgroup K. For a representation

$$\varphi\colon K = (\mathbb{F}')^* \to \mathbb{C}^*,$$

the character values of the induced representation of dimension $[G : K] = q^2 - 1$ are

$$\mathrm{Ind}(\varphi): \quad q(q-1)\varphi(x) \quad 0 \quad 0 \quad \varphi(\zeta) + \varphi(\zeta)^q$$

Here again $\zeta = x + y\sqrt{\varepsilon} \in K = (\mathbb{F}')^*$. Note that $\mathrm{Ind}(\varphi^q) \cong \mathrm{Ind}(\varphi)$, so the representations $\mathrm{Ind}(\varphi)$ for $\varphi^q \neq \varphi$ give $\frac{1}{2}q(q-1)$ different representations.

However, these represenations are not irreducible: the character χ of $\text{Ind}(\varphi)$ satisfies $(\chi, \chi) = q - 1$ if $\varphi^q \neq \varphi$, and otherwise $(\chi, \chi) = q$. We will have to work a little harder to get irreducible representations from these $\text{Ind}(\varphi)$.

Another attempt to find more representations is to look inside tensor products of representations we know. We have $V_\alpha \otimes U_\gamma = V_{\alpha\gamma}$, and $W_{\alpha,\beta} \otimes U_\gamma \cong W_{\alpha\gamma,\beta\gamma}$, so there are no new ones to be found this way. But tensor products of the V_α's and $W_{\alpha,\beta}$'s are more promising. For example, $V \otimes W_{\alpha,1}$ has character values:

$$V \otimes W_{\alpha,1}: \qquad q(q+1)\alpha(x) \qquad 0 \qquad \alpha(x) + \alpha(y) \qquad 0$$

We can calculate some inner products of these characters with each other to estimate how many irreducible representations each contains, and how many they have in common. For example,

$$(\chi_{V\otimes W_{\alpha,1}}, \chi_{W_{\alpha,1}}) = 2,$$
$$(\chi_{\text{Ind}(\varphi)}, \chi_{W_{\alpha,1}}) = 1 \quad \text{if } \varphi|_{\mathbb{F}^*} = \alpha,$$
$$(\chi_{V\otimes W_{\alpha,1}}, \chi_{V\otimes W_{\alpha,1}}) = q + 3,$$
$$(\chi_{V\otimes W_{\alpha,1}}, \chi_{\text{Ind}(\varphi)}) = q \quad \text{if } \varphi|_{\mathbb{F}^*} = \alpha,$$

Comparing with the formula $(\chi_{\text{Ind}(\varphi)}, \chi_{\text{Ind}(\varphi)}) = q - 1$, one deduces that $V \otimes W_{\alpha,1}$ and $\text{Ind}(\varphi)$ contain many of the same representations. With any luck, $\text{Ind}(\varphi)$ and $W_{\alpha,1}$ should both be contained in $V \otimes W_{\alpha,1}$. This guess is easily confirmed; the virtual character

$$\chi_\varphi = \chi_{V\otimes W_{\alpha,1}} - \chi_{W_{\alpha,1}} - \chi_{\text{Ind}(\varphi)}$$

takes values $(q-1)\alpha(x)$, $-\alpha(x)$, 0, and $-(\varphi(\zeta) + \varphi(\zeta)^q)$ on the four types of conjugacy classes. Therefore, $(\chi_\varphi, \chi_\varphi) = 1$, and $\chi_\varphi(1) = q - 1 > 0$, so χ_φ is, in fact, the character of an irreducible subrepresentation of $V \otimes W_{\alpha,1}$ of dimension $q - 1$. We denote this representation by X_φ. These $\frac{1}{2}q(q-1)$ representations, for $\varphi \neq \varphi^q$, and with $X_\varphi = X_{\varphi^q}$, therefore complete the list of irreducible representations for $GL_2(\mathbb{F})$. The character table is

	1	$q^2 - 1$	$q^2 + q$	$q^2 - q$
$GL_2(\mathbb{F}_q)$	$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = \zeta$
U_α	$\alpha(x^2)$	$\alpha(x^2)$	$\alpha(xy)$	$\alpha(\zeta^q)$
V_α	$q\alpha(x^2)$	0	$\alpha(xy)$	$-\alpha(\zeta^q)$
$W_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	0
X_φ	$(q-1)\varphi(x)$	$-\varphi(x)$	0	$-(\varphi(\zeta) + \varphi(\zeta^q))$

Exercise 5.6. Find the multiplicity of each irreducible representation in the representations $V \otimes W_{\alpha,1}$ and $\text{Ind}(\varphi)$.

Exercise 5.7. Find the character table of $PGL_2(\mathbb{F}) = GL_2(\mathbb{F})/\mathbb{F}^*$. Note that its characters are just the characters of $GL_2(\mathbb{F})$ that take the same values on elements equivalent mod $\mathbb{F}^*$.

We turn next to the subgroup $SL_2(\mathbb{F}_q)$ of 2×2 matrices of determinant one, with q odd. The conjugacy classes, together with the number of elements in each conjugacy class, and the number of conjugacy classes of each type, are

	Representative	No. Elements in Class	No. Classes
(1)	$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	1
(2)	$-e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	1	1
(3)	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\frac{q^2 - 1}{2}$	1
(4)	$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$	$\frac{q^2 - 1}{2}$	1
(5)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\frac{q^2 - 1}{2}$	1
(6)	$\begin{pmatrix} -1 & \varepsilon \\ 0 & -1 \end{pmatrix}$	$\frac{q^2 - 1}{2}$	1
(7)	$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, x \neq \pm 1$	$q(q+1)$	$\frac{q-3}{2}$
(8)	$\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}, x \neq \pm 1$	$q(q-1)$	$\frac{q-1}{2}$

The verifications are very much as we did for $GL_2(\mathbb{F}_q)$. In (7), the classes of $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ and $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$ are the same. In (8), the classes for (x, y) and $(x, -y)$ are the same; as before, a better labeling is by the element ζ in the cyclic group

$$C = \{\zeta \in (\mathbb{F}')^* : \zeta^{q+1} = 1\};$$

the elements ± 1 are not used, and the classes of ζ and ζ^{-1} are the same.

The total number of conjugacy classes is $q + 4$, so we turn to the task of finding $q + 4$ irreducible representations. We first see what we get by restricting representations from $GL_2(\mathbb{F}_q)$. Since we know the characters, there is no problem working this out, and we simply state the results:

(1) The U_α all restrict to the trivial representation U. Hence, if we restrict any representation, we will get the same for all tensor products by U_α's.

(2) The restriction V of the V_α's is irreducible.

(3) The restriction W_α of $W_{\alpha,1}$ is irreducible if $\alpha^2 \neq 1$, and $W_\alpha \cong W_\beta$ when $\beta = \alpha$ or $\beta = \alpha^{-1}$. These give $\frac{1}{2}(q-3)$ irreducible representations of dimension $q+1$.

(3′) Let τ denote the character of $\mathbb{F}^*$ with $\tau^2 = 1$, $\tau \neq 1$. The restriction of $W_{\tau,1}$ is the sum of two distinct irreducible representations, which we denote W' and W''.

(4) The restriction of X_φ depends only on the restriction of φ to the subgroup C, and φ and φ^{-1} determine the same representation. The representation is irreducible if $\varphi^2 \neq 1$. This gives $\frac{1}{2}(q-1)$ irreducible representations of dimension $q-1$.

(4′) If ψ denotes the character of C with $\psi^2 = 1$, $\psi \neq 1$, the restriction of X_ψ is the sum of two distinct irreducible representations, which we denote X' and X''.

Altogether this list gives $q+4$ distinct irreducible representations, and it is therefore the complete list. To finish the character table, the problem is to describe the four representations W', W'', X', and X''. Since we know the sum of the squares of the dimensions of all representations, we can deduce that the sum of the squares of these four representations is q^2+1, which is only possible if the first two have dimension $\frac{1}{2}(q+1)$ and the other two $\frac{1}{2}(q-1)$. This is similar to what we saw happens for restrictions of representations to subgroups of index two. Although the index here is larger, we can use what we know about index two subgroups by finding a subgroup H of index two in $GL_2(\mathbb{F}_q)$ that contains $SL_2(\mathbb{F}_q)$, and analyzing the restrictions of these four representations to H.

For H we take the matrices in $GL_2(\mathbb{F}_q)$ whose determinant is a square. The representatives of the conjugacy classes are the same as those for $GL_2(\mathbb{F}_q)$, including, of course, only those representatives whose determinant is a square, but we must add classes represented by the elements $\begin{pmatrix} x & \varepsilon \\ 0 & x \end{pmatrix}$, $x \in \mathbb{F}^*$. These are conjugate to the elements $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$ in $GL_2(\mathbb{F}_q)$, but not in H. These are the $q-1$ split conjugacy classes. The procedure of the preceding section can be used to work out all the representations of H, but we need only a little of this.

Note that the sign representation U' from G/H is U_τ, so that $W_{\tau,1} \cong W_{\tau,1} \otimes U'$ and $X_\psi \cong X_\psi \otimes U'$; their restrictions to H split into sums of conjugate irreducible representations of half their dimensions. This shows these representations stay irreducible on restriction from H to $SL_2(\mathbb{F}_q)$, so that W' and W'' are conjugate representations of dimension $\frac{1}{2}(q+1)$, and X' and X'' are conjugate representations of dimension $\frac{1}{2}(q-1)$. In addition, we know that their character values on all nonsplit conjugacy classes are the same as half the characters of the representations $W_{\tau,1}$ and X_ψ, respectively. This is all the information we need to finish the character table. Indeed, the only values not covered by this discussion are

	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & \varepsilon \\ 0 & -1 \end{pmatrix}$
W'	s	t	s'	t'
W''	t	s	t'	s'
X'	u	v	u'	v'
X''	v	u	v'	u'

The first two rows are determined as follows. We know that $s + t = \chi_{W_{\tau,1}}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = 1$. In addition, since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if q is congruent to 1 modulo 4, and to $\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$ otherwise, and since $\chi(g^{-1}) = \overline{\chi(g)}$ for any character, we conclude that s and t are real if $q \equiv 1 \bmod(4)$, and $s = \bar{t}$ if $q \equiv 3 \bmod(4)$. In addition, since $-e$ acts as the identity or minus the identity for any irreducible representation (Schur's lemma),

$$\chi(-g) = \chi(g) \cdot \chi(1)/\chi(-e)$$

for any irreducible character χ. This gives the relations $s' = \tau(-1)s$ and $t' = \tau(-1)t$. Finally, applying the equation $(\chi, \chi) = 1$ to the character of W' gives a formula for $s\bar{t} + t\bar{s}$. Solving these equations gives $s, t = \frac{1}{2} \pm \frac{1}{2}\sqrt{\omega q}$, where $\omega = \tau(-1)$ is 1 or -1 according as $q \equiv 1$ or $3 \bmod(4)$. Similarly one computes that u and v are $-\frac{1}{2} \pm \frac{1}{2}\sqrt{\omega q}$. This concludes the computations needed to write out the character table.

Exercise 5.8. By considering the action of $SL_2(\mathbb{F}_q)$ on the set $\mathbb{P}^1(\mathbb{F}_q)$, show that $SL_2(\mathbb{F}_2) \cong \mathfrak{S}_3$, $PSL_2(\mathbb{F}_3) \cong \mathfrak{A}_4$, and $SL_2(\mathbb{F}_4) \cong \mathfrak{A}_5$.

Exercise 5.9*. Use the character table for $SL_2(\mathbb{F}_q)$ to show that $PSL_2(\mathbb{F}_q)$ is a simple group if q is odd and greater than 3.

Exercise 5.10. Compute the character table of $PSL_2(\mathbb{F}_q)$, either by regarding it as a quotient of $SL_2(\mathbb{F}_q)$, or as a subgroup of index two in $PGL_2(\mathbb{F}_q)$.

Exercise 5.11*. Find the conjugacy classes of $GL_3(\mathbb{F}_q)$, and compute the characters of the permutation representations obtained by the action of $GL_3(\mathbb{F}_q)$ on (i) the projective plane $\mathbb{P}^2(\mathbb{F}_q)$ and (ii) the "flag variety" consisting of a point on a line in $\mathbb{P}^2(\mathbb{F}_q)$. Show that the first is irreducible and that the second is a sum of the trivial representation, two copies of the first representation, and an irreducible representation.

Although the characters of the above groups were found by the early pioneers in representation theory, actually producing the representations in a natural way is more difficult. There has been a great deal of work extending this story to $GL_n(\mathbb{F}_q)$ and $SL_n(\mathbb{F}_q)$ for $n > 2$ (cf. [Gr]), and for corresponding groups, called finite Chevalley groups, related to other Lie groups. For some hints in this direction see [Hu3], as well as [Ti2]. Since all but a finite number of finite simple groups are now known to arise this way (or are cyclic or alternating groups, whose characters we already know), such representations play a fundamental role in group theory. In recent work their Lie-theoretic origins have been exploited to produce their representations, but to tell this story would go far beyond the scope of these lecture(r)s.

LECTURE 6

Weyl's Construction

In this lecture we introduce and study an important collection of functors generalizing the symmetric powers and exterior powers. These are defined simply in terms of the Young symmetrizers c_λ introduced in §4: given a representation V of an arbitrary group G, we consider the dth tensor power of V, on which both G and the symmetric group on d letters act. We then take the image of the action of c_λ on $V^{\otimes d}$; this is again a representation of G, denoted $\mathbb{S}_\lambda(V)$. This gives us a way of generating new representations, whose main application will be to Lie groups: for example, we will generate all representations of $SL_n\mathbb{C}$ by applying these to the standard representation $\mathbb{C}^n$ of $SL_n\mathbb{C}$. While it may be easiest to read this material while the definitions of the Young symmetrizers are still fresh in the mind, the construction will not be used again until §15, so that this lecture can be deferred until then.

§6.1: Schur functors and their characters
§6.2: The proofs

§6.1. Schur Functors and Their Characters

For any finite-dimensional complex vector space V, we have the canonical decomposition

$$V \otimes V = \mathrm{Sym}^2 V \otimes \wedge^2 V.$$

The group $GL(V)$ acts on $V \otimes V$, and this is, as we shall soon see, the decomposition of $V \otimes V$ into a direct sum of irreducible $GL(V)$-representations. For the next tensor power,

$$V \otimes V \otimes V = \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus \text{ another space.}$$

We shall see that this other space is a sum of two copies of an irreducible

$\mathrm{GL}(V)$-representation. Just as $\mathrm{Sym}^d V$ and $\bigwedge^d V$ are images of symmetrizing operators from $V^{\otimes d} = V \otimes V \otimes \cdots \otimes V$ to itself, so are the other factors. The symmetric group $\mathfrak{S}_d$ acts on $V^{\otimes d}$, say on the right, by permuting the factors

$$(v_1 \otimes \cdots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.$$

This action commutes with the left action of $\mathrm{GL}(V)$. For any partition λ of d we have from the last lecture a Young symmetrizer c_λ in $\mathbb{C}\mathfrak{S}_d$. We denote the image of c_λ on $V^{\otimes d}$ by $\mathbb{S}_\lambda V$:

$$\mathbb{S}_\lambda V = \mathrm{Im}(c_\lambda|_{V^{\otimes d}})$$

which is again a representation of $\mathrm{GL}(V)$. We call the functor[1] $V \rightsquigarrow \mathbb{S}_\lambda V$ the *Schur functor* or *Weyl module*, or simply *Weyl's construction*, corresponding to λ. It was Schur who made the correspondence between representations of symmetric groups and representations of general linear groups, and Weyl who made the construction we give here.[2] We will give other descriptions later, cf. Exercise 6.14 and §15.5.

For example, the partition $d = d$ corresponds to the functor $V \rightsquigarrow \mathrm{Sym}^d V$, and the partition $d = 1 + \cdots + 1$ to the functor $V \rightsquigarrow \bigwedge^d V$.

We find something new for the partition $3 = 2 + 1$. The corresponding symmetrizer c_λ is

$$c_{(2,1)} = 1 + e_{(12)} - e_{(13)} - e_{(132)},$$

so the image of c_λ is the subspace of $V^{\otimes 3}$ spanned by all vectors

$$v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_3 \otimes v_1 \otimes v_2.$$

If $\bigwedge^2 V \otimes V$ is embedded in $V^{\otimes 3}$ by mapping

$$(v_1 \wedge v_3) \otimes v_2 \mapsto v_1 \otimes v_2 \otimes v_3 - v_3 \otimes v_2 \otimes v_1,$$

then the image of c_λ is the subspace of $\bigwedge^2 V \otimes V$ spanned by all vectors

$$(v_1 \wedge v_3) \otimes v_2 + (v_2 \wedge v_3) \otimes v_1.$$

It is not hard to verify that these vectors span the kernel of the canonical map from $\bigwedge^2 V \otimes V$ to $\bigwedge^3 V$, so we have

$$\mathbb{S}_{(2,1)} V = \mathrm{Ker}(\textstyle\bigwedge^2 V \otimes V \to \bigwedge^3 V).$$

(This gives the missing factor in the decomposition of $V^{\otimes 3}$.)

Note that some of the $\mathbb{S}_\lambda V$ can be zero if V has small dimension. We will see that this is the case precisely when the number of rows in the Young diagram of λ is greater than the dimension of V.

[1] The functoriality means simply that a linear map $\varphi: V \to W$ of vector spaces determines a linear map $\mathbb{S}_\lambda(\varphi): \mathbb{S}_\lambda V \to \mathbb{S}_\lambda W$, with $\mathbb{S}_\lambda(\varphi \circ \psi) = \mathbb{S}_\lambda(\varphi) \circ \mathbb{S}_\lambda(\psi)$ and $\mathbb{S}_\lambda(\mathrm{Id}_V) = \mathrm{Id}_{\mathbb{S}_\lambda V}$.

[2] The notion goes by a variety of names and notations in the literature, depending on the context. Constructions differ markedly when not over a field of characteristic zero; and many authors now parametrize them by the conjugate partitions. Our choice of notation is guided by the correspondence between these functors and Schur polynomials, which we will see are their characters.

When $G = \mathrm{GL}(V)$, and for important subgroups $G \subset \mathrm{GL}(V)$, these $\mathbb{S}_\lambda V$ give many of the irreducible representations of G; we will come back to this later in the book. For now we can use our knowledge of symmetric group representations to prove a few facts about them—in particular, we show that they decompose the tensor powers $V^{\otimes d}$, and that they are irreducible representations of $\mathrm{GL}(V)$. We will also compute their characters; this will eventually be seen to be a special case of the Weyl character formula.

Any endomorphism g of V gives rise to an endomorphism of $\mathbb{S}_\lambda V$. In order to tell what representations we get, we will need to compute the trace of this endomorphism on $\mathbb{S}_\lambda V$; we denote this trace by $\chi_{\mathbb{S}_\lambda V}(g)$. For the computation, let $x_1, \ldots, x_k$ be the eigenvalues of g on V, $k = \dim V$. Two cases are easy. For $\lambda = (d)$,

$$\mathbb{S}_{(d)} V = \mathrm{Sym}^d V, \qquad \chi_{\mathbb{S}_{(d)} V}(g) = H_d(x_1, \ldots, x_k), \tag{6.1}$$

where $H_d(x_1, \ldots, x_k)$ is the complete symmetric polynomial of degree d. The definition of these symmetric polynomials is given in (A.1) of Appendix A. The truth of (6.1) is evident when g is a diagonal matrix, and its truth for the dense set of diagonalizable endomorphisms implies it for all endomorphisms; or one can see it directly by using the Jordan canonical form of g. For $\lambda = (1, \ldots, 1)$, we have similarly

$$\mathbb{S}_{(1,\ldots,1)} V = \textstyle\bigwedge^d V, \qquad \chi_{\mathbb{S}_{(1,\ldots,1)} V}(g) = E_d(x_1, \ldots, x_k), \tag{6.2}$$

with $E_d(x_1, \ldots, x_k)$ the elementary symmetric polynomial [see (A.3)]. The polynomials H_d and E_d are special cases of the *Schur polynomials*, which we denote by $S_\lambda = S_\lambda(x_1, \ldots, x_k)$. As λ varies over the partitions of d into at most k parts, these polynomials S_λ form a basis for the symmetric polynomials of degree d in these k variables. Schur polynomials are defined and discussed in Appendix A, especially (A.4)–(A.6). The above two formulas can be written

$$\chi_{\mathbb{S}_\lambda V}(g) = S_\lambda(x_1, \ldots, x_k) \quad \text{for } \lambda = (d) \text{ and } \lambda = (1, \ldots, 1).$$

We will show that this equation is valid for all λ:

Theorem 6.3. (1) *Let $k = \dim V$. Then $\mathbb{S}_\lambda V$ is zero if $\lambda_{k+1} \neq 0$. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$, then*

$$\dim \mathbb{S}_\lambda V = S_\lambda(1, \ldots, 1) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

(2) *Let m_λ be the dimension of the irreducible representation V_λ of $\mathfrak{S}_d$ corresponding to λ. Then*

$$V^{\otimes d} \cong \bigoplus_\lambda \mathbb{S}_\lambda V^{\oplus m_\lambda}.$$

(3) *For any $g \in \mathrm{GL}(V)$, the trace of g on $\mathbb{S}_\lambda V$ is the value of the Schur polynomial on the eigenvalues $x_1, \ldots, x_k$ of g on V:*

$$\chi_{\mathbb{S}_\lambda V}(g) = S_\lambda(x_1, \ldots, x_k).$$

(4) *Each* $\mathbb{S}_\lambda V$ *is an irreducible representation of* $\mathrm{GL}(V)$.

This theorem will be proved in the next section. Other formulas for the dimension of $\mathbb{S}_\lambda V$ are given in Exercises A.30 and A.31. The following is another:

Exercise 6.4*. Show that

$$\dim \mathbb{S}_\lambda V = \frac{m_\lambda}{d!} \prod (k - i + j) = \prod \frac{(k - i + j)}{h_{ij}},$$

where the products are over the d pairs (i, j) that number the row and column of boxes for λ, and h_{ij} is the hook number of the corresponding box.

Exercise 6.5. Show that $V^{\otimes 3} \cong \mathrm{Sym}^3 V \oplus \Lambda^3 V \oplus (\mathbb{S}_{(2,1)} V)^{\oplus 2}$, and

$$V^{\otimes 4} \cong \mathrm{Sym}^4 V \oplus \Lambda^4 V \oplus (\mathbb{S}_{(3,1)} V)^{\oplus 3} \oplus (\mathbb{S}_{(2,2)} V)^{\oplus 2} \oplus (\mathbb{S}_{(2,1,1)} V)^{\oplus 3}.$$

Compute the dimensions of each of the irreducible factors.

The proof of the theorem actually gives the following corollary:

Corollary 6.6. *If* $c \in \mathbb{C}\mathfrak{S}_d$, *and* $(\mathbb{C}\mathfrak{S}_d) \cdot c = \bigoplus_\lambda V_\lambda^{\oplus r_\lambda}$ *as representations of* $\mathfrak{S}_d$, *then there is a corresponding decomposition of* $\mathrm{GL}(V)$*-spaces*:

$$V^{\otimes d} \cdot c = \bigoplus_\lambda \mathbb{S}_\lambda V^{\oplus r_\lambda}.$$

If $x_1, \ldots, x_k$ *are the eigenvalues of an endomorphism of* V, *the trace of the induced endomorphism of* $V^{\otimes d} \cdot c$ *is* $\sum r_\lambda S_\lambda(x_1, \ldots, x_k)$.

If λ and μ are different partitions, each with at most $k = \dim V$ parts, the irreducible $\mathrm{GL}(V)$-spaces $\mathbb{S}_\lambda V$ and $\mathbb{S}_\mu V$ are not isomorphic. Indeed, their characters are the Schur polynomials S_λ and S_μ, which are different. More generally, at least for those representations of $\mathrm{GL}(V)$ which can be decomposed into a direct sum of copies of the represenations $\mathbb{S}_\lambda V$'s, *the representations are completely determined by their characters*. This follows immediately from the fact that the Schur polynomials are linearly independent.

Note, however, that we cannot hope to get *all* finite-dimensional irreducible representations of $\mathrm{GL}(V)$ this way, since the duals of these representations are not included. We will see in Lecture 15 that this is essentially the only omission. Note also that although the operation that takes representations of $\mathfrak{S}_d$ to representations of $\mathrm{GL}(V)$ preserves direct sums, the situation with respect to other linear algebra constructions such as tensor products is more complicated.

One important application of Corollary 6.6 is to the decomposition of a tensor product $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ of two Weyl modules, with, say, λ a partition of

d and μ a partition of m. The result is

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V \cong \bigoplus_\nu N_{\lambda\mu\nu} \mathbb{S}_\nu V; \tag{6.7}$$

here the sum is over partitions ν of $d + m$, and $N_{\lambda\mu\nu}$ are numbers determined by the *Littlewood–Richardson rule*. This is a rule that gives $N_{\lambda\mu\nu}$ as the number of ways to expand the Young diagram of λ, using μ in an appropriate way, to achieve the Young diagram for ν; see (A.8) for the precise formula. Two important special cases are easier to use and prove since they involve only the simpler Pieri formula (A.7). For $\mu = (m)$, we have

$$\mathbb{S}_\lambda V \otimes \mathrm{Sym}^m V \cong \bigoplus_\nu \mathbb{S}_\nu V, \tag{6.8}$$

the sum over all ν whose Young diagram is obtained by adding m boxes to the Young diagram of λ, with no two in the same column. Similarly for $\mu = (1, \ldots, 1)$,

$$\mathbb{S}_\lambda V \otimes \Lambda^m V = \bigoplus \mathbb{S}_\pi V, \tag{6.9}$$

the sum over all partitions π whose Young diagram is obtained from that of λ by adding m boxes, with no two in the same row.

To prove these formulas, we need only observe that

$$\begin{aligned} \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V &= V^{\otimes n} \cdot c_\lambda \otimes V^{\otimes m} \cdot c_\mu \\ &= V^{\otimes n} \otimes V^{\otimes m} \cdot (c_\lambda \otimes c_\mu) = V^{\otimes (n+m)} \cdot c, \end{aligned}$$

with $c = c_\lambda \otimes c_\mu \in \mathbb{C}\mathfrak{S}_d \otimes \mathbb{C}\mathfrak{S}_m = \mathbb{C}(\mathfrak{S}_d \times \mathfrak{S}_m) \subset \mathbb{C}\mathfrak{S}_{d+m}$. This proves that $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ has a decomposition as in Corollary 6.6, and the coefficients are given by knowing the decomposition of the corresponding character. The character of a tensor product is the product of the characters of the factors; so this amounts to writing the product $S_\lambda S_\mu$ of Schur polynomials as a linear combination of Schur polynomials. This is done in Appendix A, and formulas (6.7), (6.8), and (6.9) follow from (A.8), (A.7), and Exercise A.32 (v), respectively.

For example, from $\mathrm{Sym}^d V \otimes V = \mathrm{Sym}^{d+1} V \oplus \mathbb{S}_{(d,1)} V$, it follows that

$$\mathbb{S}_{(d,1)} V = \mathrm{Ker}(\mathrm{Sym}^d V \otimes V \to \mathrm{Sym}^{d+1} V),$$

and similarly for the conjugate partition,

$$\mathbb{S}_{(2,1,\ldots,1)} V = \mathrm{Ker}(\Lambda^d V \otimes V \to \Lambda^{d+1} V).$$

Exercise 6.10*. One can also derive the preceding decompositions of tensor products directly from corresponding decompositions of representations of symmetric groups. Show that, in fact, $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ corresponds to the "inner product" representation $V_\lambda \circ V_\mu$ of $\mathfrak{S}_{d+m}$ described in (4.41).

Exercise 6.11*. (a) The Littlewood–Richardson rule also comes into the decomposition of a Schur functor of a direct sum of vector spaces V and W. This generalizes the well-known identities

$$\mathrm{Sym}^n(V \oplus W) = \bigoplus_{a+b=n} (\mathrm{Sym}^a V \otimes \mathrm{Sym}^b W),$$

$$\Lambda^n(V \oplus W) = \bigoplus_{a+b=n} (\Lambda^a V \otimes \Lambda^b W).$$

Prove the general decomposition over $\mathrm{GL}(V) \times \mathrm{GL}(W)$:

$$\mathbb{S}_\nu(V \oplus W) = \bigoplus N_{\lambda\mu\nu}(\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu W),$$

the sum over all partitions λ, μ such that the sum of the numbers partitioned by λ and μ is the number partitioned by ν. (To be consistent with Exercise 2.36 one should use the notation $\boxtimes$ for these "external" tensor products.)

(b) Similarly prove the formula for the Schur functor of a tensor product:

$$\mathbb{S}_\nu(V \otimes W) = \bigoplus C_{\lambda\mu\nu}(\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu W),$$

where the coefficients $C_{\lambda\mu\nu}$ are defined in Exercise 4.51. In particular show that

$$\mathrm{Sym}^d(V \otimes W) = \bigoplus \mathbb{S}_\lambda V \otimes \mathbb{S}_\lambda W,$$

the sum over all partitions λ of d with at most dim V or dim W rows. Replacing W by W^*, this gives the decomposition for the space of polynomial functions of degree d on the space $\mathrm{Hom}(V, W)$ over $\mathrm{GL}(V) \times \mathrm{GL}(W)$. For variations on this theme, see [Ho3]. Similarly,

$$\Lambda^d(V \otimes W) = \bigoplus \mathbb{S}_\lambda V \otimes \mathbb{S}_{\lambda'} W,$$

the sum over partitions λ of d with at most dim V rows and at most dim W columns.

Exercise 6.12. Regarding

$$\mathrm{GL}_n \mathbb{C} = \mathrm{GL}_n \mathbb{C} \times \{1\} \subset \mathrm{GL}_n \mathbb{C} \times \mathrm{GL}_m \mathbb{C} \subset \mathrm{GL}_{n+m} \mathbb{C},$$

the preceding exercise shows how the restriction of a representation decomposes:

$$\mathrm{Res}(\mathbb{S}_\nu(\mathbb{C}^{n+m})) = \sum (N_{\lambda\mu\nu} \dim \mathbb{S}_\mu(\mathbb{C}^m))\mathbb{S}_\lambda(\mathbb{C}^n).$$

In particular, for $m = 1$, Pieri's formula gives

$$\mathrm{Res}(\mathbb{S}_\nu(\mathbb{C}^{n+1})) = \bigoplus \mathbb{S}_\lambda(\mathbb{C}^n),$$

the sum over all λ obtained from ν by removing any number of boxes from its Young diagram, with no two in any column.

Exercise 6.13*. Show that for any partition $\mu = (\mu_1, \ldots, \mu_r)$ of d,

$$\Lambda^{\mu_1} V \otimes \Lambda^{\mu_2} V \otimes \cdots \otimes \Lambda^{\mu_r} V \cong \bigoplus_\lambda K_{\lambda\mu} \mathbb{S}_{\lambda'} V,$$

where $K_{\lambda\mu}$ is the Kostka number and λ' the conjugate of λ.

Exercise 6.14*. Let $\mu = \lambda'$ be the conjugate partition. Put the factors of the dth tensor power $V^{\otimes d}$ in one-to-one correspondence with the squares of the Young diagram of λ. Show that $\mathbb{S}_\lambda V$ is the image of this composite map:

$$\bigotimes_i (\Lambda^{\mu_i} V) \to \bigotimes_i (\otimes^{\mu_i} V) \to V^{\otimes d} \to \bigotimes_j (\otimes^{\lambda_j} V) \to \bigotimes_j (\mathrm{Sym}^{\lambda_j} V),$$

the first map being the tensor product of the obvious inclusions, the second grouping the factors of $V^{\otimes d}$ according to the columns of the Young diagram, the third grouping the factors according to the rows of the Young diagram, and the fourth the obvious quotient map. Alternatively, $\mathbb{S}_\lambda V$ is the image of a composite map

$$\bigotimes_i (\mathrm{Sym}^{\lambda_i} V) \to \bigotimes_i (\otimes^{\lambda_i} V) \to V^{\otimes d} \to \bigotimes_j (\otimes^{\mu_j} V) \to \bigotimes_j (\Lambda^{\mu_j} V).$$

In particular, $\mathbb{S}_\lambda V$ can be realized as a subspace of tensors in $V^{\otimes d}$ that are invariant by automorphisms that preserve the rows of a Young tableau of λ, *or* a subspace that is anti-invariant under those that preserve the columns, but not both, cf. Exercise 4.48.

Problem 6.15*. The preceding exercise can be used to describe a basis for the space $\mathbb{S}_\lambda V$. Let $v_1, \ldots, v_k$ be a basis for V. For each semistandard tableau T on λ, one can use it to write down an element v_T in $\bigotimes_i (\Lambda^{\mu_i} V)$; v_T is a tensor product of wedge products of basis elements, the ith factor in $\Lambda^{\mu_i} V$ being the wedge product (in order) of those basis vectors whose indices occur in the ith column of T. The fact to be proved is that the images of these elements v_T under the first composite map of the preceding exercise form a basis for $\mathbb{S}_\lambda V$.

At the end of Lecture 15, using more representation theory than we have at the moment, we will work out a simple variation of the construction of $\mathbb{S}_\lambda V$ which will give quick proofs of refinements of the preceding exercise and problem.

Exercise 6.16*. The Pieri formula gives a decomposition

$$\mathrm{Sym}^d V \otimes \mathrm{Sym}^d V = \bigoplus \mathbb{S}_{(d+a, d-a)} V,$$

the sum over $0 \le a \le d$. The left-hand side decomposes into a direct sum of $\mathrm{Sym}^2(\mathrm{Sym}^d V)$ and $\Lambda^2(\mathrm{Sym}^d V)$. Show that, in fact,

$$\mathrm{Sym}^2(\mathrm{Sym}^d V) = \mathbb{S}_{(2d,0)} V \oplus \mathbb{S}_{(2d-2,2)} V \oplus \mathbb{S}_{(2d-4,4)} V \oplus \cdots,$$

$$\Lambda^2(\mathrm{Sym}^d V) = \mathbb{S}_{(2d-1,1)} V \oplus \mathbb{S}_{(2d-3,3)} V \oplus \mathbb{S}_{(2d-5,5)} V \oplus \cdots.$$

Similarly using the dual form of Pieri to decompose $\Lambda^d V \otimes \Lambda^d V$ into the sum $\bigoplus \mathbb{S}_\lambda V$, the sum over all $\lambda = (2, \ldots, 2, 1, \ldots, 1)$ consisting of $d - a$ 2's and 2a 1's, $0 \le a \le d$, show that $\mathrm{Sym}^2(\Lambda^d V)$ is the sum of those factors with a even, and $\Lambda^2(\Lambda^d V)$ is the sum of those with a odd.

Exercise 6.17*. If λ and μ are any partitions, we can form the composite functor $\mathbb{S}_\mu(\mathbb{S}_\lambda V)$. The original "plethysm" problem—which remains very difficult in general—is to decompose these composites:

$$\mathbb{S}_\mu(\mathbb{S}_\lambda V) = \bigoplus_\nu M_{\lambda\mu\nu} \mathbb{S}_\nu V,$$

the sum over all partitions ν of dm, where λ is a partition of d and μ is a partition of m. The preceding exercise carried out four special cases of this.

(a) Show that there always *exists* such a decomposition for some non-negative integers $M_{\lambda\mu\nu}$ by constructing an element c in $\mathbb{C}\mathfrak{S}_{dm}$, depending on λ and μ, such that $\mathbb{S}_\mu(\mathbb{S}_\lambda V)$ is $V^{\otimes dm} \cdot c$.

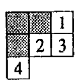
(b) Compute $\mathrm{Sym}^2(\mathbb{S}_{(2,2)} V)$ and $\Lambda^2(\mathbb{S}_{(2,2)} V)$.

Exercise 6.18* "Hermite reciprocity." Show that if $\dim V = 2$ there are isomorphisms

$$\mathrm{Sym}^p(\mathrm{Sym}^q V) \cong \mathrm{Sym}^q(\mathrm{Sym}^p V)$$

of $GL(V)$-representations, for all p and q.

Exercise 6.19*. Much of the story about Young diagrams and representations of symmetric and general linear groups can be generalized to *skew Young diagrams*, which are the differences of two Young diagrams. If λ and μ are partitions with $\mu_i \le \lambda_i$ for all i, λ/μ denotes the complement of the Young diagram for μ in that of λ. For example, if $\lambda = (3, 3, 1)$ and $\mu = (2, 1)$, λ/μ is the numbered part of



To each λ/μ we have a *skew Schur function* $S_{\lambda/\mu}$, which can be defined by any of several generalizations of constructions of ordinary Schur functions. Using the notation of Appendix A, the following definitions are equivalent:

(i) $$S_{\lambda/\mu} = |H_{\lambda_i - \mu_j - i + j}|,$$

(ii) $$S_{\lambda/\mu} = |E_{\lambda'_i - \mu'_j - i + j}|,$$

(iii) $$S_{\lambda/\mu} = \sum m_a x_1^{a_1} \cdot \ldots \cdot x_k^{a_k},$$

where m_a is the number of ways to number the boxes of λ/μ with a_1 1's, a_2 2's, $\ldots$, a_k k's, with nondecreasing rows and strictly increasing columns.

In terms of ordinary Schur polynomials, we have

(iv) $$S_{\lambda/\mu} = \sum N_{\mu\nu\lambda} S_\nu,$$

where $N_{\mu\nu\lambda}$ is the Littlewood–Richardson number.

Each λ/μ determines elements $a_{\lambda/\mu}$, $b_{\lambda/\mu}$, and Young symmetrizers $c_{\lambda/\mu} = a_{\lambda/\mu} b_{\lambda/\mu}$ in $A = \mathbb{C}\mathfrak{S}_d$, $d = \sum \lambda_i - \mu_i$, exactly as in §4.1, and hence a representation denoted $V_{\lambda/\mu} = Ac_{\lambda/\mu}$ of $\mathfrak{S}_d$. Equivalently, $V_{\lambda/\mu}$ is the image of the map $Ab_{\lambda/\mu} \to Aa_{\lambda/\mu}$ given by right multiplication by $a_{\lambda/\mu}$, or the image of the map $Aa_{\lambda/\mu} \to Ab_{\lambda/\mu}$ given by right multiplication by $b_{\lambda/\mu}$. The decomposition of $V_{\lambda/\mu}$ into irreducible representations is

(v) $$V_{\lambda/\mu} = \sum N_{\mu\nu\lambda} V_\nu.$$

Similarly there are *skew Schur functors* $\mathbb{S}_{\lambda/\mu}$, which take a vector space V to the image of $c_{\lambda/\mu}$ on $V^{\otimes d}$; equivalently, $\mathbb{S}_{\lambda/\mu} V$ is the image of a natural map (generalizing that in the Exercise 6.14)

(vi) $$\bigotimes_i (\Lambda^{\lambda'_i - \mu'_i} V) \to V^{\otimes d} \to \bigotimes_j (\mathrm{Sym}^{\lambda_j - \mu_j} V),$$

or

(vii) $$\bigotimes_i (\mathrm{Sym}^{\lambda_i - \mu_i} V) \to V^{\otimes d} \to \bigotimes_j (\Lambda^{\lambda'_j - \mu'_j} V).$$

Given a basis $v_1, \ldots, v_k$ for V and a standard tableau T on λ/μ, one can write down an element v_T in $\bigotimes_i (\Lambda^{\lambda'_j - \mu'_j} V)$; for example, corresponding to the displayed tableau, $v_T = v_4 \otimes v_2 \otimes (v_1 \wedge v_3)$. A key fact, generalizing the result of Exercise 6.15, is that the images of these elements under the map (vi) form a basis for $\mathbb{S}_{\lambda/\mu} V$.

The character of $\mathbb{S}_{\lambda/\mu} V$ is given by the Schur function $S_{\lambda/\mu}$: if g is an endomorphism of V with eigenvalues $x_1, \ldots, x_k$, then

(viii) $$\chi_{\mathbb{S}_{\lambda/\mu} V}(g) = S_{\lambda/\mu}(x_1, \ldots, x_k).$$

In terms of basic Schur functors,

(ix) $$\mathbb{S}_{\lambda/\mu} V \cong \sum N_{\mu\nu\lambda} \mathbb{S}_\nu V.$$

Exercise 6.20*. (a) Show that if $\lambda = (p, q)$, $\mathbb{S}_{(p,q)} V$ is the kernel of the contraction map

$$c_{p,q}\colon \mathrm{Sym}^p V \otimes \mathrm{Sym}^q V \to \mathrm{Sym}^{p+1} V \otimes \mathrm{Sym}^{q-1} V.$$

(b) If $\lambda = (p, q, r)$, show that $\mathbb{S}_{(p,q,r)} V$ is the intersection of the kernels of two contraction maps $c_{p,q} \otimes 1_r$ and $1_p \otimes c_{p,r}$, where 1_i denotes the identity map on $\mathrm{Sym}^i V$.

In general, for $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mathbb{S}_\lambda V \subset \mathrm{Sym}^{\lambda_1} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k} V$ is the intersection of the kernels of the $k - 1$ maps

$$\psi_i = 1_{\lambda_1} \otimes \cdots \otimes 1_{\lambda_{i-1}} \otimes c_{\lambda_i, \lambda_{i+1}} \otimes 1_{\lambda_{i+2}} \otimes \cdots \otimes 1_{\lambda_k}, \quad 1 \le i \le k - 1.$$

(c) For $\lambda = (p, 1, \ldots, 1)$, show that $\mathbb{S}_\lambda V$ is the kernel of the contraction map:

$$\mathbb{S}_{(p,1,\ldots,1)} V = \mathrm{Ker}(\mathrm{Sym}^p V \otimes \Lambda^{d-p} V \to \mathrm{Sym}^{p+1} V \otimes \Lambda^{d-p-1} V).$$

In general, for any choice of a between 1 and $k - 1$, the intersection of the kernels of all ψ_i except ψ_a is $\mathbb{S}_\sigma V \otimes \mathbb{S}_\tau V$, where $\sigma = (\lambda_1, \ldots, \lambda_a)$ and $\tau = (\lambda_{a+1}, \ldots, \lambda_k)$; so $\mathbb{S}_\lambda V$ is the kernel of a contraction map defined on $\mathbb{S}_\sigma V \otimes \mathbb{S}_\tau V$. For example, if a is $k - 1$, and we set $r = \lambda_k$, Pieri's formula writes $\mathbb{S}_\sigma V \otimes \mathrm{Sym}^r V$ as a direct sum of $\mathbb{S}_\lambda V$ and other factors $\mathbb{S}_\nu V$; the general assertion in (b) is equivalent to the claim that $\mathbb{S}_\lambda V$ is the only factor that is in the kernel of the contraction, ie.,

$$\mathbb{S}_\lambda V = \mathrm{Ker}(\mathbb{S}_{(\lambda_1, \ldots, \lambda_{k-1})} V \otimes \mathrm{Sym}^r V \to V^{\otimes(d-r+1)} \otimes \mathrm{Sym}^{r-1} V).$$

These results correspond to writing the representations $V_\lambda \subset U_\lambda$ of the symmetric group as the intersection of kernels of maps to $U_{\lambda_1, \ldots, \lambda_i+1, \lambda_{i+1}-1, \ldots, \lambda_k}$.

Exercise 6.21. The functorial nature of Weyl's construction has many consequences, which are not explored in this book. For example, if E_* is a complex of vector spaces, the tensor product $E_*^{\otimes d}$ is also a complex, and the symmetric group $\mathfrak{S}_d$ acts on it; when factors in E_p and E_q are transposed past each other, the usual sign $(-1)^{pq}$ is inserted. The image of the Young symmetrizer c_λ is a complex $\mathbb{S}_\lambda(E_*)$, sometimes called a *Schur complex*. Show that if E_* is the complex $E_{-1} = V \to E_0 = V$, with the boundary map the identity map, and $\lambda = (d)$, then $\mathbb{S}_\lambda(E_*)$ is the Koszul complex

$$0 \to \Lambda^d \to \Lambda^{d-1} \otimes S^1 \to \Lambda^{d-2} \otimes S^2 \to \cdots \to \Lambda^1 \otimes S^{d-1} \to S^d \to 0,$$

where $\Lambda^i = \Lambda^i V$, and $S^j = \mathrm{Sym}^j V$.

§6.2. The Proofs

We need first a small piece of the general story about semisimple algebras, which we work out by hand. For the moment G can be any finite group, although our application is for the symmetric group. If U is a right module over $A = \mathbb{C}G$, let

$$B = \mathrm{Hom}_G(U, U) = \{\varphi: U \to U: \varphi(v \cdot g) = \varphi(v) \cdot g, \forall v \in U, g \in G\}.$$

Note that B acts on U on the left, commuting with the right action of A; B is called the *commutator* algebra. If $U = \bigoplus U_i^{\oplus n_i}$ is an irreducible decomposition with U_i nonisomorphic irreducible right A-modules, then by Schur's Lemma 1.7

$$B = \bigoplus_i \mathrm{Hom}_G(U_i^{\oplus n_i}, U_i^{\oplus n_i}) = \bigoplus_i M_{n_i}(\mathbb{C}),$$

where $M_{n_i}(\mathbb{C})$ is the ring of $n_i \times n_i$ complex matrices.

If W is any left A-module, the tensor product

$$U \otimes_A W = U \otimes_{\mathbb{C}} W/\text{subspace generated by } \{va \otimes w - v \otimes aw\}$$

is a left B-module by acting on the first factor: $b \cdot (v \otimes w) = (b \cdot v) \otimes w$.

Lemma 6.22. *Let U be a finite-dimensional right A-module.*

(i) *For any $c \in A$, the canonical map $U \otimes_A Ac \to Uc$ is an isomorphism of left B-modules.*

(ii) *If $W = Ac$ is an irreducible left A-module, then $U \otimes_A W = Uc$ is an irreducible left B-module.*

(iii) *If $W_i = Ac_i$ are the distinct irreducible left A-modules, with m_i the dimension of W_i, then*

$$U \cong \bigoplus_i (U \otimes_A W_i)^{\oplus m_i} \cong \bigoplus_i (Uc_i)^{\oplus m_i}$$

is the decomposition of U into irreducible left B-modules.

Proof. Note first that Ac is a direct summand of A as a left A-module; this is a consequence of the semisimplicity of all representations of G (Proposition 1.5). To prove (i), consider the commutative diagram

$$\begin{array}{ccccc} U \otimes_A A & \xrightarrow{\cdot c} & U \otimes_A Ac & \hookrightarrow & U \otimes_A A \\ \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{\cdot c} & U \cdot c & \hookrightarrow & U \end{array},$$

where the vertical maps are the maps $v \otimes a \mapsto v \cdot a$; since the left horizontal maps are surjective, the right ones injective, and the outside vertical maps are isomorphisms, the middle vertical map must be an isomorphism.

For (ii), consider first the case where U is an irreducible A-module, so $B = \mathbb{C}$. It suffices to show that $\dim U \otimes_A W \leq 1$. For this we use Proposition 3.29 to identify A with a direct sum $\bigoplus_{i=1}^r M_{m_i}\mathbb{C}$ of r matrix algebras. We can identify W with a minimal left ideal of A. Any minimal ideal in the sum of matrix algebras is isomorphic to one which consists of r-tuples of matrices which are zero except in one factor, and in this factor are all zero except for one column. Similarly, U can be identified with the right ideal of r-tuples which are zero except in one factor, and in that factor all are zero except in one row. Then $U \otimes_A W$ will be zero unless the factor is the same for U and W, in which case $U \otimes_A W$ can be identified with the matrices which are zero except in one row and column of that factor. This completes the proof when U is irreducible. For the general case of (ii), decompose $U = \bigoplus_i U_i^{\oplus n_i}$ into a sum of irreducible right A-modules, so $U \otimes_A W = \bigoplus_i (U_i \otimes_A W)^{\oplus n_i} = \mathbb{C}^{\oplus n_k}$ for some k, which is visibly irreducible over $B = \bigoplus M_{n_j}(\mathbb{C})$.

Part (iii) follows, since the isomorphism $A \cong \bigoplus W_i^{\oplus m_i}$ determines an isomorphism

$$U \cong U \otimes_A A \cong U \otimes_A (\textstyle\bigoplus_i W_i^{\oplus m_i}) \cong \bigoplus_i (U \otimes_A W_i)^{\oplus m_i}. \qquad \square$$

To prove Theorem 6.3, we will apply Lemma 6.22 to the right $\mathbb{C}\mathfrak{S}_d$-module $U = V^{\otimes d}$. That lemma shows how to decompose U as a B-module, where B

is the algebra of all endomorphisms of U that commute with all permutations of the factors. The endomorphisms of U induced by endomorphisms of V are certainly in this algebra B. Although B is generally much larger than $\mathrm{End}(V)$, we have

Lemma 6.23. *The algebra B is spanned as a linear subspace of* $\mathrm{End}(V^{\otimes d})$ *by* $\mathrm{End}(V)$. *A subspace of $V^{\otimes d}$ is a sub-B-module if and only if it is invariant by* $\mathrm{GL}(V)$.

PROOF. Note that if W is any finite-dimensional vector space, then $\mathrm{Sym}^d W$ is the subspace of $W^{\otimes d}$ spanned by all $w^d = d!w \otimes \cdots \otimes w$ as w runs through W. Applying this to $W = \mathrm{End}(V) = V^* \otimes V$ proves the first statement, since $\mathrm{End}(V^{\otimes d}) = (V^*)^{\otimes d} \otimes V^{\otimes d} = W^{\otimes d}$, with compatible actions of $\mathfrak{S}_d$. The second follows from the fact that $\mathrm{GL}(V)$ is dense in $\mathrm{End}(V)$. □

We turn now to the proof of Theorem 6.3. Note that $\mathbb{S}_\lambda V$ is Uc_λ, so parts (2) and (4) follow from Lemmas 6.22 and 6.23. We use the same methods to give a rather indirect but short proof of part (3); for a direct approach see Exercise 6.28. From Lemma 6.22 we have an isomorphism of $\mathrm{GL}(V)$-modules:

$$\mathbb{S}_\lambda V \cong V^{\otimes d} \otimes_A V_\lambda \tag{6.24}$$

with $V_\lambda = A \cdot c_\lambda$. Similarly for $U_\lambda = A \cdot a_\lambda$, and since the image of right multiplication by a_λ on $V^{\otimes d}$ is the tensor product of symmetric powers, we have

$$\mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k} V \cong V^{\otimes d} \otimes_A U_\lambda. \tag{6.25}$$

But we have an isomorphism $U_\lambda \cong \bigoplus_\mu K_{\mu\lambda} V_\mu$ of A-modules by Young's rule (4.39), so we deduce an isomorphism of $\mathrm{GL}(V)$-modules

$$\mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k} V \cong \bigoplus_\mu K_{\mu\lambda} \mathbb{S}_\mu V. \tag{6.26}$$

By what we saw before the statement of the theorem, the trace of g on the left-hand side of (6.26) is the product $H_\lambda(x_1, \ldots, x_k)$ of the complete symmetric polynomials $H_{\lambda_i}(x_1, \ldots, x_k)$. Let $\mathbb{S}_\lambda(g)$ denote the endomorphism of $\mathbb{S}_\lambda V$ determined by an endomorphism g of V. We therefore have

$$H_\lambda(x_1, \ldots, x_k) = \Sigma_\mu K_{\mu\lambda} \,\mathrm{Trace}(\mathbb{S}_\mu(g)).$$

But these are precisely the relations between the functions H_λ and the Schur polynomials S_μ [see formula (A.9)], and these relations are invertible, since the matrix $(K_{\mu\lambda})$ of coefficients is triangular with 1's on the diagonal. It follows that $\mathrm{Trace}(\mathbb{S}_\lambda(g)) = S_\lambda(x_1, \ldots, x_k)$, which proves part (3).

Note that if $\lambda = (\lambda_1, \ldots, \lambda_d)$ with $d > k$ and $\lambda_{k+1} \neq 0$, this same argument shows that the trace is $S_\lambda(x_1, \ldots, x_k, 0, \ldots, 0)$, which is zero, for example by (A.6). For g the identity, this shows that $\mathbb{S}_\lambda V = 0$ in this case. From part (3) we also get

$$\dim \mathbb{S}_\lambda V = S_\lambda(1, \ldots, 1), \tag{6.27}$$

and computing $S_\lambda(1, \ldots, 1)$ via Exercise A.30(ii) yields part (1). □

Exercise 6.28. If you have given an independent proof of Problem 6.15, part (3) of Theorem 6.3 can be seen directly. The basis elements v_T for $\mathbb{S}_\lambda V$ specified in Problem 6.15 are eigenvectors for a diagonal matrix with entries $x_1, \ldots, x_k$, with eigenvalue $X^a = x_1^{a_1} \cdot \ldots \cdot x_k^{a_k}$, where the tableau T has a_1 1's, a_2 2's, ..., a_k k's. The trace is therefore $\sum K_{\lambda a} X^a$, where $K_{\lambda a}$ is the number of ways to number the boxes of the Young diagram of λ with a_1 1's, a_2 2's, ..., a_k k's. This is just the expression for S_λ obtained in Exercise A.31(a).

We conclude this lecture with a few of the standard elaborations of these ideas, in exercise form; they are not needed in these lectures.

Exercise 6.29*. Show that, in the context of Lemma 6.22, if U is a faithful A-module, then A is the commutator of its commutator B:

$$A = \{\psi : U \to U : \psi(bv) = b\psi(v), \forall v \in U, b \in B\}.$$

If U is not faithful, the canonical map from A to its bicommutator is surjective. Conclude that, in Theorem 6.3, the algebra of endomorphisms of $V^{\otimes d}$ that commute with $\mathrm{GL}(V)$ is spanned by the permutations in $\mathfrak{S}_d$.

Exercise 6.30. Show that, in Lemma 6.22, there is a natural one-to-one correspondence between the irreducible right A-modules U_i that occur in U and the irreducible left B-modules V_i. Show that there is a canonical decomposition

$$U = \bigoplus_i (V_i \otimes_{\mathbb{C}} U_i)$$

as a left B-module and as a right A-module. This shows again that the number of times V_i occurs in U is the dimension of U_i, and dually that the number of times U_i occurs is the dimension of V_i. Deduce the canonical decomposition

$$V^{\otimes d} = \bigoplus \mathbb{S}_\lambda V \otimes_{\mathbb{C}} V_\lambda,$$

the sum over partitions λ of d into at most $k = \dim V$ parts; this decomposition is compatible with the actions of $\mathrm{GL}(V)$ and $\mathfrak{S}_d$. In particular, the number of times V_λ occurs in the representation $V^{\otimes d}$ of $\mathfrak{S}_d$ is the dimension of $\mathbb{S}_\lambda V$.

Exercise 6.31. Let e be an idempotent in the group algebra $A = \mathbb{C}G$, and let $U = eA$ be the corresponding right A-module. Let $E = eAe$, a subalgebra of A. The algebra structure in A makes eA a left E-module. Show that this defines an isomorphism of $\mathbb{C}$-algebras

$$E = eAe \cong \mathrm{Hom}_A(eA, eA) = \mathrm{Hom}_G(U, U) = B.$$

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Exercise 6.32. If H is a subgroup of G, and $e \in \mathbb{C}H$ is an idempotent, corresponding to a representation $W = \mathbb{C}H \cdot e$ of H, show that $\mathbb{C}G \cdot e$ is the induced representation $\mathrm{Ind}_H^G(W)$. For example, if $\vartheta: H \to \mathbb{C}^*$ is a one-dimensional representation, then

$$\mathrm{Ind}_H^G(\vartheta) = \mathbb{C}G \cdot e_\vartheta, \quad \text{where } e_\vartheta = \frac{1}{|G|} \sum_{g \in G} \overline{\vartheta(g)} e_g.$$

PART II

LIE GROUPS AND LIE ALGEBRAS

From a naive point of view, Lie groups seem to stand at the opposite end of the spectrum of groups from finite ones.[1] On the one hand, as abstract groups they seem enormously complicated: for example, being of uncountable order, there is no question of giving generators and relations. On the other hand, they do come with the additional data of a topology and a manifold structure; this makes it possible—and, given the apparent hopelessness of approaching them purely as algebraic objects, necessary—to use geometric concepts to study them.

Lie groups thus represent a confluence of algebra, topology, and geometry, which perhaps accounts in part for their ubiquity in modern mathematics. It also makes the subject a potentially intimidating one: to have to understand, both individually and collectively, all these aspects of a single object may be somewhat daunting.

Happily, just because the algebra and the geometry/topology of a Lie group are so closely entwined, there is an object we can use to approach the study of Lie groups that extracts much of the structure of a Lie group (primarily its algebraic structure) while seemingly getting rid of the topological complexity. This is, of course, the *Lie algebra*. The Lie algebra is, at least according to its definition, a purely algebraic object, consisting simply of a vector space with bilinear operation; and so it might appear that in associating to a Lie group its Lie algebra we are necessarily giving up a lot of information about the group. This is, in fact, not the case: as we shall see in many cases (and perhaps all of the most important ones), encoded in the algebraic structure of a Lie algebra is almost all of the geometry of the group. In particular, we will

[1] In spite of this there are deep, if only partially understood, relations between finite and Lie groups, extending even to their simple group classifications.

see by the end of Lecture 8 that there is a very close relationship between representations of the Lie group we start with and representations of the Lie algebra we associate to it; and by the end of the book we will make that correspondence exact.

We said that passing from the Lie group to its Lie algebra represents a simplification because it eliminates whatever nontrivial topological structure the group may have had; it "flattens out," or "linearizes," the group. This, in turn, allows for a further simplification: since a Lie algebra is just a vector space with bilinear operation, it makes perfect sense, if we are asked to study a real Lie algebra (or one over any subfield of $\mathbb{C}$) to tensor with the complex numbers. Thus, we may investigate first the structure and representations of *complex Lie algebras*, and then go back to apply this knowledge to the study of real ones. In fact, this turns out to be a feasible approach, in every respect: the structure of complex Lie algebras tends to be substantially simpler than that of real Lie algebras; and knowing the representations of the complex Lie algebra will solve the problem of classifying the representations of the real one.

There is one further reduction to be made: some very elementary Lie algebra theory allows us to narrow our focus further to the study of *semisimple Lie algebras*. This is a subset of Lie algebras analogous to simple groups in that they are in some sense atomic objects, but better behaved in a number of ways: a semisimple Lie algebra is a direct sum of simple ones; there are easy criteria for the semisimplicity of a given Lie algebra; and, most of all, their representation theory can be approached in a completely uniform manner. Moreover, as in the case of finite groups, there is a complete classification theorem for simple Lie algebras.

We may thus describe our approach to the representation theory of Lie groups by the sequence of objects

Lie group

$\rightsquigarrow$ Lie algebra

$\rightsquigarrow$ complex Lie algebra

$\rightsquigarrow$ semisimple complex Lie algebra.

We describe this progression in Lectures 7–9. In Lectures 7 and 8 we introduce the definitions of and some basic facts about Lie groups and Lie algebras. Lecture 8 ends with a description of the exponential map, which allows us to establish the close connection between the first two objects above. We then do, in Lecture 9, the very elementary classification theory of Lie algebras that motivates our focus on semisimple complex Lie algebras, and at least state the classification theorem for these. This establishes the fact that the second, third, and fourth objects above have essentially the same irreducible representations. (This lecture may also serve to give a brief taste of some general theory, which is mostly postponed to later lectures or appendices.) In Lecture 10 we discuss examples of Lie algebras in low dimensions.

From that point on we will proceed to devote ourselves almost exclusively to the study of semisimple complex Lie algebras and their representations. We do this, we have to say, in an extremely inefficient manner: we start with a couple of very special cases, which occupy us for three lectures (11–13); enunciate the general paradigm in Lecture 14; carry this out for the classical Lie algebras in Lectures 15–20; and (finally) finish off the general theory in Lectures 21–26. Thus, it will not be until the end that we go back and use the knowledge we have gained to say something about the original problem. In view of this long interlude, it is perhaps a good idea to enunciate one more time our basic

Point of View: The primary objects of interest are Lie groups and their representations; these are what actually occur in real life and these are what we want to understand. The notion of a complex Lie algebras is introduced primarily as a tool in this study; it is an essential tool[2] and we should consider ourselves incredibly lucky to have such a wonderfully effective one; but in the end it is for us a means to an end.

The special cases worked out in Lectures 11–13 are the Lie algebras of SL_2 and SL_3. Remarkably, most of the structure shared by all semisimple Lie algebras can be seen in these examples. We should probably point out that much of what we do by hand in these cases could be deduced from the Weyl construction we saw in Lecture 6 (as we will do generally in Lecture 15), but we mainly ignore this, in order to work from a "Lie algebra" point of view and motivate the general story.

[2] Perhaps not logically so; cf. Adams' book [Ad].

LECTURE 7

Lie Groups

In this lecture we introduce the definitions and basic examples of Lie groups and Lie algebras. We assume here familiarity with the definition of differentiable manifolds and maps between them, but no more; in particular, we do not mention vector fields, differential forms, Riemannian metrics, or any other tensors. Section 7.3, which discusses maps of Lie groups that are covering space maps of the underlying manifolds, may be skimmed and referred back to as needed, though working through it will help promote familiarity with basic examples of Lie groups.

§7.1: Lie groups: definitions
§7.2: Examples of Lie groups
§7.3: Two constructions

§7.1. Lie Groups: Definitions

You probably already know what a Lie group is; it is just a set endowed simultaneously with the compatible structures of a group and a $\mathscr{C}^\infty$ manifold. "Compatible" here means that the multiplication and inverse operations in the group structure

$$\times: G \times G \to G$$

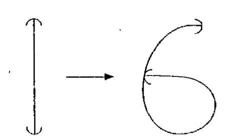
and

$$\iota: G \to G$$

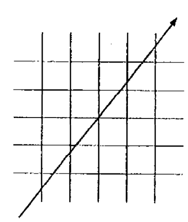
are actually differentiable maps (logically, this is equivalent to the single requirement that the map $G \times G \to G$ sending (x, y) to $x \cdot y^{-1}$ is $\mathscr{C}^\infty$).

A *map*, or *morphism*, between two Lie groups G and H is just a map $\rho: G \to H$ that is both differentiable and a group homomorphism. In general, qualifiers applied to Lie groups refer to one or another of the two structures, usually without much ambiguity; thus, *abelian* refers to the group structure, *n-dimensional* or *connected* refers to the manifold structure. Sometimes a condition on one structure turns out to be equivalent to a condition on the other; for example, we will see below that to say that a map of connected Lie groups $\varphi: G \to H$ is a surjective map of groups is equivalent to saying that the differential $d\varphi$ is surjective at every point.

One area where there is some potential confusion is in the definition of a Lie subgroup. This is essentially a difficulty inherited directly from manifold theory, where we have to make a distinction between a *closed submanifold* of a manifold M, by which we mean a subset $X \subset M$ that inherits a manifold structure from M (i.e., that may be given, locally in M, by setting a subset of the local coordinates equal to zero), and an *immersed submanifold*, by which we mean the image of a manifold X under a one-to-one map with injective differential everywhere—that is, a map that is an embedding *locally in* X. The distinction is necessary simply because the underlying topological space structure of an immersed submanifold may not agree with the topological structure induced by the inclusion of X in M. For example, the map from X to M could be the immersion of an open interval in $\mathbb{R}$ into the plane $\mathbb{R}^2$ as a figure "6":



Another standard example of this, which is also an example in the category of groups, would be to take M to be the two-dimensional real torus $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$, and X the image in M of a line $V \subset \mathbb{R}^2$ having irrational slope:



The upshot of this is that we define a *Lie subgroup* (or *closed Lie subgroup*, if we want to emphasize the point) of a Lie group G to be a subset that is

simultaneously a subgroup and a *closed* submanifold; and we define an *immersed subgroup* to be the image of a Lie group H under an injective morphism to G. (That a one-to-one morphism of Lie groups has everywhere injective differential will follow from discussions later in this lecture.)

The definition of a *complex Lie group* is exactly analogous, the words "differentiable manifold" being replaced by "complex manifold" and all related notions revised accordingly. Similarly, to define an *algebraic group* one replaces "differentiable manifold" by "algebraic variety" and "differentiable map" by "regular morphism." As we will see, the category of complex Lie groups is in many ways markedly different from that of real Lie groups (for example, there are many fewer complex Lie groups than real ones). Of course, the study of algebraic groups in general is quite different from either of these since an algebraic group comes with a field of definition that may or may not be a subfield of $\mathbb{C}$ (it may, for that matter, have positive characteristic). In practice, though, while the two are not the same (we will see examples of this in Lecture 10, for example), the category of algebraic groups over $\mathbb{C}$ behaves very much like the category of complex Lie groups.

§7.2. Examples of Lie Groups

The basic example of a Lie group is of course the *general linear group* $\mathrm{GL}_n\mathbb{R}$ of invertible $n \times n$ real matrices; this is an open subset of the vector space of all $n \times n$ matrices, and gets its manifold structure accordingly (so that the entries of the matrix are coordinates on $\mathrm{GL}_n\mathbb{R}$). That the multiplication map $\mathrm{GL}_n\mathbb{R} \times \mathrm{GL}_n\mathbb{R} \to \mathrm{GL}_n\mathbb{R}$ is differentiable is clear; that the inverse map $\mathrm{GL}_n\mathbb{R} \to \mathrm{GL}_n\mathbb{R}$ is follows from Cramer's formula for the inverse. Occasionally $\mathrm{GL}_n\mathbb{R}$ will come to us as the group of automorphisms of an n-dimensional real vector space V; when we want to think of $\mathrm{GL}_n\mathbb{R}$ in this way (e.g., without choosing a basis for V and thereby identifying G with the group of matrices), we will write it as $\mathrm{GL}(V)$ or $\mathrm{Aut}(V)$. A *representation* of a Lie group G, of course, is a morphism from G to $\mathrm{GL}(V)$.

Most other Lie groups are defined initially as subgroups of GL_n (though they may appear in other contexts as subgroups of other general linear groups, which is, of course, the subject matter of these lectures). For the most part, such subgroups may be described either by equations on the entries of an $n \times n$ matrix, or as the subgroup of automorphisms of $V \cong \mathbb{R}^n$ preserving some structure on $\mathbb{R}^n$. For example, we have:

the *special linear group* $\mathrm{SL}_n\mathbb{R}$ of automorphisms of $\mathbb{R}^n$ preserving the volume element; equivalently, $n \times n$ matrices A with determinant 1.

the group B_n of *upper-triangular* matrices; equivalently, the subgroup of automorphisms of $\mathbb{R}^n$ preserving the flag[1]

[1] In general, a *flag* is a sequence of subspaces of a fixed vector space, each properly contained in the next; it is a *complete* flag if each has one dimension larger than the preceding, and *partial* otherwise.

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots V_{n-1} \subset V_n = \mathbb{R}^n,$$

where V_i is the span of the standard basis vectors $e_1, \ldots, e_i$. Note that choosing a different basis and correspondingly a different flag yields a different subgroup of $\mathrm{GL}_n\mathbb{R}$, but one isomorphic to (indeed, conjugate to) B_n. Somewhat more generally, for any sequence of positive integers $a_1, \ldots, a_k$ with sum n we can look at the group of block-upper-triangular matrices; this is the subgroup of automorphisms of $\mathbb{R}^n$ preserving a partial flag

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = \mathbb{R}^n,$$

where the dimension of V_i is $a_1 + \cdots + a_i$. If the subspace V_i is spanned by the first $a_1 + \cdots + a_i$ basis vectors, the group will be the set of matrices of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \begin{matrix} \}a_1 \\ \}a_2 \\ \\ \}a_k \end{matrix}$$

The group N_n of *upper-triangular unipotent* matrices (that is, upper triangular with 1's on the diagonal); equivalently, the subgroup of automorphisms of $\mathbb{R}^n$ preserving the complete flag $\{V_i\}$ where V_i is the span of the standard basis vectors $e_1, \ldots, e_i$, and acting as the identity on the successive quotients V_{i+1}/V_i. As before, we can, for any sequence of positive integers $a_1, \ldots, a_k$ with sum n, look at the group of block-upper-triangular unipotent matrices; this is the subgroup of automorphisms of $\mathbb{R}^n$ preserving a partial flag and acting as the identity on successive quotients, i.e., matrices of the form

$$\begin{pmatrix} I & * & * & * \\ 0 & I & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & I \end{pmatrix} \begin{matrix} \}a_1 \\ \}a_2 \\ \\ \}a_k \end{matrix}$$

Next, there are the subgroups of $\mathrm{GL}_n\mathbb{R}$ defined as the group of transformations of $V = \mathbb{R}^n$ of determinant 1 preserving some bilinear form $Q\colon V \times V \to V$. If the bilinear form Q is symmetric and positive definite, the group we get is called the (*special*) *orthogonal group* $\mathrm{SO}_n\mathbb{R}$ (sometimes written $\mathrm{SO}(n)$; see p. 100). If Q is symmetric and nondegenerate but not definite—e.g., if it has k positive eigenvalues and l negative—the group is denoted $\mathrm{SO}_{k,l}\mathbb{R}$ or $\mathrm{SO}(k, l)$; note that $\mathrm{SO}(k, l) \cong \mathrm{SO}(l, k)$. If Q is skew-symmetric and nondegenerate, the group is called the *symplectic group* and denoted $\mathrm{Sp}_n\mathbb{R}$; note that in this case n must be even.

The equations that define the subgroup of $\mathrm{GL}_n\mathbb{R}$ preserving a bilinear form Q are easy to write down. If we represent Q by a matrix M—that is, we write

$$Q(v, w) = {}^t v \cdot M \cdot w$$

for all $v, w \in \mathbb{R}^n$—then the condition

$$Q(Av, Aw) = Q(v, w)$$

translates into the condition that

$${}^t v \cdot {}^t A \cdot M \cdot A \cdot w = {}^t v \cdot M \cdot w$$

for all v and w; this is equivalent to saying that

$${}^t A \cdot M \cdot A = M.$$

Thus, for example, if Q is the symmetric form $Q(v, w) = {}^t v \cdot w$ given by the identity matrix $M = I_n$, the group $\mathrm{SO}_n\mathbb{R}$ is just the group of $n \times n$ real matrices A of determinant 1 such that ${}^t A = A^{-1}$.

Exercise 7.1*. Show that in the case of $\mathrm{Sp}_{2n}\mathbb{R}$ the requirement that the transformations have determinant 1 is redundant; whereas in the case of $\mathrm{SO}_n\mathbb{R}$ if we do not require the transformations to have determinant 1 the group we get (denoted $\mathrm{O}_n\mathbb{R}$, or sometimes $\mathrm{O}(n)$) is disconnected.

Exercise 7.2*. Show that $\mathrm{SO}(k, l)$ has two connected components if k and l are both positive. The connected component containing the identity is often denoted $\mathrm{SO}^+(k, l)$. (Composing with a projection onto $\mathbb{R}^k$ or $\mathbb{R}^l$, we may associate to an automorphism $A \in \mathrm{SO}(k, l)$ automorphisms of $\mathbb{R}^k$ and $\mathbb{R}^l$; $\mathrm{SO}^+(k, l)$ will consist of those $A \in \mathrm{SO}(k, l)$ whose associated automorphisms preserve the orientations of $\mathbb{R}^k$ and $\mathbb{R}^l$.)

Note that if the form Q is degenerate, a transformation preserving Q will carry its kernel

$$\mathrm{Ker}(Q) = \{v \in V\colon Q(v, w) = 0\ \forall w \in V\}$$

into itself; so that the group we get is simply the group of matrices preserving the subspace $\mathrm{Ker}(Q)$ and preserving the induced nondegenerate form $\tilde{Q}$ on the quotient $V/\mathrm{Ker}(Q)$. Likewise, if Q is a general bilinear form, that is, neither symmetric nor skew-symmetric, a linear transformation preserving Q will preserve the symmetric and skew-symmetric parts of Q individually, so we just get an intersection of the subgroups encountered already. At any rate, we usually limit our attention to nondegenerate forms that are either symmetric or skew-symmetric.

Of course, the group $\mathrm{GL}_n\mathbb{C}$ of complex linear automorphisms of a complex vector space $V = \mathbb{C}^n$ can be viewed as subgroup of the general linear group $\mathrm{GL}_{2n}\mathbb{R}$; it is, thus, a real Lie group as well, as is the subgroup $\mathrm{SL}_n\mathbb{C}$ of $n \times n$ complex matrices of determinant 1. Similarly, the subgroups $\mathrm{SO}_n\mathbb{C} \subset \mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C} \subset \mathrm{SL}_{2n}\mathbb{C}$ of transformations of a complex vector space preserving a symmetric and skew-symmetric nondegenerate *bilinear* form, respectively, are real as well as complex Lie subgroups. Note that since all nondegenerate bilinear symmetric forms on a complex vector space are isomorphic (in particular, there is no such thing as a signature), there is only one complex orthogonal subgroup $\mathrm{SO}_n\mathbb{C} \subset \mathrm{SL}_n\mathbb{C}$ up to conjugation; there are no analogs of the groups $\mathrm{SO}_{k,l}\mathbb{R}$.

Another example we can come up with here is the *unitary group* U_n or $\mathrm{U}(n)$, defined to be the group of complex linear automorphisms of an n-dimensional complex vector space V preserving a positive definite *Hermitian* inner product H on V. (A *Hermitian* form H is required to be conjugate linear in the first[2] factor, and linear in the second: $H(\lambda v, \mu w) = \bar{\lambda} H(v, w)\mu$, and $H(w, v) = \overline{H(v, w)}$; it is *positive definite* if $H(v, v) > 0$ for $v \neq 0$.)

Just as in the case of the subgroups SO and Sp, it is easy to write down the equations for $\mathrm{U}(n)$: for some $n \times n$ matrix M we can write the form H as

$$H(v, w) = {}^t\bar{v} \cdot M \cdot w, \quad \forall v, w \in \mathbb{C}^n$$

(note that for H to be conjugate symmetric, M must be conjugate symmetric, i.e., ${}^tM = \bar{M}$); then the group $\mathrm{U}(n)$ is just the group of $n \times n$ complex matrices A satisfying

$${}^t\bar{A} \cdot M \cdot A = M.$$

In particular, if H is the "standard" Hermitian inner product $H(v, w) = {}^t\bar{v} \cdot w$ given by the identity matrix, $\mathrm{U}(n)$ will be the group of $n \times n$ complex matrices A such that ${}^t\bar{A} = A^{-1}$.

Exercise 7.3. Show that if H is a Hermitian form on a complex vector space V, then the real part $R = \mathrm{Re}(H)$ of H is a symmetric form on the underlying real space, and the imaginary part $C = \mathrm{Im}(H)$ is a skew-symmetric real form; these are related by $C(v, w) = R(iv, w)$. Both R and C are invariant by multiplication by i: $R(iv, iw) = R(v, w)$. Show conversely that any such real symmetric R is the real part of a unique Hermitian H. Show that if H is standard, so is R, and C corresponds to the matrix $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Deduce that

$$\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{Sp}_{2n}\mathbb{R}.$$

Note that the determinant of a unitary matrix can be any complex number of modulus 1; the *special unitary group*, $\mathrm{SU}(n)$, is the subgroup of $\mathrm{U}(n)$ of automorphisms with determinant 1. The subgroup of $\mathrm{GL}_n\mathbb{C}$ preserving an indefinite Hermitian inner product with k positive eigenvalues and l negative ones is denoted $\mathrm{U}_{k,l}$ or $\mathrm{U}(k, l)$; the subgroup of those of determinant 1 is denoted $\mathrm{SU}_{k,l}$ or $\mathrm{SU}(k, l)$.

In a similar vein, the group $\mathrm{GL}_n\mathbb{H}$ of quaternionic linear automorphisms of an n-dimensional vector space V over the ring $\mathbb{H}$ of quaternions is a real

[2] This choice of which factor is linear and which conjugate linear is less common than the other. It makes little difference in what follows, but it does have the small advantage of being compatible with the natural choice for quaternions.

Lie subgroup of the group $\mathrm{GL}_{4n}\mathbb{R}$, as are the further subgroups of $\mathbb{H}$-linear transformations of V preserving a bilinear form. Since $\mathbb{H}$ is not commutative, care must be taken with the conventions here, and it may be worth a little digression to go through this now. We take the vector spaces V to be right $\mathbb{H}$-modules; $\mathbb{H}^n$ is the space of column vectors with *right* multiplication by scalars. In this way the $n \times n$ matrices with entries in $\mathbb{H}$ act in the usual way on $\mathbb{H}^n$ on the left. Scalar multiplication on the left (only) is $\mathbb{H}$-linear.

View $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C} \cong \mathbb{C}^2$. Then left multiplication by elements of $\mathbb{H}$ give $\mathbb{C}$-linear endomorphisms of $\mathbb{C}^2$, which determines a mapping $\mathbb{H} \to M_2\mathbb{C}$ to the 2×2 complex matrices. In particular, $\mathbb{H}^* = \mathrm{GL}_1\mathbb{H} \hookrightarrow \mathrm{GL}_2\mathbb{C}$. Similarly $\mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n = \mathbb{C}^{2n}$, so we have an embedding $\mathrm{GL}_n\mathbb{H} \hookrightarrow \mathrm{GL}_{2n}\mathbb{C}$. Note that a $\mathbb{C}$-linear mapping $\varphi\colon \mathbb{H}^n \to \mathbb{H}^n$ is $\mathbb{H}$-linear exactly when it commutes with j: $\varphi(vj) = \varphi(v)j$. If $v = v_1 + jv_2$, then $v \cdot j = -\bar{v}_2 + j\bar{v}_1$, so multiplication by j takes $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ to $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}$. It follows that if J is the matrix of the preceding exercise, then

$$\mathrm{GL}_n\mathbb{H} = \{A \in \mathrm{GL}_{2n}\mathbb{C}\colon AJ = J\bar{A}\}.$$

Those matrices with real determinant 1 form a subgroup $\mathrm{SL}_n\mathbb{H}$.

A *Hermitian form* (or "symplectic scalar product") on a quaternionic vector space V is an $\mathbb{R}$-bilinear form $K\colon V \times V \to \mathbb{H}$ that is conjugate $\mathbb{H}$-linear in the first factor and $\mathbb{H}$-linear in the second: $K(v\lambda, w\mu) = \bar{\lambda} K(v, w)\mu$, and satisfies $K(w, v) = \overline{K(v, w)}$. It is *positive definite* if $K(v, v) > 0$ for $v \neq 0$. (The conjugate $\bar{\lambda}$ of a quaternion $\lambda = a + bi + cj + dk$ is defined to be $a - bi - cj - dk$.) The *standard* Hermitian form on $\mathbb{H}^n$ is $\Sigma \bar{v}_i w_i$. The group of automorphisms of an n-dimensional quaternionic space preserving such a form is called the *compact symplectic* group and denoted $\mathrm{Sp}(n)$ or $\mathrm{U}_{\mathbb{H}}(n)$. The *standard* Hermitian form on $\mathbb{H}^n$ is $\Sigma \bar{v}_i w_i$.

Exercise 7.4. Regarding V as a complex vector space, show that every quaternionic Hermitian form K has the form

$$K(v, w) = H(v, w) + jQ(v, w),$$

where H is a complex Hermitian form and Q is a skew-symmetric complex linear form on V, with H and Q related by $Q(v, w) = H(vj, w)$, and H satisfying the condition $H(vj, wj) = \overline{H(v, w)}$. Conversely, any such Hermitian H is the complex part of a unique K. If K is standard, so is H, and Q is given by the same matrix as in Exercise 7.3. Deduce that

$$\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{Sp}_{2n}\mathbb{C}.$$

This shows that the two notions of "symplectic" are compatible.

More generally, if K is not positive definite, but has signature (p, q), say the standard $\sum_{i=1}^{p} \bar{v}_i w_i - \sum_{i=p+1}^{p+q} \bar{v}_i w_i$, the automorphisms preserving it form a group $\mathrm{U}_{p,q}\mathbb{H}$. Or if the form is a skew Hermitian form (satisfying the same linearity conditions, but with $K(w, v) = -\overline{K(v, w)}$, the group is denoted $U_n^*\mathbb{H}$.

Exercise 7.5. Identify, among all the real Lie groups described above, which ones are compact.

Complex Lie Groups

So far, all of our examples have been examples of real Lie groups. As for *complex* Lie groups, these are fewer in number. The general linear group $GL_n\mathbb{C}$ is one, and again, all the elementary examples come to us as subgroups of the general linear group $GL_n\mathbb{C}$. There is, for example, the subgroup $SO_n\mathbb{C}$ of automorphisms of an n-dimensional complex vector space V having determinant 1 and preserving a nondegenerate symmetric bilinear form Q (note that Q no longer has a signature); and likewise the subgroup $Sp_n\mathbb{C}$ of transformations of determinant 1 preserving a skew-symmetric bilinear form.

Exercise 7.6. Show that the subgroup $SU(n) \subset SL_n\mathbb{C}$ is *not* a complex Lie subgroup. (It is not enough to observe that the defining equations given above are not holomorphic.)

Exercise 7.7. Show that none of the complex Lie groups described above is compact.

We should remark here that both of these exercises are immediate consequences of the general fact that *any compact complex Lie group is abelian*; we will prove this in the next lecture. A *representation* of a complex Lie group G is a map of complex Lie groups from G to $GL(V) = GL_n\mathbb{C}$ for an n-dimensional complex vector space V; note that such a map is required to be complex analytic.

Remarks on Notation

A common convention is to use a notation without subscripts or mention of ground field to denote the *real groups*:

$$O(n),\quad SO(n),\quad SO(p, q),\quad U(n),\quad SU(n),\quad SU(p, q),\quad Sp(n)$$

and to use subscripts for the algebraic groups GL_n, SL_n, SO_n, and Sp_n. This, of course, introduces some anomalies: for example, $SO_n\mathbb{R}$ is $SO(n)$, but $Sp_n\mathbb{R}$ is not $Sp(n)$; but some violation of symmetry seems inevitable in any notation. The notations $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ are often used in place of our $GL_n\mathbb{R}$ or $GL_n\mathbb{C}$, and similarly for SL, SO, and Sp.

Also, where we have written Sp_{2n}, some write Sp_n. In practice, it seems that those most interested in algebraic groups or Lie algebras use the former notation, and those interested in compact groups the latter. Other common notations are $U^*(2n)$ in place of our $GL_n\mathbb{H}$, $Sp(p, q)$ for our $U_{p,q}\mathbb{H}$, and $O^*(2n)$ for our $U_n^*\mathbb{H}$.

Exercise 7.8. Find the dimensions of the various real Lie groups $GL_n\mathbb{R}$, $SL_n\mathbb{R}$, B_n, N_n, $SO_n\mathbb{R}$, $SO_{k,l}\mathbb{R}$, $Sp_{2n}\mathbb{R}$, $U(n)$, $SU(n)$, $GL_n\mathbb{C}$, $SL_n\mathbb{C}$, $GL_n\mathbb{H}$, and $Sp(n)$ introduced above.

§7.3. Two Constructions

There are two constructions, in some sense inverse to one another, that arise frequently in dealing with Lie groups (and that also provide us with further examples of Lie groups). They are expressed in the following two statements.

Proposition 7.9. *Let G be a Lie group, H a connected manifold, and $\varphi: H \to G$ a covering space map.*[3] *Let e' be an element lying over the identity e of G. Then there is a unique Lie group structure on H such that e' is the identity and φ is a map of Lie groups; and the kernel of φ is in the center of H.*

Proposition 7.10. *Let H be a Lie group, and $\Gamma \subset Z(H)$ a discrete subgroup of its center. Then there is a unique Lie group structure on the quotient group $G = H/\Gamma$ such that the quotient map $H \to G$ is a Lie group map.*

The proof of the second proposition is straightforward. To prove the first, one shows that the multiplication on G lifts uniquely to a map $H \times H \to H$ which takes (e', e') to e', and verifies that this product satisfies the group axioms. In fact, it suffices to do this when H is the universal covering of G, for one can then apply the second proposition to intermediate coverings. □

Exercise 7.11*. (a) Show that any discrete normal subgroup of a connected Lie group G is in the center $Z(G)$.

(b) If $Z(G)$ is discrete, show that $G/Z(G)$ has trivial center.

These two propositions motivate a definition: we say that a Lie group map between two Lie groups G and H is an *isogeny* if it is a covering space map of the underlying manifolds; and we say two Lie groups G and H are *isogenous* if there is an isogeny between them (in either direction). Isogeny is not an equivalence relation, but generates one; observe that every isogeny equivalence class has an initial member (that is, one that maps to every other one by an isogeny)—that is, just the universal covering space $\tilde{G}$ of any one—and, if the center of this universal cover is discrete, as will be the case for all our semisimple groups, a final object $\tilde{G}/Z(\tilde{G})$ as well. For any group G in such an equivalence class, we will call $\tilde{G}$ the *simply connected form* of the group G, and $\tilde{G}/Z(\tilde{G})$ (if it exists) the *adjoint form* (we will see later a more general definition of adjoint form).

[3] This means that φ is a continuous map with the property that every point of G has a neighborhood U such that $\varphi^{-1}(U)$ is a disjoint union of open sets each mapping homeomorphically to U.

Exercise 7.12. If $H \to G$ is a covering of connected Lie groups, show that $Z(G)$ is discrete if and only if $Z(H)$ is discrete, and then $H/Z(H) = G/Z(G)$. Therefore, if $Z(G)$ is discrete, the adjoint form of G exists and is $G/Z(G)$.

To apply these ideas to some of the examples discussed, note that the center of SL_n (over $\mathbb{R}$ or $\mathbb{C}$) is just the subgroup of multiples of the identity by an nth root of unity; the quotient may be denoted $PSL_n\mathbb{R}$ or $PSL_n\mathbb{C}$. In the complex case, $PSL_n\mathbb{C}$ is isomorphic to the quotient of $GL_n\mathbb{C}$ by its center $\mathbb{C}^*$ of scalar matrices, and so one often writes $PGL_n\mathbb{C}$ instead of $PSL_n\mathbb{C}$. The center of the group SO_n is the subgroup $\{\pm I\}$ when n is even, and trivial when n is odd; in the former case the quotient will be denoted $PSO_n\mathbb{R}$ or $PSO_n\mathbb{C}$. Finally the center of the group Sp_{2n} is similarly the subgroup $\{\pm I\}$, and the quotient is denoted $PSp_{2n}\mathbb{R}$ or $PSp_{2n}\mathbb{C}$.

Exercise 7.13*. Realize $PGL_n\mathbb{C}$ as a matrix group, i.e., find an embedding (faithful representation) $PGL_n\mathbb{C} \hookrightarrow GL_N\mathbb{C}$ for some N. Do the same for the other quotients above.

In the other direction, whenever we have a Lie group that is not simply connected, we can ask what its universal covering space is. This is, for example, how the famous *spin groups* arise: as we will see, the orthogonal groups $SO_n\mathbb{R}$ and $SO_n\mathbb{C}$ have fundamental group $\mathbb{Z}/2$, and so by the above there exist connected, two-sheeted covers of these groups. These are denoted $Spin_n\mathbb{R}$ and $Spin_n\mathbb{C}$, and will be discussed in Lecture 20; for the time being, the reader may find it worthwhile (if frustrating) to try to realize these as matrix groups. The last exercises of this section sketch a few steps in this direction which can be done now by hand.

Exercise 7.14. Show that the universal covering of $U(n)$ can be identified with the subgroup of the product $U(n) \times \mathbb{R}$ consisting of pairs (g, t) with $\det(g) = e^{\pi i t}$.

Exercise 7.15. We have seen in Exercise 7.4 that

$$SU(2) = Sp(2) = \{q \in \mathbb{H}: q\bar{q} = 1\}.$$

Identifying $\mathbb{R}^3$ with the imaginary quaternions (with basis i, j, k), show that, for $q\bar{q} = 1$, the map $v \mapsto qv\bar{q}$ maps $\mathbb{R}^3$ to itself, and is an isometry. Verify that the resulting map

$$SU(2) = Sp(2) \to SO(3)$$

is a $2:1$ covering map. Since the equation $q\bar{q} = 1$ describes a 3-sphere, $SU(2)$ is the universal covering of $SO(3)$; and $SO(3)$ is the adjoint form of $SU(2)$.

Exercise 7.16. Let $M_2\mathbb{C} = \mathbb{C}^4$ be the space of 2×2 matrices, with symmetric form $Q(A, B) = \frac{1}{2}\,\mathrm{Trace}(AB^\dagger)$, where $B^\dagger$ is the adjoint of the matrix B; the quadratic form associated to Q is the determinant. For g and h in $SL_2\mathbb{C}$, the mapping $A \mapsto gAh^{-1}$ is in $SO_4\mathbb{C}$. Show that this gives a $2:1$ covering

$$SL_2\mathbb{C} \times SL_2\mathbb{C} \to SO_4\mathbb{C},$$

which, since $SL_2\mathbb{C}$ is simply connected, realizes the universal covering of $SO_4\mathbb{C}$.

Exercise 7.17. Identify $\mathbb{C}^3$ with the space of traceless matrices in $M_2\mathbb{C}$, so $g \in SL_2\mathbb{C}$ acts by $A \mapsto gAg^{-1}$. Show that this gives a $2:1$ covering

$$SL_2\mathbb{C} \to SO_3\mathbb{C},$$

which realizes the universal covering of $SO_3\mathbb{C}$.

LECTURE 8

Lie Algebras and Lie Groups

In this crucial lecture we introduce the definition of the Lie algebra associated to a Lie group and its relation to that group. All three sections are logically necessary for what follows; §8.1 is essential. We use here a little more manifold theory: specifically, the differential of a map of manifolds is used in a fundamental way in §8.1, the notion of the tangent vector to an arc in a manifold is used in §8.2 and §8.3, and the notion of a vector field is introduced in an auxiliary capacity in §8.3. The Campbell-Hausdorff formula is introduced only to establish the First and Second Principles of §8.1 below; if you are willing to take those on faith the formula (and exercises dealing with it) can be skimmed. Exercises 8.27–8.29 give alternative descriptions of the Lie algebra associated to a Lie group, but can be skipped for now.

§8.1: Lie algebras: motivation and definition
§8.2: Examples of Lie algebras
§8.3: The exponential map

§8.1. Lie Algebras: Motivation and Definition

Given that we want to study the representations of a Lie group, how do we go about it? As we have said, the notions of generators and relations is hardly relevant here. The answer, of course, is that we have to use the continuous structure of the group. The first step in doing this is

Exercise 8.1. Let G be a connected Lie group, and $U \subset G$ any neighborhood of the identity. Show that U generates G.

This statement implies that any map $\rho: G \to H$ between connected Lie groups will be determined by what it does on any open set containing the

identity in G, i.e., ρ is determined by its germ at $e \in G$. In fact, we can extend this idea a good bit further: later in this lecture we will establish the

First Principle: Let G and H be Lie groups, with G connected. A map $\rho: G \to H$ is uniquely determined by its differential $d\rho_e: T_e G \to T_e H$ at the identity.

This is, of course, great news: we can completely describe a homomorphism of Lie groups by giving a linear map between two vector spaces. It is not really worth that much, however, unless we can give at least some answer to the next, obvious question: *which maps between these two vector spaces actually arise as differentials of group homomorphisms?* The answer to this is expressed in the *Second Principle* below, but it will take us a few pages to get there. To start, we have to ask ourselves what it means for a map to be a homomorphism, and in what ways this may be reflected in the differential.

To begin with, the definition of a homomorphism is simply a $\mathscr{C}^\infty$ map ρ such that

$$\rho(gh) = \rho(g) \cdot \rho(h)$$

for all g and h in G. To express this in a more confusing way, we can say that *a homomorphism respects the action of a group on itself by left or right multiplication*: that is, for any $g \in G$ we denote by $m_g: G \to G$ the differentiable map given by multiplication by g, and observe that a $\mathscr{C}^\infty$ map $\rho: G \to H$ of Lie groups will be a homomorphism if it carries m_g to $m_{\rho(g)}$ in the sense that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & H \\ \downarrow{\scriptstyle m_g} & & \downarrow{\scriptstyle m_{\rho(g)}} \\ G & \xrightarrow[\rho]{} & H \end{array}$$

commutes.

The problem with this characterization is that, since the maps m_g have no fixed points, it is hard to associate to them any operation on the tangent space to G at one point. This suggests looking, not at the diffeomorphisms m_g, but at the automorphisms of G given by conjugation. Explicitly, for any $g \in G$ we define the map

$$\Psi_g: G \to G$$

by

$$\Psi_g(h) = g \cdot h \cdot g^{-1}.$$

(Ψ_g is actually a Lie group map, but that is not relevant for our present purposes.) It is now equally the case that *a homomorphism ρ respects the action of a group G on itself by conjugation*: that is, it will carry Ψ_g into $\Psi_{\rho(g)}$ in the sense that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & H \\ \Psi_g \downarrow & & \downarrow \Psi_{\rho(g)} \\ G & \xrightarrow[\rho]{} & H \end{array}$$

commutes. We have, in other words, a natural map

$$\Psi: G \to \operatorname{Aut}(G).$$

The advantage of working with Ψ_g is that it fixes the identity element $e \in G$; we can therefore extract some of its structure by looking at its differential at e: we set

$$\operatorname{Ad}(g) = (d\Psi_g)_e: T_eG \to T_eG. \tag{8.2}$$

This is a representation

$$\operatorname{Ad}: G \to \operatorname{Aut}(T_eG) \tag{8.3}$$

of the group G on its own tangent space, called the *adjoint representation* of the group. This gives a third characterization[1]: *a homomorphism ρ respects the adjoint action of a group G on its tangent space T_eG at the identity*. In other words, for any $g \in G$ the actions of $\operatorname{Ad}(g)$ on T_eG and $\operatorname{Ad}(\rho(g))$ on T_eH must commute with the differential $(d\rho)_e: T_eG \to T_eH$, i.e., the diagram

$$\begin{array}{ccc} T_eG & \xrightarrow{(d\rho)_e} & T_eH \\ \operatorname{Ad}(g) \downarrow & & \downarrow \operatorname{Ad}(\rho(g)) \\ T_eG & \xrightarrow[(d\rho)_e]{} & T_eH \end{array}$$

commutes; equivalently, for any tangent vector $v \in T_eG$,

$$d\rho(\operatorname{Ad}(g)(v)) = \operatorname{Ad}(\rho(g))(d\rho(v)). \tag{8.4}$$

This is nice, but does not yet answer our question, for preservation of the adjoint representation $\operatorname{Ad}: G \to \operatorname{Aut}(T_eG)$ still involves the map ρ on the group G itself, and so is not purely a condition on the differential $(d\rho)_e$. We have instead to go one step further, and *take the differential of the map* Ad. The group $\operatorname{Aut}(T_eG)$ being just an open subset of the vector space of endomorphisms of T_eG, its tangent space at the identity is naturally identified with $\operatorname{End}(T_eG)$; taking the differential of the map Ad we arrive at a map

$$\operatorname{ad}: T_eG \to \operatorname{End}(T_eG). \tag{8.5}$$

This is essentially a trilinear gadget on the tangent space T_eG; that is, we can view the image $\operatorname{ad}(X)(Y)$ of a tangent vector Y under the map $\operatorname{ad}(X)$ as a

[1] "Characterization" is not the right word here (or in the preceding case), since we do not mean an equivalent condition, but rather something implied by the condition that ρ be a homomorphism.

function of the two variables X and Y, so that we get a bilinear map

$$T_eG \times T_eG \to T_eG.$$

We use the notation [,] for this bilinear map; that is, for a pair of tangent vectors X and Y to G at e, we write

$$[X, Y] \stackrel{\text{def}}{=} \operatorname{ad}(X)(Y). \tag{8.6}$$

As desired, the map ad involves only the tangent space to the group G at e, and so gives us our final characterization: *the differential $(d\rho)_e$ of a homomorphism ρ on a Lie group G respects the adjoint action of the tangent space to G on itself*. Explicitly, the fact that ρ and $d\rho_e$ respect the adjoint representation implies in turn that the diagram

$$\begin{array}{ccc} T_eG & \xrightarrow{(d\rho)_e} & T_eH \\ \operatorname{ad}(v) \downarrow & & \downarrow \operatorname{ad}(d\rho(v)) \\ T_eG & \xrightarrow[(d\rho)_e]{} & T_eH \end{array}$$

commutes; i.e., for any pair of tangent vectors X and Y to G at e,

$$d\rho_e(\operatorname{ad}(X)(Y)) = \operatorname{ad}(d\rho_e(X))(d\rho_e(Y)). \tag{8.7}$$

or, equivalently,

$$d\rho_e([X, Y]) = [d\rho_e(X), d\rho_e(Y)]. \tag{8.8}$$

All this may be fairly confusing (if it is not, you probably do not need to be reading this book). Two things, however, should be borne in mind. They are:

(i) *It is not so bad*, in the sense that we can make the bracket operation, as defined above, reasonably explicit. We do this first for the general linear group $G = GL_n\mathbb{R}$. Note that in this case conjugation extends to the ambient linear space $E = \operatorname{End}(\mathbb{R}^n) = M_n\mathbb{R}$ of $GL_n\mathbb{R}$ by the same formula: $\operatorname{Ad}(g)(M) = gMg^{-1}$, and this ambient space is identified with the tangent space T_eG; differentiation in E is usual differentiation of matrices. For any pair of tangent vectors X and Y to $GL_n\mathbb{R}$ at e, let $\gamma: I \to G$ be an arc with $\gamma(0) = e$ and tangent vector $\gamma'(0) = X$. Then our definition of $[X, Y]$ is that

$$[X, Y] = \operatorname{ad}(X)(Y) = \left.\frac{d}{dt}\right|_{t=0} (\operatorname{Ad}(\gamma(t))(Y)).$$

Applying the product rule to $\operatorname{Ad}(\gamma(t))(Y) = \gamma(t)Y\gamma(t)^{-1}$, this is

$$\begin{aligned} &= \gamma'(0)\cdot Y\cdot\gamma(0) + \gamma(0)\cdot Y\cdot(-\gamma(0)^{-1}\cdot\gamma'(0)\cdot\gamma(0)^{-1}) \\ &= X\cdot Y - Y\cdot X, \end{aligned}$$

which, of course, explains the bracket notation. In general, any time a Lie group is given as a subgroup of a general linear group $GL_n\mathbb{R}$, we can view its tangent space T_eG at the identity as a subspace of the space of endomorphisms of $\mathbb{R}^n$; and since bracket is preserved by (differentials of) maps of Lie groups, the bracket operation on T_eG will coincide with the commutator.

(ii) *Even if it were that bad, it would be worth it.* This is because it turns out that the bracket operation is exactly the answer to the question we raised before. Precisely, later in this lecture we will prove the

Second Principle: Let G and H be Lie groups, with G connected and simply connected. A linear map $T_eG \to T_eH$ is the differential of a homomorphism $\rho: G \to H$ if and only if it preserves the bracket operation, in the sense of (8.8) above.

We are now almost done: maps between Lie groups are classified by maps between vector spaces preserving the structure of a bilinear map from the vector space to itself. We have only one more question to answer: when does a vector space with this additional structure actually arise as the tangent space at the identity to a Lie group, with the adjoint or bracket product? Happily, we have the answer to this as well. First, though it is far from clear from our initial definition, it follows from our description of the bracket as a commutator that *the bracket is skew-symmetric*, i.e, $[X, Y] = -[Y, X]$. Second, it likewise follows from the description of $[X, Y]$ as a commutator that it satisfies the *Jacobi identity*: for any three tangent vectors X, Y, and Z,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

We thus make the

Definition 8.9. A *Lie algebra* $\mathfrak{g}$ is a vector space together with a skew-symmetric bilinear map

$$[\ ,\ ]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity.

We should take a moment out here to make one important point. Why, you might ask, do we define the bracket operation in terms of the relatively difficult operations Ad and ad, instead of just defining $[X, Y]$ to be the commutator $X \cdot Y - Y \cdot X$? The answer is that the "composition" $X \cdot Y$ of elements of a Lie algebra is not well defined. Specifically, any time we embed a Lie group G in a general linear group $GL(V)$, we get a corresponding embedding of its Lie algebra $\mathfrak{g}$ in the space $\text{End}(V)$, and can talk about the composition $X \cdot Y \in \text{End}(V)$ of elements of $\mathfrak{g}$ in this context; but it must be borne in mind that this composition $X \cdot Y$ will depend on the embedding of $\mathfrak{g}$, and for that matter need not even be an element of $\mathfrak{g}$. *Only* the commutator $X \cdot Y - Y \cdot X$ is always an element of $\mathfrak{g}$, independent of the representation. The terminology sometimes heightens the confusion: for example, when we speak of embedding a Lie algebra in the algebra $\text{End}(V)$ of endomorphisms of V, the word *algebra* may mean two very different things. In general, when we want

to refer to the endomorphisms of a vector space V (resp. $\mathbb{R}^n$) as a Lie algebra, we will write $\mathfrak{gl}(V)$ (resp. $\mathfrak{gl}_n\mathbb{R}$) instead of $\text{End}(V)$ (resp. $M_n\mathbb{R}$).

To return to our discussion of Lie algebras, a *map* of Lie algebras is a linear map of vector spaces preserving the bracket, in the sense of (8.8); notions like *Lie subalgebra* are defined accordingly. We note in passing one thing that will turn out to be significant: the definition of Lie algebra does not specify the field. Thus, we have real Lie algebras, complex Lie algebras, etc., all defined in the same way; and in addition, given a real Lie algebra $\mathfrak{g}$ we may associate to it a complex Lie algebra, whose underlying vector space is $\mathfrak{g} \otimes \mathbb{C}$ and whose bracket operation is just the bracket on $\mathfrak{g}$ extended by linearity.

Exercise 8.10*. The skew-commutativity and Jacobi identity also follow from the naturality of the bracket (8.8), without using an embedding in $\mathfrak{gl}(V)$:

(a) Deduce the skew-commutativity $[X, X] = 0$ from that fact that any X can be written the image of a vector by $d\rho_e$ for some homomorphism $\rho: \mathbb{R} \to G$. (See §8.3 for the existence of ρ.)

(b) Given that the bracket is skew-commutative, verify that the Jacobi identity is equivalent to the assertion that

$$\text{ad} = d(\text{Ad})_e: \mathfrak{g} \to \text{End}(\mathfrak{g})$$

preserves the bracket. In particular, ad is a map of Lie algebras.

To sum up our progress so far: taking for the moment on faith the statements made, we have seen that

(i) the tangent space $\mathfrak{g}$ at the identity to a Lie group G is naturally endowed with the structure of a Lie algebra;

(ii) if G and H are Lie groups with G connected and simply connected, the maps from G to H are in one-to-one correspondence with maps of the associated Lie algebras, by associating to $\rho: G \to H$ its differential $(d\rho)_e: \mathfrak{g} \to \mathfrak{h}$.

Of course, we make the

Definition 8.11. A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V is simply a map of Lie algrbras

$$\rho: \mathfrak{g} \to \mathfrak{gl}(V) = \text{End}(V),$$

i.e., a linear map that preserves brackets, or an action of $\mathfrak{g}$ on V such that

$$[X, Y](v) = X(Y(v)) - Y(X(v)).$$

Statement (ii) above implies in particular that *representations of a connected and simply connected Lie group are in one-to-one correspondence with repre-*

sentations of its Lie algebra. This is, then, the first step of the series of reductions outlined in the introduction to Part II.

At this point, a few words are in order about the relation between representations of a Lie group and the corresponding representations of its Lie algebra. The first remark to make is about tensors. Recall that if V and W are representations of a Lie group G, then we define the representation $V \otimes W$ to be the vector space $V \otimes W$ with the action of G described by

$$g(v \otimes w) = g(v) \otimes g(w).$$

The definition for representations of a Lie algebra, however, is quite different. For one thing, if $\mathfrak{g}$ is the Lie algebra of G, so that the representation of G on the vector spaces V and W induces representations of $\mathfrak{g}$ on these spaces, we want the tensor product of the representations V and W of $\mathfrak{g}$ to be the representation induced by the action of G on $V \otimes W$ above. But now suppose that $\{\gamma_t\}$ is an arc in G with $\gamma_0 = e$ and tangent vector $\gamma'_0 = X \in \mathfrak{g}$. Then by definition the action of X on V is given by

$$X(v) = \frac{d}{dt}\bigg|_{t=0} \gamma_t(v)$$

and similarly for $w \in W$; it follows that the action of X on the tensor product $v \otimes w$ is

$$\begin{aligned} X(v \otimes w) &= \frac{d}{dt}\bigg|_{t=0} (\gamma_t(v) \otimes \gamma_t(w)) \\ &= \left(\frac{d}{dt}\bigg|_{t=0} \gamma_t(v)\right) \otimes w + v \otimes \left(\frac{d}{dt}\bigg|_{t=0} \gamma_t(w)\right), \end{aligned}$$

so

$$X(v \otimes w) = X(v) \otimes w + v \otimes X(w). \tag{8.12}$$

This, then, is how we *define* the action of a Lie algebra $\mathfrak{g}$ on the tensor product of two representations of $\mathfrak{g}$. This describes as well other tensors: for example, if V is a representation of the group G, $v \in V$ is any vector and $v^2 \in \mathrm{Sym}^2 V$ its square, then for any $g \in G$,

$$g(v^2) = g(v)^2.$$

On the other hand, if V is a representation of the Lie algebra $\mathfrak{g}$ and $X \in \mathfrak{g}$ is any element, we have

$$X(v^2) = 2 \cdot v \cdot X(v). \tag{8.13}$$

One further example: if $\rho\colon G \to \mathrm{GL}(V)$ is a representation of the group G, the dual representation $\rho'\colon G \to \mathrm{GL}(V^*)$ is defined by setting

$$\rho'(g) = {}^t\rho(g^{-1})\colon V^* \to V^*.$$

Differentiating this, we find that if $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of a Lie algebra $\mathfrak{g}$, the dual representation of $\mathfrak{g}$ on V^* will be given by

$$\rho'(X) = {}^t\rho(-X) = -{}^t\rho(X)\colon V^* \to V^*. \tag{8.14}$$

A second and related point to be made concerns terminology. Obviously, when we speak of the action of a group G on a vector space V preserving some extra structure on V, we mean that literally: for example, if we have a quadratic form Q on V, to say that G preserves Q means just that

$$Q(g(v), g(w)) = Q(v, w), \quad \forall g \in G \text{ and } v, w \in V.$$

Equivalently, we mean that the associated action of G on the vector space $\mathrm{Sym}^2 V^*$ fixes the element $Q \in \mathrm{Sym}^2 V^*$. But by the above calculation, the action of the associated Lie algebra $\mathfrak{g}$ on V satisfies

$$Q(v, X(w)) + Q(X(v), w) = 0, \quad \forall X \in \mathfrak{g} \text{ and } v, w \in V \tag{8.15}$$

or, equivalently, $Q(v, X(v)) = 0$ for all $X \in \mathfrak{g}$ and $v \in V$; in other words, *the induced action on* $\mathrm{Sym}^2 V^*$ *kills the element* Q. By way of terminology, then, *we will in general say that the action of a Lie algebra on a vector space preserves some structure when a corresponding Lie group action does.*

The next section will be spent in giving examples. In §8.3 we will establish the basic relations between Lie groups and their Lie algebras, to the point where we can prove the First and Second Principles above. The further statement that any Lie algebra is the Lie algebra of some Lie group will follow from the statement (see Appendix E) that every Lie algebra may be embedded in $\mathfrak{gl}_n\mathbb{R}$.

Exercise 8.16*. Show that if G is connected the image of $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint form of the group G when that exists.

Exercise 8.17*. Let V be a representation of a connected Lie group G and $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ the corresponding map of Lie algebras. Show that a subspace W of V is invariant by G if and only if it is carried into itself under the action of the Lie algebra $\mathfrak{g}$, i.e., $\rho(X)(W) \subset W$ for all X in $\mathfrak{g}$. Hence, V is irreducible over G if and only if it is irreducible over $\mathfrak{g}$.

§8.2. Examples of Lie Algebras

We start with the Lie algebras associated to each of the groups mentioned in Lecture 7. Each of these groups is given as a subgroup of $\mathrm{GL}(V) = \mathrm{GL}_n\mathbb{R}$, so their Lie algebras will be subspaces of $\mathrm{End}(V) = \mathfrak{gl}_n\mathbb{R}$.

Consider first the special linear group $\mathrm{SL}_n\mathbb{R}$. If $\{A_t\}$ is an arc in $\mathrm{SL}_n\mathbb{R}$ with $A_0 = I$ and tangent vector $A'_0 = X$ at $t = 0$, then by definition we have for any basis $e_1, \ldots, e_n$ of $V = \mathbb{R}^n$,

$$A_t(e_1) \wedge \cdots \wedge A_t(e_n) \equiv e_1 \wedge \cdots \wedge e_n.$$

Taking the derivative and evaluating at $t = 0$ we have by the product rule

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} (A_t(e_1) \wedge \cdots \wedge A_t(e_n)) \\ &= \sum e_1 \wedge \cdots \wedge X(e_i) \wedge \cdots \wedge e_n \\ &= \mathrm{Trace}(X)\cdot(e_1 \wedge \cdots \wedge e_n). \end{aligned}$$

The tangent vectors to $\mathrm{SL}_n\mathbb{R}$ are thus all endomorphisms of trace 0; comparing dimensions we can see that the Lie algebra $\mathfrak{sl}_n\mathbb{R}$ is exactly the vector space of traceless $n \times n$ matrices.

The orthogonal and symplectic cases are somewhat simpler. For example, the orthogonal group $\mathrm{O}_n\mathbb{R}$ is defined to be the automorphisms A of an n-dimensional vector space V preserving a quadratic form Q, so that if $\{A_t\}$ is an arc in $\mathrm{O}_n\mathbb{R}$ with $A_0 = I$ and $A_0' = X$ we have for every pair of vectors v, $w \in V$

$$Q(A_t(v), A_t(w)) \equiv Q(v, w).$$

Taking derivatives, we see that

$$Q(X(v), w) + Q(v, X(w)) = 0 \tag{8.18}$$

for all $v, w \in V$; this is exactly the condition that describes the orthogonal Lie algebra $\mathfrak{so}_n\mathbb{R} = \mathfrak{o}_n\mathbb{R}$. In coordinates, if the quadratic form Q is given on $V = \mathbb{R}^n$ as

$$Q(v, w) = {}^t v \cdot M \cdot w \tag{8.19}$$

for some symmetric $n \times n$ matrix M, then as we have seen the condition on $A \in \mathrm{GL}_n\mathbb{R}$ to be in $\mathrm{O}_n\mathbb{R}$ is that

$${}^tA \cdot M \cdot A = M. \tag{8.20}$$

Differentiating, the condition on an $n \times n$ matrix X to be in the Lie algebra $\mathfrak{so}_n\mathbb{R}$ of the orthogonal group is that

$${}^tX \cdot M + M \cdot X = 0. \tag{8.21}$$

Note that if M is the identity matrix—i.e., Q is the "standard" quadratic form $Q(v, w) = {}^tv \cdot w$ on $\mathbb{R}^n$—then this says that *$\mathfrak{so}_n\mathbb{R}$ is the subspace of skew-symmetric $n \times n$ matrices*. To put it intrinsically, in terms of the identification of V with V^* given by the quadratic form Q, and the consequent identification $\mathrm{End}(V) = V \otimes V^* = V \otimes V$, the Lie algebra $\mathfrak{so}_n\mathbb{R} \subset \mathrm{End}(V)$ is just the subspace $\wedge^2 V \subset V \otimes V$ of skew-symmetric tensors:

$$\mathfrak{so}_n\mathbb{R} = \wedge^2 V \subset \mathrm{End}(V) = V \otimes V. \tag{8.22}$$

All of the above, with the exception of the last paragraph, works equally well to describe the Lie algebra $\mathfrak{sp}_{2n}\mathbb{R}$ of the Lie group $\mathrm{Sp}_{2n}\mathbb{R}$ of transformations preserving a skew-symmetric bilinear form Q; that is, $\mathfrak{sp}_{2n}\mathbb{R}$ is the subspace of endomorphisms of V satisfying (8.18) for every pair of vectors v, $w \in V$, or, if Q is given by a skew-symmetric $2n \times 2n$ matrix M as in (8.19), the space of matrices satisfying (8.21). The one statement that has to be substantially modified is the last one of the last paragraph: because Q is skew-symmetric, condition (8.18) is equivalent to saying that

$$Q(X(v), w) = Q(X(w), v)$$

for all $v, w \in V$; thus, in terms of the identification of V with V^* given by Q, the Lie algebra $\mathfrak{sp}_{2n}\mathbb{R} \subset \mathrm{End}(V) = V \otimes V^* = V \otimes V$ is the subspace $\mathrm{Sym}^2 V \subset V \otimes V$:

$$\mathfrak{sp}_{2n}\mathbb{R} = \mathrm{Sym}^2 V \subset \mathrm{End}(V) = V \otimes V. \tag{8.23}$$

Exercise 8.24*. With Q a standard skew form, say of Exercise 7.3, describe $\mathrm{Sp}_{2n}\mathbb{R}$ and its Lie algebra $\mathfrak{sp}_{2n}\mathbb{R}$ (as subgroup of $\mathrm{GL}_{2n}\mathbb{R}$ and subalgebra of $\mathfrak{gl}_{2n}\mathbb{R}$). Do a corresponding calculation for $\mathrm{SO}_{k,l}\mathbb{R}$.

One more similar example is that of the Lie algebra $\mathfrak{u}_n$ of the unitary group $\mathrm{U}(n)$; by a similar calculation we find that the Lie algebra of complex linear endomorphisms of $\mathbb{C}^n$ preserving a Hermitian inner product H is just the space of matrices X satisfying

$$H(X(v), w) + H(v, X(w)) = 0, \quad \forall v, w \in V;$$

if H is given by $H(v, w) = {}^t\bar{v} \cdot w$, this amounts to saying that X is conjugate skew-symmetric, i.e., that ${}^t\bar{X} = -X$.

Exercise 8.25. Find the Lie algebras of the real Lie groups $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{SL}_n\mathbb{H}$—the elements in $\mathrm{GL}_n\mathbb{H}$ whose real determinant is 1.

Exercise 8.26. Show that the Lie algebras of the Lie groups B_n and N_n introduced in §7.2 are the algebra $\mathfrak{b}_n\mathbb{R}$ of upper triangular $n \times n$ matrices and the algebra $\mathfrak{n}_n\mathbb{R}$ of strictly upper triangular $n \times n$ matrices, respectively.

If G is a complex Lie group, its Lie algebra is a complex Lie algebra. Just as in the real case, we have the complex Lie algebras $\mathfrak{gl}_n\mathbb{C}$, $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_m\mathbb{C}$, and $\mathfrak{sp}_{2n}\mathbb{C}$ of the Lie groups $\mathrm{GL}_n\mathbb{C}$, $\mathrm{SL}_n\mathbb{C}$, $\mathrm{SO}_m\mathbb{C}$, and $\mathrm{Sp}_{2n}\mathbb{C}$.

Exercise 8.27. Let A be any (real or complex) algebra, not necessarily finite dimensional, or even associative. A *derivation* is a linear map $D: A \to A$ satisfying the Leibnitz rule $D(ab) = aD(b) + D(a)b$.

(a) Show that the derivations $\mathrm{Der}(A)$ form a Lie algebra under the bracket $[D, E] = D \circ E - E \circ D$. If A is finite dimensional, so is $\mathrm{Der}(A)$.
(b) The group of automorphisms of A is a closed subgroup G of the group $\mathrm{GL}(A)$ of linear automorphisms of A. Show that the Lie algebra of G is $\mathrm{Der}(A)$.
(c) If the algebra A is a Lie algebra, the map $A \to \mathrm{Der}(A)$, $X \mapsto D_X$, where $D_X(Y) = [X, Y]$, is a map of Lie algebras.

Exercise 8.28*. If $\mathfrak{g}$ is a Lie algebra, the Lie algebra automorphisms of $\mathfrak{g}$ form a Lie subgroup $\mathrm{Aut}(\mathfrak{g})$ of the general linear group $\mathrm{GL}(\mathfrak{g})$.

(a) Show that the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is $\mathrm{Der}(\mathfrak{g})$. If G is a simply connected Lie group with Lie algebra $\mathfrak{g}$, the map $\mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ by $\varphi \mapsto d\varphi$ is one-to-one and onto, giving $\mathrm{Aut}(G)$ the structure of a Lie group with Lie algebra $\mathrm{Der}(\mathfrak{g})$.

(b) Show that the automorphism group of any connected Lie group is a Lie subgroup of the automorphism group of its Lie algebra.

Exercise 8.29*. For any manifold M, the C^∞ vector fields on M form a Lie algebra $\mathfrak{v}(M)$, as follows: a vector field v can be identified with a derivation of the ring A of C^∞ functions on M, with $v(f)$ the function whose value at a point x of M is the value of the tangent vector v_x on f at x. Show that the vector fields on M form a Lie algebra, in fact a Lie subalgebra of the Lie algebra $\mathrm{Der}(A)$. If a Lie group G acts on M, the G-invariant vector fields form a Lie subalgebra $\mathfrak{v}_G M$ of $\mathfrak{v}(M)$. If the action is transitive, the invariant vector fields form a finite-dimensional Lie algebra.

If G is a Lie group, $\mathfrak{v}_G(G) = T_eG$ becomes a Lie algebra by the above process. Show that this bracket agrees with that defined using the adjoint map (8.6). This gives another proof that the bracket is skew-symmetric and satisfies Jacobi's identity.

§8.3. The Exponential Map

The essential ingredient in studying the relationship between a Lie group G and its Lie algebra $\mathfrak{g}$ is the exponential map. This may be defined in very straightforward fashion, using the notion of *one-parameter subgroups*, which we study next. Suppose that $X \in \mathfrak{g}$ is any element, viewed simply as a tangent vector to G at the identity. For any element $g \in G$, denote by $m_g: G \to G$ the map of manifolds given by multiplication on the left by g. Then we can define a vector field v_X on all of G simply by setting

$$v_X(g) = (m_g)_*(X).$$

This vector field is clearly invariant under left translation (i.e., it is carried into itself under the diffeomorphism m_g for all g); and it is not hard to see that this gives an identification of $\mathfrak{g}$ with the space of all left-invariant vector fields on G. Under this identification, the bracket operation on the Lie algebra $\mathfrak{g}$ corresponds to Lie bracket of vector fields; indeed, this may be adopted as the definition of the Lie algebra associated to a Lie group (cf. Exercise 8.29). For our present purposes, however, all we need to know is that v_X exists and is left-invariant.

Given any vector field v on a manifold M and a point $p \in M$, a basic theorem from differential equations allows us to integrate the vector field. This gives a differentiable map $\varphi: I \to M$, defined on some open interval I containing 0, with $\varphi(0) = p$, whose tangent vector at any point is the vector assigned to that point by v, i.e., such that

$$\varphi'(t) = v(\varphi(t))$$

for all t in I. The map φ is uniquely characterized by these properties. Now suppose the manifold in question is a Lie group G, the vector field the field v_X associated to an element $X \in \mathfrak{g}$, and p the identity. We arrive then at a map $\varphi: I \to G$; we claim that, at least where φ is defined, it is a *homomorphism*, i.e., $\varphi(s + t) = \varphi(s)\varphi(t)$ whenever s, t, and $s + t$ are in I. To prove this, fix s and let t vary; that is, consider the two arcs α and β given by $\alpha(t) = \varphi(s)\cdot\varphi(t)$ and $\beta(t) = \varphi(s + t)$. Of course, $\alpha(0) = \beta(0)$; and by the invariance of the vector field v_X, we see that the tangent vectors satisfy $\alpha'(t) = v_X(\alpha(t))$ and $\beta'(t) = v_X(\beta(t))$ for all t. By the uniqueness of the integral curve of a vector field on a manifold, we deduce that $\alpha(t) = \beta(t)$ for all t.

From the fact that $\varphi(s + t) = \varphi(s)\varphi(t)$ for all s and t near 0, it follows that φ extends uniquely to all of $\mathbb{R}$, defining a homomorphism

$$\varphi_X: \mathbb{R} \to G$$

with $\varphi_X'(t) = v_X(\varphi)(t) = (m_{\varphi(t)})_*(X)$ for all t.

Exercise 8.30. Establish the *product rule* for derivatives of arcs in a Lie group G: if α and β are arcs in G and $\gamma(t) = \alpha(t)\cdot\beta(t)$, then

$$\gamma'(t) = dm_{\alpha(t)}(\beta'(t)) + dn_{\beta(t)}(\alpha'(t)),$$

where for any $g \in G$, the map m_g (resp. n_g): $G \to G$ is given by left (resp. right) multiplication by g. Use this to give another proof that φ is a homomorphism.

Exercise 8.31. Show that φ_X is uniquely determined by the fact that it is a homomorphism of $\mathbb{R}$ to G with tangent vector $\varphi_X'(0)$ at the identity equal to X. Deduce that if $\psi: G \to H$ is a map of Lie groups, then $\varphi_{\psi_* X} = \psi \circ \varphi_X$.

The Lie group map $\varphi_X: \mathbb{R} \to G$ is called the *one-parameter subgroup of G with tangent vector X at the identity*. The construction of these one-parameter subgroups for each X amounts to the verification of the Second Principle of §8.1 for homomorphisms from $\mathbb{R}$ to G. The fact that there exists such a one-parameter subgroup of G with any given tangent vector at the identity is crucial. For example, it is not hard to see (we will do this in a moment) that these one-parameter subgroups fill up a neighborhood of the identity in G, which immediately implies the First Principle of §8.1. To carry this out, we define the *exponential map*

$$\exp: \mathfrak{g} \to G$$

by

$$\exp(X) = \varphi_X(1). \tag{8.32}$$

Note that by the uniqueness of φ_X, we have

$$\varphi_{(\lambda X)}(t) = \varphi_X(\lambda t);$$

so that the exponential map restricted to the lines through the origin in $\mathfrak{g}$ gives the one-parameter subgroups of G. Indeed, Exercise 8.31 implies the characterization:

Proposition 8.33. *The exponential map is the unique map from $\mathfrak{g}$ to G taking 0 to e whose differential at the origin*

$$(\exp_*)_0 \colon T_0\mathfrak{g} = \mathfrak{g} \to T_eG = \mathfrak{g}$$

is the identity, and whose restrictions to the lines through the origin in $\mathfrak{g}$ are one-parameter subgroups of G.

This in particular implies (cf. Exercise 8.31) that the exponential map is natural, in the sense that for any map $\psi \colon G \to H$ of Lie groups the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\psi_*} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\psi]{} & H \end{array}$$

commutes.

Now, since the differential of the exponential map at the origin in $\mathfrak{g}$ is an isomorphism, the image of exp will contain a neighborhood of the identity in G. If G is connected, this will generate all of G; from this follows the First Principle: *if G is connected, then the map ψ is determined by its differential $(d\psi)_e$ at the identity.*

Using (8.32), we can write down the exponential map very explicitly in the case of $\mathrm{GL}_n\mathbb{R}$, and hence for any subgroup of $\mathrm{GL}_n\mathbb{R}$. We just use the standard power series for the function e^x, and set, for $X \in \mathrm{End}(V)$,

$$\exp(X) = 1 + X + \frac{X^2}{2} + \frac{X^3}{6} + \cdots. \tag{8.34}$$

Observe that this converges and is invertible, with inverse $\exp(-X)$. Clearly, the differential of this map from $\mathfrak{g}$ to G at the origin is the identity; and by the standard power series computation, the restriction of the map to any line through the origin in $\mathfrak{g}$ is a one-parameter subgroup of G. Thus, the map coincides with the exponential as defined originally; and by naturality the same is true for any subgroup of G. (Note that, as we have pointed out, the individual terms in the expression on the right of (8.34) are very much dependent of the particular embedding of G in a general linear group $\mathrm{GL}(V)$ and correspondingly of $\mathfrak{g}$ in $\mathrm{End}(V)$, even though the *sum* on the right in (8.34) is not.)

This explicit form of the exponential map allows us to give substance to

the assertion that "the group structure of G is encoded in the Lie algebra." Explicitly, we claim that not only do the exponentials $\exp(X)$ generate G, but for X and Y in a sufficiently small neighborhood of the origin in $\mathfrak{g}$, we can write down the product $\exp(X)\cdot\exp(Y)$ as an exponential. To do this, we introduce first the "inverse" of the exponential map: for $g \in G \subset \mathrm{GL}_n\mathbb{R}$, we set

$$\log(g) = (g - I) - \frac{(g - I)^2}{2} + \frac{(g - I)^3}{3} - \cdots \in \mathfrak{gl}_n\mathbb{R}.$$

Of course, this will be defined only for g sufficiently close to the identity in G; but where it is defined it will be an inverse to the exponential map.

Now, we define a new bilinear operation on $\mathfrak{gl}_n\mathbb{R}$: we set

$$X * Y = \log(\exp(X)\cdot\exp(Y)).$$

We have to be careful what we mean by this, of course; we substitute for g in the expression above for $\log(g)$ the quantity

$$\begin{aligned} \exp(X)\cdot\exp(Y) &= \left(I + X + \frac{X^2}{2} + \cdots\right)\cdot\left(I + Y + \frac{Y^2}{2} + \cdots\right) \\ &= I + (X + Y) + \left(\frac{X^2}{2} + X\cdot Y + \frac{Y^2}{2}\right) + \cdots, \end{aligned}$$

being careful, of course, to preserve the order of the factors in each product. Doing this, we arrive at

$$\begin{aligned} X * Y &= (X + Y) + \left(-\frac{(X + Y)^2}{2} + \left(\frac{X^2}{2} + X\cdot Y + \frac{Y^2}{2}\right)\right) + \cdots \\ &= X + Y + \tfrac{1}{2}[X, Y] + \cdots. \end{aligned}$$

Observe in particular that the terms of degree 2 in X and Y do not involve the squares of X and Y or the product $X\cdot Y$ alone, but only the commutator. In fact, this is true of each term in the formula, i.e., the quantity $\log(\exp(X)\cdot\exp(Y))$ can be expressed purely in terms of X, Y, and the bracket operation; the resulting formula is called the *Campbell–Hausdorff formula* (although the actual formula in closed form was given by Dynkin). To degree three, it is

$$X * Y = X + Y + \tfrac{1}{2}[X, Y] \pm \tfrac{1}{12}[X, [X, Y]] \pm \tfrac{1}{12}[Y, [Y, X]] + \cdots.$$

Exercise 8.35*. Verify (and find the correct signs in) the cubic term of the Campbell–Hausdorff formula.

Exercise 8.36. Prove the assertion of the last paragraph that the power series $\log(\exp(X)\cdot\exp(Y))$ can be expressed purely in terms of X, Y, and the bracket operation.

Exercise 8.37. Show that for X and Y sufficiently small, the power series $\log(\exp(X)\cdot\exp(Y))$ converges.

Exercise 8.38*. (a) Show that there is a constant C such that for $X, Y \in \mathfrak{gl}_n$, $X * Y = X + Y + [X, Y] + E$, where $\|E\| \leq C(\|X\| + \|Y\|)^3$.

(b) Show that $\exp(X + Y) = \lim_{n\to\infty}(\exp(X/n)\cdot\exp(Y/n))^n$.

(c) Show that

$$\exp([X, Y]) = \lim_{n\to\infty}\left(\exp\left(\frac{X}{n}\right)\cdot\exp\left(\frac{Y}{n}\right)\cdot\exp\left(-\frac{X}{n}\right)\cdot\exp\left(-\frac{Y}{n}\right)\right)^{n^2}.$$

Exercise 8.39. Show that if G is a subgroup of $\mathrm{GL}_n\mathbb{R}$, the elements of its Lie algebra are the "infinitesimal transformations" of G in the sense of von Neumann, i.e., they are the matrices in $\mathfrak{gl}_n\mathbb{R}$ which can be realized as limits

$$\lim_{t\to 0}\frac{A_t - I}{\varepsilon_t}, \quad A_t \in G, \varepsilon_t > 0, \varepsilon_t \to 0.$$

Exercise 8.40. Show that exp is surjective for $G = \mathrm{GL}_n\mathbb{C}$ but not for $G = \mathrm{GL}_n^+\mathbb{R}$ if $n > 1$, or for $G = \mathrm{SL}_2\mathbb{C}$.

By the Campbell–Hausdorff formula, we can not only identify all the elements of G in a neighborhood of the identity, but we can also say what their pairwise products are, thus making precise the sense in which $\mathfrak{g}$ and its bracket operation determines G and its group law locally. Of course, we have not written a closed-form expression for the Campbell–Hausdorff formula; but, as we will see shortly, its very existence is significant. (For such a closed form, see [Se1, I§4.8].)

We now consider another very natural question, namely, when a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra of (i.e., tangent space at the identity to) an immersed subgroup of G. Obviously, a necessary condition is that $\mathfrak{h}$ is closed under the bracket operation; we claim here that this is sufficient as well:

Proposition 8.41. *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Then the subgroup of the group G generated by $\exp(\mathfrak{h})$ is an immersed subgroup H with tangent space $T_eH = \mathfrak{h}$.*

PROOF. Note that the subgroup generated by $\exp(\mathfrak{h})$ is the same as the subgroup generated by $\exp(U)$, where U is any neighborhood of the origin in $\mathfrak{h}$. It will suffice, then (see Exercise 8.42), to show that the image of $\mathfrak{h}$ under the exponential map is "locally" closed under multiplication, i.e., that for a sufficiently small disc $\Delta \subset \mathfrak{h}$, the product $\exp(\Delta)\cdot\exp(\Delta)$ (that is, the set of pairwise products $\exp(X)\cdot\exp(Y)$ for $X, Y \in \Delta$) is contained in the image of $\mathfrak{h}$ under the exponential map.

We will do this under the hypothesis that G may be realized as a subgroup of a general linear group $\mathrm{GL}_n\mathbb{R}$, so that we can use the formula (8.34) for the exponential map. This is a harmless assumption, given the statement (to be proved in Appendix E) that any finite-dimensional Lie algebra may be embedded in the Lie algebra $\mathfrak{gl}_n\mathbb{R}$: the subgroup of $\mathrm{GL}_n\mathbb{R}$ generated by $\exp(\mathfrak{g})$ will be a group isogenous to G, and, as the reader can easily check, proving the proposition for a group isogenous to G is equivalent to proving it for G.

It thus suffices to prove the assertion in case the group G is $\mathrm{GL}_n\mathbb{R}$. But this is exactly the content of the Campbell–Hausdorff formula. □

When applied to an embedding of a Lie algebra $\mathfrak{g}$ into $\mathfrak{gl}_n$, we see, in particular, that *every finite-dimensional Lie algebra is the Lie algebra of a Lie group*. From what we have seen, this Lie group is unique if we require it to be simply connected, and then all others are obtained by dividing this simply connected model by a discrete subgroup of its center.

Exercise 8.42*. Suppose G_0 is an open neighborhood of the identity in a Lie group G such that $G_0 \cdot G_0 \subset G_0$ and $G_0^{-1} = G_0$. Suppose H_0 is a closed submanifold of G_0 such that $H_0 \cdot H_0 \subset H_0$ and $H_0^{-1} = H_0$. Show that the subgroup H of G generated by H_0 is an immersed Lie subgroup of G.

As a fairly easy consequence of this proposition, we can finally give a proof of the Second Principle stated in §8.1, which we may restate as

Second Principle. *Let G and H be Lie groups with G simply connected, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. A linear map $\alpha: \mathfrak{g} \to \mathfrak{h}$ is the differential of a map $A: G \to H$ of Lie groups if and only if α is a map of Lie algebras.*

PROOF. To see this, consider the product $G \times H$. Its Lie algebra is just $\mathfrak{g} \oplus \mathfrak{h}$. Let $\mathfrak{j} \subset \mathfrak{g} \oplus \mathfrak{h}$ be the graph of the map α. Then the hypothesis that α is a map of Lie algebras is equivalent to the statement that $\mathfrak{j}$ is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$; and given this, by the proposition there exists an immersed Lie subgroup $J \subset G \times H$ with tangent space $T_eJ = \mathfrak{j}$.

Look now at the map $\pi: J \to G$ given by projection on the first factor. By hypothesis, the differential of this map $d\pi_e: \mathfrak{j} \to \mathfrak{g}$ is an isomorphism, so that the map $J \to G$ is an isogeny; but since G is simply connected *it follows that π is an isomorphism*. The projection $\eta: G \cong J \to H$ on the second factor is then a Lie group map whose differential at the identity is α. □

Exercise 8.43*. If $\mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of Lie algebras with kernel $\mathfrak{h}$, show that the kernel H of the corresponding map of simply connected Lie groups $G \to G'$ is a closed subgroup of G with Lie group $\mathfrak{h}$. This does not extend to non-normal subgroups, i.e., to the situation when $\mathfrak{h}$ is not the kernel of a homomorphism: give an example of an immersed subgroup of a simply connected Lie group G whose image in G is not closed.

Exercise 8.44. Use the ideas of this lecture to prove the assertion that a compact complex connected Lie group G must be abelian:

(a) Verify that the map $\mathrm{Ad}\colon G \to \mathrm{Aut}(T_eG) \subset \mathrm{End}(T_eG)$ is holomorphic, and, therefore (by the maximum principle), constant.
(b) Deduce that if Ψ_g is conjugation by g, then $d\Psi_g$ is the identity, so $\Psi_g(\exp(X)) = \exp(d\Psi_g(X)) = \exp(X)$ for all $X \in T_eG$, which implies that G is abelian.
(c) Show that the exponential map from T_eG to G is surjective, with the kernel a lattice Λ, so $G = T_eG/\Lambda$ is a complex torus.

LECTURE 9

Initial Classification of Lie Algebras

In this lecture we define various subclasses of Lie algebras: nilpotent, solvable, semisimple, etc., and prove basic facts about their representations. The discussion is entirely elementary (largely because the hard theorems are stated without proof for now); there are no prerequisites beyond linear algebra. Apart from giving these basic definitions, the purpose of the lecture is largely to motivate the narrowing of our focus to semisimple algebras that will take place in the sequel. In particular, the first part of §9.3 is logically the most important for what follows.

§9.1: Rough classification of Lie algebras
§9.2: Engel's Theorem and Lie's Theorem
§9.3: Semisimple Lie algebras
§9.4: Simple Lie algebras

§9.1. Rough Classification of Lie Algebras

We will give, in this section, a preliminary sort of classification of Lie algebras, reflecting the degree to which a given Lie algebra $\mathfrak{g}$ fails to be abelian. As we have indicated, the goal ultimately is to narrow our focus onto *semisimple* Lie algebras.

Before we begin, two definitions, both completely straightforward: First, we define the *center* $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ to be the subspace of $\mathfrak{g}$ of elements $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$. Of course, we say $\mathfrak{g}$ is *abelian* if all brackets are zero.

Exercise 9.1. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. Show that the subgroup of G generated by exponentiating the Lie subalgebra $Z(\mathfrak{g})$ is the connected component of the identity in the center $Z(G)$ of G.

Next, we say that a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is an *ideal* if it satisfies the condition

$$[X, Y] \in \mathfrak{h} \quad \text{for all } X \in \mathfrak{h},\ Y \in \mathfrak{g}.$$

Just as connected subgroups of a Lie group correspond to subalgebras of its Lie algebra, the notion of ideal in a Lie algebra corresponds to the notion of normal subgroup, in the following sense:

Exercise 9.2. Let G be a connected Lie group, $H \subset G$ a connected subgroup and $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras. Show that H is a normal subgroup of G if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Observe also that the bracket operation on $\mathfrak{g}$ induces a bracket on the quotient space $\mathfrak{g}/\mathfrak{h}$ if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

This, in turns, motivates the next bit of terminology: we say that a Lie algebra $\mathfrak{g}$ is *simple* if $\dim \mathfrak{g} > 1$ and it contains no nontrivial ideals. By the last exercise, this is equivalent to saying that the adjoint form G of the Lie algebra $\mathfrak{g}$ has no nontrivial normal Lie subgroups.

Now, to attempt to classify Lie algebras, we introduce two descending chains of subalgebras. The first is the *lower central series* of subalgebras $\mathscr{D}_k\mathfrak{g}$, defined inductively by

$$\mathscr{D}_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$$

and

$$\mathscr{D}_k\mathfrak{g} = [\mathfrak{g}, \mathscr{D}_{k-1}\mathfrak{g}].$$

Note that the subalgebras $\mathscr{D}_k\mathfrak{g}$ are in fact ideals in $\mathfrak{g}$. The other series is called the *derived series* $\{\mathscr{D}^k\mathfrak{g}\}$; it is defined by

$$\mathscr{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$$

and

$$\mathscr{D}^k\mathfrak{g} = [\mathscr{D}^{k-1}\mathfrak{g}, \mathscr{D}^{k-1}\mathfrak{g}].$$

Exercise 9.3. Use the Jacobi identity to show that $\mathscr{D}^k\mathfrak{g}$ is also an ideal in $\mathfrak{g}$. More generally, if $\mathfrak{h}$ is an ideal in a Lie algebra $\mathfrak{g}$, show that $[\mathfrak{h}, \mathfrak{h}]$ is also an ideal in $\mathfrak{g}$; hence all $\mathscr{D}^k\mathfrak{h}$ are ideals in $\mathfrak{g}$.

Observe that we have $\mathscr{D}^k\mathfrak{g} \subset \mathscr{D}_k\mathfrak{g}$ for all k, with equality when $k = 1$; we often write simply $\mathscr{D}\mathfrak{g}$ for $\mathscr{D}_1\mathfrak{g} = \mathscr{D}^1\mathfrak{g}$ and call this the *commutator subalgebra*. We now make the

Definitions

(i) We say that $\mathfrak{g}$ is *nilpotent* if $\mathscr{D}_k\mathfrak{g} = 0$ for some k.
(ii) We say that $\mathfrak{g}$ is *solvable* if $\mathscr{D}^k\mathfrak{g} = 0$ for some k.

(iii) We say that $\mathfrak{g}$ is *perfect* if $\mathscr{D}\mathfrak{g} = \mathfrak{g}$ (this is not a concept we will use much).
(iv) We say that $\mathfrak{g}$ is *semisimple* if $\mathfrak{g}$ has no nonzero solvable ideals.

The standard example of a nilpotent Lie algebra is the algebra $\mathfrak{n}_n\mathbb{R}$ of strictly upper-triangular $n \times n$ matrices; in this case the kth subalgebra $\mathscr{D}_k\mathfrak{g}$ in the lower central series will be the subspace $\mathfrak{n}_{k+1,n}\mathbb{R}$ of matrices $A = (a_{i,j})$ such that $a_{i,j} = 0$ whenever $j \le i + k$, i.e., that are zero below the diagonal and within a distance k of it in each column or row. (In terms of a complete flag $\{V_i\}$ as in §7.2, these are just the endomorphisms that carry V_i into V_{i-k-1}.) It follows also that any subalgebra of the Lie algebra $\mathfrak{n}_n\mathbb{R}$ is likewise nilpotent; we will show later that any nilpotent Lie algebra is isomorphic to such a subalgebra. We will also see that if a Lie algebra $\mathfrak{g}$ is represented on a vector space V, such that each element acts as a nilpotent endomorphism, there is a basis for V such that, identifying $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n\mathbb{R}$, $\mathfrak{g}$ maps to the subalgebra $\mathfrak{n}_n\mathbb{R} \subset \mathfrak{gl}_n\mathbb{R}$.

Similarly, a standard example of a solvable Lie algebra is the space $\mathfrak{b}_n\mathbb{R}$ of upper-triangular $n \times n$ matrices; in this Lie algebra the commutator $\mathscr{D}\mathfrak{b}_n\mathbb{R}$ is the algebra $\mathfrak{n}_n\mathbb{R}$ and the derived series is, thus, $\mathscr{D}^k\mathfrak{b}_n\mathbb{R} = \mathfrak{n}_{2^{k-1},n}\mathbb{R}$. Again, it follows that any subalgebra of the algebra $\mathfrak{b}_n\mathbb{R}$ is likewise solvable; and we will prove later that, conversely, *any* representation of a solvable Lie algebra on a vector space V consists, in terms of a suitable basis, entirely of upper-triangular matrices (i.e., given a solvable Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$, there exists a basis for V such that under the corresponding identification of $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n\mathbb{R}$, the subalgebra $\mathfrak{g}$ is contained in $\mathfrak{b}_n\mathbb{R} \subset \mathfrak{gl}_n\mathbb{R}$).

It is clear from the definitions that the properties of being nilpotent or solvable are inherited by subalgebras or homomorphic images. We will see that the same is true for semisimplicity in the case of homomorphic images, though not for subalgebras.

Note that $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}$ has a sequence of Lie *subalgebras* $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_k = 0$, such that $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$ and $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian. Indeed, if this is the case, one sees by induction that $\mathscr{D}^i\mathfrak{g} \subset \mathfrak{g}_i$ for all i. (One may also refine such a sequence to one where each quotient $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is one dimensional.) It follows from this description that if $\mathfrak{h}$ is an ideal in a Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ *is solvable if and only if* $\mathfrak{h}$ *and* $\mathfrak{g}/\mathfrak{h}$ *are solvable Lie algebras.* (The analogous assertion for nilpotent Lie algebras is false: the ideal $\mathfrak{n}_n$ is nilpotent in the Lie algebra $\mathfrak{b}_n$ of upper-triangular matrices, and the quotient is the nilpotent algebra $\mathfrak{d}_n$ of diagonal matrices, but $\mathfrak{b}_n$ is not nilpotent.) If $\mathfrak{g}$ is the Lie algebra of a connected Lie group G, then $\mathfrak{g}$ is solvable if and only if there is a sequence of connected subgroups, each normal in G (or in the next in the sequence), such that the quotients are abelian.

In particular, the sum of two solvable ideals in a Lie algebra $\mathfrak{g}$ is again solvable [note that $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b} \cong \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$]. It follows that the sum of all solvable ideals in $\mathfrak{g}$ is a maximal solvable ideal, called the *radical* of $\mathfrak{g}$ and denoted $\mathrm{Rad}(\mathfrak{g})$. The quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple. Any Lie algebra $\mathfrak{g}$ thus fits into an exact sequence

$$0 \to \mathrm{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) \to 0 \qquad (9.4)$$

where the first algebra is solvable and the last is semisimple. With this somewhat shaky justification (but see Proposition 9.17), we may say that to study the representation theory of an arbitrary Lie algebra, we have to understand individually the representation theories of solvable and semisimple Lie algebras. Of these, the former is relatively easy, at least as regards irreducible representations. The basic fact about them—that any irreducible representation of a solvable Lie algebra is one dimensional—will be proved later in this lecture. The representation theory of semisimple Lie algebras, on the other hand, is extraordinarily rich, and it is this subject that will occupy us for most of the remainder of the book.

Another easy consequence of the definitions is the fact that *a Lie algebra is semisimple if and only if it has no nonzero abelian ideals.* Indeed, the last nonzero term in the derived sequence of ideals $\mathscr{D}^k\mathrm{Rad}(\mathfrak{g})$ would be an abelian ideal in $\mathfrak{g}$ (cf. Exercise 9.3). A semisimple Lie algebra can have no center, so *the adjoint representation of a semisimple Lie algebra is faithful.*

It is a fact that the sequence (9.4) splits, in the sense that there are subalgebras of $\mathfrak{g}$ that map isomorphically onto $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$. The existence of such a *Levi decomposition* is part of the general theory we are postponing. To show that an arbitrary Lie algebra has a faithful representation (*Ado's theorem*), one starts with a faithful representation of the center, and then builds a representation of the radical step by step, inserting a string of ideals between the center and the radical. Then one uses a splitting to get from a faithful representation on the radical to some representation on all of $\mathfrak{g}$; the sum of this representation and the adjoint representation is then a faithful representation. See Appendix E for details.

One reason for the terminology simple/semisimple will become clear later in this lecture, when we show that a semisimple Lie algebra is a direct sum of simple ones.

Exercise 9.5. Every semisimple Lie algebra is perfect. Show that the Lie group of Euclidean motions of $\mathbb{R}^3$ has a Lie algebra $\mathfrak{g}$ which is perfect, i.e., $\mathscr{D}\mathfrak{g} = \mathfrak{g}$, but $\mathfrak{g}$ is not semisimple. More generally, if $\mathfrak{h}$ is semisimple, and V is an irreducible representation of $\mathfrak{h}$, the twisted product

$$\mathfrak{g} = \{(v, X) | v \in V, X \in \mathfrak{h}\} \quad \text{with } [(v, X), (w, Y)] = (Xw - Yv, [X, Y])$$

is a Lie algebra with $\mathscr{D}\mathfrak{g} = \mathfrak{g}$, $\mathrm{Rad}(\mathfrak{g}) = V$ abelian, and $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) = \mathfrak{h}$.

Exercise 9.6. (a) Show that the following are equivalent for a Lie algebra $\mathfrak{g}$: (i) $\mathfrak{g}$ is nilpotent. (ii) There is a chain of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = 0$ with $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ contained in the center of $\mathfrak{g}/\mathfrak{g}_{i+1}$. (iii) There is an integer n such that

$$\mathrm{ad}(X_1) \circ \mathrm{ad}(X_2) \circ \cdots \circ \mathrm{ad}(X_n)(Y) = [X_1, [X_2, \ldots, [X_n, Y] \ldots]] = 0$$

for all $X_1, \ldots, X_n, Y$ in $\mathfrak{g}$.

(b) Conclude that a connected Lie group G is nilpotent if and only if it can be realized as a succession of central extensions of abelian Lie groups.

Exercise 9.7*. If G is connected and nilpotent, show that the exponential map $\exp: \mathfrak{g} \to G$ is surjective, making $\mathfrak{g}$ the universal covering space of G.

Exercise 9.8. Show that the following are equivalent for a Lie algebra $\mathfrak{g}$: (i) $\mathfrak{g}$ is solvable. (ii) There is a chain of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = 0$ with $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ abelian. (iii) There is a chain of subalgebras $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = 0$ such that $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$, and $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian.

§9.2. Engel's Theorem and Lie's Theorem

We will now prove the statement made above about representations of solvable Lie algebras always being upper triangular. This may give the reader an idea of how the general theory proceeds, before we go back to the concrete examples that are our main concern. The starting point is

Theorem 9.9 (Engel's Theorem). *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be any Lie subalgebra such that every $X \in \mathfrak{g}$ is a nilpotent endomorphism of V. Then there exists a nonzero vector $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{g}$.*

Note this implies that there exists a basis for V in terms of which the matrix representative of each $X \in \mathfrak{g}$ is strictly upper triangular: since $\mathfrak{g}$ kills v, it will act on the quotient $\bar{V}$ of V by the span of v, and by induction we can find a basis $\bar{v}_2, \ldots, \bar{v}_n$ for $\bar{V}$ in terms of which this action is strictly upper triangular. Lifting $\bar{v}_i$ to any $v_i \in V$ and setting $v_1 = v$ then gives a basis for V as desired.

PROOF OF THEOREM 9.9. One observation before we start is that if $X \in \mathfrak{gl}(V)$ is any nilpotent element, then the adjoint action $\mathrm{ad}(X): \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is nilpotent. This is straightforward: to say that X is nilpotent is to say that there exists a flag of subspaces $0 \subset V_1 \subset V_2 \subset \cdots \subset V_k \subset V_{k+1} = V$ such that $X(V_i) \subset V_{i-1}$; we can then check that for any endomorphism Y of V the endomorphism $\mathrm{ad}(X)^m(Y)$ carries V_i into V_{i+k-m}.

We now proceed by induction on the dimension of $\mathfrak{g}$. The first step is to show that, under the hypotheses of the problem, $\mathfrak{g}$ *contains an ideal* $\mathfrak{h}$ *of codimension one.* In fact, let $\mathfrak{h} \subset \mathfrak{g}$ be any maximal proper subalgebra; we claim that $\mathfrak{h}$ has codimension one and is an ideal. To see this, we look at the adjoint representation of $\mathfrak{g}$; since $\mathfrak{h}$ is a subalgebra the adjoint action $\mathrm{ad}(\mathfrak{h})$ of $\mathfrak{h}$ on $\mathfrak{g}$ preserves the subspace $\mathfrak{h} \subset \mathfrak{g}$ and so acts on $\mathfrak{g}/\mathfrak{h}$. Moreover, by our observation above, for any $X \in \mathfrak{h}$ $\mathrm{ad}(X)$ acts nilpotently on $\mathfrak{gl}(V)$, hence on $\mathfrak{g}$, hence on $\mathfrak{g}/\mathfrak{h}$. Thus, by induction, there exists a nonzero element $\bar{Y} \in \mathfrak{g}/\mathfrak{h}$ killed by $\mathrm{ad}(X)$ for all $X \in \mathfrak{h}$; equivalently, there exists an element $Y \in \mathfrak{g}$ not in $\mathfrak{h}$ such that $\mathrm{ad}(X)(Y) \in \mathfrak{h}$ for all $X \in \mathfrak{h}$. But this is to say that the subspace $\mathfrak{h}'$ of $\mathfrak{g}$ spanned by $\mathfrak{h}$ and Y is a Lie subalgebra of $\mathfrak{g}$, in which $\mathfrak{h}$ sits as an ideal of codimension one; by the maximality of $\mathfrak{h}$ we have $\mathfrak{h}' = \mathfrak{g}$ and we are done.

We return now to the representation of $\mathfrak{g}$ on V. We may apply the induction hypothesis to the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ found in the preceding paragraph to conclude that there exists a nonzero vector $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{h}$; let $W \subset V$ be the subspace of all such vectors $v \in V$. Let Y be any element of $\mathfrak{g}$ not in $\mathfrak{h}$; since $\mathfrak{h}$ and Y span $\mathfrak{g}$, it will suffice to show that there exists a (nonzero) vector $v \in W$ such that $Y(v) = 0$. Now for any vector $w \in W$ and any $X \in \mathfrak{h}$, we have

$$X(Y(w)) = Y(X(w)) + [X, Y](w).$$

The first term on the right is zero because by hypothesis $w \in W$, $X \in \mathfrak{h}$ and so $X(w) = 0$; likewise, the second term is zero because $[X, Y] = \mathrm{ad}(X)(Y) \in \mathfrak{h}$. Thus, $X(Y(w)) = 0$ for all $X \in \mathfrak{h}$; we deduce that $Y(w) \in W$. But this means that the action of Y on V carries the subspace W into itself; since Y acts nilpotently on V, it follows that there exists a vector $v \in W$ such that $Y(v) = 0$. □

Exercise 9.10*. Show that a Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}(X)$ is a nilpotent endomorphism of $\mathfrak{g}$ for every $X \in \mathfrak{g}$.

Engel's theorem, in turn, allows us to prove the basic statement made above that every representation of a solvable Lie group can be put in upper-triangular form. This is implied by

Theorem 9.11 (Lie's Theorem). *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a complex solvable Lie algebra. Then there exists a nonzero vector $v \in V$ that is an eigenvector of X for all $X \in \mathfrak{g}$.*

Exercise 9.12. Show that this implies the existence of a basis for V in terms of which the matrix representative of each $X \in \mathfrak{g}$ is upper triangular.

PROOF OF THEOREM 9.11. Once more, the first step in the argument is to assert that $\mathfrak{g}$ contains an ideal $\mathfrak{h}$ of codimension one. This time, since $\mathfrak{g}$ is solvable we know that $\mathscr{D}\mathfrak{g} \neq \mathfrak{g}$, so that the quotient $\mathfrak{a} = \mathfrak{g}/\mathscr{D}\mathfrak{g}$ is a nonzero abelian Lie algebra; the inverse image in $\mathfrak{g}$ of any codimension one subspace of $\mathfrak{a}$ will then be a codimension one ideal in $\mathfrak{g}$.

Still following the lines of the previous argument, we may by induction assume that there is a vector $v_0 \in V$ that is an eigenvector for all $X \in \mathfrak{h}$. Denote the eigenvalue of X corresponding to v_0 by $\lambda(X)$. We then consider the subspace $W \subset V$ of all vectors satisfying the same relation, i.e., we set

$$W = \{v \in V: X(v) = \lambda(X)\cdot v \ \forall X \in \mathfrak{h}\}.$$

Let Y now be any element of $\mathfrak{g}$ not in $\mathfrak{h}$. As before, it will suffice to show that Y carries some vector $v \in W$ into a multiple of itself, and for this it is enough

to show that Y carries W into itself. We prove this in a general context in the following lemma.

Lemma 9.13. *Let $\mathfrak{h}$ be an ideal in a Lie algebra $\mathfrak{g}$. Let V be a representation of $\mathfrak{g}$, and $\lambda: \mathfrak{h} \to \mathbb{C}$ a linear function. Set*

$$W = \{v \in V: X(v) = \lambda(X)\cdot v \ \forall X \in \mathfrak{h}\}.$$

Then $Y(W) \subset W$ for all $Y \in \mathfrak{g}$.

PROOF. Let w be any nonzero element of W; to test whether $Y(w) \in W$ we let X be any element of $\mathfrak{h}$ and write

$$\begin{aligned} X(Y(w)) &= Y(X(w)) + [X, Y](w) \\ &= \lambda(X)\cdot Y(w) + \lambda([X, Y])\cdot w \end{aligned} \tag{9.14}$$

since $[X, Y] \in \mathfrak{h}$. This differs from our previous calculation in that the second term on the right is not immediately seen to be zero; indeed, $Y(w)$ will lie in W if and only if $\lambda([X, Y]) = 0$ for all $X \in \mathfrak{h}$.

To verify this, we introduce another subspace of V, namely, the span U of the images w, $Y(w)$, $Y^2(w)$, ... of w under successive applications of Y. This subspace is clearly preserved by Y; we claim that any $X \in \mathfrak{h}$ carries U into itself as well. It is certainly the case that $\mathfrak{h}$ carries w into a multiple of itself, and hence into U, and (9.14) says that $\mathfrak{h}$ carries $Y(w)$ into a linear combination of $Y(w)$ and w, and so into U. In general, we can see that $\mathfrak{h}$ carries $Y^k(w)$ into U by induction: for any $X \in \mathfrak{h}$ we write

$$X(Y^k(w)) = Y(X(Y^{k-1}(w))) + [X, Y](Y^{k-1}(w)). \tag{9.15}$$

Since $X(Y^{k-1}(w)) \in U$ by induction the first term on the right is in U, and since $[X, Y] \in \mathfrak{h}$ the second term is in U as well.

In fact, we see something more from (9.14) and (9.15): it follows that, in terms of the basis w, $Y(w)$, $Y^2(w)$, ... for U, the action of any $X \in \mathfrak{h}$ is upper triangular, with diagonal entries all equal to $\lambda(X)$. In particular, for any $X \in \mathfrak{h}$ the trace of the restriction of X to U is just the dimension of U times $\lambda(X)$. On the other hand, for any element $X \in \mathfrak{h}$ the commutator $[X, Y]$ acts on U, and being the commutator of two endomorphisms of U the trace of this action is zero. It follows then that $\lambda([X, Y]) = 0$, and we are done. □

Exercise 9.16. Show that any irreducible representation of a solvable Lie algebra $\mathfrak{g}$ is one dimensional, and $\mathscr{D}\mathfrak{g}$ acts trivially.

At least for *irreducible* representations, Lie's theorem implies they will all be known for an arbitrary Lie algebra when they are known for the semisimple case. In fact, we have:

Proposition 9.17. *Let $\mathfrak{g}$ be a complex Lie algebra, $\mathfrak{g}_{ss} = \mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$. Every irreducible representation of $\mathfrak{g}$ is of the form $V = V_0 \otimes L$, where V_0 is an irreducible* *representation of* $\mathfrak{g}_{ss}$ [*i.e., a representation of* $\mathfrak{g}$ *that is trivial on* $\text{Rad}(\mathfrak{g})$], *and* L *is a one-dimensional representation.*

PROOF. By Lie's theorem there is a $\lambda \in (\text{Rad}(\mathfrak{g}))^*$ such that

$$W = \{v \in V: X(v) = \lambda(X)\cdot v \ \forall X \in \text{Rad}(\mathfrak{g})\}$$

is not zero. Apply the preceding lemma, with $\mathfrak{h} = \text{Rad}(\mathfrak{g})$. Since V is irreducible, we must have $W = V$. Now extend λ in any way to a linear function on $\mathfrak{g}$, and let L be the one-dimensional representation of $\mathfrak{g}$ determined by λ; in other words, $Y(z) = \lambda(Y)\cdot z$ for all $Y \in \mathfrak{g}$ and $z \in L$. Then $V \otimes L^*$ is a representation that is trivial on $\text{Rad}(\mathfrak{g})$, so it comes from a representation of $\mathfrak{g}_{ss}$, as required. □

Exercise 9.18. Show that if $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ that maps isomorphically onto $\mathfrak{g}/\text{Rad}(\mathfrak{g})$, then any irreducible representation of $\mathfrak{g}$ restricts to an irreducible representation of $\mathfrak{g}'$, and any irreducible representation of $\mathfrak{g}'$ extends to a representation of $\mathfrak{g}$.

§9.3. Semisimple Lie Algebras

As is clear from the above, many of the aspects of the representation theory of finite groups that were essential to our approach are no longer valid in the context of general Lie algebras and Lie groups. Most obvious of these is complete reducibility, which we have seen fails for Lie groups; another is the fact that not only can the action of elements of a Lie group or algebra on a vector space be nondiagonalizable, the action of some element of a Lie algebra may be diagonalizable under one representation and not under another.

That is the bad news. The good news is that, if we just restrict ourselves to semisimple Lie algebras, everything is once more as well behaved as possible. For one thing, we have complete reducibility again:

Theorem 9.19 (Complete Reducibility). *Let* V *be a representation of the semisimple Lie algebra* $\mathfrak{g}$ *and* $W \subset V$ *a subspace invariant under the action of* $\mathfrak{g}$. *Then there exists a subspace* $W' \subset V$ *complementary to* W *and invariant under* $\mathfrak{g}$.

The proof of this basic result will be deferred to Appendix C.

The other question, the diagonalizability of elements of a Lie algebra under a representation, requires a little more discussion. Recall first the statement of *Jordan decomposition*: any endomorphism X of a complex vector space V can be uniquely written in the form

$$X = X_s + X_n$$

where X_s is diagonalizable, X_n is nilpotent, and the two commute. Moreover, X_s and X_n may be expressed as polynomials in X.

Now, suppose that $\mathfrak{g}$ is an arbitrary Lie algebra, $X \in \mathfrak{g}$ any element, and $\rho: \mathfrak{g} \to \mathfrak{gl}_n\mathbb{C}$ any representation. We have seen that the image $\rho(X)$ need not be diagonalizable; we may still ask how $\rho(X)$ behaves with respect to the Jordan decomposition. The answer is that, in general, absolutely nothing need be true. For example, just taking $\mathfrak{g} = \mathbb{C}$, we see that under the representation

$$\rho_1: t \mapsto (t)$$

every element is diagonalizable, i.e., $\rho(X)_s = \rho(X)$; under the representation

$$\rho_2: t \mapsto \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

every element is nilpotent [i.e., $\rho(X)_s = 0$]; whereas under the representation

$$\rho_3: t \mapsto \begin{pmatrix} t & t \\ 0 & 0 \end{pmatrix}$$

not only are the images $\rho(X)$ neither diagonalizable nor nilpotent, the diagonalizable and nilpotent parts of $\rho(X)$ are not even in the image $\rho(\mathfrak{g})$ of the representation.

If we assume the Lie algebra $\mathfrak{g}$ is semisimple, however, the situation is radically different. Specifically, we have

Theorem 9.20 (Preservation of Jordan Decomposition). *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. For any element* $X \in \mathfrak{g}$, *there exist* X_s *and* $X_n \in \mathfrak{g}$ *such that for any representation* $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ *we have*

$$\rho(X)_s = \rho(X_s) \quad \text{and} \quad \rho(X)_n = \rho(X_n).$$

In other words, if we think of ρ as injective and $\mathfrak{g}$ accordingly as a Lie subalgebra of $\mathfrak{gl}(V)$, *the diagonalizable and nilpotent parts of any element* X *of* $\mathfrak{g}$ *are again in* $\mathfrak{g}$ *and are independent of the particular representation* ρ.

The proofs we will give of the last two theorems both involve introducing objects that are not essential for the rest of this book, and we therefore relegate them to Appendix C. It is worth remarking, however, that another approach was used classically by Hermann Weyl; this is the famous *unitary trick*, which we will describe briefly.

A Digression on "The Unitary Trick"

Basically, the idea is that the statements above (complete reducibility, preservation of Jordan decomposition) can be proved readily for the representations of a compact Lie group. To prove complete reducibility, for example, we can proceed more or less just as in the case of a finite group: if the compact group G acts on a vector space, we see that there is a Hermitian metric on V invariant under the action of G by taking an arbitrary metric on V and averaging its images under the action of G. If G fixes a subspace $W \subset V$, it will then fix as well its orthogonal complement $W^\perp$ with respect to this metric. (Alternatively, we can choose an arbitrary complement W' to W, not necessarily fixed by G, and average over G the projection map to $g(W')$ with kernel W; this average will have image invariant under G.)

How does this help us analyze the representation of a semisimple Lie algebra? The key fact here (to be proved in Lecture 26) is that *if* $\mathfrak{g}$ *is any complex semisimple Lie algebra, there exists a (unique) real Lie algebra* $\mathfrak{g}_0$ *with complexification* $\mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{g}$, *such that the simply connected form of the Lie algebra* $\mathfrak{g}_0$ *is a compact Lie group G*. Thus, restricting a given representation of $\mathfrak{g}$ to $\mathfrak{g}_0$, we can exponentiate to obtain a representation of G, for which complete reducibility holds; and we can deduce from this the complete reducibility of the original representation. For example, while it is certainly not true that any representation ρ of the Lie group $SL_n\mathbb{R}$ on a vector space V admits an invariant Hermitian metric (in fact, it cannot, unless it is the trivial representation), we can

(i) let ρ' be the corresponding (complex) representation of the Lie algebra $\mathfrak{sl}_n\mathbb{R}$;
(ii) by linearity extend the representation ρ' of $\mathfrak{sl}_n\mathbb{R}$ to a representation ρ'' of $\mathfrak{sl}_n\mathbb{C}$;
(iii) restrict to a representation ρ''' of the subalgebra $\mathfrak{su}_n \subset \mathfrak{sl}_n\mathbb{C}$;
(iv) exponentiate to obtain a representation ρ'''' of the unitary group SU_n.

We can now argue that

If a subspace $W \subset V$ is invariant under the action of $SL_n\mathbb{R}$,

it must be invariant under $\mathfrak{sl}_n\mathbb{R}$; and since $\mathfrak{sl}_n\mathbb{C} = \mathfrak{sl}_n\mathbb{R} \otimes \mathbb{C}$, it follows that
it will be invariant under $\mathfrak{sl}_n\mathbb{C}$; so of course
it will be invariant under $\mathfrak{su}_n$; and hence
it will be invariant under SU_n.

Now, since SU_n is compact, there will exist a complementary subspace W' preserved by SU_n; we argue that

W' will then be invariant under $\mathfrak{su}_n$; and since $\mathfrak{sl}_n\mathbb{C} = \mathfrak{su}_n \otimes \mathbb{C}$, it follows that
it will be invariant under $\mathfrak{sl}_n\mathbb{C}$. Restricting, we see that
it will be invariant under $\mathfrak{sl}_n\mathbb{R}$, and exponentiating,
it will be invariant under $SL_n\mathbb{R}$.

Similarly, if one wants to know that the diagonal elements of $SL_n\mathbb{R}$ act semisimply in any representation, or equivalently that the diagonal elements of $\mathfrak{sl}_n\mathbb{R}$ act semisimply, one goes through the same reasoning, coming down to the fact that the group of diagonal elements in $\mathfrak{su}_n$ is abelian and *compact*.

In general, most of the theorems about the finite-dimensional representation of semisimple Lie algebras admit proofs along two different lines: either algebraically, using just the structure of the Lie algebra; or by the unitary trick, that is, by associating to a representation of such a Lie algebra a representation of a compact Lie group and working with that. Which is preferable depends very much on taste and context; in this book we will for the most part go with the algebraic proofs, though in the case of the Weyl character formula in Part IV the proof via compact groups is so much more appealing it has to be mentioned.

The following exercises include a few applications of these two theorems.

Exercise 9.21*. Show that a Lie algebra $\mathfrak{g}$ is semisimple if and only if every finite-dimensional representation is semisimple, i.e., every invariant subspace has a complement.

Exercise 9.22. Use Weyl's unitary trick to show that, for $n > 2$, all representations of $SO_n\mathbb{C}$ are semisimple, so that, in particular, the Lie algebras $\mathfrak{so}_n\mathbb{C}$ are semisimple. Do the same for $Sp_{2n}\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$, $n \geq 1$. Where does the argument break down for $SO_2\mathbb{C}$?

Exercise 9.23. Show that a real Lie algebra $\mathfrak{g}$ is solvable if and only if the complex Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is solvable. Similarly for nilpotent and semisimple.

Exercise 9.24*. If $\mathfrak{h}$ is an ideal in a Lie algebra $\mathfrak{g}$, show that $\mathfrak{g}$ is semisimple if and only if $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are semisimple. Deduce that every semisimple Lie algebra is a direct sum of simple Lie algebras.

Exercise 9.25*. A Lie algebra is called *reductive* if its radical is equal to its center. A Lie group is reductive if its Lie algebra is reductive. For example, $GL_n\mathbb{C}$ is reductive. Show that the following are true for a reductive Lie algebra $\mathfrak{g}$: (i) $\mathscr{D}\mathfrak{g}$ is semisimple; (ii) the adjoint representation of $\mathfrak{g}$ is semisimple; (iii) $\mathfrak{g}$ is a product of a semisimple and an abelian Lie algebra; (iv) $\mathfrak{g}$ has a finite-dimensional faithful semisimple representation. In fact, each of these conditions is equivalent to $\mathfrak{g}$ being reductive.

§9.4. Simple Lie Algebras

There is one more basic fact about Lie algebras to be stated here; though its proof will have to be considerably deferred, it informs our whole approach to the subject. This is the complete classification of simple Lie algebras:

Theorem 9.26. *With five exceptions, every simple complex Lie algebra is isomorphic to either* $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$, *or* $\mathfrak{sp}_{2n}\mathbb{C}$ *for some n.*

The five exceptions can all be explicitly described, though none is particularly simple except in name; they are denoted $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$. We will give a construction of each later in the book (§22.3). The algebras $\mathfrak{sl}_n\mathbb{C}$ (for $n > 1$), $\mathfrak{so}_n\mathbb{C}$ (for $n > 2$), and $\mathfrak{sp}_{2n}\mathbb{C}$ are commonly called the *classical Lie algebras* (and the corresponding groups the *classical Lie groups*); the other five algebras are called, naturally enough, the *exceptional Lie algebras*.

The nature of the classification theorem for simple Lie algebras creates a dilemma as to how we approach the subject: many of the theorems about simple Lie algebras can be proved either in the abstract, or by verifying them in turn for each of the particular algebras listed in the classification theorem. Another alternative is to declare that we are concerned with understanding only the representations of the classical algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$, and $\mathfrak{sp}_{2n}\mathbb{C}$, and verify any relevant theorems just in these cases.

Of these three approaches, the last is in many ways the least satisfactory; it is, however, the one that we shall for the most part take. Specifically, what we will do, starting in Lecture 11, is the following:

(i) Analyze in Lectures 11–13 a couple of examples, namely, $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$, on what may appear to be an ad hoc basis.
(ii) On the basis of these examples, propose in Lecture 14 a general paradigm for the study of representations of a simple (or semisimple) Lie algebra.
(iii) Proceed in Lectures 15–20 to carry out this analysis for the classical algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$, and $\mathfrak{sp}_{2n}\mathbb{C}$.
(iv) Give in Part IV and the appendices proofs for general simple Lie algebras of the facts discovered in the preceding sections for the classical ones (as well as one further important result, the *Weyl character formula*).

We can at least partially justify this seemingly inefficient approach by saying that even if one makes a beeline for the general theorems about the structure and representation theory of a simple Lie algebra, to apply these results in practice we would still need to carry out the sort of explicit analysis of the individual algebras done in Lectures 11–20. This is, however, a fairly bald rationalization: the fact is, the reason we are doing it this way is that this is the only way we have ever been able to understand any of the general results.

LECTURE 10

Lie Algebras in Dimensions One, Two, and Three

Just to get a sense of what a Lie algebra is and what groups might be associated to it, we will classify here all Lie algebras of dimension three or less. We will work primarily with complex Lie algebras and Lie groups, but will mention the real case as well. Needless to say, this lecture is logically superfluous; but it is easy, fun, and serves a didactic purpose, so why not read it anyway. The analyses of both the Lie algebras and the Lie groups are completely elementary, with one exception: the classification of the complex Lie groups associated to abelian Lie algebras involves the theory of complex tori, and should probably be skipped by anyone not familiar with this subject.

§10.1: Dimensions one and two
§10.2: Dimension three, rank one
§10.3: Dimension three, rank two
§10.4: Dimension three, rank three

§10.1. Dimensions One and Two

To begin with, any one-dimensional Lie algebra $\mathfrak{g}$ is clearly abelian, that is, $\mathbb{C}$ with all brackets zero.

The simply connected Lie group with this Lie algebra is just the group $\mathbb{C}$ under addition; and other connected Lie groups that have $\mathfrak{g}$ as their Lie algebra must all be quotients of $\mathbb{C}$ by discrete subgroups $\Lambda \subset \mathbb{C}$. If Λ has rank one, then the quotient is just $\mathbb{C}^*$ under multiplication. If Λ has rank two, however, G may be any one of a continuously varying family of *complex tori of dimension one* (or *Riemann surfaces of genus one*, or *elliptic curves over* $\mathbb{C}$). The set of isomorphism classes of such tori is parametrized by the complex plane with coordinate j, where the function j on the set of lattices $\Lambda \subset \mathbb{C}$ is as described in, e.g., [Ahl].

Over the real numbers, the situation is completely straightforward: the only real Lie algebra of dimension one is again $\mathbb{R}$ with trivial bracket; the simply connected Lie group associated to it is $\mathbb{R}$ under addition; and the only other connected real Lie group with this Lie algebra is $\mathbb{R}/\mathbb{Z} \cong S^1$.

Dimension Two

Here we have to consider two cases, depending on whether $\mathfrak{g}$ is abelian or not.

Case 1: $\mathfrak{g}$ *abelian*. This is very much like the previous case; the simply connected two-dimensional abelian complex Lie group is just $\mathbb{C}^2$ under addition, and the remaining connected Lie groups with Lie algebra $\mathfrak{g}$ are just quotients of $\mathbb{C}^2$ by discrete subgroups. Such a subgroup $\Lambda \subset \mathbb{C}^2$ can have rank 1, 2, 3, or 4, and we analyze these possibilities in turn (the reader who has seen enough complex tori in the preceding example may wish to skip directly to Case 2 at this point).

If the rank of Λ is 1, we can complete the generator of Λ to a basis for $\mathbb{C}^2$, so that $\Lambda = \mathbb{Z}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $G \cong \mathbb{C}^* \times \mathbb{C}$. If the rank of Λ is 2, there are two possibilities: either Λ lies in a one-dimensional complex subspace of $\mathbb{C}^2$ or it does not. If it does not, a pair of generators for Λ will also be a basis for $\mathbb{C}^2$ over $\mathbb{C}$, so that $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, and $G \cong \mathbb{C}^* \times \mathbb{C}^*$. If on the other hand Λ does lie in a complex line in $\mathbb{C}^2$, so that we have $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}\tau e_1$ for some $\tau \in \mathbb{C}\setminus\mathbb{R}$, then $G = E \times \mathbb{C}$ will be the product of the torus $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ and $\mathbb{C}$; the remarks above apply to the classification of these (see Exercise 10.1).

The cases where Λ has rank 3 or 4 are a little less clear. To begin with, if the rank of Λ is 3, the main question to ask is whether any rank 2 sublattice Λ' of Λ lies in a complex line. If it does, then we can assume this sublattice is saturated (i.e., a pair of generators for Λ' can be completed to a set of generators for Λ) and write $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}\tau e_1 \oplus \mathbb{Z}e_2$, so that we will have $G = E \times \mathbb{C}^*$, where E is a torus as above.

Exercise 10.1*. For two one-dimensional complex tori E and E', show that the complex Lie groups $G = E \times \mathbb{C}$ and $G' = E' \times \mathbb{C}$ are isomorphic if and only if $E \cong E'$. Similarly for $E \times \mathbb{C}^*$ and $E' \times \mathbb{C}^*$.

If, on the other hand, no such sublattice of Λ exists, the situation is much more mysterious. One way we can try to represent G is by choosing a generator for Λ and considering the projection of $\mathbb{C}^2$ onto the quotient of $\mathbb{C}^2$ by the line spanned by this generator; thus, if we write $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}(\alpha e_1 + \beta e_2)$ then (assuming β is not real) we have maps

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2/\mathbb{C}e_1 = \mathbb{C} \\ \downarrow & & \downarrow \\ G = \mathbb{C}^2/\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}(\alpha e_1 + \beta e_2) & \longrightarrow & \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\beta) \end{array}$$

expressing G as a bundle over a torus $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\beta)$, with fibers isomorphic

to $\mathbb{C}^*$. This expression of G does not, however, help us very much to describe the family of all such groups. For one thing, the elliptic curve E is surely not determined by the data of G: if we just exchange e_1 and e_2, for example, we replace E by $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\alpha)$, which, of course, need not even be isogenous to E. Indeed, this yields an example of different algebraic groups isomorphic as complex Lie groups: expressing G as a $\mathbb{C}^*$ bundle in this way gives it the structure of an algebraic variety, which, in turn, determines the elliptic curve E (for example, the field of rational functions on G will be the field of rational functions on E with one variable adjoined). Thus, different expressions of the complex Lie group G as a $\mathbb{C}^*$ bundle yield nonisomorphic algebraic groups.

Finally, the case where Λ has rank 4 remains completely mysterious. Among such two-dimensional complex tori are the *abelian varieties*; these are just the tori that may be embedded in complex projective space (and hence may be realized as algebraic varieties). For polarized abelian varieties (that is, abelian varieties with equivalence class of embedding in projective space) there exists a reasonable moduli theory; but the set of abelian varieties forms only a countable dense union in the set of all complex tori (indeed, the general complex torus possesses no nonconstant meromorphic functions whatsoever). No satisfactory theory of moduli is known for these objects.

Needless to say, the foregoing discussion of the various abelian complex Lie groups in dimension two is completely orthogonal to our present purposes. We hope to make the point, however, that even in this seemingly trivial case there lurk some fairly mysterious phenomena. Of course, none of this occurs in the real case, where the two-dimensional abelian simply connected real Lie group is just $\mathbb{R} \times \mathbb{R}$ and any other connected two-dimensional abelian real Lie group is the quotient of this by a sublattice $\Lambda \subset \mathbb{R} \times \mathbb{R}$ of rank 1 or 2, which is to say either $\mathbb{R} \times S^1$ or $S^1 \times S^1$.

Case 2: $\mathfrak{g}$ *not abelian*. Viewing the Lie bracket as a linear map $[\ ,\ ]\colon \wedge^2\mathfrak{g} \to \mathfrak{g}$, we see that if it is not zero, it must have one-dimensional image. We can thus choose a basis $\{X, Y\}$ for $\mathfrak{g}$ as vector space with X spanning the image of $[\ ,\ ]$; after multiplying Y by an appropriate scalar we will have $[X, Y] = X$, which of course determines $\mathfrak{g}$ completely. There is thus a unique nonabelian two-dimensional Lie algebra $\mathfrak{g}$ over either $\mathbb{R}$ or $\mathbb{C}$.

What are the complex Lie groups with Lie algebra $\mathfrak{g}$? To find one, we start with the adjoint representation of $\mathfrak{g}$, which is faithful: we have

$$\mathrm{ad}(X)\colon X \mapsto 0, \quad Y \mapsto X, \qquad \mathrm{ad}(Y)\colon X \mapsto -X, \quad Y \mapsto 0$$

or in matrix notation, in terms of the basis $\{X, Y\}$ for $\mathfrak{g}$,

$$\mathrm{ad}(X) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathrm{ad}(Y) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

These generate the algebra $\mathfrak{g} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \subset \mathfrak{gl}_2\mathbb{C}$; we may exponentiate to arrive at the adjoint form

$$G_0 = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0\right\} \subset \mathrm{GL}_2\mathbb{C}.$$

Topologically this group is homeomorphic to $\mathbb{C} \times \mathbb{C}^*$. To take its universal cover, we write a general member of G_0 as

$$\begin{pmatrix} e^t & s \\ 0 & 1 \end{pmatrix}.$$

The product of two such matrices is given by

$$\begin{pmatrix} e^t & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{t'} & s' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{t+t'} & s + e^t s' \\ 0 & 1 \end{pmatrix},$$

so we may realize the universal cover G of G_0 as the group of pairs $(t, s) \in \mathbb{C} \times \mathbb{C}$ with group law

$$(t, s) \cdot (t', s') = (t + t', s + e^t s').$$

The center of G is just the subgroup

$$Z(G) = \{(2\pi i n, 0)\} \cong \mathbb{Z},$$

so that the connected groups with Lie algebra $\mathfrak{g}$ form a partially ordered tower

$$\begin{array}{l} G \\ \downarrow \\ \vdots \\ \downarrow \\ G_n = G/n\mathbb{Z} = \{(a, b) \in \mathbb{C}^* \times \mathbb{C}; (a, b)\cdot(a', b') = (aa', b + a^n b')\}. \\ \downarrow \\ \vdots \\ \downarrow \\ G_0 \end{array}$$

Exercise 10.2*. Show that for $n \neq m$ the two groups G_n and G_m are not isomorphic.

Finally, in the real case things are simpler: when we exponentiate the adjoint representation as above, the Lie group we arrive at is already simply connected, and so is the unique connected real Lie group with this Lie algebra.

§10.2. Dimension Three, Rank 1

As in the case of dimension two, we look at the Lie bracket as a linear map from $\Lambda^2\mathfrak{g}$ to $\mathfrak{g}$ and begin our classification by considering the rank of this map (that is, the dimension of $\mathscr{D}\mathfrak{g}$), which may be either 0, 1, 2, or 3. For the case of rank 0, we refer back to the discussion of abelian Lie groups above. We begin with the case of rank 1.

Here the kernel of the map $[\ ,\ ]: \Lambda^2\mathfrak{g} \to \mathfrak{g}$ is two dimensional, which means that for some $X \in \mathfrak{g}$ it consists of all vectors of the form $X \wedge Y$ with Y ranging over all of $\mathfrak{g}$ (X here will just be the vector corresponding to the hyperplane $\ker([\ ,\ ]) \subset \Lambda^2\mathfrak{g}$ under the natural (up to scalars) duality between a three-dimensional vector space and its exterior square). Completing X to a basis $\{X, Y, Z\}$ of $\mathfrak{g}$, we can write $\mathfrak{g}$ in the form

$$[X, Y] = [X, Z] = 0,$$
$$[Y, Z] = \alpha X + \beta Y + \gamma Z$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$. If either β or γ is nonzero, we may now rechoose our basis, replacing Y by a multiple of the linear combination $\alpha X + \beta Y + \gamma Z$ and either leaving Z alone (if $\beta \neq 0$) or replacing Z by Y (if $\gamma \neq 0$). We will then have.

$$[X, Y] = [X, Z] = 0,$$
$$[Y, Z] = Y$$

from which we see that $\mathfrak{g}$ is just the product of the one-dimensional abelian Lie algebra $\mathbb{C}X$ with the non-abelian two-dimensional Lie algebra $\mathbb{C}Y \oplus \mathbb{C}Z$ described in the preceding discussion. We may thus ignore this case and assume that in fact we have $\beta = \gamma = 0$; replacing X by αX we then have the Lie algebra

$$[X, Y] = [X, Z] = 0,$$
$$[Y, Z] = X.$$

How do we find the Lie groups with this Lie algebra? As before, we need to start with a faithful representation of $\mathfrak{g}$, but here the adjoint representation is useless, since X is in its kernel. We can, however, arrive at a representation of $\mathfrak{g}$ by considering the equations defining $\mathfrak{g}$: we want to find a pair of endomorphisms Y and Z on some vector space that do not commute, but that do commute with their commutator $X = [Y, Z]$; thus,

$$Y(YZ - ZY) - (YZ - ZY)Y = Y^2Z - 2YZY + ZY^2 = 0$$

and similarly for $[Z, [Y, Z]]$. One simple way to find such a pair of endomorphisms is make all three terms Y^2Z, YZY, and Z^2Y in the above equation zero, e.g., by making Y and Z both have square zero, and to have $YZ = 0$ while $ZY \neq 0$. For example, on a three-dimensional vector space with basis e_1, e_2, and e_3 we could take Y to be the map carrying e_3 to e_2 and killing e_1 and e_2, and Z the map carrying e_2 to e_1 and killing e_1 and e_3; we then have $YZ = 0$ while ZY sends e_3 to e_1. We see then that $\mathfrak{g}$ *is just the Lie algebra* $\mathfrak{n}_3$ *of strictly upper-triangular* 3×3 *matrices.* When we exponentiate we arrive at the group

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, a, b, c \in \mathbb{C} \right\}$$

which is simply connected. Now the center of G is the subgroup

$$Z(G) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b \in \mathbb{C} \right\} \cong \mathbb{C},$$

so the discrete subgroups of $Z(G)$ are just lattices Λ of rank 1 or 2; thus any connected group with Lie algebra $\mathfrak{g}$ is either G, $G/\mathbb{Z}$, or $G/(\mathbb{Z} \times \mathbb{Z})$—that is, an extension of $\mathbb{C} \times \mathbb{C}$ by either $\mathbb{C}$, $\mathbb{C}^*$, or a torus E.

Exercise 10.3. Show that G/Λ is determined up to isomorphism by the one-dimensional $Z(G)/\Lambda$.

A similar analysis holds in the real case: just as before, $\mathfrak{n}_3$ is the unique real Lie algebra of dimension three with commutator subalgebra of dimension one; its simply connected form is the group G of unipotent 3×3 matrices and (the center of this group being $\mathbb{R}$) the only other group with this Lie algebra is the quotient $H = G/\mathbb{Z}$.

Incidentally, the group H represents an interesting example of a group that cannot be realized as a matrix group, i.e., that admits no faithful finite-dimensional representations. One way to see this is to argue that in any irreducible finite-dimensional representation V the center S^1 of H, being compact and abelian, must be diagonalizable; and so under the corresponding representation of the Lie algebra $\mathfrak{g}$ the element X must be carried to a diagonalizable endomorphism of V. But now if $v \in V$ is any eigenvector for X with eigenvalue λ, we also have, arguing as in §9.2,

$$X(Y(v)) = Y(X(v)) = Y(\lambda v) = \lambda Y(v)$$

and similarly $X(Z(v)) = \lambda Z(v)$, i.e., both $Y(v)$ and $Z(v)$ are also eigenvectors for X with eigenvalue λ. Since Y and Z generate $\mathfrak{g}$ and the representation V is irreducible, it follows that X must act as a scalar multiple $\lambda \cdot I$ of the identity; but since $X = [Y, Z]$ is a commutator and so has trace 0, it follows that $\lambda = 0$.

Exercise 10.4*. Show that if G is a simply connected Lie group, and its Lie algebra is solvable, then G cannot contain any nontrivial compact subgroup (in particular, it contains no elements of finite order).

The group H does, however, have an important infinite-dimensional representation. This arises from the representation of the Lie algebra $\mathfrak{g}$ on the space V of $\mathscr{C}^\infty$ functions on the real line $\mathbb{R}$ with coordinate x, in which Y, Z, and X are the operators

$$Y: f \mapsto \pi i x \cdot f,$$

$$Z: f \mapsto \frac{df}{dx}$$

and $X = [Y, Z]$ is $-\pi i$ times the identity. Exponentiating, we see that e^{tY} acts on a function f by multiplying it by the function $(\cos tx + i \cdot \sin tx)$; e^{tZ} sends f to the function F_t where $F_t(x) = f(t + x)$, and e^{tX} sends f to the scalar multiple $e^{-\pi i t} \cdot f$.

§10.3. Dimension Three, Rank 2

In this case, write the commutator subalgebra $\mathscr{D}\mathfrak{g} \subset \mathfrak{g}$ as the span of two elements Y and Z. The commutator of Y and Z can then be written

$$[Y, Z] = \alpha Y + \beta Z.$$

But now the endomorphism $\operatorname{ad}(Y)$ of $\mathfrak{g}$ carries $\mathfrak{g}$ into $\mathscr{D}\mathfrak{g}$, kills Y, and sends Z to $\alpha Y + \beta Z$, and so has trace β; on the other hand, since $\operatorname{ad}(Y)$ is a commutator in $\operatorname{End}(\mathfrak{g})$, it must have trace 0. Thus, β, and similarly α, must be zero; i.e., the subalgebra $\mathscr{D}\mathfrak{g}$ must be abelian. It follows from this that for any element $X \in \mathfrak{g}$ not in $\mathscr{D}\mathfrak{g}$, the map

$$\operatorname{ad}(X): \mathscr{D}\mathfrak{g} \to \mathscr{D}\mathfrak{g}$$

must be an isomorphism. We may now distinguish two possibilities: either $\operatorname{ad}(X)$ is diagonalizible or it is not.

(Note that for the first time we see a case where the classification of the real Lie algebra will be more complicated than that of the complex: in the real case we will have to deal with the third possibility that $\operatorname{ad}(X)$ is diagonalizible over $\mathbb{C}$ but not over $\mathbb{R}$, i.e., that it has two complex conjugate eigenvalues. Though we have not seen it much in these low-dimensional examples, in fact it is generally the case that the real picture is substantially more complicated than the complex one, for essentially just this reason.)

Possibility A: $\operatorname{ad}(X)$ *is diagonalizable*. In this case it is natural to use as a basis for $\mathscr{D}\mathfrak{g}$ a pair of eigenvectors Y, Z for $\operatorname{ad}(X)$; and by multiplying X by a suitable scalar we can assume that one of the eigenvalues (both of which are nonzero) is 1. We thus have the equations for $\mathfrak{g}$

$$[X, Y] = Y, \qquad [X, Z] = \alpha Z, \qquad [Y, Z] = 0 \tag{10.5}$$

for some $\alpha \in \mathbb{C}^*$.

Exercise 10.6. Show that two Lie algebras $\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}$ corresponding to two different scalars in the structure equations (10.5) are isomorphic if and only if $\alpha = \alpha'$ or

$\alpha = 1/\alpha'$. Observe that we have for the first time a continuously varying family of nonisomorphic complex Lie algebras.

To find the groups with these Lie algebras we go to the adjoint representation, which here is faithful. Explicitly, $\mathrm{ad}(Y)$ carries X to $-Y$ and kills Y and Z; $\mathrm{ad}(Z)$ carries X to $-\alpha Z$ and also kills Y and Z; and $\mathrm{ad}(X)$ carries Y to itself, Z to αZ, and kills X. A general member $aX - bY - cZ$ of the Lie algebra is thus represented (with respect to the basis $\{Y, Z, X\}$ for $\mathfrak{g}$) by the matrix

$$\begin{pmatrix} a & 0 & b \\ 0 & \alpha a & \alpha c \\ 0 & 0 & 0 \end{pmatrix}.$$

Exponentiating, we find that a group with Lie algebra $\mathfrak{g}$ is

$$G = \left\{ \begin{pmatrix} e^t & 0 & u \\ 0 & e^{\alpha t} & v \\ 0 & 0 & 1 \end{pmatrix}, t, u, v \in \mathbb{C} \right\} \subset \mathrm{GL}_3\mathbb{C}.$$

Here we run across a very interesting circumstance. If the complex number α is not rational, then the exponential map from $\mathfrak{g}$ to G is one-to-one, and hence a homeomorphism; thus, in particular, G is simply connected. If, on the other hand, α is rational, G will have nontrivial fundamental group. To see this, observe that we always have an exact sequence of groups

$$1 \to B \to G \to A \to 1,$$

where

$$A = \left\{ \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{\alpha t} & 0 \\ 0 & 0 & 1 \end{pmatrix}, t \in \mathbb{C} \right\}$$

and

$$B = \left\{ \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}, u, v \in \mathbb{C} \right\} \cong \mathbb{C} \times \mathbb{C}.$$

Now when $\alpha \notin \mathbb{Q}$, the group $A \cong \mathbb{C}$ is simply connected; but when $\alpha \in \mathbb{Q}$—whatever its denominator—we have $A \cong \mathbb{C}^*$ and correspondingly $\pi_1(G) = \mathbb{Z}$.

Exercise 10.7. Show that G has no center, and hence when $\alpha \neq \mathbb{Q}$, it is the unique connected group with Lie algebra $\mathfrak{g}$. For $\alpha \in \mathbb{Q}$, describe the universal covering of G and classify all groups with Lie algebra $\mathfrak{g}$.

Observe that in this case, even though we have a continuously varying family of Lie algebras $\mathfrak{g}_\alpha$, we have no corresponding continuously varying family of the adjoint (linear) Lie groups; the simply-connected forms do form a family, however.

Possibility B: $\mathrm{ad}(X)$ *is not diagonalizable.* In this case the natural thing to do is to choose a basis $\{Y, Z\}$ of $\mathscr{D}\mathfrak{g}$ with respect to which $\mathrm{ad}(X)$ is in Jordan normal form; replacing X by a multiple, we may assume both its eigenvalues are 1 so that we will have the Lie algebra

$$[X, Y] = Y, \qquad [X, Z] = Y + Z, \qquad [Y, Z] = 0. \tag{10.8}$$

With respect to the basis $\{Y, Z, X\}$ for $\mathfrak{g}$, then, the adjoint action of the general element $aX - bY - cZ$ of the Lie algebra is represented by the matrix

$$\begin{pmatrix} a & a & b+c \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix}$$

and exponentiating we find that the corresponding group is

$$G = \left\{ \begin{pmatrix} e^t & te^t & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix}, t, u, v \in \mathbb{C} \right\}.$$

Exercise 10.9. Show that this group has no center, and hence is the unique connected complex Lie group with its Lie algebra.

Note that the real Lie groups obtained by exponentiating the adjoint action of the Lie algebras given by (10.5) and (10.8) are all homeomorphic to $\mathbb{R}^3$ and have no center, and so are the only connected real Lie groups with these Lie algebras.

Exercise 10.10. Complete the analysis of real Lie groups in Case 2 by considering the third possibility mentioned above: that $\mathrm{ad}(X)$ acts on $\mathscr{D}\mathfrak{g}$ with distinct complex conjugate eigenvalues. Observe that in this way we arrive at our first example of two nonisomorphic real Lie algebras whose tensor products with $\mathbb{C}$ are isomorphic.

§10.4. Dimension Three, Rank 3

Our analysis of this final case begins, as in the preceding one, by looking for eigenvectors of the adjoint action of a suitable element $X \in \mathfrak{g}$. Specifically, we claim that we can find an element $H \in \mathfrak{g}$ such that $\mathrm{ad}(H)\colon \mathfrak{g} \to \mathfrak{g}$ has an eigenvector with nonzero eigenvalue. To see this, observe first that for any nonzero $X \in \mathfrak{g}$, the rank of $\mathrm{ad}(X)$ must be 2; in particular, we must have $\mathrm{Ker}(\mathrm{ad}(X)) = \mathbb{C}X$. Now start with any $X \in \mathfrak{g}$. Either $\mathrm{ad}(X)$ has an eigenvector with nonzero eigenvalue or it is nilpotent; if it is nilpotent, then there exists a vector $Y \in \mathfrak{g}$, not in the kernel of $\mathrm{ad}(X)$ but in the kernel of $\mathrm{ad}(X)^2$—that is, such that $\mathrm{ad}(X)(Y) = \alpha X$ for some nonzero $\alpha \in \mathbb{C}$. But then of course $\mathrm{ad}(Y)(X) = -\alpha X$, so that X is an eigenvector for $\mathrm{ad}(Y)$ with nonzero eigenvalue.

So: choose H and $X \in \mathfrak{g}$ so that X is an eigenvector with nonzero eigenvalue for $\mathrm{ad}(H)$, and write $[H, X] = \alpha X$. Since $H \in \mathscr{D}\mathfrak{g}$, $\mathrm{ad}(H)$ is a commutator in $\mathrm{End}(\mathfrak{g})$, and so has trace 0; it follows that $\mathrm{ad}(H)$ must have a third eigenvector Y with eigenvalue $-\alpha$. To describe the structure of $\mathfrak{g}$ completely it now remains to find the commutator of X and Y; but this follows from the Jacobi identity. We have

$$\begin{aligned}[H,[X, Y]] &= -[X,[Y, H]] - [Y,[H, X]]\\ &= -[X, \alpha Y] - [Y, \alpha X]\\ &= 0,\end{aligned}$$

from which we deduce that $[X, Y]$ must be a multiple of H; since it must be a nonzero multiple, we can multiply X or Y by a scalar to make it 1. Similarly multiplying H by a scalar we can assume α is 1 or any other nonzero scalar. Thus, there is only one possible complex Lie algebra $\mathfrak{g}$ of this type. One could look for endomorphisms H, X, and Y whose commutators satisfy these relations, as we did before. Or we may simply realize that the three-dimensional Lie algebra $\mathfrak{sl}_2\mathbb{C}$ has not yet been seen, so it must be this last possibility. In fact, a natural basis for $\mathfrak{sl}_2\mathbb{C}$ is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

whose Lie algebra is given by

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{10.11}$$

What groups other than $SL_2\mathbb{C}$ have Lie algebra $\mathfrak{sl}_2\mathbb{C}$? To begin with, the group $SL_2\mathbb{C}$ is simply connected: for example, the map $SL_2\mathbb{C} \to \mathbb{C}^2 - \{(0, 0)\}$ sending a matrix to its first row expresses the topological space $SL_2\mathbb{C}$ as a bundle with fiber $\mathbb{C}$ over $\mathbb{C}^2 - \{(0, 0)\}$. Also, it is not hard to see that the center of $SL_2\mathbb{C}$ is just the subgroup $\{\pm I\}$ of scalar matrices, so that the only other connected group with Lie algebra $\mathfrak{sl}_2\mathbb{C}$ is the quotient $PSL_2\mathbb{C} = Sl_2\mathbb{C}/\{\pm I\}$.

As in the preceding case, the analysis of real three-dimensional Lie algebras $\mathfrak{g}$ with $\mathscr{D}\mathfrak{g} = \mathfrak{g}$ involves one additional possibility. At the outset of the argument above, we started with an arbitrary $H \in \mathfrak{g}$ and said that if $\mathrm{ad}(H)$ had no eigenvector other than H itself, then it would have to be nilpotent. Of course, in the real case it is also possible that $\mathrm{ad}(H)$ has two distinct complex conjugate eigenvalues λ and $\bar{\lambda}$. Since $\mathrm{ad}(H)$ is a commutator in $\mathrm{End}(\mathfrak{g})$ and so has trace 0, λ will have to be purely imaginary in this case; and so multiplying H by a real scalar we can assume that its eigenvalues are i and $-i$. It follows then that we can find $X, Y \in \mathfrak{g}$ with

$$[H, X] = Y \qquad \text{and} \qquad [H, Y] = -X.$$

Using the Jacobi identity as before we may conclude that the commutator of X and Y is a multiple of H; after multiplying each of X and Y by a real scalar we can assume that it is either H or $-H$. Finally, if $[X, Y] = -H$, then we observe that we are in the case we considered before: $\mathrm{ad}(Y)$ will have $X + H$ as an eigenvector with nonzero eigenvalue, and following our previous analysis we may conclude that $\mathfrak{g} \cong \mathfrak{sl}_2\mathbb{R}$. Thus, we are left with the sole additional possibility that $\mathfrak{g}$ has structure equations

$$[H, X] = Y, \qquad [H, Y] = -X, \qquad [X, Y] = H. \tag{10.12}$$

This, finally, we may recognize as the Lie algebra $\mathfrak{su}_2$ of the real Lie group $SU(2)$ (as you may recall, the isomorphism $\mathfrak{su}_2 \otimes \mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$ was used in the last lecture).

What are the real Lie groups with Lie algebras $\mathfrak{sl}_2\mathbb{R}$ and $\mathfrak{su}_2$? To start, the center of the group $SL_2\mathbb{R}$ is again just the scalar matrices $\{\pm I\}$, so the only group dominated by $SL_2\mathbb{R}$ is the quotient $PSL_2\mathbb{R}$. On the other hand, unlike the complex case $SL_2\mathbb{R}$ is not simply connected: now the map associating to a 2×2 matrix its first row expresses $SL_2\mathbb{R}$ as a bundle with fiber $\mathbb{R}$ over $\mathbb{R}^2 - \{(0, 0)\}$, so that $\pi_1(SL_2\mathbb{R}) = \mathbb{Z}$. More precisely $PSL_2\mathbb{R}$ maps to the real projective line $\mathbb{P}^1\mathbb{R}$, which is homeomorphic to the circle, with fiber homeomorphic to $\mathbb{R}^2$, so $\pi_1(PSL_2\mathbb{R}) = \mathbb{Z}$. We thus have a tower of covering spaces of $PSL_2\mathbb{R}$, consisting of the simply-connected group $\tilde{S}$ with center $\mathbb{Z}$ and its quotients $\tilde{S}_n = \tilde{S}/n\mathbb{Z}$ (not all of these are covers of $SL_2\mathbb{R}$, despite the diagram below).

A note: In §10.2 we encountered a real Lie group with no faithful finite-dimensional representations; only its universal cover could be represented as a matrix group. Here we find in some sense the opposite phenomenon: the groups $\tilde{S}$ and $\tilde{S}_n$ have no faithful finite-dimensional representations, all finite-dimensional representations factoring through $SL_2\mathbb{R}$ or $PSL_2\mathbb{R}$. This fact will be proved as a consequence of our discussion of the representations of the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ in the next lecture.

What about groups with Lie algebra $\mathfrak{su}_2$? To begin with, there is $SU(2)$, which (again via the map sending a matrix to its first row vector) is homeomorphic to S^3 and thus simply connected. The center of this group is again $\{\pm I\}$, so that the quotient $PSU(2)$ is the only other group with Lie algebra $\mathfrak{su}_2$. (Alternatively, we may realize $SU(2)$ as the group of unit quaternions, cf. Exercise 7.15.)

Finally, we remark that there are other representations of the real and complex Lie groups discussed above. As we will see, the Lie algebra $\mathfrak{so}_3\mathbb{C}$ is isomorphic to $\mathfrak{sl}_2\mathbb{C}$, which induces an isomorphism between the corresponding adjoint forms $PSL_2\mathbb{C}$ and $SO_3\mathbb{C}$ (and between the simply-connected forms $SL_2\mathbb{C}$ and the spin group $\mathrm{Spin}_3\mathbb{C}$). This in turn suggests two more real forms of this group: $SO_3\mathbb{R}$ and $SO^+(2, 1)$. In fact, it is not hard to see that $SO_3\mathbb{R} \cong PSU(2)$, while $SO^+(2, 1) \cong PSL_2\mathbb{R}$. Lastly the isomorphism $\mathfrak{su}_{1,1} \otimes \mathbb{C} \cong$

$\mathfrak{su}_2 \otimes \mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$ implies that the real Lie algebra $\mathfrak{su}_{1,1}$ is isomorphic to either $\mathfrak{su}_2$ or $\mathfrak{sl}_2\mathbb{R}$; in fact, the latter is the case and this induces an isomorphism of groups $\mathrm{SU}_{1,1} \cong \mathrm{SL}_2\mathbb{R}$. We summarize the isomorphisms mentioned in the diagram below:

$$\begin{array}{ccccc}
\tilde{S} & & & & \\
\downarrow & & & & \\
\tilde{S}_n & & \mathrm{Spin}_3\mathbb{C} & & \mathrm{Spin}_3\mathbb{R} \\
\downarrow & & \| & & \| \\
\mathrm{SU}(1,1) = \mathrm{SL}_2\mathbb{R} & \hookrightarrow & \mathrm{SL}_2\mathbb{C} & \hookleftarrow & \mathrm{SU}(2) = \{\text{unit quaternions}\} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{PSL}_2\mathbb{R} & \hookrightarrow & \mathrm{PSL}_2\mathbb{C} & \hookleftarrow & \mathrm{PSU}(2) = \{\text{unit quaternions}\}/\pm 1 \\
\| & & \| & & \| \\
\mathrm{SO}^+(2,1) & \hookrightarrow & \mathrm{SO}_3\mathbb{C} & \hookleftarrow & \mathrm{SO}_3\mathbb{R}
\end{array} \qquad (10.13)$$

Note also the coincidences:

$$\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C}), \qquad \mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}), \qquad (10.14)$$

which follow from the fact that Sp refers to preserving a skew-symmetric bilinear form, and for 2×2 matrices the determinant is such a form.

Exercise 10.15. Identify the Lie algebras $\mathfrak{so}_3$, $\mathfrak{su}_2$, $\mathfrak{su}_{1,1}$, $\mathfrak{so}_{2,1}$, and verify the assertions made about the corresponding Lie groups in the diagram.

Exercise 10.16. For each of the Lie algebras encountered in this lecture, compute the lower central series and the derived series, and say whether the algebra is nilpotent, solvable, simple, or semisimple.

Exercise 10.17. The following are Lie groups of dimension two or three, so must appear on our list. Find them: (i) the group of affine transformations of the line ($x \mapsto ax + b$, under composition); (ii) the group of upper-triangular 2×2 matrices; (iii) the group of orientation preserving Euclidean transformations of the plane (compositions of translations and rotations).

Exercise 10.18. Locate $\mathbb{R}^3$ with the usual cross-product on our list of Lie algebras. More generally, consider the family of Lie algebras parametrized by real quadruples (a, b, c, d), each with basis X, Y, Z with bracket given by

$$[X, Y] = aZ + dY, \qquad [Y, Z] = bX, \qquad [Z, X] = cY - dZ.$$

Classify this Lie algebra as (a, b, c, d) varies in $\mathbb{R}^4$, showing in particular that every three-dimensional Lie algebra can be written in this way.

Exercise 10.19. Realize the isomorphism of $\mathrm{SU}(1, 1)$ with $\mathrm{SL}_2\mathbb{R}$ by identifying them with the groups of complex automorphisms of the unit disk and the upper half-plane, respectively.

Exercise 10.20. Classify all Lie algebras of dimension four and rank 1; in particular, show that they are all direct sums of Lie algebras described above.

Exercise 10.21. Show more generally that there exists a Lie algebra of dimension m and rank 1 that is not a direct sum of smaller Lie algebras if and only if m is odd; in case m is odd show that this Lie algebra is unique and realize it as a Lie subalgebra of $\mathfrak{sl}_n\mathbb{C}$.

(No 5) Lectures 11 ÷ 13

LECTURE 11

Representations of $\mathfrak{sl}_2\mathbb{C}$

This is the first of four lectures—§11–14—that comprise in some sense the heart of the book. In particular, the naive analysis of §11.1, together with the analogous parts of §12 and §13, form the paradigm for the study of finite-dimensional representations of all semisimple Lie algebras and groups. §11.2 is less central; in it we show how the analysis carried out in §11.1 can be used to explicitly describe the tensor products of irreducible representations. §11.3 is least important; it indicates how we can interpret geometrically some of the results of the preceding section. The discussions in §11.1 and §11.2 are completely elementary (we do use the notion of symmetric powers of a vector space, but in a non-threatening way). §11.3 involves a fair amount of classical projective geometry, and can be skimmed or skipped by those not already familiar with the relevant basic notions from algebraic geometry.

§11.1: The irreducible representations
§11.2: A little plethysm
§11.3: A little geometric plethysm

§11.1. The Irreducible Representations

We start our discussion of representations of semisimple Lie algebras with the simplest case, that of $\mathfrak{sl}_2\mathbb{C}$. As we will see, while this case does not exhibit any of the complexity of the more general case, the basic idea that informs the whole approach is clearly illustrated here.

This approach is one already mentioned above, in connection with the representations of the symmetric group on three letters. The idea in that case was that given a representation of our group on a vector space V we first restrict the representation to the abelian subgroup generated by a 3-cycle τ. We obtain a decomposition

$$V = \bigoplus V_\alpha$$

of V into eigenspaces for the action of τ; the commutation relations satisfied by the remaining elements σ of the group with respect to τ implied that such σ simply permuted these subspaces V_α, so that the representation was in effect determined by the collection of eigenvalues of τ.

Of course, circumstances in the case of Lie algebra representations are quite different: to name two, it is no longer the case that the action of an abelian object on any vector space admits such a decomposition; and even if such a decomposition exists we certainly cannot expect that the remaining elements of our Lie algebra will simply permute its summands. Nevertheless, the idea remains essentially a good one, as we shall now see.

To begin with, we choose the basis for the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ from the last lecture:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfying

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{11.1}$$

These seem like a perfectly natural basis to choose, but in fact the choice is dictated by more than aesthetics; there is, as we will see, a nearly canonical way of choosing a basis of a semisimple Lie algebra (up to conjugation), which will yield this basis in the present circumstance and which will share many of the properties we describe below.

In any event, let V be an irreducible finite-dimensional representation of $\mathfrak{sl}_2\mathbb{C}$. We start by trotting out one of the facts that we quoted in Lecture 9, the preservation of Jordan decomposition; in the present circumstances it implies that

$$\textit{The action of } H \textit{ on } V \textit{ is diagonalizable.} \tag{11.2}$$

We thus have, as indicated, a decomposition

$$V = \bigoplus V_\alpha, \tag{11.3}$$

where the α run over a collection of complex numbers, such that for any vector $v \in V_\alpha$ we have

$$H(v) = \alpha \cdot v.$$

The next question is obviously how X and Y act on the various spaces V_α. We claim that X and Y must each carry the subspaces V_α into other subspaces $V_{\alpha'}$. In fact, we can be more specific: if we want to know where the image of a given vector $v \in V_\alpha$ under the action of X sits in relation to the decomposition (11.3), we have to know how H acts on $X(v)$; this is given by the

Fundamental Calculation (first time):

$$H(X(v)) = X(H(v)) + [H, X](v)$$
$$= X(\alpha \cdot v) + 2X(v)$$
$$= (\alpha + 2) \cdot X(v);$$

i.e., *if v is an eigenvector for H with eigenvalue α, then $X(v)$ is also an eigenvector for H, with eigenvalue $\alpha + 2$.* In other words, we have

$$X: V_\alpha \to V_{\alpha+2}.$$

The action of Y on each V_α is similarly calculated; we have $Y(V_\alpha) \subset V_{\alpha-2}$.

Observe that as an immediate consequence of this and the irreducibility of V, *all the complex numbers α that appear in the decomposition* (11.3) *must be congruent to one another* mod 2: for any α_0 that actually occurs, the subspace

$$\bigoplus_{n \in \mathbb{Z}} V_{\alpha_0 + 2n}$$

would be invariant under $\mathfrak{sl}_2\mathbb{C}$ and hence equal to all of V. Moreover, by the same token, the V_α that appear must form an unbroken string of numbers of the form $\beta, \beta + 2, \ldots, \beta + 2k$. We denote by n the last element in this sequence; at this point we just know n is a complex number, but we will soon see that it must be an integer.

To proceed with our analysis, we have the following picture of the action of $\mathfrak{sl}_2\mathbb{C}$ on the vector space V:

Choose any nonzero vector $v \in V_n$; since $V_{n+2} = (0)$, we must have $X(v) = 0$. We ask now what happens when we apply the map Y to the vector v. To begin with, we have

Claim 11.4. *The vectors $\{v, Y(v), Y^2(v), \ldots\}$ span V.*

Proof. From the irreducibility of V it is enough to show that the subspace $W \subset V$ spanned by these vectors is carried into itself under the action of $\mathfrak{sl}_2\mathbb{C}$. Clearly, Y preserves W, since it simply carries the vector $Y^m(v)$ into $Y^{m+1}(v)$. Likewise, since the vector $Y^m(v)$ is in V_{n-2m}, we have $H(Y^m(v)) = (n - 2m) \cdot Y^m(v)$, so H preserves the subspace W. Thus, it suffices to check that $X(W) \subset W$, i.e., that for each m, X carries $Y^m(v)$ into a linear combination of the $Y^i(v)$. We check this in turn for $m = 0, 1, 2$, etc.

To begin with, we have $X(v) = 0 \in W$. To see what X does to $Y(v)$, we use

the commutation relations for $\mathfrak{sl}_2\mathbb{C}$: we have

$$X(Y(v)) = [X, Y](v) + Y(X(v))$$
$$= H(v) + Y(0)$$
$$= n \cdot v.$$

Next, we see that

$$X(Y^2(v)) = [X, Y](Y(v)) + Y(X(Y(v)))$$
$$= H(Y(v)) + Y(n \cdot v)$$
$$= (n - 2) \cdot Y(v) + n \cdot Y(v).$$

The pattern now is clear: X carries each vector in the sequence $v, Y(v), Y^2(v), \ldots$ into a multiple of the previous vector. Explicitly, we have

$$X(Y^m(v)) = (n + (n - 2) + (n - 4) + \cdots + (n - 2m + 2)) \cdot Y^{m-1}(v),$$

or

$$X(Y^m(v)) = m(n - m + 1) \cdot Y^{m-1}(v), \tag{11.5}$$

as can readily be verified by induction. □

There are a number of corollaries of the calculation in the above Claim. To begin with, we make the observation that

$$\textit{all the eigenspaces } V_\alpha \textit{ of } H \textit{ are one dimensional.} \tag{11.6}$$

Second, since we have in the course of the proof written down a basis for V and said exactly where each of H, X, and Y takes each basis vector, the representation V is completely determined by the one complex number n that we started with; in particular, of course, we have that

V is determined by the collection of α occurring in the decomposition $V = \bigoplus V_\alpha$. (11.7)

To complete our analysis, we have to use one more time the finite dimensionality of V. This tells us that there is a lower bound on the α for which $V_\alpha \neq (0)$ as well as an upper one, so that we must have $Y^k(v) = 0$ for sufficiently large k. But now if m is the smallest power of Y annihilating v, then from the relation (11.5),

$$0 = X(Y^m(v)) = m(n - m + 1) \cdot Y^{m-1}(v),$$

and the fact that $Y^{m-1}(v) \neq 0$, we conclude that $n - m + 1 = 0$; in particular, it follows that *n is a non-negative integer.* The picture is thus that the eigenvalues α of H on V form a string of integers differing by 2 and symmetric about the origin in $\mathbb{Z}$. In sum, then, we see that there is a unique representation $V^{(n)}$ for each non-negative integer n; the representation *$V^{(n)}$ is $(n + 1)$-dimensional, with H having eigenvalues $n, n - 2, \ldots, -n + 2, -n$.*

Note that the existence part of this statement may be deduced by checking that the actions of H, X, and Y as given above in terms of the basis v, Yv, $Y^2(v), \ldots, Y^n(v)$ for V do indeed satisfy all the commutation relations for $\mathfrak{sl}_2\mathbb{C}$. Alternatively, we will exhibit them in a moment. Note also that by the symmetry of the eigenvalues we may deduce the useful fact that *any representation V of $\mathfrak{sl}_2\mathbb{C}$ such that the eigenvalues of H all have the same parity and occur with multiplicity one is necessarily irreducible*; more generally, *the number of irreducible factors in an arbitrary representation V of $\mathfrak{sl}_2\mathbb{C}$ is exactly the sum of the multiplicities of* 0 *and* 1 *as eigenvalues of H.*

We can identify in these terms some of the standard representations of $\mathfrak{sl}_2\mathbb{C}$. To begin with, the trivial one-dimensional representation $\mathbb{C}$ is clearly just $V^{(0)}$. As for the standard representation of $\mathfrak{sl}_2\mathbb{C}$ on $V = \mathbb{C}^2$, if x and y are the standard basis for $\mathbb{C}^2$, then we have $H(x) = x$ and $H(y) = -y$, so that $V = \mathbb{C}\cdot x \oplus \mathbb{C}\cdot y = V_{-1} \oplus V_1$ is just the representation $V^{(1)}$ above. Similarly, a basis for the symmetric square $W = \mathrm{Sym}^2 V = \mathrm{Sym}^2\mathbb{C}^2$ is given by $\{x^2, xy, y^2\}$, and we have

$$H(x\cdot x) = x\cdot H(x) + H(x)\cdot x = 2x\cdot x,$$
$$H(x\cdot y) = x\cdot H(y) + H(x)\cdot y = 0,$$
$$H(y\cdot y) = y\cdot H(y) + H(y)\cdot y = -2y\cdot y,$$

so the representation $W = \mathbb{C}\cdot x^2 \oplus \mathbb{C}\cdot xy \oplus \mathbb{C}\cdot y^2 = W_{-2} \oplus W_0 \otimes W_2$ is the representation $V^{(2)}$ above. More generally, the nth symmetric power $\mathrm{Sym}^n V$ of V has basis $\{x^n, x^{n-1}y, \ldots, y^n\}$, and we have

$$\begin{aligned} H(x^{n-k}y^k) &= (n-k)\cdot H(x)\cdot x^{n-k-1}y^k + k\cdot H(y)\cdot x^{n-k}y^{k-1} \\ &= (n-2k)\cdot x^{n-k}y^k \end{aligned}$$

so that the eigenvalues of H on $\mathrm{Sym}^n V$ are exactly $n, n-2, \ldots, -n$. By the observation above that a representation for which all eigenvalues of H occur with multiplicity 1 must be irreducible, it follows that $\mathrm{Sym}^n V$ is irreducible, and hence that

$$V^{(n)} = \mathrm{Sym}^n V.$$

In sum then, we can say simply that

Any irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ is a symmetric power of the standard representation $V \cong \mathbb{C}^2$. (11.8)

Observe that when we exponentiate the image of $\mathfrak{sl}_2\mathbb{C}$ under the embedding $\mathfrak{sl}_2\mathbb{C} \to \mathfrak{sl}_{n+1}\mathbb{C}$ corresponding to the representation $\mathrm{Sym}^n V$, we arrive at the group $\mathrm{SL}_2\mathbb{C}$ when n is odd and $\mathrm{PGL}_2\mathbb{C}$ when n is even. Thus, *the representations of the group $\mathrm{PGL}_2\mathbb{C}$ are exactly the even powers $\mathrm{Sym}^{2n}V$.*

Exercise 11.9. Use the analysis of the representations of $\mathfrak{sl}_2\mathbb{C}$ to prove the statement made in the previous lecture that the universal cover $\widetilde{S}$ of $\mathrm{SL}_2\mathbb{R}$ has no finite-dimensional representations.

§11.2. A Little Plethysm

Clearly, knowing the eigenspace decomposition of given representations tells us the eigenspace decomposition of all their tensor, symmetric, and alternating products and powers: for example, if $V = \bigoplus V_\alpha$ and $W = \bigoplus W_\beta$ then $V \otimes W = \bigoplus (V_\alpha \otimes W_\beta)$ and $V_\alpha \otimes W_\beta$ is an eigenspace for H with eigenvalue $\alpha + \beta$. We can use this to describe the decomposition of these products and powers into irreducible representations of the algebra $\mathfrak{sl}_2\mathbb{C}$.

For example, let $V \cong \mathbb{C}^2$ be the standard representation of $\mathfrak{sl}_2\mathbb{C}$; and suppose we want to study the representation $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V$; we ask in particular whether if it irreducible and, if not, how it decomposes. We have seen that the eigenvalues of $\mathrm{Sym}^2 V$ are 2, 0, and -2, and those of $\mathrm{Sym}^3 V$ are 3, 1, -1, and -3. The 12 eigenvalues of the tensor product $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V$ are thus 5 and -5, 3 and -3 (taken twice), and 1 and -1 (taken three times); we may represent them by the diagram

The eigenvector with eigenvalue 5 will generate a subrepresentation of the tensor product isomorphic to $\mathrm{Sym}^5 V$, which will account for one occurrence of each of the eigenvalues 5, 3, 1, -1, -3, and -5. Similarly, the complement of $\mathrm{Sym}^5 V$ in the tensor product will have eigenvalues 3 and -3, and 1 and -1 (taken twice), and so will contain a copy of the representation $\mathrm{Sym}^3 V$, which will account for one occurrence of the eigenvalues 3, 1, -1 and -3; and the complement of these two subrepresentations will be simply a copy of V. We have, thus,

$$\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V \cong \mathrm{Sym}^5 V \oplus \mathrm{Sym}^3 V \oplus V.$$

Note that the projection map

$$\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V \to \mathrm{Sym}^5 V$$

on the first factor is just multiplication of polynomials; the other two projections do not admit such obvious interpretations.

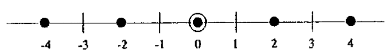
Exercise 11.10. Find, in a similar way, the decomposition of the tensor product $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^5 V$.

Exercise 11.11*. Show, in general, that for $a > b$ we have

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V = \mathrm{Sym}^{a+b} V \oplus \mathrm{Sym}^{a+b-2} V \oplus \cdots \oplus \mathrm{Sym}^{a-b} V.$$

As indicated, we can also look at symmetric and exterior powers of given representations; in many ways this is more interesting. For example, let

$V \cong \mathbb{C}^2$ be as above the standard representation of $\mathfrak{sl}_2\mathbb{C}$, and let $W = \mathrm{Sym}^2 V$ be its symmetric square; i.e., in the notation introduced above, take $W = V^{(2)}$. We ask now whether the symmetric square of W is irreducible, and if not what its decomposition is. To answer this, observe that W has eigenvalues -2, 0, and 2, each occurring once, so that the symmetric square of W will have eigenvalues the pairwise sums of these numbers—that is, -4, -2, 0 (occurring twice), 2, and 4. We may represent $\mathrm{Sym}^2 V$ by the diagram:



From this, it is clear that the representation $\mathrm{Sym}^2 W$ must decompose into one copy of the representation $V^{(4)} = \mathrm{Sym}^4 V$, plus one copy of the trivial (one-dimensional) representation:

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V)) = \mathrm{Sym}^4 V \oplus \mathrm{Sym}^0 V. \tag{11.12}$$

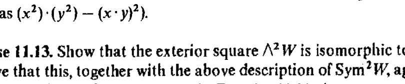
Indeed, we can see this directly: we have a natural map

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V)) \to \mathrm{Sym}^4 V$$

obtained simply by evaluation; this will have a one-dimensional kernel (if x and y are as above the standard basis for V we can write a generator of this kernel as $(x^2)\cdot(y^2) - (x\cdot y)^2$).

Exercise 11.13. Show that the exterior square $\Lambda^2 W$ is isomorphic to W itself. Observe that this, together with the above description of $\mathrm{Sym}^2 W$, agrees with the decomposition of $W \otimes W$ given in Exercise 11.11 above.

We can, in a similar way, describe the decomposition of all the symmetric powers of the representation $W = \mathrm{Sym}^2 V$. For example, the third symmetric power $\mathrm{Sym}^3 W$ has eigenvalues given by the triple sums of the set $\{-2, 0, 2\}$; these are -6, -4, -2 (twice), 0 (twice), 2 (twice), 4, and 6; diagrammatically,



Again, there is no ambiguity about the decomposition; this collection of eigenspaces can only come from the direct sum of $\mathrm{Sym}^6 V$ with $\mathrm{Sym}^2 V$, so we must have

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V) = \mathrm{Sym}^6 V \oplus \mathrm{Sym}^2 V.$$

As before, we can see at least part of this directly: we have a natural evaluation map

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V) \to \mathrm{Sym}^6 V,$$

and the eigenspace decomposition tells us that the kernel is the irreducible representation $\mathrm{Sym}^2 V$.

Exercise 11.14. Use the eigenspace decomposition to establish the formula

$$\mathrm{Sym}^n(\mathrm{Sym}^2 V) = \bigoplus_{\alpha=0}^{[n/2]} \mathrm{Sym}^{2n-4\alpha} V$$

for all n.

§11.3. A Little Geometric Plethysm

We want to give some geometric interpretations of these and similar decompositions of higher tensor powers of representations of $\mathfrak{sl}_2\mathbb{C}$. One big difference is that instead of looking at the action of either the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ or the groups $\mathrm{SL}_2\mathbb{C}$ or $\mathrm{PGL}_2\mathbb{C}$ on a representation W, we look at the action of the group $\mathrm{PGL}_2\mathbb{C}$ on the associated projective space[1] $\mathbb{P}W$. In this context, it is natural to look at various geometric objects associated to the action: for example, we look at closures of orbits of the action, which all turn out to be algebraic varieties, i.e., definable by polynomial equations. In particular, our goal in the following will be to describe the symmetric and exterior powers of W in terms of the action of $\mathrm{PGL}_2\mathbb{C}$ on the projective spaces $\mathbb{P}W$ and various loci in $\mathbb{P}W$.

The main point is that while the action of $\mathrm{PGL}_2\mathbb{C}$ on the projective space $\mathbb{P}V \cong \mathbb{P}^1$ associated to the standard representation V is transitive, its action on the spaces $\mathbb{P}(\mathrm{Sym}^n V) \cong \mathbb{P}^n$ for $n > 1$ is not. Rather, the action will preserve various orbits whose closures are algebraic subvarieties of $\mathbb{P}^n$—for example, the locus of points

$$C = \{[v\cdot v\cdot\ldots\cdot v] : v \in V\} \subset \mathbb{P}(\mathrm{Sym}^n V)$$

corresponding to nth powers in $\mathrm{Sym}^n V$ will be an algebraic curve in $\mathbb{P}(\mathrm{Sym}^n V) \cong \mathbb{P}^n$, called the *rational normal curve*; and this curve will be carried into itself by any element of $\mathrm{PGL}_2\mathbb{C}$ acting on $\mathbb{P}^n$ (more about this in a moment). Thus, a knowledge of the geometry of these subvarieties of $\mathbb{P}W$ may illuminate the representation W, and vice versa. This approach is particularly useful in describing the symmetric powers of W, since these powers can be viewed as the vector spaces of homogeneous polynomials on the projective space $\mathbb{P}(W^*)$ (or, mod scalars, as hypersurfaces in that projective space). Decomposing these symmetric powers should therefore correspond to some interesting projective geometry.

[1] $\mathbb{P}W$ here denotes the projective space of lines through the origin in W, or the quotient space of $W\setminus\{0\}$ by multiplication by nonzero scalars; we write $[w]$ for the point in $\mathbb{P}W$ determined by the nonzero vector w. For $W = \mathbb{C}^{m+1}$, $[z_0, \ldots, z_m]$ is the point in $\mathbb{P}^m = \mathbb{P}W$ determined by a point $(z_0, \ldots, z_m)$ in $\mathbb{C}^{m+1}$.

Digression on Projective Geometry

First, as we have indicated, we want to describe representations of Lie groups in terms of the corresponding actions on projective spaces. The following fact from algebraic geometry is therefore of some moral (if not logical) importance:

Fact 11.15. The group of automorphisms of projective space $\mathbb{P}^n$—either as algebraic variety or as complex manifold—is just the group $PGL_{n+1}\mathbb{C}$.

For a proof, see [Ha]. (For the Riemann sphere $\mathbb{P}^1$ at least, this should be a familiar fact from complex analysis.)

For any vector space W of dimension $n+1$, $\mathrm{Sym}^k W^*$ is the space of homogeneous polynomials of degree k on the projective space $\mathbb{P}^n = \mathbb{P}W$ of lines in W; dually, $\mathrm{Sym}^k W$ will be the space of homogeneous polynomials of degree k on the projective space $\mathbb{P}^n = \mathbb{P}(W^*)$ of lines in W^*, or of hyperplanes in W. Thus, the projective space $\mathbb{P}(\mathrm{Sym}^k W)$ is the space of hypersurfaces of degree k in $\mathbb{P}^n = \mathbb{P}(W^*)$. (Because of this duality, we usually work with objects in the projective space $\mathbb{P}(W^*)$ rather than the dual space $\mathbb{P}W$ in order to derive results about symmetric powers $\mathrm{Sym}^k W$; this may seem initially more confusing, but we believe it is ultimately less so.)

For any vector space V and any positive integer n, we have a natural map, called the *Veronese embedding*

$$\mathbb{P}V^* \hookrightarrow \mathbb{P}(\mathrm{Sym}^n V^*)$$

that maps the line spanned by $v \in V^*$ to the line spanned by $v^n \in \mathrm{Sym}^n V^*$. We will encounter the Veronese embedding of higher-dimensional vector spaces in later lectures; here we are concerned just with the case where V is two dimensional, so $\mathbb{P}V^* = \mathbb{P}^1$. In this case we have a map

$$\iota_n : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n = \mathbb{P}(\mathrm{Sym}^n V^*)$$

whose image is called the *rational normal curve* $C = C_n$ *of degree* n. Choosing bases $\{\alpha, \beta\}$ for V^* and $\{\ldots[n!/k!(n-k)!]\alpha^k\beta^{n-k}\ldots\}$ for $\mathrm{Sym}^n V^*$ and expanding out $(x\alpha + y\beta)^n$ we see that in coordinates this map may be given as

$$[x, y] \mapsto [x^n, x^{n-1}y, x^{n-2}y^2, \ldots, xy^{n-1}, y^n].$$

From the definition, the action of $PGL_2\mathbb{C}$ on $\mathbb{P}^n$ preserves C_n; conversely, since any automorphism of $\mathbb{P}^n$ fixing C_n pointwise is the identity, from Fact 11.15 it follows that *the group G of automorphisms of $\mathbb{P}^n$ that preserve C_n is precisely* $PGL_2\mathbb{C}$. (Note that conversely if W is any $(n+1)$-dimensional representation of $SL_2\mathbb{C}$ and $\mathbb{P}W \cong \mathbb{P}^n$ contains a rational normal curve of degree n preserved by the action of $PGL_2\mathbb{C}$, then we must have $W \cong \mathrm{Sym}^n V$; we leave this as an exercise.[2])

When $n = 2$, C is the *plane conic* defined by the equation

[2] Note that any confusion between $\mathbb{P}W$ and $\mathbb{P}W^*$ is relatively harmless for us here, since the representations $\mathrm{Sym}^n V$ are isomorphic to their duals.

$$F(Z_0, Z_1, Z_2) = Z_0Z_2 - Z_1^2 = 0.$$

For $n = 3$, C is the *twisted cubic curve* in $\mathbb{P}^3$, and is defined by three quadratic polynomials

$$Z_0Z_2 - Z_1^2, \quad Z_0Z_3 - Z_1Z_2, \quad \text{and} \quad Z_1Z_3 - Z_2^2.$$

More generally, the rational normal curve is the common zero locus of the 2×2 minors of the matrix

$$M = \begin{pmatrix} Z_0 Z_1 \ldots Z_{n-1} \\ Z_1 Z_2 \ldots \ Z_n \end{pmatrix},$$

that is, the locus where the rank of M is 1.

Back to Plethysm

We start with Example (11.12). We can interpret the decomposition given there (or rather the decomposition of the representation of the corresponding Lie group $SL_2\mathbb{C}$) geometrically via the Veronese embedding $\iota_2 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$. As noted, $SL_2\mathbb{C}$ acts on $\mathbb{P}^2 = \mathbb{P}(\mathrm{Sym}^2 V^*)$ as the group of motions of $\mathbb{P}^2$ carrying the conic curve C_2 into itself. Its action on the space $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ of quadratic polynomials on $\mathbb{P}^2$ thus must preserve the one-dimensional subspace $\mathbb{C}\cdot F$ spanned by the polynomial F above that defines the conic C_2. At the same time, we see that pullback via ι_2 defines a map from the space of quadratic polynomials on $\mathbb{P}^2$ to the space of quartic polynomials on $\mathbb{P}^1$, which has kernel $\mathbb{C}\cdot F$; thus, we have an exact sequence

$$0 \to \mathbb{C} = \mathrm{Sym}^0 V \to \mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V \to 0,$$

which implies the decomposition of $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ described above.

Note that what comes to us at first glance is not actually the direct sum decomposition (11.12) of $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$, but just the exact sequence above. The splitting of this sequence of $SL_2\mathbb{C}$-modules, guaranteed by the general theory, is less obvious. For example, we are saying that given a conic curve C in the plane $\mathbb{P}^2$, there is a subspace U_C of the space of all conics in $\mathbb{P}^2$, complementary to the one-dimensional subspace spanned by C itself and invariant under the action of the group of motions of the plane $\mathbb{P}^2$ carrying C into itself. Is there a geometric description of this space? Yes: the following proposition gives one.

Proposition 11.16. *The subrepresentation $\mathrm{Sym}^4 V \subset \mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is the space of conics spanned by the family of double lines tangent to the conic $C = C_2$.*

Proof. One way to prove this is to simply write out this subspace in coordinates: in terms of homogeneous coordinates Z_i on $\mathbb{P}^2$ as above, the tangent line to the conic C at the point $[1, \alpha, \alpha^2]$ is the line

$$L_\alpha = \{Z : \alpha^2 Z_0 - 2\alpha Z_1 + Z_2 = 0\}.$$

The double line $2L_\alpha$ is, thus, the conic with equation

$$\alpha^4 Z_0^2 - 4\alpha^3 Z_0 Z_1 + 2\alpha^2 Z_0 Z_2 + 4\alpha^2 Z_1^2 - 4\alpha Z_1 Z_2 + Z_2^2 = 0.$$

The subspace these conics generate is thus spanned by Z_0^2, $Z_0 Z_1$, $Z_1 Z_2$, Z_2^2, and $Z_0 Z_2 + 2Z_1^2$. By construction, this is invariant under the action of $SL_2\mathbb{C}$, and it is visibly complementary to the trivial subrepresentation $\mathbb{C}\cdot F = \mathbb{C}\cdot(Z_0 Z_2 - Z_1^2)$.

For those familiar with some algebraic geometry, it may not be necessary to write all this down in coordinates: we could just observe that the map from the conic curve C to the projective space $\mathbb{P}(\mathrm{Sym}^2(\mathrm{Sym}^2 V))$ of conics in $\mathbb{P}^2$ sending each point $p \in C$ to the square of the tangent line to C at p is the restriction to C of the quadratic Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$, and so has image a quartic rational normal curve. This spans a four-dimensional projective subspace of $\mathbb{P}(\mathrm{Sym}^2(\mathrm{Sym}^2 V))$, which must correspond to a subrepresentation isomorphic to $\mathrm{Sym}^4 V$. □

We will return to this notion in Exercise 11.26 below.

We can, in a similar way, describe the decomposition of all the symmetric powers of the representation $W = \mathrm{Sym}^2 V$; in the general setting, the geometric interpretation becomes quite handy. For example, we have seen that the third symmetric power decomposes

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V) = \mathrm{Sym}^6 V \oplus \mathrm{Sym}^2 V.$$

This is immediate from the geometric description: the space of cubics in the plane $\mathbb{P}^2$ naturally decomposes into the space of cubics vanishing on the conic $C = C_2$, plus a complementary space isomorphic (via the pullback map ι_2^*) to the space of sextic polynomials on $\mathbb{P}^1$; moreover, since a cubic vanishing on C_2 factors into the quadratic polynomial F and a linear factor, the space of cubics vanishing on the conic curve $C \subset \mathbb{P}^2$ may be identified with the space of lines in $\mathbb{P}^2$.

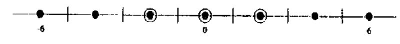
One more special case: from the general formula (11.14), we have

$$\mathrm{Sym}^4(\mathrm{Sym}^2 V) \cong \mathrm{Sym}^8 V \oplus \mathrm{Sym}^4 V \oplus \mathrm{Sym}^0 V.$$

Again, this is easy to see from the geometric picture: the space of quartic polynomials on $\mathbb{P}^2$ consists of the one-dimensional space of quartics spanned by the square of the defining equation F of C itself, plus the space of quartics vanishing on C modulo multiples of F^2, plus the space of quartics modulo those vanishing on C. (We use the word "plus," suggesting a direct sum, but as before only an exact sequence is apparent).

Exercise 11.17. Show that, in general, the order of vanishing on C defines a filtration on the space of polynomials of degree n in $\mathbb{P}^2$, whose successive quotients are the direct sum factors on the right hand side of the decomposition of Exercise 11.14.

We can similarly analyze symmetric powers of the representation $U = \mathrm{Sym}^3 V$. For example, since U has eigenvalues -3, -1, 1, and 3, the symmetric square of U has eigenvalues -6, -4, -2 (twice), 0 (twice), 2 (twice), 4, and 6; diagrammatically, we have



This implies that

$$\mathrm{Sym}^2(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^6 V \oplus \mathrm{Sym}^2 V. \tag{11.18}$$

We can interpret this in terms of the twisted cubic $C = C_3 \subset \mathbb{P}^3$ as follows: the space of quadratic polynomials on $\mathbb{P}^3$ contains, as a subrepresentation, the three-dimensional vector space of quadrics containing the curve C itself; and the quotient is isomorphic, via the pullback map ι_3^*, to the space of sextic polynomials on $\mathbb{P}^1$.

Exercise 11.19*. By the above, the action of $SL_2\mathbb{C}$ on the space of quadric surfaces containing the twisted cubic curve C is the same as its action on $\mathbb{P}(\mathrm{Sym}^2 V^*) \cong \mathbb{P}^2$. Make this explicit by associating to every quadric containing C a polynomial of degree 2 on $\mathbb{P}^1$, up to scalars.

Exercise 11.20*. The direct sum decomposition (11.18) says that there is a linear space of quadric surfaces in $\mathbb{P}^3$ preserved under the action of $SL_2\mathbb{C}$ and complementary to the space of quadrics containing C. Describe this space.

Exercise 11.21. The projection map from $\mathrm{Sym}^2(\mathrm{Sym}^3 V)$ to $\mathrm{Sym}^2 V$ given by the decomposition (11.18) above may be viewed as a *quadratic* map from the vector space $\mathrm{Sym}^3 V$ to the vector space $\mathrm{Sym}^2 V$. Show that it may be given in these terms as the *Hessian*, that is, by associating to a homogeneous cubic polynomial in two variables the determinant of the 2×2 matrix of its second partials.

Exercise 11.22. The map in the preceding exercise may be viewed as associating to an unordered triple of points $\{p, q, r\}$ in $\mathbb{P}^1$ an unordered pair of points $\{s, t\} \subset \mathbb{P}^1$. Show that this pair of points is the pair of fixed points of the automorphism of $\mathbb{P}^1$ permuting the three points p, q, and r cyclically.

Exercise 11.23*. Show that

$$\mathrm{Sym}^3(\mathrm{Sym}^3 V) = \mathrm{Sym}^9 V \oplus \mathrm{Sym}^5 V \oplus \mathrm{Sym}^3 V,$$

and interpret this in terms of the geometry of the twisted cubic curve. In particular, show that the space of cubic surfaces containing the curve is the direct sum of the last two factors, and identify the subspace of cubics corresponding to the last factor.

Exercise 11.24. Analyze the representation $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ similarly. In particular, show that it contains a trivial one-dimensional subrepresentation.

The trivial subrepresentation of $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ found in the last exercise has an interesting interpretation. To say that $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ has such an invariant one-dimensional subspace is to say that *there exists a quartic surface in $\mathbb{P}^3$ preserved under all motions of $\mathbb{P}^3$ carrying the rational normal curve $C = C_3$ into itself.* What is this surface? The answer is simple: it is the *tangent developable* to the twisted cubic, that is, the surface given as the union of the tangent lines to C.

Exercise 11.25*. Show that the representation $\mathrm{Sym}^3(\mathrm{Sym}^4 V)$ contains a trivial subrepresentation, and interpret this geometrically.

Problem 11.26. Another way of interpreting the direct sum decomposition of $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ geometrically is to say that given a conic curve $C \subset \mathbb{P}^2$ and given four points on C, we can find a conic $C' = C'(C; p_1, \ldots, p_4) \subset \mathbb{P}^2$ intersecting C in exactly these points, in a way that is preserved by the action of the group $\mathrm{PGL}_3\mathbb{C}$ of all motions of $\mathbb{P}^2$ (i.e., for any motion $A\colon \mathbb{P}^2 \to \mathbb{P}^2$ of the plane, we have $A(C'(C; p_1, \ldots, p_4)) = C'(AC; Ap_1, \ldots, Ap_4)$). What is a description of this process? In particular, show that the cross-ratio of the four points p_i on the curve C' must be a function of the cross-ratio of the p_i on C, and find this function. Observe also that this process gives an endomorphism of the pencil

$$\{C \subset \mathbb{P}^2 : p_1, \ldots, p_4 \in C\} \cong \mathbb{P}^1$$

of conics passing through any four points $p_i \in \mathbb{P}^2$. What is the degree of this endomorphism?

The above questions have all dealt with the symmetric powers of $\mathrm{Sym}^n V$. There are also interesting questions about the exterior powers of $\mathrm{Sym}^n V$. To start with, consider the exterior square $\Lambda^2(\mathrm{Sym}^3 V)$. The eigenvalues of this representation are just the pairwise sums of distinct elements of $\{3, 1, -1, -3\}$, that is, 4, 2, 0 (twice), -2, and -4; we deduce that

$$\Lambda^2(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^4 V \oplus \mathrm{Sym}^0 V. \tag{11.27}$$

Observe in particular that according to this there is a skew-symmetric bilinear form on the space $U = \mathrm{Sym}^3 V$ preserved (up to scalars) by the action of $\mathrm{SL}_2\mathbb{C}$. What is this form? One way of describing it would be in terms of the twisted cubic: the map from C to the dual projective space $(\mathbb{P}^3)^*$ sending each point $p \in C$ to the osculating plane to C at p extends to a skew-symmetric linear isomorphism of $\mathbb{P}^3$ with $(\mathbb{P}^3)^*$.

Exercise 11.28. Show that a line in $\mathbb{P}^3$ is isotropic for this form if and only if, viewed as an element of $\mathbb{P}(\Lambda^2 U)$, it lies in the linear span of the locus of tangent lines to the twisted cubic.

Exercise 11.29. Show that the projection on the first factor in the decomposition (11.27) is given explicitly by the map

$$F \wedge G \mapsto F \cdot dG - G \cdot dF$$

and say precisely what this means.

Exercise 11.30. Show that, in general, the representation $\Lambda^2(\mathrm{Sym}^n V)$ has as a direct sum factor the representation $\mathrm{Sym}^{2n-2} V$, and that the projection on this factor is given as in the preceding exercise. Find the remaining factors of $\Lambda^2(\mathrm{Sym}^n V)$, and interpret them.

More on Rational Normal Curves

Exercise 11.31. Analyze in general the representations $\mathrm{Sym}^2(\mathrm{Sym}^n V)$; show, using eigenvalues, that we have

$$\mathrm{Sym}^2(\mathrm{Sym}^n V) = \bigoplus_{\alpha \geq 0} \mathrm{Sym}^{2n-4\alpha} V.$$

Exercise 11.32*. Interpret the space $\mathrm{Sym}^2(\mathrm{Sym}^n V)$ of the preceding exercise as the space of quadrics in the projective space $\mathbb{P}^n$, and use the geometry of the rational normal curve $C = C_n \subset \mathbb{P}^n$ to interpret the decomposition of this representation into irreducible factors. In particular, show that direct sum

$$\bigoplus_{\alpha \geq 1} \mathrm{Sym}^{2n-4\alpha} V$$

is the space of quadratic polynomials vanishing on the rational normal curve; and that the direct sum

$$\bigoplus_{\alpha \geq 2} \mathrm{Sym}^{2n-4\alpha} V$$

is the space of quadrics containing the *tangential developable* of the rational normal curve, that is, the union of the tangent lines to C. Can you interpret the sums for $\alpha \geq k$ for $k > 2$?

Exercise 11.33. Note that by Exercise 11.11, the tensor power

$$\mathrm{Sym}^n V \otimes \mathrm{Sym}^n V$$

always contains a copy of the trivial representation; and that by Exercises 11.30 and 11.31, this trivial subrepresentation will lie in $\mathrm{Sym}^2(\mathrm{Sym}^n V)$ if n is even and in $\Lambda^2(\mathrm{Sym}^n V)$ if n is odd. Show that in either case, the bilinear form on $\mathrm{Sym}^n V$ preserved by $\mathrm{SL}_2\mathbb{C}$ may be described as the isomorphism of $\mathbb{P}^n$ with $(\mathbb{P}^n)^*$ carrying each point p of the rational normal curve $C \subset \mathbb{P}^n$ into the osculating hyperplane to C at p.

Comparing Exercises 11.14 and 11.31, we see that $\mathrm{Sym}^2(\mathrm{Sym}^n V) \cong \mathrm{Sym}^n(\mathrm{Sym}^2 V)$; apparently coincidentally. This is in fact a special case of a more general theorem (cf. Exercise 6.18):

Exercise 11.34. (Hermite Reciprocity). Use the eigenvalues of H to prove the isomorphism

$$\mathrm{Sym}^k(\mathrm{Sym}^n V) \cong \mathrm{Sym}^n(\mathrm{Sym}^k V).$$

Can you exhibit explicitly a map between these two?

Note that in the examples of Hermite reciprocity we have seen, it seems completely coincidental: for example, the fact that the representations $\mathrm{Sym}^3(\mathrm{Sym}^4 V)$ and $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ both contain a trivial representation corresponds to the facts that the tangential developable of the twisted cubic in $\mathbb{P}^3$ has degree 4, while the chordal variety of the rational normal quartic in $\mathbb{P}^4$ has degree 3.

Exercise 11.35*. Show that $\wedge^m(\mathrm{Sym}^n V) \cong \mathrm{Sym}^m(\mathrm{Sym}^{n+1-m} V)$.

We will see in Lecture 23 that there is a unique closed orbit in $\mathbb{P}(W)$ for any irreducible representation W. For now, we can do the following special case.

Exercise 11.36. Show that the unique closed orbit of the action of $SL_2\mathbb{C}$ on the projectivization of any irreducible representation is isomorphic to $\mathbb{P}^1$ (these are the *rational normal curves* introduced above).

LECTURE 12

Representations of $\mathfrak{sl}_3\mathbb{C}$, Part I

This lecture develops results for $\mathfrak{sl}_3\mathbb{C}$ analogous to those of §11.1 (though not in exactly the same order). This involves generalizing some of the basic terms of §11 (e.g., the notions of eigenvalue and eigenvector have to be redefined), but the basic ideas are in some sense already in §11. Certainly no techniques are involved beyond those of §11.1.

We come now to a second important stage in the development of the theory: in the following, we will take our analysis of the representations of $\mathfrak{sl}_2\mathbb{C}$ and see how it goes over in the next case, the algebra $\mathfrak{sl}_3\mathbb{C}$. As we will see, a number of the basic constructions need to be modified, or at least rethought. There are, however, two pieces of good news that should be borne in mind. First, we will arrive, by the end of the following lecture, at a classification of the representations of $\mathfrak{sl}_3\mathbb{C}$ that is every bit as detailed and explicit as the classification we arrived at previously for $\mathfrak{sl}_2\mathbb{C}$. Second, once we have redone our analysis in this context, *we will need to introduce no further concepts to carry out the classification of the finite-dimensional representations of all remaining semisimple Lie algebras.*

We will proceed by analogy with the previous lecture. To begin with, we started out our analysis of $\mathfrak{sl}_2\mathbb{C}$ with the basis $\{H, X, Y\}$ for the Lie algebra; we then proceeded to decompose an arbitrary representation V of $\mathfrak{sl}_2\mathbb{C}$ into a direct sum of eigenspaces for the action of H. What element of $\mathfrak{sl}_3\mathbb{C}$ in particular will play the role of H? The answer—and this is the first and perhaps most wrenching change from the previous case—is that no one element really allows us to see what is going on.[1] Instead, we have to replace

[1] This is not literally true: as we will see from the following analysis, if H is any diagonal matrix whose entries are independent over $\mathbb{Q}$, then the action of H on any representation V of $\mathfrak{sl}_3\mathbb{C}$ determines the representation (i.e., if we know the eigenvalues of H we know V). But (as we will also see) trying to carry this out in practice would be sheer perversity.

the single element $H \in \mathfrak{sl}_2\mathbb{C}$ with a *subspace* $\mathfrak{h} \subset \mathfrak{sl}_3\mathbb{C}$, namely, the two-dimensional subspace of all diagonal matrices. The idea is a basic one; it comes down to the observation that *commuting diagonalizable matrices are simultaneously diagonalizable*. This translates in the present circumstances to the statement that any finite-dimensional representation V of $\mathfrak{sl}_3\mathbb{C}$ admits a decomposition $V = \bigoplus V_\alpha$, where every vector $v \in V_\alpha$ is an eigenvector for every element $H \in \mathfrak{h}$.

At this point some terminology is clearly in order, since we will be dealing with the action not of a single matrix H but rather a vector space $\mathfrak{h}$ of them. To begin with, by an *eigenvector* for $\mathfrak{h}$ we will mean, reasonably enough, a vector $v \in V$ that is an eigenvector for every $H \in \mathfrak{h}$. For such a vector v we can write

$$H(v) = \alpha(H)\cdot v, \tag{12.1}$$

where $\alpha(H)$ is a scalar depending linearly on H, i.e., $\alpha \in \mathfrak{h}^*$. This leads to our second notion: by an *eigenvalue* for the action of $\mathfrak{h}$ we will mean an element $\alpha \in \mathfrak{h}^*$ such that there exists a nonzero element $v \in V$ satisfying (12.1); and by the *eigenspace* associated to the eigenvalue α we will mean the subspace of all vectors $v \in V$ satisfying (12.1). Thus we may phrase the statement above as

(12.2) *Any finite-dimensional representation V of $\mathfrak{sl}_3\mathbb{C}$ has a decomposition*

$$V = \bigoplus V_\alpha,$$

where V_α is an eigenspace for $\mathfrak{h}$ and α ranges over a finite subset of $\mathfrak{h}^*$.

This is, in fact, a special case of a more general statement: for any semisimple Lie algebra $\mathfrak{g}$, we will be able to find an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$, such that the action of $\mathfrak{h}$ on any $\mathfrak{g}$-module V will be diagonalizable, i.e., we will have a direct sum decomposition of V into eigenspaces V_α for $\mathfrak{h}$.

Having decided what the analogue for $\mathfrak{sl}_3\mathbb{C}$ of $H \in \mathfrak{sl}_2\mathbb{C}$ is, let us now consider what will play the role of X and Y. The key here is to look at the commutation relations

$$[H, X] = 2X \quad \text{and} \quad [H, Y] = -2Y$$

in $\mathfrak{sl}_2\mathbb{C}$. The correct way to interpret these is as saying that *X and Y are eigenvectors for the adjoint action of H on $\mathfrak{sl}_2\mathbb{C}$*. In our present circumstances, then, we want to look for eigenvectors (in the new sense) for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3\mathbb{C}$. In other words, we apply (12.2) to the adjoint representation of $\mathfrak{sl}_3\mathbb{C}$ to obtain a decomposition

$$\mathfrak{sl}_3\mathbb{C} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_\alpha\right), \tag{12.3}$$

where α ranges over a finite subset of $\mathfrak{h}^*$ and $\mathfrak{h}$ acts on each space $\mathfrak{g}_\alpha$ by scalar multiplication, i.e., for any $H \in \mathfrak{h}$ and $Y \in \mathfrak{g}_\alpha$,

$$[H, Y] = \mathrm{ad}(H)(Y) = \alpha(H)\cdot Y.$$

This is probably easier to carry out in practice than it is to say; we are being

longwinded here because once this process is understood it will be straightforward to apply it to the other Lie algebras. In any case, to do it in the present circumstances, we just observe that multiplication of a matrix M on the left by a diagonal matrix D with entries a_i multiplies the ith row of M by a_i, while multiplication on the right multiplies the ith column by a_i; if the entries of M are $m_{i,j}$, the entries of the commutator $[D, M]$ are thus $(a_i - a_j)m_{i,j}$. We see then that the commutator $[D, M]$ will be a multiple of M for all D if and only if all but one entry of M are zero. Thus, if we let $E_{i,j}$ be the 3×3 matrix whose (i, j)th entry is 1 and all of whose other entries are 0, we see that the $E_{i,j}$ exactly generate the eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$.

Explicitly, we have

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}$$

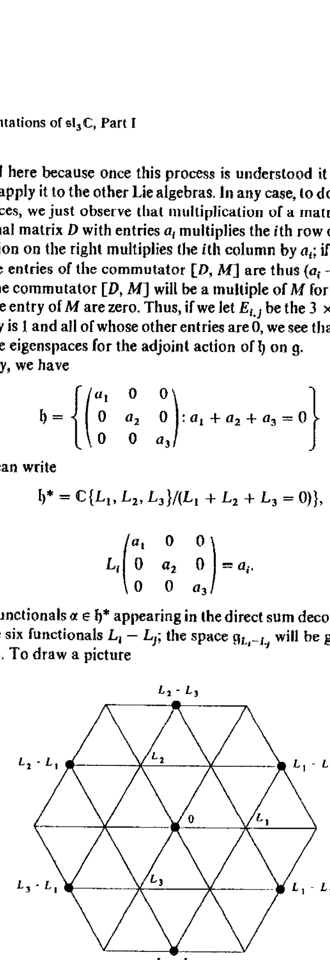
and so we can write

$$\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, L_3\}/(L_1 + L_2 + L_3 = 0)\},$$

where

$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i.$$

The linear functionals $\alpha \in \mathfrak{h}^*$ appearing in the direct sum decomposition (12.3) are thus the six functionals $L_i - L_j$; the space $\mathfrak{g}_{L_i - L_j}$ will be generated by the element $E_{i,j}$. To draw a picture



(12.4)

The virtue of this decomposition and the corresponding picture is that we can read off from it pretty much the entire structure of the Lie algebra. Of course, the action of $\mathfrak{h}$ on $\mathfrak{g}$ is clear from the picture: $\mathfrak{h}$ carries each of the subspaces $\mathfrak{g}_\alpha$ into itself, acting on each $\mathfrak{g}_\alpha$ by scalar multiplication by the linear functional represented by the corresponding dot. Beyond that, though, we can also see, much as in the case of representations of $\mathfrak{sl}_2\mathbb{C}$, how the rest of the Lie algebra acts. Basically, we let X be any element of $\mathfrak{g}_\alpha$ and ask where $\operatorname{ad}(X)$ sends a given vector $Y \in \mathfrak{g}_\beta$; the answer as before comes from knowing how $\mathfrak{h}$ acts on $\operatorname{ad}(X)(Y)$. Explicitly, we let H be an arbitrary element of $\mathfrak{h}$ and as on page 148 we make the

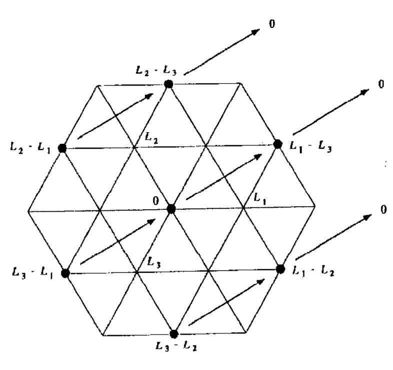
Fundamental Calculation (second time):

$$\begin{aligned}[H, [X, Y]] &= [X, [H, Y]] + [[H, X], Y] \\ &= [X, \beta(H)\cdot Y] + [\alpha(H)\cdot X, Y] \\ &= (\alpha(H) + \beta(H))\cdot [X, Y].\end{aligned}$$

In other words, $[X, Y] = \operatorname{ad}(X)(Y)$ *is again an eigenvector for* $\mathfrak{h}$, *with eigenvalue* $\alpha + \beta$. Thus,

$$\operatorname{ad}(\mathfrak{g}_\alpha)\colon \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta};$$

in particular, the action of $\operatorname{ad}(\mathfrak{g}_\alpha)$ preserves the decomposition (12.3) in the sense that it carries each eigenspace $\mathfrak{g}_\beta$ into another. We can interpret this in terms of the diagram (12.4) of eigenspaces by saying that each $\mathfrak{g}_\alpha$ acts, so to speak, by "translation"; that is, it carries each space $\mathfrak{g}_\beta$ corresponding to a dot in the diagram into the subspace $\mathfrak{g}_{\alpha+\beta}$ corresponding to that dot translated by α. For example, the action of $\mathfrak{g}_{L_1-L_3}$ may be pictured as



(12.5)

i.e., it carries $\mathfrak{g}_{L_2-L_1}$ into $\mathfrak{g}_{L_2-L_3}$; $\mathfrak{g}_{L_3-L_1}$ into $\mathfrak{h}$; $\mathfrak{h}$ into $\mathfrak{g}_{L_1-L_3}$, $\mathfrak{g}_{L_3-L_2}$ into $\mathfrak{g}_{L_1-L_2}$, and kills $\mathfrak{g}_{L_2-L_3}$, $\mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_1-L_2}$. Of course, not all the data can be read off of the diagram, at least on the basis on what we have said so far. For example, we do not at present see from the diagram the kernel of $\operatorname{ad}(\mathfrak{g}_{L_1-L_3})$ on $\mathfrak{h}$, though we will see later how to read this off as well. We do, however, have at least a pretty good idea of who is doing what to whom.

Pretty much the same picture applies to any representation V of $\mathfrak{sl}_3\mathbb{C}$: we start from the eigenspace decomposition $V = \bigoplus V_\alpha$ for the action of $\mathfrak{h}$ that we saw in (12.2). Next, the commutation relations for $\mathfrak{sl}_3\mathbb{C}$ tell us exactly how the remaining summands of the decomposition (12.3) of $\mathfrak{sl}_3\mathbb{C}$ act on the space V, and again we will see that each of the spaces $\mathfrak{g}_\alpha$ acts by carrying one eigenspace V_β into another. As usual, for any $X \in \mathfrak{g}_\alpha$ and $v \in V_\beta$ we can tell where X will send v if we know how an arbitrary element $H \in \mathfrak{h}$ will act on $X(v)$. This we can determine by making the

Fundamental Calculation (third time):

$$\begin{aligned}H(X(v)) &= X(H(v)) + [H, X](v) \\ &= X(\beta(H)\cdot v) + (\alpha(H)\cdot X)(v) \\ &= (\alpha(H) + \beta(H))\cdot X(v).\end{aligned}$$

We see from this that $X(v)$ *is again an eigenvector for the action of* $\mathfrak{h}$, *with eigenvalue* $\alpha + \beta$; in other words, the action of $\mathfrak{g}_\alpha$ carries V_β to $V_{\alpha+\beta}$. We can thus represent the eigenspaces V_α of V by dots in a plane diagram so that each $\mathfrak{g}_\alpha$ acts again "by translation," as we did for representations of $\mathfrak{sl}_2\mathbb{C}$ in the preceding lecture and the adjoint representation of $\mathfrak{sl}_3\mathbb{C}$ above. Just as in the case of $\mathfrak{sl}_2\mathbb{C}$ (page 148), we have

Observation 12.6. *The eigenvalues α occurring in an irreducible representation of $\mathfrak{sl}_3\mathbb{C}$ differ from one other by integral linear combinations of the vectors $L_i - L_j \in \mathfrak{h}^*$.*

Note that these vectors $L_i - L_j$ generate a lattice in $\mathfrak{h}^*$, which we will denote by Λ_R, and that all the α lie in some translate of this lattice.

At this point, we should begin to introduce some of the terminology that appears in this subject. The basic object here, the eigenvalue $\alpha \in \mathfrak{h}^*$ of the action of $\mathfrak{h}$ on a representation V of $\mathfrak{g}$, is called a *weight* of the representation; the corresponding eigenvectors in V_α are called, naturally enough, *weight vectors* and the spaces V_α themselves *weight spaces*. Clearly, the weights that occur in the adjoint representation are special; these are called the *roots* of the Lie algebra and the corresponding subspaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ *root spaces*; by convention, zero is not a root. The lattice $\Lambda_R \subset \mathfrak{h}^*$ generated by the roots α is called the *root lattice*.

To see what the next step should be, we go back to the analysis of representations of $\mathfrak{sl}_2\mathbb{C}$. There, at this stage we continued our analysis by going to an extremal eigenspace V_α and taking a vector $v \in V_\alpha$. The point was that since V_α was extremal, the operator X, which would carry V_α to $V_{\alpha+2}$, would have to kill v; so that v would be then both an eigenvector for H and in the kernel of X. We then saw that these two facts allowed us to completely describe the representation V in terms of images of v.

What would be the appropriately analogous setup in the case of $\mathfrak{sl}_3\mathbb{C}$? To start at the beginning, there is the question of what we mean by *extremal*: in the case of $\mathfrak{sl}_2\mathbb{C}$, since we knew that all the eigenvalues were scalars differing by integral multiples of 2, there was not much ambiguity about what we meant by this. In the present circumstance this does involve a priori a choice (though as we shall see the choice does not affect the outcome): we have to choose a direction, and look for the farthest α in that direction appearing in the decomposition (12.3). What this means is that we should choose a linear functional

$$l\colon \Lambda_R \to \mathbb{R},$$

extend it by linearity to a linear functional $l\colon \mathfrak{h}^* \to \mathbb{C}$, and then for any representation V we should go to the eigenspace V_α for which the real part of $l(\alpha)$ is maximal.[2] Of course, to avoid ambiguity we should choose l to be irrational with respect to the lattice Λ_R, that is, to have no kernel.

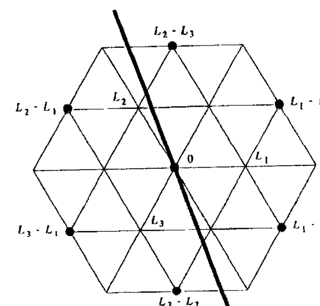
What is the point of this? The answer is that, just as in the case of a representation V of $\mathfrak{sl}_2\mathbb{C}$ we found in this way a vector $v \in V$ that was simultaneously in the kernel of the operator X and an eigenvector for H, in the present case what we will find is a vector $v \in V_\alpha$ that is an eigenvector for $\mathfrak{h}$, *and at the same time in the kernel of the action of $\mathfrak{g}_\beta$ for every β such that* $l(\beta) > 0$—that is, *that is killed by half the root spaces $\mathfrak{g}_\beta$* (specifically, the root spaces corresponding to dots in the diagram (12.4) lying in a half plane). This will likewise give us a nearly complete description of the representation V.

To carry this out explicitly, choose our functional l to be given by

$$l(a_1L_1 + a_2L_2 + a_3L_3) = aa_1 + ba_2 + ca_3,$$

where $a + b + c = 0$ and $a > b > c$, so that the spaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ for which we have $l(\alpha) > 0$ are then exactly $\mathfrak{g}_{L_1-L_3}$, $\mathfrak{g}_{L_2-L_3}$, and $\mathfrak{g}_{L_1-L_2}$; they correspond to matrices with one nonzero entry above the diagonal.

[2] The real-versus-complex business is a red herring since (it will turn out very shortly) all the eigenvalues α actually occurring in any representation will in fact be in the real (in fact, the rational) linear span of Λ_R.



(12.7)

Thus, for $i < j$, the matrices $E_{i,j}$ *generate the positive root spaces*, and the $E_{j,i}$ *generate the negative root spaces*. We set

$$H_{i,j} = [E_{i,j}, E_{j,i}] = E_{i,i} - E_{j,j}. \tag{12.8}$$

Now let V be any irreducible, finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$. The upshot of all the above is the

Lemma 12.9. *There is a vector $v \in V$ with the properties that*

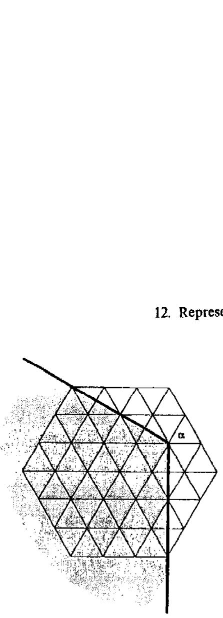
(i) *v is an eigenvector for $\mathfrak{h}$, i.e. $v \in V_\alpha$ for some α; and*
(ii) *v is killed by $E_{1,2}$, $E_{1,3}$, and $E_{2,3}$.*

For any representation V of $\mathfrak{sl}_3\mathbb{C}$, a vector $v \in V$ with these properties is called a *highest weight vector*.

In the case of $\mathfrak{sl}_2\mathbb{C}$, having found an eigenvector v for H killed by X, we argued that the images of v under successive applications of Y generated the representation. The situation here is the same: analogous to Claim 11.4 we have

Claim 12.10. *Let V be an irreducible representation of $\mathfrak{sl}_3\mathbb{C}$, and $v \in V$ a highest weight vector. Then V is generated by the images of v under successive applications of the three operators $E_{2,1}$, $E_{3,1}$, and $E_{3,2}$.*

Before we check the claim, we note three immediate consequences. First, it says that all the eigenvalues $\beta \in \mathfrak{h}^*$ occurring in V lie in a sort of $\frac{1}{3}$-plane with corner at α:



Second, we see that the dimension of V_α itself is 1, so that v is the unique eigenvector with this eigenvalue (up to scalars, of course). (We will see below that in fact v is the unique highest weight vector of V up to scalars; see Proposition 12.11.) Lastly, it says that the spaces $V_{\alpha+n(L_2-L_1)}$ and $V_{\alpha+n(L_3-L_2)}$ are all at most one dimensional, since they must be spanned by $(E_{2,1})^n(v)$ and $(E_{3,2})^n(v)$, respectively.

Proof of Claim 12.10. This is formally the same as the proof of the corresponding statement for $\mathfrak{sl}_2\mathbb{C}$: we argue that the subspace W of V spanned by images of v under the subalgebra of $\mathfrak{sl}_3\mathbb{C}$ generated by $E_{2,1}$, $E_{3,1}$, and $E_{3,2}$ is, in fact, preserved by all of $\mathfrak{sl}_3\mathbb{C}$ and hence must be all of V. To do this we just have to check that $E_{1,2}$, $E_{2,3}$, and $E_{1,3}$ carry W into itself (in fact it is enough to do this for the first two, the third being their commutator), and this is straightforward. To begin with, v itself is in the kernel of $E_{1,2}$, $E_{2,3}$, and $E_{1,3}$, so there is no problem there. Next we check that $E_{2,1}(v)$ is kept in W: we have

$$\begin{aligned} E_{1,2}(E_{2,1}(v)) &= (E_{2,1}(E_{1,2}(v)) + [E_{1,2}, E_{2,1}](v) \\ &= \alpha([E_{1,2}, E_{2,1}])\cdot v \end{aligned}$$

since $E_{1,2}(v) = 0$ and $[E_{1,2}, E_{2,1}] \in \mathfrak{h}$; and

$$\begin{aligned} E_{2,3}(E_{2,1}(v)) &= (E_{2,1}(E_{2,3}(v)) + [E_{2,3}, E_{2,1}](v) \\ &= 0 \end{aligned}$$

since $E_{2,3}(v) = 0$ and $[E_{2,3}, E_{2,1}] = 0$. A similar computation shows that $E_{3,2}(v)$ is also carried into V by $E_{1,2}$ and $E_{2,3}$.

More generally, we may argue the claim by a sort of induction: we let w_n denote any word of length n or less in the letters $E_{2,1}$ and $E_{3,2}$ and take W_n to be the vector space spanned by the vectors $w_n(v)$ for all such words; note that W is the union of the spaces W_n, since $E_{3,1}$ is the commutator of $E_{3,2}$ and $E_{2,1}$. We claim that $E_{1,2}$ and $E_{2,3}$ carry W_n into W_{n-1}. To see this, we can

write w_n as either $E_{2,1} \circ w_{n-1}$ or $E_{3,2} \circ w_{n-1}$; in either case $w_{n-1}(v)$ will be an eigenvector for $\mathfrak{h}$ with eigenvalue β for some β. In the former case we have

$$\begin{aligned} E_{1,2}(w_n(v)) &= E_{1,2}(E_{2,1}(w_{n-1}(v))) \\ &= E_{2,1}(E_{1,2}(w_{n-1}(v))) + [E_{1,2}, E_{2,1}](w_{n-1}(v)) \\ &\in E_{2,1}(W_{n-2}) + \beta([E_{1,2}, E_{2,1}])\cdot w_{n-1}(v) \\ &\subset W_{n-1} \end{aligned}$$

since $[E_{1,2}, E_{2,1}] \in \mathfrak{h}$; and

$$\begin{aligned} E_{2,3}(w_n(v)) &= E_{2,3}(E_{2,1}(w_{n-1}(v))) \\ &= E_{2,1}(E_{2,3}(w_{n-1}(v))) + [E_{2,3}, E_{2,1}](w_{n-1}(v)) \\ &\in E_{2,1}(W_{n-2}) \\ &\subset W_{n-1} \end{aligned}$$

since $[E_{2,3}, E_{2,1}] = 0$. Essentially the same calculation covers the latter case $w_n = E_{3,2} \circ w_{n-1}$, establishing the claim. □

This argument shows a little more; in fact, it proves

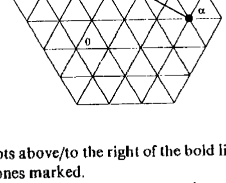
Proposition 12.11. *If V is any representation of $\mathfrak{sl}_3\mathbb{C}$ and $v \in V$ is a highest weight vector, then the subrepresentation W of V generated by the images of v by successive applications of the three operators $E_{2,1}$, $E_{3,1}$, and $E_{3,2}$ is irreducible.*

Proof. Let α be the weight of v. The above shows that W is a subrepresentation, and it is clear that W_α is one dimensional. If W were not irreducible, we would have $W = W' \oplus W''$ for some representations W' and W''. But since projection to W' and W'' commute with the action of $\mathfrak{h}$, we have $W_\alpha = W'_\alpha \oplus W''_\alpha$. This shows that one of these spaces is zero, which implies that v belongs to W' or W'', and hence that W is W' or W''. □

As a corollary of this proposition we see that any irreducible representation of $\mathfrak{sl}_3\mathbb{C}$ has a unique highest weight vector, up to scalars; more generally, the set of highest weight vectors in V forms a union of linear subspaces Ψ_W corresponding to the irreducible subrepresentations W of V, with the dimension of Ψ_W equal to the number of times W appears in the direct sum decomposition of V into irreducibles.

What do we do next? Well, let us continue to look at the border vectors $(E_{2,1})^k(v)$. We call these border vectors because they live in (and, as we saw, span) a collection of eigenspaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{\alpha+L_2-L_1}$, $\mathfrak{g}_{\alpha+2(L_2-L_1)}$, ... that correspond to points on the boundary of the diagram above of possible eigenvalues of V. We also know that they span an uninterrupted string of nonzero eigenspaces $\mathfrak{g}_{\alpha+k(L_2-L_1)} \cong \mathbb{C}$, $k = 0, 1, \ldots,$ until we get to the first m such that

$(E_{2,1})^m(v) = 0$; after that we have $\mathfrak{g}_{\alpha+k(L_2-L_1)} = (0)$ for all $k \geq m$. The picture is thus:



(12.12)

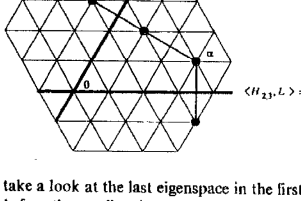
where we have no dots above/to the right of the bold line, and no dots on that line other than the ones marked.

The obvious question now is how long the string of dots along this line is. One way to answer this would be to make a calculation analogous to the one in the preceding lecture: use the computation made above to say explicitly for any k what multiple of $(E_{2,1})^{k-1}(v)$ the image of $(E_{2,1})^k(v)$ under the map $E_{1,2}$ is, and use the fact that $(E_{2,1})^m(v) = 0$ to determine m. It will be simpler—and more useful in general—if instead we just use what we have already learned about representations of $\mathfrak{sl}_2\mathbb{C}$. The point is, *the elements $E_{1,2}$ and $E_{2,1}$, together with their commutator $[E_{1,2}, E_{2,1}] = H_{1,2}$, span a subalgebra of $\mathfrak{sl}_3\mathbb{C}$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$ via an isomorphism carrying $E_{1,2}$, $E_{2,1}$ and $H_{1,2}$ to the elements X, Y and H.* We will denote this subalgebra by $\mathfrak{s}_{L_1-L_2}$ (the notation may appear awkward, but this is a special case of a general construction). By the description we have already given of the action of $\mathfrak{sl}_3\mathbb{C}$ on the representation V in terms of the decomposition $V = \bigoplus V_\alpha$, we see that the subalgebra $\mathfrak{s}_{L_1-L_2}$ will shift eigenspaces V_α only in the direction of $L_2 - L_1$; in particular, the direct sum of the eigenspaces in question, namely the subspace

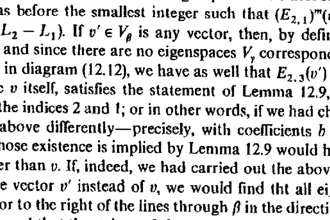
$$W = \bigoplus_k \mathfrak{g}_{\alpha+k(L_2-L_1)} \tag{12.13}$$

of V will be preserved by the action of $\mathfrak{s}_{L_1-L_2}$. In other words, W is a representation of $\mathfrak{s}_{L_1-L_2} \cong \mathfrak{sl}_2\mathbb{C}$ and we may deduce from this that *the eigenvalues of $H_{1,2}$ on W are integral, and symmetric with respect to zero.* Leaving aside the integrality for the moment, this says that the string of dots in diagram (12.12) must be symmetric with respect to the line $\langle H_{1,2}, L\rangle = 0$ in the plane $\mathfrak{h}^*$. Happily (though by no means coincidentally, as we shall see), this line is perpendicular to the line spanned by $L_1 - L_2$ in the picture we have drawn; so we can say simply that *the string of dots occurring in diagram* (12.12) *is preserved under reflection in the line $\langle H_{1,2}, L\rangle = 0$.*

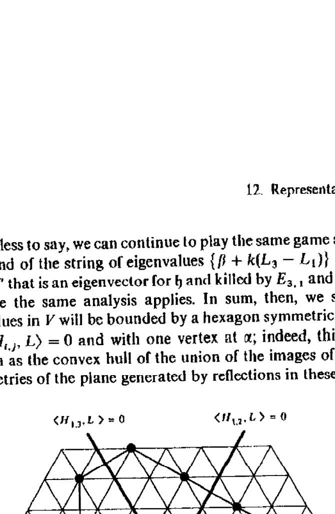
In general, for any $i \neq j$ the elements $E_{i,j}$ and $E_{j,i}$, together with their commutator $[E_{i,j}, E_{j,i}] = H_{i,j}$, span a subalgebra $\mathfrak{s}_{L_i-L_j}$ of $\mathfrak{sl}_3\mathbb{C}$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$ via an isomorphism carrying $E_{i,j}$, $E_{j,i}$, and $H_{i,j}$ to the elements X, Y, and H. (Note that $H_{i,j} = -H_{j,i}$.) Analyzing the action of the subalgebra $\mathfrak{s}_{L_2-L_3}$ in particular then shows that the string of dots corresponding to the eigenspaces $\mathfrak{g}_{\alpha+k}(L_3 - L_2)$ is likewise preserved under reflection in the line $\langle H_{2,3}, L\rangle = 0$ in $\mathfrak{h}^*$. The picture is thus



Let us now take a look at the last eigenspace in the first string, that is, V_β where m is as before the smallest integer such that $(E_{2,1})^m(v) = 0$ and $\beta = \alpha + (m-1)(L_2 - L_1)$. If $v' \in V_\beta$ is any vector, then, by definition, we have $E_{2,1}(v') = 0$; and since there are no eigenspaces V_γ corresponding to γ above the bold line in diagram (12.12), we have as well that $E_{2,3}(v') = E_{1,3}(v') = 0$. Thus, v', like v itself, satisfies the statement of Lemma 12.9, except for the exchange of the indices 2 and 1; or in other words, if we had chosen the linear functional l above differently—precisely, with coefficients $b > a > c$—then the vector whose existence is implied by Lemma 12.9 would have turned out to be v' rather than v. If, indeed, we had carried out the above analysis with respect to the vector v' instead of v, we would find tht all eigenvalues of V occur below or to the right of the lines through β in the directions of $L_1 - L_2$ and $L_3 - L_1$, and that the strings of eigenvalues occurring on these two lines were symmetric about the lines $\langle H_{1,2}, L\rangle = 0$ and $\langle H_{1,3}, L\rangle = 0$, respectively. The picture now is



Needless to say, we can continue to play the same game all the way around: at the end of the string of eigenvalues $\{\beta + k(L_3 - L_1)\}$ we will arrive at a vector v'' that is an eigenvector for $\mathfrak{h}$ and killed by $E_{3,1}$ and $E_{2,1}$, and to which therefore the same analysis applies. In sum, then, we see that the set of eigenvalues in V will be bounded by a hexagon symmetric with respect to the lines $\langle H_{i,j}, L\rangle = 0$ and with one vertex at α; indeed, this characterizes the hexagon as the convex hull of the union of the images of α under the group of isometries of the plane generated by reflections in these three lines.



(12.14)

We will see in a moment that the set of eigenvalues will include all the points congruent to α modulo the lattice Λ_R generated by the $L_i - L_j$ lying on the boundary of this hexagon, and that each of these eigenvalues will occur with multiplicity one.

The use of the subalgebras $\mathfrak{s}_{L_i - L_j}$ does not stop here. For one thing, observe that as an immediate consequence of our analysis of $\mathfrak{sl}_2\mathbb{C}$, all the eigenvalues of the elements $H_{i,j}$ must be integers; it is not hard to see that this means that all the eigenvalues occurring in (12.2) must be integral linear combinations of the L_i, i.e., in terms of the diagrams above, all dots must lie in the lattice Λ_W of interstices (as indeed we have been drawing them). Thus, we have

Proposition 12.15. *All the eigenvalues of any irreducible finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$ must lie in the lattice $\Lambda_W \subset \mathfrak{h}^*$ generated by the L_i and be congruent modulo the lattice $\Lambda_R \subset \mathfrak{h}^*$ generated by the $L_i - L_j$.*

This is exactly analogous to he situation of the previous lecture: there we saw that the eigenvalues of H in any irreducible, finite-dimensional representation of $\mathfrak{sl}_2\mathbb{C}$ lay in the lattice $\Lambda_W \cong \mathbb{Z}$ of linear forms on $\mathbb{C}H$ integral on H, and were congruent to one another modulo the sublattice $\Lambda_R = 2 \cdot \mathbb{Z}$ generated

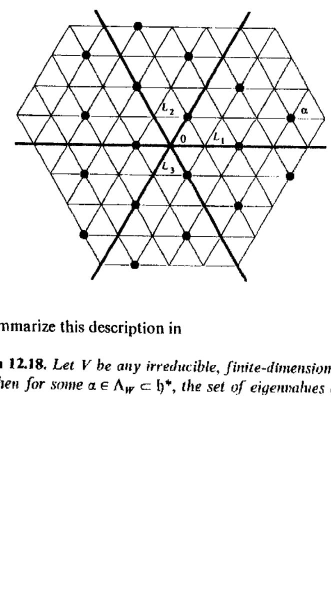
by the eigenvalues of H under the adjoint representation. Note that in the case of $\mathfrak{sl}_2\mathbb{C}$ we have $\Lambda_W/\Lambda_R \cong \mathbb{Z}/2$, while in the present case we have $\Lambda_W/\Lambda_R \cong \mathbb{Z}/3$; we will see later how this reflects a general pattern. The lattice Λ_W is called the *weight lattice*.

Exercise 12.16. Show that the two conditions that the eigenvalues of V are congruent to one another modulo Λ_R and are preserved under reflection in the three lines $\langle H_{i,j}, L\rangle = 0$ imply that they all lie in Λ_W, and that, in fact, this characterizes Λ_W.

To continue, we can go into the interior of the diagram (12.14) of eigenvalues of V by observing that the direct sums (12.13) are not the only visible subspaces of V preserved under the action of the subalgebras $\mathfrak{s}_{L_i - L_j}$; more generally, for any $\beta \in \mathfrak{h}^*$ appearing in the decomposition (12.2) and any i, j the direct sum

$$W = \bigoplus_k \mathfrak{g}_{\beta + k(L_i - L_j)}$$

will be a representation of $\mathfrak{s}_{L_i - L_j}$ (not necessarily irreducible, of course); in particular it follows that the values of k for which $V_{\beta + k(L_i - L_j)} \neq (0)$ form an unbroken string of integers. Observing that if β is any of the "extremal" eigenvalues pictured in diagram (12.14), then this string will include another; so that all eigenvalues congruent to the dots pictured in diagram (12.14) and lying in their convex hull must also occur. Thus, the complete diagram of eigenvalues will look like



(12.17)

We can summarize this description in

Proposition 12.18. *Let V be any irreducible, finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$. Then for some $\alpha \in \Lambda_W \subset \mathfrak{h}^*$, the set of eigenvalues occurring in V is* *exactly the set of linear functionals congruent to* α *modulo the lattice* Λ_R *and lying in the hexagon with vertices the images of* α *under the group generated by reflections in the lines* $\langle H_{i,j}, L\rangle = 0$.

Remark. We did, in the analysis thus far, make one apparently arbitrary choice when we defined the notion of "extremal" eigenvalue by choosing a linear functional l on $\mathfrak{h}^*$. We remark here that, in fact, the choice was not as broad as might at first have appeared. Indeed, given the fact that the configuration of eigenvalues occurring in any irreducible finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$ is always either a triangle or a hexagon, the "extremal" eigenvalue picked out by l will always turn out to be one of the three or six vertices of this figure; in other words, if we define the linear functional l to take $a_1L_1 + a_2L_2 + a_3L_3$ to $aa_1 + ba_2 + ca_3$, then only the ordering of the three real numbers a, b, and c matters. Indeed, in hindsight this choice was completely analogous to the choice we made (implicitly) in the case of $\mathfrak{sl}_2\mathbb{C}$ in choosing one of the two directions along the real line.

We said at the outset of this lecture that our goal was to arrive at a description of representations of $\mathfrak{sl}_3\mathbb{C}$ as complete as that for $\mathfrak{sl}_2\mathbb{C}$. We have now, certainly, as complete a description of the possible configurations of eigenvalues; but clearly much more is needed. Specifically, we should have

an existence and uniqueness theorem;

an explicit construction of each representations, analogous to the statement that every representation of $\mathfrak{sl}_2\mathbb{C}$ is a symmetric power of the standard; and

for the purpose of analyzing tensor products of representations of $\mathfrak{sl}_3\mathbb{C}$, we need a description not just of the set of eigenvalues, but of the multiplicities with which they occur.

(Note that the last question is one that has no analogue in the case of $\mathfrak{sl}_2\mathbb{C}$: in both cases, any irreducible representation is generated by taking a single eigenvector $v \in V_\alpha$ and pushing it around by elements of $\mathfrak{g}_\alpha$; but whereas in the previous case there was only one way to get from V_α to V_β—that is, by applying Y over and over again—in the present circumstance there will be more than one way of getting, for example, from V_α to $V_{\alpha+L_3-L_1}$; and these may yield independent eigenvectors.) This has been, however, already too long a lecture, and so we will defer these questions, along with all examples, to the next.

LECTURE 13

Representations of $\mathfrak{sl}_3\mathbb{C}$, Part II: Mainly Lots of Examples

In this lecture we complete the analysis of the irreducible representations of $\mathfrak{sl}_3\mathbb{C}$, culminating in §13.2 with the answers to all three of the questions raised at the end of the last lecture: we explicitly construct the unique irreducible representation with given highest weight, and in particular determine its multiplicities. The latter two sections correspond to §11.2 and §11.3 in the lecture on $\mathfrak{sl}_2\mathbb{C}$. In particular, §13.4, like §11.3, involves some projective algebraic geometry and may be skipped by those to whom this is unfamiliar.

§13.1: Examples
§13.2: Description of the irreducible representations
§13.3: A little more plethysm
§13.4: A little more geometric plethysm

§13.1. Examples

This lecture will be largely concerned with studying examples, giving constructions and analyzing tensor products of representations of $\mathfrak{sl}_3\mathbb{C}$. We start, however, by at least stating the basic existence and uniqueness theorem that provides the context for this analysis.

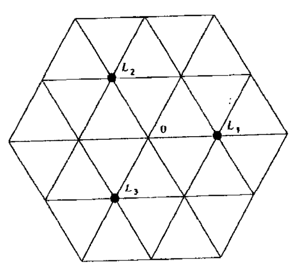
To state this, recall from the previous lecture than any irreducible, finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$ has a vector, unique up to scalars, that is simultaneously an eigenvector for the subalgebra $\mathfrak{h}$ and killed by the three subspaces $\mathfrak{g}_{L_1-L_2}$, $\mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_2-L_3}$. We called such a vector a *highest weight vector* of the representation V; its associated eigenvalue will, of course, be called the *highest weight* of V. More generally, in any finite-dimensional representation W of $\mathfrak{sl}_3\mathbb{C}$, any vector $v \in W$ with these properties will be called a highest weight vector; we saw that it will generate an irreducible subrepresentation V of W. Finally, from the description given in the last lecture of the possible configurations of eigenvalues for a representation of $\mathfrak{sl}_3\mathbb{C}$, we see that any highest weight vector must lie in the $(\frac{1}{6})$-plane described by the inequalities $\langle H_{1,2}, L\rangle \geq 0$ and $\langle H_{2,3}, L\rangle \geq 0$, i.e., it must be of the form $(a+b)L_1 + bL_2 = aL_1 - bL_3$ for some pair of non-negative integers a and b. We can now state

Theorem 13.1. *For any pair of natural numbers a, b there exists a unique irreducible, finite-dimensional representation $\Gamma_{a,b}$ of $\mathfrak{sl}_3\mathbb{C}$ with highest weight $aL_1 - bL_3$.*

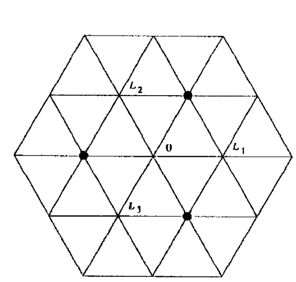
We will defer the proof of this theorem until the second section of this lecture, not so much because it is in any way difficult but simply because it is time to get to some examples. We will remark, however, that whereas in the case of $\mathfrak{sl}_2\mathbb{C}$ the analysis that led to the concept of highest weight vector immediately gave the uniqueness part of the analogous theorem, here to establish uniqueness we will be forced to resort to a more indirect trick. The proof of existence, by contrast, will be very much like that of the corresponding statement for $\mathfrak{sl}_2\mathbb{C}$: we will construct the representations $\Gamma_{a,b}$ out of the standard representation by multilinear algebra.

For the time being, though, we would like to apply the analysis of the previous lecture to some of the obvious representations of $\mathfrak{sl}_3\mathbb{C}$, partly to gain some familiarity with what goes on and partly in the hopes of seeing a general multilinear-algebraic construction.

We begin with the standard representation of $\mathfrak{sl}_3\mathbb{C}$ on $V \cong \mathbb{C}^3$. Of course, the eigenvectors for the action of $\mathfrak{h}$ are just the standard basis vectors e_1, e_2, and e_3; they have eigenvalues L_1, L_2, and L_3, respectively. The weight diagram for V is thus



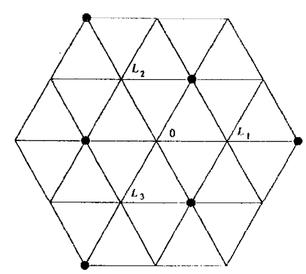
Next, consider the dual representation V^*. The eigenvalues of the dual of a representation of a Lie algebra are just the negatives of the eigenvalues of the original, so the diagram of V^* is



Alternatively, of course, we can just observe that the dual basis vectors e_i^* are eigenvectors with eigenvalues $-L_i$.

Note that while in the case of $\mathfrak{sl}_2\mathbb{C}$ the weights of any representation were symmetric about the origin, and correspondingly each representation was isomorphic to its dual, the same is not true here (that the diagrams for V and V^* look the same is a reflection of the fact that the two representations are carried into one another by an automorphism of $\mathfrak{sl}_3\mathbb{C}$, namely, the automorphism $X \mapsto -{}^tX$). Observe also that V^* is also isomorphic to the representation $\wedge^2 V$, whose weights are the pairwise sums of the distinct weights of V; and that likewise V is isomorphic as representation to $\wedge^2 V^*$.

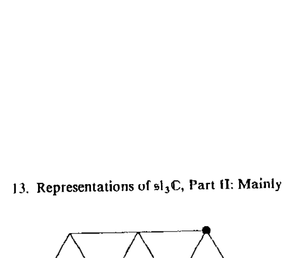
Next, consider the degree 2 tensor products of V and V^*. Since the weights of the symmetric square of a representation are the pairwise sums of he weights of he original, the weight diagram of $\mathrm{Sym}^2 V$ will look like



and likewise the symmetric square $\mathrm{Sym}^2 V^*$ has weights $\{-2L_i, -L_i - L_j\} = \{-2L_i - 2L_j, L_k\}$:

We see immediately from these diagrams that $\mathrm{Sym}^2 V$ and $\mathrm{Sym}^2 V^*$ are irreducible, since neither collection of weights is the union of two collections arising from representations of $\mathfrak{sl}_3\mathbb{C}$.

As for the tensor product $V \otimes V^*$, its weights are just the sums of the weights $\{L_i\}$ of V with those $\{-L_i\}$ of V^*, that is, the linear functionals $L_i - L_j$ (each occurring once, with weight vector $e_i \otimes e_j^*$) and 0 (occurring with multiplicity three, with weight vectors $e_i \otimes e_i^*$). We can represent these weights by the diagram



where the triple circle is intended to convey the fact that the weight space V_0 is three dimensional. By contrast with the last two examples, this representation is not irreducible: there is a linear map

$$V \otimes V^* \to \mathbb{C}$$

given simply by the contraction

$$v \otimes u^* \mapsto \langle v, u^* \rangle = u^*(v)$$

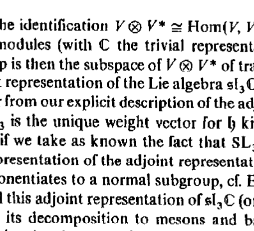
(or, in terms of the identification $V \otimes V^* \cong \mathrm{Hom}(V, V)$, by the *trace*) that is a map of $\mathfrak{sl}_3\mathbb{C}$-modules (with $\mathbb{C}$ the trivial representation, of course). The kernel of this map is then the subspace of $V \otimes V^*$ of traceless matrices, which is just the adjoint representation of the Lie algebra $\mathfrak{sl}_3\mathbb{C}$ and is irreducible (we can see this either from our explicit description of the adjoint representation—for example, $E_{1,3}$ is the unique weight vector for $\mathfrak{h}$ killed by $\mathfrak{g}_{L_1-L_2}$, $\mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_2-L_3}$—or, if we take as known the fact that $\mathrm{SL}_3\mathbb{C}$ is simple, from the fact that a subrepresentation of the adjoint representation is an *ideal* in a Lie algebra, and exponentiates to a normal subgroup, cf. Exercise 8.43.)

(Physicists call this adjoint representation of $\mathfrak{sl}_3\mathbb{C}$ (or SU(3)) the "eightfold way," and relate its decomposition to mesons and baryons. The standard representation V is related to "quarks" and V^* to "antiquarks." See [S-W], [Mack].)

(We note that, in general, if V is any faithful representation of a Lie algebra, the adjoint representation will appear as a subrepresentation of the tensor $V \otimes V^*$.)

Let us continue now with some of the triple tensor products of V and V^*, which will be the last specific cases we look at. To begin with, we have the symmetric cubes $\mathrm{Sym}^3 V$ and $\mathrm{Sym}^3 V^*$, with weight diagrams

and



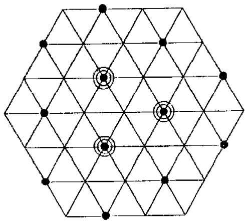
respectively. In general it is clear that, in terms of the description given in the preceding lecture of the possible weight diagrams of irreducible representations of $\mathfrak{sl}_3\mathbb{C}$, the symmetric powers of V and V^* will be exactly the representations with triangular, as opposed to hexagonal, diagrams.

It also follows from the above description and the fact that the weights of the symmetric powers $\mathrm{Sym}^n V$ occur with multiplicity 1 that $\mathrm{Sym}^n V$ and $\mathrm{Sym}^n V^*$ are all irreducible, i.e., we have, in the notation of Theorem 13.1,

$$\mathrm{Sym}^n V = \Gamma_{n,0} \quad \text{and} \quad \mathrm{Sym}^n V^* = \Gamma_{0,n}.$$

By way of notation, we will often write $\mathrm{Sym}^n V$ in place of $\Gamma_{n,0}$.

Consider now the mixed tensor $\mathrm{Sym}^2 V \otimes V^*$. Its weights are the sums of the weights of $\mathrm{Sym}^2 V$—that is, the pairwise sums of the L_i—with the weights of V^*; explicitly, these are $L_i + L_j - L_k$ and $2L_i - L_j$ (each occurring once) and the L_i themselves (each occurring three times, as $L_i + L_j - L_j$). Diagrammatically, the representation looks like



Now, we know right off the bat that this is not irreducible: we have a natural map

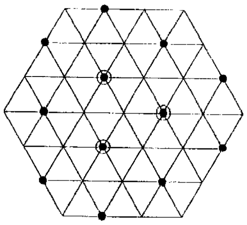
$$\iota\colon \mathrm{Sym}^2 V \otimes V^* \to V$$

given again by contraction, that is, by the map

$$vw \otimes u^* \mapsto \langle v, u^* \rangle \cdot w + \langle w, u^* \rangle \cdot v,$$

which is a map of $\mathfrak{sl}_3\mathbb{C}$-modules.[1] What does the kernel of this map look like? Of course, its weight diagram is

[1] Another way to see that $\mathrm{Sym}^2 V \otimes V^*$ is not irreducible is to observe that if a representation W is generated by a highest weight vector v of weight $2L_1 - L_3$, as $\mathrm{Sym}^2 V \otimes V^*$ must be if it is irreducible, the eigenvalue L_1 can be taken with multiplicity at most 2, the corresponding eigenspace being generated by $E_{1,2} \circ E_{2,3}(v)$ and $E_{2,3} \circ E_{1,2}(v)$.



and we know one other thing: certainly any vector in the weight space of $2L_1 - L_3$—that is to say, of course, any multiple of the vector $e_1^2 \otimes e_3^*$—is killed by $\mathfrak{g}_{L_1-L_2}$, $\mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_2-L_3}$, so that the kernel of ι will contain an irreducible representation $\Gamma = \Gamma_{2,1}$ with $2L_1 - L_3$ as its highest weight. Since Γ must then assume every weight of $\mathrm{Ker}(\iota)$, there are exactly two possibilities: either $\mathrm{Ker}(\iota) = \Gamma$, which assumes the weights L_i with multiplicity 2; or all the weights of Γ occur with multiplicity one and $\mathrm{Ker}(\iota) \cong \Gamma \oplus V$.

How do we settle this issue? There are at least three ways. To begin with, we can try to analyze directly the structure of the kernel of ι. An alternative approach would be to determine a priori with what multiplicities the weights of $\Gamma_{a,b}$ are taken. Certainly it is clear that a formula giving us the latter information will be tremendously valuable—it would for one thing clear up the present confusion instantly—and indeed there exist several such, one of which, the *Weyl character formula*, we will prove later in the book. (We will also prove the *Kostant multiplicity formula*, which can be applied to deduce directly the independence statement we arrive at below.) As a third possibility, we can identify the representations $\Gamma_{a,b}$ as Weyl modules and appeal to Lecture 6. Rather than invoke such general formulas at present, however, we will take the first approach here. This is straightforward: in terms of the notation we have been using, the highest weight vector for the representation $\Gamma \subset \mathrm{Sym}^2 V \otimes V^*$ is the vector $e_1^2 \otimes e_3^*$, and so the eigenspace $\Gamma_{L_1} \subset \Gamma$ with eigenvalue L_1 will be spanned by the images of this vector under the two compositions $E_{2,1} \circ E_{3,2}$ and $E_{3,2} \circ E_{2,1}$. These are, respectively,

$$\begin{aligned} E_{2,1} \circ E_{3,2}(e_1^2 \otimes e_3^*) &= E_{2,1}(E_{3,2}(e_1^2) \otimes e_3^* + e_1^2 \otimes E_{3,2}(e_3^*)) \\ &= E_{2,1}(-e_1^2 \otimes e_2^*) \\ &= -2(e_1 \cdot e_2) \otimes e_2^* + e_1^2 \otimes e_1^* \end{aligned}$$

and

$$\begin{aligned} E_{3,2} \circ E_{2,1}(e_1^2 \otimes e_3^*) &= E_{3,2}(E_{2,1}(e_1^2) \otimes e_3^* + e_1^2 \otimes E_{2,1}(e_3^*)) \\ &= E_{3,2}((2e_1 \cdot e_2) \otimes e_3^*) \\ &= 2(e_1 \cdot e_3) \otimes e_3^* - 2(e_1 \cdot e_2) \otimes e_2^*. \end{aligned}$$

Since these are independent, we conclude that the weight L_1 does occur in Γ with multiplicity 2, and hence that the kernel of ι is irreducible, i.e.,

$$\mathrm{Sym}^2 V \otimes V^* \cong \Gamma_{2,1} \oplus V.$$

§13.2. Description of the Irreducible Representations

At this point, rather than go on with more examples we should state some of the general principles that have emerged so far. The first and most important (though pretty obvious) is the basic

Observation 13.2. *If the representations V and W have highest weight vectors v and w with weights α and β, respectively, then the vector $v \otimes w \in V \otimes W$ is a highest weight vector of weight $\alpha + \beta$.*

Of course, there are numerous generalizations of this: the vector $v^n \in \mathrm{Sym}^n V$ is a highest weight vector of weight $n\alpha$, etc.[2] Just the basic statement above, however, enables us to give the

PROOF OF THEOREM 13.1. First, the existence statement follows immediately from the observation: the representation $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ will contain an irreducible subrepresentation $\Gamma_{a,b}$ with highest weight $aL_1 - bL_3$.

The uniqueness part is only slightly harder (if less explicit): Given irreducible representations V and W with highest weight α, let $v \in V$ and $w \in W$ be highest weight vectors with weight α. Then (v, w) is again a highest weight vector in the representation $V \oplus W$ with highest weight α; let $U \subset V \oplus W$ be the irreducible subrepresentation generated by (v, w). The projection maps $\pi_1: U \to V$ and $\pi_2: U \to W$, being nonzero maps between irreducible representations of $\mathfrak{sl}_3\mathbb{C}$, must be isomorphisms, and we deduce that $V \cong W$. □

Exercise 13.3*. Let $\mathbb{S}_\lambda$ be the Schur functor introduced in Lecture 6. What can you say about the highest weight vectors in the representation $\mathbb{S}_\lambda(V)$ obtained by applying it to a given representation V?

To continue our discussion of tensor products like $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ in general, as we indicated we would like to make more explicit the construction of the representation $\Gamma_{a,b}$, which we know to be lying in $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$. To begin with, we have in general a contraction map

$$\iota_{a,b}: \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* \to \mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^*$$

analogous to the map ι introduced above; we can describe this map either (in fancy language) as the dual of the map from $\mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^*$ to $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ given by multiplication by the identity element in $V \otimes V^* = \mathrm{Hom}(V, V)$; or, concretely, by sending

$$(v_1 \cdot \ldots \cdot v_a) \otimes (v_1^* \cdot \ldots \cdot v_b^*)$$
$$\mapsto \sum \langle v_i, v_j^* \rangle (v_1 \cdot \ldots \cdot \hat{v}_i \cdot \ldots \cdot v_a) \otimes (v_1^* \cdot \ldots \cdot \hat{v}_j^* \cdot \ldots \cdot v_b^*).$$

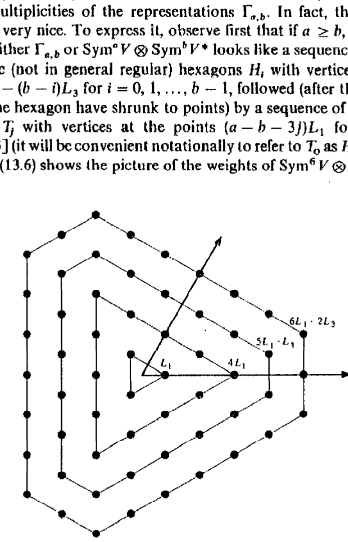
Clearly this map is surjective, and, since the target does not have eigenvalue $aL_1 - bL_3$, the subrepresentation $\Gamma_{a,b} \subset \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ must lie in the kernel. In fact, we have, just as in the case of $\mathrm{Sym}^2 V \otimes V^*$ above,

Claim 13.4. *The kernel of the map $\iota_{a,b}$ is the irreducible representation $\Gamma_{a,b}$.*

We will defer the proof of this for a moment and consider some of its consequences. To begin with, we can deduce from this assertion the complete decomposition of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$: we must have (if, say, $b \le a$)

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* = \bigoplus_{i=0}^{b} \Gamma_{a-i,b-i}. \qquad (13.5)$$

Since we know, a priori, all the multiplicities of the eigenvalues of the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$, this will, in turn, determine (inductively at least) all the multiplicities of the representations $\Gamma_{a,b}$. In fact, the answer turns out to be very nice. To express it, observe first that if $a \ge b$, the weight diagram of either $\Gamma_{a,b}$ or $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ looks like a sequence of b shrinking concentric (not in general regular) hexagons H_i with vertices at the points $(a-i)L_1 - (b-i)L_3$ for $i = 0, 1, \ldots, b-1$, followed (after the shorter three sides of the hexagon have shrunk to points) by a sequence of $[(a-b)/3] + 1$ triangles T_j with vertices at the points $(a-b-3j)L_1$ for $j = 0, 1, \ldots, [(a-b)/3]$ (it will be convenient notationally to refer to T_0 as H_b occasionally). Diagram (13.6) shows the picture of the weights of $\mathrm{Sym}^6 V \otimes \mathrm{Sym}^2 V^*$:



(13.6)

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[2] One slightly less obvious statement is this: if the weights of V are $\alpha_1, \alpha_2, \alpha_3 \ldots$ with $l(\alpha_1) > l(\alpha_2) > \ldots$, then $\bigwedge^n V$ possesses a highest weight vector weight $\alpha_1 + \cdots + \alpha_n$. Note that since the ordering of the α_i may in fact depend on the choice of l (even with the restriction $a > b > c$ on the coefficients of l as above), this may in some cases imply the existence of several subrepresentations of $\bigwedge^n V$.

(Note that by the decomposition (13.5), the weights of the highest weight vectors in $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ will be $aL_1 - bL_3, (a-1)L_1 - (b-1)L_3, \ldots, (a-b)L_1$, as shown in the diagram.)

An examination of the representation $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ shows that it has multiplicity $(i+1)(i+2)/2$ on the hexagon H_i, and then a constant multiplicity $(b+1)(b+2)/2$ on all the triangles T_j; and it follows from the decomposition (13.5), in general, that the representation $\Gamma_{a,b}$ has multiplicity $(i+1)$ on H_i and b on T_j. In English, *the multiplicities of $\Gamma_{a,b}$ increase by one on each of the concentric hexagons of the eigenvalue diagram and are constant on the triangles.* Note in particular that the description of $\Gamma_{2,1}$ in the preceding section is a special case of this.

Proof of Claim 13.4. We remark first that the claim will be implied by the Weyl character formula or by the description via Weyl's construction in Lecture 15; so the reader who wishes to can skip the following without dire consequences to the logical structure of the book. Otherwise, observe first that the claim is equivalent to asserting the decomposition (13.5); this, in turn, is equivalent to the statement that the representation $W = \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ has exactly $b+1$ irreducible components (still assuming $a \geq b$). The irreducible factors in a representation correspond to the highest weight vectors in the representation up to scalars; so in sum the claim is equivalent to the assertion that *the eigenspace W_α of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ contains a unique highest weight vector (up to scalars) if α is of the form $(a-i)L_1 - (b-i)L_3$ for $i \leq b$, and none otherwise*; this is what we shall prove.

To begin with, the "none otherwise" part of the statement follows (given the other) just from looking at the diagram: if, for example, any of the eigenspaces W_α corresponding to a point α on a hexagon H_i (other than the vertex $(a-i)L_1 - (b-i)L_3$ of H_i) possessed a highest weight vector, the multiplicity of α in W would be strictly greater than of $(a-i)L_1 - (b-i)L_3$, which we know is not the case; similarly, the fact that the multiplicities of W in the triangular part of the eigenvalue diagram are constant implies that there can be no highest weight vectors with eigenvalue on a T_j for $j \geq 1$. Thus, we just have to check that the weight spaces W_α for $\alpha = (a-i)L_1 - (b-i)L_3$ contain only the one highest weight vector we know is there; and we do this by explicit calculation.

To start, for any monomial index $I = (i_1, i_2, i_3)$ of degree $\sum i_\gamma = i$, we denote by $e^I \in \mathrm{Sym}^i V$ the corresponding monomial $\prod(e_\gamma^{i_\gamma})$ and define $(e^*)^I \in \mathrm{Sym}^i V^*$ similarly. We can then write any element of the weight space $W_{(a-i)L_1-(b-i)L_3}$ of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ as

$$v = \sum c_I \cdot (e_1^{a+b-i} \cdot e^I) \otimes ((e_3^*)^{b-i} \cdot (e^*)^I).$$

In these terms, it is easy to write down the action of the two operators $E_{1,2}$

and $E_{2,3}$. First, $E_{1,2}$ kills both $e_1 \in V$ and $e_3^* \in V^*$, so that we have

$$\begin{aligned} E_{1,2}((e_1^{a+b-i} \cdot e^I) &\otimes ((e_3^*)^{b-i} \cdot (e^*)^I)) \\ &= i_2(e_1^{a+b-i} \cdot e^{I'}) \otimes ((e_3^*)^{b-i} \cdot (e^*)^I) \\ &\quad - i_1(e_1^{a+b-i} \cdot e^I) \otimes ((e_3^*)^{b-i} \cdot (e^*)^{I''}), \end{aligned}$$

where $I' = (i_1 + 1, i_2 - 1, i_3)$ and $I'' = (i_1 - 1, i_2 + 1, i_3)$ (and we adopt the convention that $e^I = 0$ if $i_\gamma < 0$ for any γ). It follows that *the vector v above is in the kernel of $E_{1,2}$ if and only if the coefficients c_I satisfy $i_2 c_I = (i_1 + 1)c_{I'}$*; and by the analogous calculation that v is in the kernel of $E_{2,3}$ if and only if $i_3 c_I = (i_2 + 1)c_J$ whenever the indices I and J are related by $j_1 = i_1$, $j_2 = i_2 + 1$, and $j_3 = i_3 - 1$. These conditions are equivalent to saying that the numbers $i_1! i_2! i_3! c_I$ are independent of I. We see, in other words, that v is a highest weight vector if and only if all the coefficients c_I are equal to $c/i_1! i_2! i_3!$ for some constant c. □

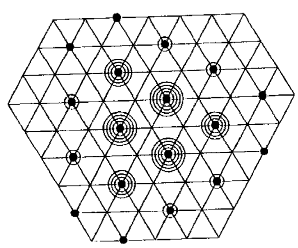
§13.3. A Little More Plethysm

We would like to consider here, as we did in the case of $\mathfrak{sl}_2\mathbb{C}$ in Lecture 11, how the tensor products and powers of the representations we have described decompose. We start with one general remark: given our knowledge of the eigenvalue diagrams of the irreducible representations of $\mathfrak{sl}_3\mathbb{C}$ (*with* multiplicities), there can be no possible ambiguity about the decomposition of any representation U given as the tensor product of representations whose eigenvalue diagrams are known. Indeed, we have an algorithm for determining the components of that decomposition, as follows:

1. Write down the eigenvalue decomposition of U.
2. Find the eigenvalue $\alpha = aL_1 - bL_3$ appearing in this diagram for which the value of $l(\alpha)$ is maximal.
3. We now know that U will contain a copy of the irreducible representation $\Gamma_\alpha = \Gamma_{a,b}$, i.e., $U \cong \Gamma_\alpha \oplus U'$ for some U'. Since we also know the eigenvalue diagram of Γ_α, we can thus write down the eigenvalue diagram of U' as well.
4. Repeat this process for U'.

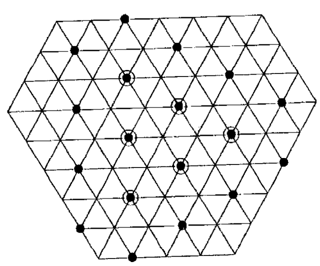
To see how this goes in practice, consider some examples of tensor products of the basic irreducible representations described so far. We have already seen how the tensor products of the symmetric powers of the standard representation V of $\mathfrak{sl}_3\mathbb{C}$ and symmetric powers of its dual decompose; let us look now at an example of a more general tensor product of irreducible representations: say V itself and the representation $\Gamma_{2,1}$. We start by writing down the weights of the tensor product: since $\Gamma_{2,1}$ has weights $2L_i - L_j$, $L_i + L_j - L_k$, and L_i

(taken twice) and V has weights L_i, the tensor product will have weights $3L_i - L_j$, $2L_i + L_j - L_k$ (taken twice), $2L_i$ (taken four times), and $L_i + L_j$ (taken five times). The diagram is thus

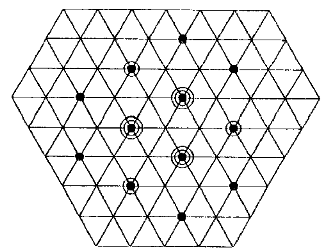


(One thing we may deduce from this diagram is that we are soon going to need a better system for presenting the data of the weights of a representation. In the future, we may simply draw one sector of the plane, and label weights with numbers to indicate multiplicities.)

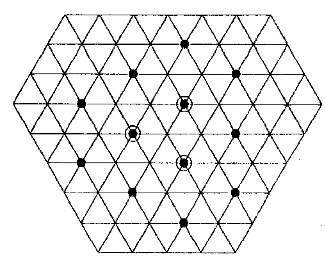
We know right off the bat that the tensor product $V \otimes \Gamma_{2,1}$ contains a copy of the irreducible representation $\Gamma_{3,1}$ with highest weight $3L_1 - L_3$. By what we have said, the weight diagram of $\Gamma_{3,1}$ is



so the complement of $\Gamma_{3,1}$ in the tensor product $V \otimes \Gamma_{2,1}$ will look like



One obvious highest weight in this representation is $2L_1 + L_2 - L_3 = L_1 - 2L_3$, so that the tensor product will contain a copy of the irreducible representation $\Gamma_{1,2}$ as well; since this has weight diagram

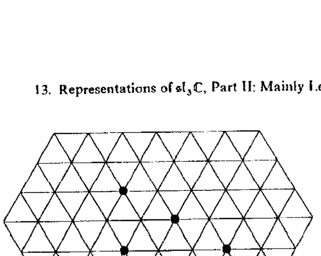


the remaining part of the tensor product will have weight diagram which we recognize as the weight diagram of the symmetric square $\mathrm{Sym}^2 V = \Gamma_{2,0}$ of the standard representation. We have, thus,

$$V \otimes \Gamma_{2,1} = \Gamma_{3,1} \oplus \Gamma_{1,2} \oplus \Gamma_{2,0}. \tag{13.7}$$

Exercise 13.8*. Find the decomposition into irreducible representations of the tensor products $V \otimes \Gamma_{1,1}$, $V \otimes \Gamma_{1,2}$ and $V \otimes \Gamma_{3,1}$. Can you find a general pattern to the outcomes?

As in the case of $\mathfrak{sl}_2\mathbb{C}$, the next thing to look at are the tensor powers—symmetric and exterior—of representations other than the standard; we look first at tensors of the symmetric square $W = \mathrm{Sym}^2 V$. First, consider the symmetric square $\mathrm{Sym}^2 W = \mathrm{Sym}^2(\mathrm{Sym}^2 V)$. We know the diagram for $\mathrm{Sym}^2 W$; it is



Now, there is only one possible decomposition of a representation whose eigenvalue diagram looks like this: we must have

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V) \cong \mathrm{Sym}^4 V \oplus \mathrm{Sym}^2 V^*.$$

Indeed, the presence of the $\mathrm{Sym}^4 V$ factor is clear: there is an obvious map

$$\varphi\colon \mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$$

obtained simply by multiplying out. The identification of the kernel of this map with the representation $\mathrm{Sym}^2 V^*$ is certainly less obvious, but can still be made explicit. We can identify V^* with $\Lambda^2 V$ as we saw, and then define a map

$$\tau\colon \mathrm{Sym}^2(\Lambda^2 V) \to \mathrm{Sym}^2(\mathrm{Sym}^2 V))$$

by sending the generator $(u \wedge v)\cdot(w \wedge z) \in \mathrm{Sym}^2(\Lambda^2 V)$ to the element $(u \cdot w)\cdot(v \cdot z) - (u \cdot z)\cdot(v \cdot w) \in \mathrm{Sym}^2(\mathrm{Sym}^2 V)$, which is clearly in the kernel of φ.

Exercise 13.9. Verify that this map is well defined and that it extends linearly to an isomorphism of $\mathrm{Sym}^2(\Lambda^2 V)$ with $\mathrm{Ker}(\varphi)$.

Exercise 13.10. Apply the techniques above to show that the representation $\Lambda^2(\mathrm{Sym}^2 V)$ is isomorphic to $\Gamma_{2,1}$.

Exercise 13.11. Apply the same techniques to determine the irreducible factors of the representation $\Lambda^3(\mathrm{Sym}^2 V)$. Note: we will return to this example in Exercise 13.22.

Exercise 13.12. Find the decomposition into irreducibles of the representations $\mathrm{Sym}^2(\mathrm{Sym}^3 V)$ and $\mathrm{Sym}^3(\mathrm{Sym}^2 V)$ (observe in particular that Hermite reciprocity has bitten the dust). Describe the projection maps to the various factors. Note: we will describe these examples further in the following section.

§13.4. A Little More Geometric Plethysm

Just as in the case of $\mathfrak{sl}_2\mathbb{C}$, some of these identifications can also be seen in geometric terms. To do this, recall from §11.3 the definition of the *Veronese embedding*: if $\mathbb{P}^2 = \mathbb{P}V^*$ is the projective space of one-dimensional subspaces of V^*, there is then a natural embedding of $\mathbb{P}^2$ in the projective space $\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2 V^*)$, obtained simply by sending the point $[v^*] \in \mathbb{P}^2$ corresponding to the vector $v^* \in V^*$ to the point $[v^{*2}] \in \mathbb{P}(\mathrm{Sym}^2 V^*)$ associated to the vector $v^{*2} = v^* \cdot v^* \in \mathrm{Sym}^2 V^*$. The image $S \subset \mathbb{P}^5$ is called the *Veronese surface*. As in the case of the rational normal curves discussed in Lecture 11, it is not hard to see that the group of automorphisms of $\mathbb{P}^5$ carrying S into itself is exactly the group $\mathrm{PGL}_3\mathbb{C}$ of automorphisms of $S = \mathbb{P}^2$.

Now, a quadratic polynomial in the homogeneous coordinates of the space $\mathbb{P}(\mathrm{Sym}^2 V^*) \cong \mathbb{P}^5$ will restrict to a quartic polynomial on the Veronese surface $S = \mathbb{P}V^*$, which corresponds to the natural evaluation map φ of the preceding section; the kernel of this map is thus the vector space of quadratic polynomials in $\mathbb{P}^5$ vanishing on the Veronese surface S, on which the group of automorphisms of $\mathbb{P}^5$ carrying S to itself obviously acts. Now, for any pair of points $P = [u^*]$, $Q = [v^*] \in S$, it is not hard to see that the cone over the Veronese surface with vertex the line $\overline{PQ} \subset \mathbb{P}^5$ (that is, the union of the 2-planes $\overline{PQR}$ as R varies over the surface S) will be a quadric hypersurface in $\mathbb{P}^5$ containing the Veronese surface; sending the generator $u^* \cdot v^* \in \mathrm{Sym}^2 V^*$ to this quadric hypersurface will then define an isomorphism of the space of such quadrics with the projective space associated to $\mathrm{Sym}^2 V^*$.

Exercise 13.13. Verify the statements made in the last paragraph: that the union of the $\overline{PQR}$ is a quadric hypersurface and that this extends to a linear isomorphism $\mathbb{P}(\mathrm{Sym}^2 V^*) \cong \mathbb{P}(\mathrm{Ker}(\varphi))$. Verify also that this isomorphism coincides with the one given in Exercise 13.9.

There is another way of representing the Veronese surface that will shed some light on this kernel. If, in terms of some coordinates e_i on V^*, we think of $\mathrm{Sym}^2 V^*$ as the vector space of symmetric 3×3 matrices, then the Veronese surface is just the locus, in the associated projective space, of rank 1 matrices up to scalars, i.e., in terms of homogeneous coordinates $Z_{i,j} = e_i \cdot e_j$ on $\mathbb{P}^5$,

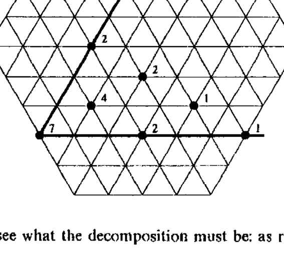
$$S = \left\{ [Z] : \mathrm{rank}\begin{pmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} \\ Z_{1,2} & Z_{2,2} & Z_{2,3} \\ Z_{1,3} & Z_{2,3} & Z_{3,3} \end{pmatrix} = 1 \right\}.$$

The vector space of quadratic polynomials vanishing on S is then generated by the 2×2 minors of the matrix $(Z_{i,j})$; in particular, for any pair of linear combinations of the rows and pair of linear combinations of the columns we get a 2×2 matrix whose determinant vanishes on S.

Exercise 13.14. Show that this is exactly the isomorphism $\mathrm{Sym}^2(\wedge^2 V) \cong \mathrm{Ker}(\varphi)$ described above.

We note in passing that if indeed the space of quadrics containing the Veronese surface, with the action of the group $\mathrm{PGL}_3\mathbb{C}$ of motions of $\mathbb{P}^5$ preserving S, is the projectivization of the representation $\mathrm{Sym}^2 V^*$, then it must contain its own Veronese surface, i.e., there must be a surface $T = \mathbb{P}(V^*) \subset \mathbb{P}(\mathrm{Ker}(\varphi))$ invariant under this action. This turns out to be just the set of *quadrics of rank* 3 containing the Veronese, that is, the quadrics whose singular locus is a 2-plane. In fact, the 2-plane will be the tangent plane to S at a point, giving the identification $T = S$.

Let us consider one more example of this type, namely, the symmetric cube $\mathrm{Sym}^3(\mathrm{Sym}^2 V)$. (We promise we will stop after this one.) As before, it is easy to write down the eigenvalues of this representation; they are just the triple sums of the eigenvalues $\{2L_i, L_i + L_j\}$ of $\mathrm{Sym}^2 V$. The diagram (we will draw here only one-sixth of the plane and indicate multiplicities with numbers rather than rings) thus looks like



from which we see what the decomposition must be: as representations we have

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V) \cong \mathrm{Sym}^6 V \oplus \Gamma_{2,2} \oplus \mathbb{C}. \qquad (13.15)$$

As before, the map to the first factor is just the obvious one; it is the identification of the kernel that is intriguing, and especially the identification of the last factor.

To see what is going on here, we should look again at the geometry of the Veronese surface $S \subset \mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2 V^*)$. The space $\mathrm{Sym}^3(\mathrm{Sym}^2 V)$ is just the space of homogeneous cubic polynomials on the ambient space $\mathbb{P}^5$, and as before the map to the first factor of the right-hand side of (13.15) is just the restriction, so that the last two factors of (13.15) represent the vector space $I(S)_3$ of cubic polynomials vanishing on S. Note that we could in fact prove (13.15) without recourse to eigenvalue diagrams from this: since the ideal of the Veronese surface is generated by the vector space $I(S)_2$ of quadratic polynomials vanishing on it, we have a surjective map

$$I(S)_2 \otimes W = \mathrm{Sym}^2 V^* \otimes \mathrm{Sym}^2 V \to I(S)_3.$$

But we already know how the left hand side decomposes: we have

$$\mathrm{Sym}^2 V^* \otimes \mathrm{Sym}^2 V = \Gamma_{2,2} \oplus \Gamma_{1,1} \oplus \mathbb{C}, \qquad (13.16)$$

so that $I(S)_3$ must be a partial direct sum of these three irreducible representations; by dimension considerations it can only be $\Gamma_{2,2} \oplus \mathbb{C}$.

This, in turn, tells us how to make the isomorphism (13.15) explicit (assuming we want to): we can define a map

$$\mathrm{Sym}^2(\wedge^2 V) \otimes \mathrm{Sym}^2 V \to \mathrm{Sym}^3(\mathrm{Sym}^2 V)$$

by sending

$$(u \wedge v)\cdot(w \wedge z) \otimes (s \cdot t) \mapsto ((u \cdot w)\cdot(v \cdot z) - (u \cdot z)\cdot(v \cdot w))\cdot(s \cdot t)$$

and then just check that this gives an isomorphism of $\Gamma_{2,2} \oplus \mathbb{C} \subset \mathrm{Sym}^2 V^* \otimes \mathrm{Sym}^2 V$ with the kernel of projection on the first factor of the right-hand side of (13.15).

What is really most interesting in this whole situation, though, is the trivial summand in the expression (13.15). To say that there is such a summand is to say that *there exists a cubic hypersurface X in $\mathbb{P}^5$ preserved under all automorphisms of $\mathbb{P}^5$ carrying S to itself.* Of course, we have already run into this one: it is the determinant of the 3×3 matrix $(Z_{i,j})$ introduced above. To express this more intrinsically, if we think of the Veronese as the set of rank 1 tensors in $\mathrm{Sym}^2 V^*$, it is just the set of tensors of rank 2 or less. This, in turn, yields another description of X: since a rank 2 tensor is just one that can be expressed as a linear combination of two rank 1 tensors, we see that X is the famous *chordal variety* of the Veronese surface: it is the union of the chords to S, and at the same time the union of all the tangent planes to S.

Exercise 13.17. Show that the only symmetric powers of $\mathrm{Sym}^2 V$ that possess trivial summands are the powers $\mathrm{Sym}^{3k}(\mathrm{Sym}^2 V)$ divisible by 3, and that the unique trivial summand in this is just the kth power of the trivial summand of $\mathrm{Sym}^3(\mathrm{Sym}^2 V)$.

Exercise 13.18. Given the isomorphism of the projectivization of the vector space $I(S)_2$—that is, the projective space of quadric hypersurfaces containing the Veronese surface—with $\mathbb{P}(\mathrm{Sym}^2 V^*)$, find the unique cubic hypersurface in $I(S)_2$ invariant under the action of $\mathrm{PGL}_3\mathbb{C}$.

Exercise 13.19. Analyze the representation $\mathrm{Sym}^2(\mathrm{Sym}^3 V)$ of $\mathfrak{sl}_3\mathbb{C}$. Interpret the direct sum factors in terms of the geometry of the Veronese embedding of $\mathbb{P}V^* = \mathbb{P}^2$ in $\mathbb{P}(\mathrm{Sym}^3 V^*) = \mathbb{P}^9$.

Exercise 13.20*. Show that the representations $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ and $\mathrm{Sym}^6(\mathrm{Sym}^3 V)$ contain trivial summands, and that the representation $\mathrm{Sym}^{12}(\mathrm{Sym}^3 V)$ contains two. Interpret these.

Exercise 13.21. Apply the techniques above to show that the representation $\wedge^2(\mathrm{Sym}^2 V)$ is isomorphic to $\Gamma_{2,1}$.

Exercise 13.22*. Apply the techniques above to analyze the representation $\wedge^3(\mathrm{Sym}^2 V)$, and in particular to interpret its decomposition into irreducible representations.

Exercise 13.23. If $\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2 V^*)$ is the ambient space of the Veronese surface, the Grassmannian $G(2, 5)$ of 2-planes in $\mathbb{P}^5$ naturally embeds in the projective space $\mathbb{P}(\wedge^3(\mathrm{Sym}^2 V))$. Describe, in terms of the decomposition in the preceding exercise, the span of the locus of tangent 2-planes to the

Veronese, and the span of the locus of 2-planes in $\mathbb{P}^5$ spanned by the images in S of lines in $\mathbb{P}V^*$.

Exercise 13.24*. Show that the unique closed orbit of the action of $\mathrm{SL}_3\mathbb{C}$ on the representation $\Gamma_{a,b}$ is either isomorphic to $\mathbb{P}^2$ (embedded as the Veronese surface) if either a or b is zero, or to the incidence correspondence

$$\Sigma = \{(p, l) \colon p \in l\} \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$$

if neither a or b is zero.

(№6) Lectures 14, 15

PART III

THE CLASSICAL LIE ALGEBRAS AND THEIR REPRESENTATIONS

As we indicated at the outset, the analysis we have just carried out of the structure of $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$ and their representations carries over to other semisimple complex Lie algebras. In Lecture 14 we codify this structure, using the pattern of the examples we have worked out so far to give a model for the analysis of arbitrary semisimple Lie algebras and stating some of the most important facts that are true in general. As usual, we postpone proofs of many of these facts until Part IV and the Appendices, the main point here being to introduce a unifying approach and language. The facts themselves will all be seen explicitly on a case-by-case basis for the classical Lie algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_n\mathbb{C}$, which are studied in some detail in Lectures 15–20.

Most of the development follows the outline we developed in Lectures 11–13, the main goal being to describe the irreducible representations as explicitly as we can, and to see the decomposition of naturally occurring representations, both algebraically and geometrically. While most of the representations are found inside tensor powers of the standard representations, for the orthogonal Lie algebras this only gives half of them, and one needs new methods to construct the other "spin" representations. This is carried out using Clifford algebras in Lecture 20.

We also make the tie with Weyl's construction of representations of $GL_n\mathbb{C}$ from Lecture 6, which arose from the representation theory of the symmetric groups. We show in Lecture 15 that these are the irreducible representations of $\mathfrak{sl}_n\mathbb{C}$; in Lecture 17 we show how to use them to construct the irreducible representations of the symplectic Lie algebras, and in Lecture 19 to give the nonspin representation of the orthogonal Lie algebras. These give useful descriptions of the irreducible representations, and powerful methods for decomposing other representations, but they are not necessary for the logical progression of the book, and many of these decompositions can also be deduced from the Weyl character formula which we will discuss in Part IV.

LECTURE 14

The General Setup: Analyzing the Structure and Representations of an Arbitrary Semisimple Lie Algebra

This is the last of the four central lectures; in the body of it, §14.1, we extract from the examples of §11–13 the basic algorithm for analyzing a general semisimple Lie algebra and its representations. It is this algorithm that we will spend the remainder of Part III carrying out for the classical algebras, and the reader who finds the general setup confusing may wish to read this lecture in parallel with, for example, Lectures 15 and 16. In particular, §14.2 is less clearly motivated by what we have worked out so far; the reader may wish to skim it for now and defer a more thorough reading until after going through some more of the examples of Lectures 15–20.

§14.1: Analyzing simple Lie algebras in general
§14.2: About the Killing form

§14.1. Analyzing Simple Lie Algebras in General

We said at the outset of Lecture 12 that once the analysis of the representations of $\mathfrak{sl}_3\mathbb{C}$ was understood, the analysis of the representations of any semisimple Lie algebra would be clear, at least in broad outline. Here we would like to indicate how that analysis will go in general, by providing an essentially algorithmic procedure for describing the representations of an arbitrary complex semisimple Lie algebra $\mathfrak{g}$. The process we give here is directly analogous, step for step, to that carried out in Lecture 12 for $\mathfrak{sl}_3\mathbb{C}$; the only difference is one change in the order of steps: having seen in the case of $\mathfrak{sl}_3\mathbb{C}$ the importance of the "distinguished" subalgebras $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C} \subset \mathfrak{g}$ and the corresponding distinguished elements $H_\alpha \in \mathfrak{s}_\alpha \subset \mathfrak{h}$, we will introduce them earlier here.

Step 0. *Verify that your Lie algebra is semisimple*; if not, none of the following will work (but see Remark 14.3). If your Lie algebra is not semisimple, pass as indicated in Lecture 9 to its semisimple part; a knowledge of the representations of this quotient algebra may not tell you everything about the representations of the original, but it will at least tell you about the irreducible representations.

Step 1. *Find an abelian subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *acting diagonally*. This is of course the analogue of looking at the specific element H in $\mathfrak{sl}_2\mathbb{C}$ and the subalgebra $\mathfrak{h}$ of diagonal matrices in the case of $\mathfrak{sl}_3\mathbb{C}$; in general, to serve an analogous function it should be an abelian subalgebra that acts diagonally on one faithful (and hence, by Theorem 9.20, on any) representation of $\mathfrak{g}$. Moreover, in order that the restriction of a representation V of $\mathfrak{g}$ to $\mathfrak{h}$ carry the greatest possible information about V, $\mathfrak{h}$ should clearly be maximal among abelian, diagonalizable subalgebras; such a subalgebra is called a *Cartan subalgebra*.

Note that while this step would seem to be somewhat less than algorithmic (in particular, while it is certainly possible to tell when a subalgebra of a given Lie algebra is abelian, and when it is diagonalizable, it is not clear how to tell whether it is maximal with respect to these properties). This defect will, however, be largely cleared up in the next step (see Remark 14.3).

Step 2. *Let* $\mathfrak{h}$ *act on* $\mathfrak{g}$ *by the adjoint representation, and decompose* $\mathfrak{g}$ *accordingly*. By the choice of $\mathfrak{h}$, its action on any representation of $\mathfrak{g}$ will be diagonalizable; applying this to the adjoint representation we arrive at a direct sum decomposition, called a *Cartan decomposition*,

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_\alpha\right), \tag{14.1}$$

where the action of $\mathfrak{h}$ preserves each $\mathfrak{g}_\alpha$ and acts on it by scalar multiplication by the linear functional $\alpha \in \mathfrak{h}^*$; that is, for any $H \in \mathfrak{h}$ and $X \in \mathfrak{g}_\alpha$ we will have

$$\mathrm{ad}(H)(X) = \alpha(H) \cdot X.$$

The second direct sum in the expression (14.1) is over a finite set of eigenvalues $\alpha \in \mathfrak{h}^*$; these eigenvalues—in the language of Lecture 12, the *weights of the adjoint representation*—are called the *roots* of the Lie algebra and the corresponding subspaces $\mathfrak{g}_\alpha$ are called the *root spaces*. Of course, $\mathfrak{h}$ itself is just the eigenspace for the action of $\mathfrak{h}$ corresponding to the eigenvalue 0 (see Remark 14.3 below); so that in some contexts—such as the following paragraph, for example—it will be convenient to adopt the convention that $\mathfrak{g}_0 = \mathfrak{h}$; but we do not usually count $0 \in \mathfrak{h}^*$ as a root. The set of all roots is usually denoted $R \subset \mathfrak{h}^*$.

As in the previous cases, we can picture the structure of the Lie algebra in terms of the diagram of its roots: by the fundamental calculation of §11.1 and Lecture 12 (which we will not reproduce here for the fourth time) we see that the adjoint action of $\mathfrak{g}_\alpha$ carries the eigenspace $\mathfrak{g}_\beta$ into another eigenspace $\mathfrak{g}_{\alpha+\beta}$.

There are a couple of things we can anticipate about how the configuration of roots (and the corresponding root spaces) will look. We will simply state them here as

Facts 14.2

(i) *each root space* $\mathfrak{g}_\alpha$ *will be one dimensional.*
(ii) R *will generate a lattice* $\Lambda_R \subset \mathfrak{h}^*$ *of rank equal to the dimension of* $\mathfrak{h}$.

(iii) R *is symmetric about the origin, i.e., if* $\alpha \in R$ *is a root, then* $-\alpha \in R$ *is a root as well.*

These facts will all be proved in general in due course; for the time being, they are just things we will observe as we do the analysis of each simple Lie algebra in turn. We mention them here simply because some of what follows will make sense only given these facts. Note in particular that by (ii), the roots all lie in (and span) a real subspace of $\mathfrak{h}^*$; all our pictures clearly will be of this real subspace.

Remark 14.3. If indeed 0 does appear as an eigenvalue of the action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$, then we may conclude from this that $\mathfrak{h}$ was not maximal to begin with: by the above, anything in the 0-eigenspace of the action of $\mathfrak{h}$ commutes with $\mathfrak{h}$ and (given the fact that the $\mathfrak{g}_\alpha$ are one dimensional) acts diagonally on $\mathfrak{g}$, so that if it not already in $\mathfrak{h}$, then $\mathfrak{h}$ could be enlarged while still retaining the properties of being abelian and diagonalizable. Similarly, the assertion in (ii) that the roots span $\mathfrak{h}^*$ follows from the fact that an element of $\mathfrak{h}$ in the annihilator of all of them would be in the center of $\mathfrak{g}$.

From what we have done so far, we get our first picture of the structure of an arbitrary irreducible finite-dimensional representation V of $\mathfrak{g}$. Specifically, V will admit a direct sum decomposition

$$V = \bigoplus V_\alpha, \tag{14.4}$$

where the direct sum runs over a finite set of $\alpha \in \mathfrak{h}^*$ and $\mathfrak{h}$ acts diagonally on each V_α by multiplication by the eigenvalue α, i.e., for any $H \in \mathfrak{h}$ and $v \in V_\alpha$ we will have

$$H(v) = \alpha(H) \cdot v.$$

The eigenvalues $\alpha \in \mathfrak{h}^*$ that appear in this direct sum decomposition are called the *weights* of V; the V_α themselves are called *weight spaces*; and the dimension of a weight space V_α will be called the *multiplicity* of the weight α in V. We will often represent V by drawing a picture of the set of its weights and thinking of each dot as representing a subspace; this picture (often with some annotation to denote the multiplicity of each weight) is called the *weight diagram* of V.

The action of the rest of the Lie algebra on V can be described in these terms: for any root β, we have

$$\mathfrak{g}_\beta \colon V_\alpha \to V_{\alpha+\beta},$$

so we can think of the action of $\mathfrak{g}_\beta$ on V as a translation in the weight diagram, shifting each of the dots over by β and mapping the weight spaces correspondingly.

Observe next that all the weights of an irreducible representation are congruent to one another modulo the root lattice Λ_R: otherwise, for any weight α of V the subspace

$$V' = \bigoplus_{\beta \in \Lambda_R} V_{\alpha+\beta}$$

would be a proper subrepresentation of V. In particular, in view of Fact 14.2(ii), this means that the weights all lie in a translate of the real subspace spanned by the roots, so that it is not so unreasonable to draw a picture of them.

Step 3. *Find the distinguished subalgebras* $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C} \subset \mathfrak{g}$. As we saw in the example of $\mathfrak{sl}_3\mathbb{C}$, a crucial ingredient in the analysis of an arbitrary irreducible finite-dimensional representation is the restriction of the representation to certain special copies of the algebra $\mathfrak{sl}_2\mathbb{C}$ contained in $\mathfrak{g}$, and the application of what we know from Lecture 11 about such representations. To generalize this to our arbitrary Lie algebra $\mathfrak{g}$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be a root space, one dimensional by (i) of Fact 14.2. Then by (iii) of Fact 14.2, there is another root space $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}$; and their commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ must be a subspace of $\mathfrak{g}_0 = \mathfrak{h}$, of dimension at most one. The adjoint action of the commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ thus carries each of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ into itself; so that *the direct sum*

$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \qquad (14.5)$$

is a subalgebra of $\mathfrak{g}$. The structure of $\mathfrak{s}_\alpha$ is not hard to describe, given two further facts that we will state here, verify in cases, and prove in general in Appendix D.

Facts 14.6.

(i) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$; *and*
(ii) $[[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0$.

Given these, it follows that *the subalgebra* $\mathfrak{s}_\alpha$ *is isomorphic* to $\mathfrak{sl}_2\mathbb{C}$. In particular, we can pick a basis $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, and $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ satisfying the standard commutation relations (9.1) for $\mathfrak{sl}_2\mathbb{C}$; X_α and Y_α are not determined by this, but H_α is, being the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ having eigenvalues 2 and -2 on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, respectively [i.e., H_α is uniquely characterized by the requirements that $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $\alpha(H_\alpha) = 2$.]

Step 4. *Use the integrality of the eigenvalues of the* H_α. The distinguished elements $H_\alpha \in \mathfrak{h}$ found above are important first of all because, by the analysis of the representations of $\mathfrak{sl}_2\mathbb{C}$ carried out in Lecture 9, in any representation of $\mathfrak{s}_\alpha$—and hence in any representation of $\mathfrak{g}$—*all eigenvalues of the action of* H_α *must be integers*. Thus, every eigenvalue $\beta \in \mathfrak{h}^*$ of every representation of $\mathfrak{g}$ must assume integer values on all the H_α. We correspondingly let Λ_W be the set of linear functionals $\beta \in \mathfrak{h}^*$ that are integer valued on all the H_α; Λ_W will be a lattice, called the *weight lattice* of $\mathfrak{g}$, with the property that

all weights of all representations of $\mathfrak{g}$ *will lie in* Λ_W.

Note, in particular, that $R \subset \Lambda_W$ and hence $\Lambda_R \subset \Lambda_W$; in fact, the root lattice will in general be a sublattice of finite index in the weight lattice.

Step 5. *Use the symmetry of the eigenvalues of the* H_α. The integrality of the eigenvalues of the H_α under any representation is only half the story; it is also true that they are symmetric about the origin in $\mathbb{Z}$. To express this, for any α we introduce the involution W_α on the vector space $\mathfrak{h}^*$ with $+1$-eigenspace the hyperplane

$$\Omega_\alpha = \{\beta \in \mathfrak{h}^* : \langle H_\alpha, \beta\rangle = 0\} \qquad (14.7)$$

and minus 1 eigenspace the line spanned by α itself.[1] In English, W_α is the reflection in the plane Ω_α with axis the line spanned by α:

$$W_\alpha(\beta) = \beta - \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}\alpha = \beta - \beta(H_\alpha)\alpha. \qquad (14.8)$$

Let $\mathfrak{W}$ be the group generated by these involutions; $\mathfrak{W}$ is called the *Weyl group* of the Lie algebra $\mathfrak{g}$.

Now suppose that V is any representation of $\mathfrak{g}$, with eigenspace decomposition $V = \bigoplus V_\beta$. The weights β appearing in this decomposition can then be broken up into equivalence classes mod α, and the direct sum

$$V_{[\beta]} = \bigoplus_{n \in \mathbb{Z}} V_{\beta+n\alpha} \qquad (14.9)$$

of the eigenspaces in a given equivalence class will be a subrepresentation of V for $\mathfrak{s}_\alpha$. It follows then that the set of weights of V congruent to any given β mod α will be invariant under the involution W_α; in particular,

The set of weights of any representation of $\mathfrak{g}$ *is invariant under the Weyl group.*

To make this more explicit, the string of weights that correspond to nonzero summands in (14.9) are, possibly after replacing β by a translate by a multiple of α:

$$\beta, \beta + \alpha, \beta + 2\alpha, \ldots, \beta + m\alpha, \quad \text{with } m = -\beta(H_\alpha). \qquad (14.10)$$

(Note that by our analysis of $\mathfrak{sl}_2\mathbb{C}$ this must be an uninterrupted string.) Indeed if we choose β and $m \geq 0$ so that (14.10) is the string corresponding to nonzero summands in (14.9), then the string of integers

$$\beta(H_\alpha), (\beta + \alpha)(H_\alpha) = \beta(H_\alpha) + 2, \ldots, (\beta + m\alpha)(H_\alpha) = \beta(H_\alpha) + 2m$$

must be symmetric about zero, so $\beta(H_\alpha) = -m$. In particular,

$$W_\alpha(\beta + k\alpha) = \beta + (-\beta(H_\alpha) - k)\alpha = \beta + (m - k)\alpha.$$

Note also that by the same analysis the multiplicities of the weights are invariant under the Weyl group.

We should mention one other fact about the Weyl group, whose proof we also postpone:

[1] Note that by the nondegeneracy assertion (ii) of Fact 14.6, the line $\mathbb{C}\cdot\alpha$ does not lie in the hyperplane Ω_α. Recall that $\langle\ ,\ \rangle$ is the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$, so $\langle H_\alpha, \beta\rangle = \beta(H_\alpha)$.

Fact 14.11. *Every element of the Weyl group is induced by an automorphism of the Lie algebra* $\mathfrak{g}$ *carrying* $\mathfrak{h}$ *to itself.*

We can even say what automorphism of $\mathfrak{g}$ does the trick: to get the involution W_α, take the adjoint action of the exponential $\exp(\pi i U_\alpha) \in G$, where G is any group with Lie algebra $\mathfrak{g}$ and U_α is a suitable element of the direct sum of the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. To prove that $\mathrm{Ad}(\exp(\pi i U_\alpha))$ actually does this requires more knowledge of $\mathfrak{g}$ than we currently possess; but it would be an excellent exercise to verify this assertion directly in each of the cases studied below. (For the general case see (23.20) and (26.15).)

Step 6. *Draw the picture* (optional). While there is no logical need to do so at this point, it will be much easier to think about what is going on in $\mathfrak{h}^*$ if we introduce the appropriate inner product, called the *Killing form*, on $\mathfrak{g}$ (hence by restriction on $\mathfrak{h}$, and hence on $\mathfrak{h}^*$). Since the introduction of the Killing form is, logically, a digression, we will defer until later in this lecture a discussion of its various definitions and properties. It will suffice for now to mention the characteristic property of the induced inner product on $\mathfrak{h}^*$: up to scalars it is the unique inner product on $\mathfrak{h}^*$ preserved by the Weyl group, i.e., in terms of which the Weyl group acts as a group of orthogonal transformations. Equivalently, it is the unique inner product (up to scalars) such that the line spanned by each root $\alpha \in \mathfrak{h}^*$ is actually perpendicular to the plane Ω_α (so that the involution W_α is just a reflection in that hyperplane). Indeed, in practice this is most often how we will compute it. In terms of the Killing form, then, we can say that *the Weyl group is just the group generated by the reflections in the hyperplanes perpendicular to the roots of the Lie algebra.*

Step 7. *Choose a direction in* $\mathfrak{h}^*$. By this we mean a real linear functional l on the lattice Λ_R irrational with respect to this lattice. This gives us a decomposition of the set

$$R = R^+ \cup R^-, \tag{14.12}$$

where $R^+ = \{\alpha : l(\alpha) > 0\}$ (the $\alpha \in R^+$ are called the *positive* roots, those in R^- *negative*); this decomposition is called an *ordering of the roots.* For most purposes, the only aspect of l that matters is the associated ordering of the roots.

The point of choosing a direction—and thereby an ordering of the roots $R = R^+ \cup R^-$—is, of course, to mimic the notion of highest weight vector that was so crucial in the cases of $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$. Specifically, we make the

Definition. Let V be any representation of $\mathfrak{g}$. A nonzero vector $v \in V$ that is both an eigenvector for the action of $\mathfrak{h}$ and in the kernel of $\mathfrak{g}_\alpha$ for all $\alpha \in R^+$ is called a *highest weight vector* of V.

Just as in the previous cases, we then have

Proposition 14.13. *For any semisimple complex Lie algebra* $\mathfrak{g}$,

(i) *every finite-dimensional representation* V *of* $\mathfrak{g}$ *possesses a highest weight vector;*

(ii) *the subspace* W *of* V *generated by the images of a highest weight vector* v *under successive applications of root spaces* $\mathfrak{g}_\beta$ *for* $\beta \in R^-$ *is an irreducible subrepresentation;*

(iii) *an irreducible representation possesses a unique highest weight vector up to scalars.*

Proof. Part (i) is immediate: we just take α to be the weight appearing in V for which the value $l(\alpha)$ is maximal and choose v any nonzero vector in the weight space V_α. Since $V_{\alpha+\beta} = (0)$ for all $\beta \in R^+$, such a vector v will necessarily be in the kernel of all root spaces $\mathfrak{g}_\beta$ corresponding to positive roots β.

Part (ii) may be proved by the same argument as in the two cases we have already discussed: we let W_n be the subspace spanned by all $w_n \cdot v$ where w_n is a word of length at most n in elements of $\mathfrak{g}_\beta$ for negative β. We then claim that for any X in any positive root space, $X \cdot W_n \subset W_n$. To see this, write a generator of W_n in the form $Y \cdot w$, $w \in W_{n-1}$, and use the commutation relation $X \cdot Y \cdot w = Y \cdot X \cdot w + [X, Y] \cdot w$; the claim follows by induction, since $[X, Y]$ is always in $\mathfrak{h}$. The subspace $W \subset V$ which is a union of all the W_n's is thus a subrepresentation; to see that it is irreducible; note that if we write $W = W' \oplus W''$, then either W' or W'' will have to contain the one-dimensional weight space W_α, and so will have to equal W.

The uniqueness of the highest weight vector of an irreducible representation follows immediately: if $v \in V_\alpha$ and $w \in V_\beta$ were two such, not scalar multiples of each other, we would have $l(\alpha) > l(\beta)$ and vice versa. □

Exercise 14.14. Show that in (ii) one need only apply those $\mathfrak{g}_\beta$ for which $\mathfrak{g}_\beta \cdot v \neq 0$. (Note: with W_n defined using only these $\mathfrak{g}_\beta$, and X in any root space, the same inductive argument shows that $X \cdot W_n \subset W_{n+1}$. On the other hand, if one uses all $\mathfrak{g}_\beta$ with β negative and primitive, as in Observation 14.16, then $X \cdot W_n \subset W_{n-1}$. One cannot combine these, however: V may *not* be generated by successively applying those $\mathfrak{g}_\beta$ with β negative, primitive, and $\mathfrak{g}_\beta \cdot v \neq 0$, e.g., the standard representation of $\mathfrak{sl}_3\mathbb{C}$.)

The weight α of the highest weight vector of an irreducible representation will be called, not unreasonably, the *highest weight* of that representation; the term *dominant weight* is also common.

We can refine part (ii) of this proposition slightly in another direction; this is not crucial but will be useful later on in estimating multiplicities of various representations. This refinement is based on

Exercise 14.15*. (a) Let $\alpha_1, \ldots, \alpha_k$ be roots of a semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{g}_{\alpha_i} \subset \mathfrak{g}$ the corresponding root spaces. Show that the subalgebra of $\mathfrak{g}$ generated by the Cartan subalgebra $\mathfrak{h}$ together with the $\mathfrak{g}_{\alpha_i}$ is exactly the direct sum $\mathfrak{h} \oplus (\bigoplus \mathfrak{g}_\alpha)$, where the direct sum is over the intersection of the set R of roots of $\mathfrak{g}$ with the semigroup $\mathbb{N}\{\alpha_1, \ldots, \alpha_k\} \subset \mathfrak{h}$ generated by the α_i.

(b) Similarly, let $\alpha_1, \ldots, \alpha_k$ be negative roots of a semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{g}_{\alpha_i} \subset \mathfrak{g}$ the corresponding root spaces. Show that the subalgebra of $\mathfrak{g}$ generated by the $\mathfrak{g}_{\alpha_i}$ is exactly the direct sum $\bigoplus \mathfrak{g}_\alpha$, where the direct sum is over the intersection of the set R of roots of $\mathfrak{g}$ with the semigroup $\mathbb{N}\{\alpha_1, \ldots, \alpha_k\} \subset \mathfrak{h}$ generated by the α_i.

(Note that by the description of the adjoint action of a Lie algebra on itself we have an obvious inclusion; the problem here is to show—given the facts above—that if $\alpha + \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$.)

From this exercise, it is clear that generating a subrepresentation W of a given representation V by successive applications of root spaces $\mathfrak{g}_\beta$ for $\beta \in R^-$ to a highest weight vector v is inefficient; we need only apply the root spaces $\mathfrak{g}_\beta$ corresponding to a set of roots β generating R^- as a semigroup. We accordingly introduce another piece of terminology: we say that a positive (resp., negative) root $\alpha \in R$ is *primitive* or *simple* if it cannot be epxressed as a sum of two positive (resp. negative) roots. (Note that, since there are only finitely many roots, every positive root can be written as a sum of primitive positive roots.) We then have

Observation 14.16. *Any irreducible representation V is generated by the images of its highest weight vector v under successive applications of root spaces $\mathfrak{g}_\beta$ where β ranges over the primitive negative roots.*

We have already seen one example of this in the case of $\mathfrak{sl}_3\mathbb{C}$, where we observed (in the proof of Claim 12.10 and in the analysis of $\mathrm{Sym}^2 V \otimes V^*$ in Lecture 13) that any irreducible representation was generated by applying the two elements $E_{2,1} \in \mathfrak{g}_{L_2-L_1}$ and $E_{3,2} \in \mathfrak{g}_{L_3-L_2}$ to a highest weight vector.

To return to our description of the weights of an irreducible representation V, we observe next that in fact *every vertex of the convex hull of the weights of V must be conjugate to α under the Weyl group.* To see this, note that by the above the set of weights is contained in the cone $\alpha + C_\alpha^-$, where C_α^- is the positive real cone spanned by the roots $\beta \in R^-$ such that $\mathfrak{g}_\beta(v) \neq 0$—that is, such that $\alpha(H_\beta) \neq 0$. Conversely, the weights of V will contain the string of weights

$$\alpha, \alpha + \beta, \alpha + 2\beta, \ldots, \alpha + (-\alpha(H_\beta))\beta \tag{14.17}$$

for any $\beta \in R^-$. Thus, any vertex of the convex hull of the set of weights of V adjacent to α must be of the form

$$\alpha - \alpha(H_\beta)\beta = W_\beta(\alpha)$$

for some β; applying the same analysis to each successive vertex gives the statement.

From the above, we deduce that the set of weights of V will lie in the convex hull of the images of α under the Weyl group. Since, moreover, we know that the intersection of this set with any set of weights of the form $\{\beta + n\gamma\}$ will be a connected string, it follows that *the set of weights of V will be exactly the weights that are congruent to α modulo the root lattice Λ_R and that lie in the convex hull of the images of α under the Weyl group.*

One more bit of terminology, and then we are done. By what we have seen (cf. (14.17)), the highest weight of any representation of V will be a weight α satisfying $\alpha(H_\gamma) \geq 0$ for every $\gamma \in R^+$. The locus $\mathscr{W}$, in the real span of the roots, of points satisfying these inequalities—in terms of the Killing form, making an acute or right angle with each of the positive roots—is called the (closed) *Weyl chamber* associated to the ordering of the roots. A Weyl chamber could also be described as the closure of a connected component of the complement of the union of the hyperplanes Ω_α. The Weyl group acts simply transitively on the set of Weyl chambers and likewise on the set of orderings of the roots. As usual, these statements will be easy to see in the cases we study, while the abstract proofs are postponed (to Appendix D).

Step 8. *Classify the irreducible, finite-dimensional representations of* $\mathfrak{g}$. Where all the above is leading should be pretty clear; it is expressed in the fundamental existence and uniqueness theorem:

Theorem 14.18. *For any α in the intersection of the Weyl chamber $\mathscr{W}$ associated to the ordering of the roots with the weight lattice Λ_W, there exists a unique irreducible, finite-dimensional representation Γ_α of $\mathfrak{g}$ with highest weight α; this gives a bijection between $\mathscr{W} \cap \Lambda_W$ and the set of irreducible representations of $\mathfrak{g}$. The weights of Γ_α will consist of those elements of the weight lattice congruent to α modulo the root lattice Λ_R and lying in the convex hull of the set of points in $\mathfrak{h}^*$ conjugate to α under the Weyl group.*

HALF-PROOF. We will give here just the proof of uniqueness, which is easy. The existence part we will demonstrate explicitly in each example in turn; and later on we will sketch some of the constructions that can be made in general.

The uniqueness part is exactly the same as for $\mathfrak{sl}_3\mathbb{C}$. If V and W are two irreducible, finite-dimensional representations of $\mathfrak{g}$ with highest weight vectors v and w, respectively, both having weight α, then the vector $(v, w) \in V \oplus W$ will again be a highest weight vector of weight α in that representation. Let $U \subset V \oplus W$ be the subrepresentation generated by (v, w); since U will again be irreducible the projection maps $\pi_1 \colon U \to V$ and $\pi_2 \colon U \to W$, being nonzero, will have to be isomorphisms. □

Another fact which we will see as we go along—and eventually prove in general—is that there are always *fundamental weights* $\omega_1, \ldots, \omega_n$ with the property that any dominant weight can be expressed uniquely as a non-negative integral linear combination of them. They can be characterized geometrically as the first weights met along the edges of the Weyl chamber, or algebraically as those elements ω_i in $\mathfrak{h}^*$ such that $\omega_i(H_{\alpha_j}) = \delta_{i,j}$, where $\alpha_1, \ldots, \alpha_n$ are the simple roots (in some order). When we have found them, we often write $\Gamma_{a_1,\ldots,a_n}$ for the irreducible representation with highest weight $a_1\omega_1 + \cdots + a_n\omega_n$; i.e.,

$$\Gamma_{a_1,\ldots,a_n} = \Gamma_{a_1\omega_1 + \cdots + a_n\omega_n}.$$

As with most of the material in this section, general proofs will be found in Lecture 21 and Appendix D.

One basic point we want to repeat here (and that we hope to demonstrate in succeeding lectures) is this: that actually carrying out this process in practice is completely elementary and straightforward. Any mathematician, stranded on a desert island with only these ideas and the definition of a particular Lie algebra $\mathfrak{g}$ such as $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$, or $\mathfrak{sp}_{2n}\mathbb{C}$, would in short order have a complete description of all the objects defined above in the case of $\mathfrak{g}$. We should say as well, however, that at the conclusion of this procedure we are left without one vital piece of information about the representations of $\mathfrak{g}$, without which we will be unable to analyze completely, for example, tensor products of known representations; this is, of course, a description of the multiplicities of the basic representations Γ_a. As we said, we will, in fact, describe and prove such a formula (the Weyl character formula); but it is of a much less straightforward character (our hypothetical shipwrecked mathematician would have to have what could only be described as a pretty good day to come up with the idea) and will be left until later. For now, we will conclude this lecture with the promised introduction to the Killing form.

§14.2. About the Killing Form

As we said, the Killing form is an inner product (symmetric bilinear form) on the Lie algebra $\mathfrak{g}$; abusing our notation, we will denote by B both the Killing form and the induced inner products on $\mathfrak{h}$ and $\mathfrak{h}^*$. B can be defined in several ways; the most common is by associating to a pair of elements $X, Y \in \mathfrak{g}$ the trace of the composition of their adjoint actions on $\mathfrak{g}$, i.e.,

$$B(X, Y) = \operatorname{Tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y): \mathfrak{g} \to \mathfrak{g}). \tag{14.19}$$

As we will see, the Killing form may be computed in practice either from this definition, or (up to scalars) by using its invariance under the group of automorphisms of $\mathfrak{g}$. We remark that this definition is not as opaque as it may seem at first. For one thing, the description of the adjoint action of the root space $\mathfrak{g}_\alpha$ as a "translation" of the root diagram—that is, carrying each root space $\mathfrak{g}_\beta$ into $\mathfrak{g}_{\alpha+\beta}$—tells us immediately that $\mathfrak{g}_\alpha$ is perpendicular to $\mathfrak{g}_\beta$ for all β other than $-\alpha$; in other words, the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \right) \tag{14.20}$$

is orthogonal. As for the restriction of B to $\mathfrak{h}$, this is more subtle, but it is not hard to write down: if X, Y are in $\mathfrak{h}$, and Z_α generates $\mathfrak{g}_\alpha$, then $\operatorname{ad}(X) \circ \operatorname{ad}(Y)(Z_\alpha) = \alpha(X)\alpha(Y)Z_\alpha$, so $B(X, Y) = \sum \alpha(X)\alpha(Y)$, the sum over the roots; viewing $B|_\mathfrak{h}$ as an element of the symmetric square $\operatorname{Sym}^2(\mathfrak{h}^*)$, we have

$$B|_\mathfrak{h} = \frac{1}{2} \sum_{\alpha \in R} \alpha^2. \tag{14.21}$$

A key fact following from this—one that, if nothing else, makes picturing $\mathfrak{h}^*$ with the inner product B involve less eyestrain—is

(14.22) *B is positive definite on the real subspace of $\mathfrak{h}$ spanned by the vectors $\{H_\alpha : \alpha \in R\}$.*

Indeed, all roots take on real values on this space (since all $\alpha(H_\beta) \in \mathbb{Z} \subset \mathbb{R}$), so for H in this real subspace of $\mathfrak{h}$, $B(H, H)$ is non-negative, and is zero only when all $\alpha(H) = 0$, which implies $H = 0$, since the roots span $\mathfrak{h}^*$.

To see that the Killing form is nondegenerate on all of $\mathfrak{g}$, we need the useful identity:

$$B([X, Y], Z) = B(X, [Y, Z]) \tag{14.23}$$

for all X, Y, Z in $\mathfrak{g}$. This follows from the identity

$$\operatorname{Trace}((\bar{X}\bar{Y} - \bar{Y}\bar{X})\bar{Z}) = \operatorname{Trace}(\bar{X}(\bar{Y}\bar{Z} - \bar{Z}\bar{Y}))$$

for any endomorphisms $\bar{X}, \bar{Y}, \bar{Z}$ of a vector space. And this, in turn, follows from

$$\operatorname{Trace}(\bar{Y}\bar{X}\bar{Z} - \bar{X}\bar{Z}\bar{Y}) = \operatorname{Trace}([\bar{Y}, \bar{X}\bar{Z}]) = 0.$$

An immediate consequence of (14.23) is that if $\mathfrak{a}$ is any ideal in a Lie algebra $\mathfrak{g}$, then its orthogonal complement $\mathfrak{a}^\perp$ with respect to B is also an ideal. In particular, if $\mathfrak{g}$ is simple, the kernel of B is zero (note that the kernel cannot be $\mathfrak{g}$ since it does not contain $\mathfrak{h}$). Since the Killing form of a direct sum is the sum of the Killing forms of the factors, it follows that *the Killing form is nondegenerate on a semisimple Lie algebra* $\mathfrak{g}$.

One of the reasons the Killing form helps to picture $\mathfrak{h}^*$ is the fact mentioned above:

Proposition 14.24. *With respect to B, the line spanned by each root α is perpendicular to the hyperplane Ω_α.*

As we observed, this is equivalent to saying that the involutions W_α above are simply reflections in hyperplanes, and in turn to saying that the whole Weyl group is orthogonal. Note also that Proposition 14.24 thereby follows immediately from the Fact 14.11: from the definition of B above, it is clearly invariant under any automorphism of $\mathfrak{g}$. Nevertheless, we would prefer not to rely on this fact; and anyway giving a direct proof of the proposition is not hard, in terms of the picture we have of the adjoint action of $\mathfrak{g}$ on itself. To prove the assertion $\alpha \perp \Omega_\alpha$, it suffices to prove the dual assertion that $H \perp H_\alpha$ for all H in the annihilator of α. But now by construction H_α is the commutator $[X_\alpha, Y_\alpha]$ of an element $X_\alpha \in \mathfrak{g}_\alpha$ and an element $Y_\alpha \in \mathfrak{g}_{-\alpha}$. Using (14.23) we have for any H in $\mathfrak{h}$,

$$\begin{aligned} B(H_\alpha, H) &= B([X_\alpha, Y_\alpha], H) = B(X_\alpha, [Y_\alpha, H]) \\ &= B(X_\alpha, \alpha(H) Y_\alpha) = \alpha(H) B(X_\alpha, Y_\alpha), \end{aligned} \tag{14.25}$$

which vanishes since $\alpha(H) = 0$.

Note that as a consequence of this, we can characterize the Weyl chamber associated to an ordering of the roots as exactly those vectors in the real span of the roots forming an acute angle with all the positive roots (or, equivalently, with all the primitive ones); the Weyl chamber is thus the cone whose faces lie in the hyperplanes perpendicular to the primitive positive roots.

Equation (14.25) leads to a formula for the isomorphism of $\mathfrak{h}$ with $\mathfrak{h}^*$ determined by the Killing form. First note that for $H = H_\alpha$ it gives

$$B(H_\alpha, H_\alpha) = 2B(X_\alpha, Y_\alpha) \neq 0,$$

for if $B(X_\alpha, Y_\alpha)$ were zero we would have $B(H_\alpha, H) = 0$ for all H, contradicting the nondegeneracy of B on $\mathfrak{h}$. The element T_α of $\mathfrak{h}$ which corresponds to $\alpha \in \mathfrak{h}^*$ by the Killing form is by definition the element of $\mathfrak{h}$ that satisfies the condition

$$B(T_\alpha, H) = \alpha(H) \quad \text{for all } H \in \mathfrak{h}. \tag{14.26}$$

Looking at (14.25), we see that $T_\alpha = H_\alpha/B(X_\alpha, Y_\alpha) = 2H_\alpha/B(H_\alpha, H_\alpha)$. This proves

Corollary 14.27. *The isomorphism of* $\mathfrak{h}^*$ *and* $\mathfrak{h}$ *determined by the Killing form* B *carries* α *to* $T_\alpha = (2/B(H_\alpha, H_\alpha))\cdot H_\alpha$.

The Killing form on $\mathfrak{h}^*$ is defined by $B(\alpha, \beta) = B(T_\alpha, T_\beta)$.

Exercise 14.28. Show that the inverse isomorphism from $\mathfrak{h}$ to $\mathfrak{h}^*$ takes H_α to $(2/B(\alpha, \alpha))\cdot\alpha$.

The orthogonality of W_α can be expressed by the formula

$$W_\alpha(\beta) = \beta - \frac{2B(\beta, \alpha)}{B(\alpha, \alpha)}\alpha.$$

Comparing with (14.8) this says:

Corollary 14.29. *If* α *and* β *are roots, then*

$$2B(\beta, \alpha)/B(\alpha, \alpha) = \beta(H_\alpha)$$

is an integer.

By the above identification of $\mathfrak{h}$ with $\mathfrak{h}^*$, (14.22) translates to

Corollary 14.30. *The Killing form* B *is positive definite on the real vector space spanned by the root lattice* Λ_R.

Note that it follows immediately from (14.22) that the Weyl group $\mathfrak{W}$ is finite, being simultaneously discrete ($\mathfrak{W}$ preserves the set R of roots of $\mathfrak{g}$ and hence the lattice Λ_R; it follows that $\mathfrak{W}$ can be realized as a subgroup of $\mathrm{GL}_n\mathbb{Z}$) and compact ($\mathfrak{W}$ preserves the Killing form, and hence is a subgroup of the orthogonal group $\mathrm{O}_n\mathbb{R}$.) Alternatively, $\mathfrak{W}$ is a subgroup of the permutation group of the set of roots.

As we observed, the Killing form on $\mathfrak{h}^*$ is preserved by the Weyl group. In fact, in case $\mathfrak{g}$ is simple, the Killing form is, up to scalars, the unique inner product preserved by the Weyl group. This will follow from

Proposition 14.31. *The space* $\mathfrak{h}^*$ *is an irreducible representation of the Weyl group* $\mathfrak{W}$.

Proof. Suppose that $\mathfrak{z} \subset \mathfrak{h}^*$ were preserved by the action of $\mathfrak{W}$. This means that every root $\alpha \in \mathfrak{h}^*$ of $\mathfrak{g}$ will either lie in the subspace $\mathfrak{z}$ or be perpendicular to it, i.e., for every $\alpha \in \mathfrak{z}$ and $\beta \notin \mathfrak{z}$ we will have $\beta(H_\alpha) = 0$. We claim then that *the subspace* $\mathfrak{g}'$ *of* $\mathfrak{g}$ *spanned by the subalgebras* $\{\mathfrak{s}_\alpha\}_{\alpha\in\mathfrak{z}}$ *will be an ideal in* $\mathfrak{g}$. Clearly it will be a subalgebra; the space spanned by the distinguished subalgebras $\mathfrak{s}_\alpha$ corresponding to the set of roots lying in any subspace of $\mathfrak{h}^*$ will be. To see that it is in fact an ideal, let $Y \in \mathfrak{g}_\beta$ be an element of a root space. Then for any $\alpha \in \mathfrak{z}$, we have

$$[Y, Z] \in \mathfrak{g}_{\alpha+\beta} = 0$$

since $\alpha + \beta$ is neither in $\mathfrak{z}$ nor perpendicular to it, and so cannot be a root; and

$$[Y, H_\alpha] = -[H_\alpha, Y] = \beta(H_\alpha)\cdot Y = 0.$$

Thus, $\mathrm{ad}(Y)$ kills $\mathfrak{g}'$; since, of course, all of H itself will preserve $\mathfrak{g}'$, it follows that $\mathfrak{g}'$ is an ideal. Thus, either all the roots lie in $\mathfrak{z}$ and so $\mathfrak{z} = \mathfrak{h}^*$, or all roots are perpendicular to $\mathfrak{z}$ and correspondingly $\mathfrak{z} = (0)$. □

Note that given Fact 14.11, we can also express the last statement by saying that (in case $\mathfrak{g}$ is simple) the Killing form on $\mathfrak{h}$ is the unique form preserved by every automorphism of the Lie algebra $\mathfrak{g}$ carrying $\mathfrak{h}$ to itself. As we will see, in practice this is most often how we will first describe the Killing form.

Exercise 14.32. Find the Killing form on the Lie algebras $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$ by explicit computation, and verify the statements made above in these cases.

Exercise 14.33*. If a semisimple Lie algebra is a direct sum of simple subalgebras, then its Killing form is the orthogonal sum of the Killing forms of the factors. Show that, conversely, if the roots of a semisimple Lie algebra lie in a collection of mutually perpendicular subspaces, then the Lie algebra decomposes accordingly.

Exercise 14.34*. Suppose $\mathfrak{g}$ is a Lie algebra that has an abelian subalgebra $\mathfrak{h}$ such that $\mathfrak{g}$ has a decomposition (14.1), satisfying the conditions of Facts 14.2 and 14.6. Show that $\mathfrak{g}$ is semisimple, and $\mathfrak{h}$ is a Cartan subalgebra.

The preceding exercise can be used instead of Weyl's unitary trick or any abstract theory to verify that the algebras we meet in the next few lectures are all semisimple. It is tempting to call such a Lie algebra "visibly semisimple."

The discussion of the geometry of the roots of a semisimple Lie algebra will be continued in Lecture 21 and completed in Appendix D. The Killing form becomes particularly useful in the general theory; for example, solvability and semisimplicity can both be characterized by properties of the Killing form (see Appendix C).

Exercise 14.35*. Show that $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$ is a maximal solvable subalgebra of $\mathfrak{g}$; $\mathfrak{b}$ is called a *Borel subalgebra*. Show that $\bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$ is a maximal nilpotent subalgebra of $\mathfrak{g}$. These will be discussed in Lecture 25.

Exercise 14.36*. Show that the Killing form on the Lie algebra $\mathfrak{gl}_m$ is given by the formula

$$B(X, Y) = 2m \operatorname{Tr}(X \circ Y) - 2 \operatorname{Tr}(X) \operatorname{Tr}(Y).$$

Find similar formulas for $\mathfrak{sl}_m$, $\mathfrak{so}_m$, and $\mathfrak{sp}_m$, showing in each case that $B(X, Y)$ is a constant multiple of $\operatorname{Tr}(X \circ Y)$.

Exercise 14.37. If G is a real Lie group, the Killing form on its Lie algebra $\mathfrak{g} = T_e G$ may not be positive definite. When it is, it determines, by left translation, a Riemannian metric on G. Show that the Killing form is positive definite for $G = \mathrm{SO}_n\mathbb{R}$, but not for $\mathrm{SL}_n\mathbb{R}$.

LECTURE 15

$\mathfrak{sl}_4\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}$

In this lecture, we will illustrate the general paradigm of the previous lecture by applying it to the Lie algebras $\mathfrak{sl}_n\mathbb{C}$; this is typical of the analyses of specific Lie algebras carried out in this Part. We start in §15.1 by describing the Cartan subalgebra, roots, root spaces, etc., for $\mathfrak{sl}_n\mathbb{C}$ in general. We then give in §15.2 a detailed account of the representations of $\mathfrak{sl}_4\mathbb{C}$, which generalizes directly to $\mathfrak{sl}_n\mathbb{C}$; in particular, we deduce the existence part of Theorem 14.18 for $\mathfrak{sl}_n\mathbb{C}$.

In §15.3 we give an explicit construction of the irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ using the Weyl construction introduced in Lecture 6; analogous constructions of the irreducible representations of the remaining classical Lie algebras will be given in §17.3 and §19.5. This section presupposes familiarity with Lecture 6 and Appendix A, but can be skipped by those willing to forego §17.3 and 19.5 as well. Section 15.4 requires essentially the same degree of knowledge of classical algebraic geometry as §§11.3 and 13.4 (it does not presuppose §15.3), but can also be skipped. Finally, §15.5 describes representations of $\mathrm{GL}_n\mathbb{C}$; this appears to involve the Weyl construction but in fact the main statement, Proposition 15.47 (and even its proof) can be understood without the preceding two sections.

§15.1: Analyzing $\mathfrak{sl}_n\mathbb{C}$
§15.2: Representations of $\mathfrak{sl}_4\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}$
§15.3: Weyl's construction and tensor products
§15.4: Some more geometry
§15.5: Representations of $\mathrm{GL}_n\mathbb{C}$

§15.1. Analyzing $\mathfrak{sl}_n\mathbb{C}$

To begin with, we have to locate a Cartan subalgebra, and this is not hard; as in the case of $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$ the subalgebra of diagonal matrices will work fine. Writing H_i for the diagonal matrix $E_{i,i}$ that takes e_i to itself and kills e_j

for $j \neq i$, we have

$$\mathfrak{h} = \{a_1 H_1 + a_2 H_2 + \cdots + a_n H_n : a_1 + a_2 + \cdots + a_n = 0\};$$

note that H_i is not in $\mathfrak{h}$. We can correspondingly write

$$\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, \ldots, L_n\}/(L_1 + L_2 + \cdots + L_n = 0),$$

where $L_i(H_j) = \delta_{i,j}$. We often write L_i for the image of L_i in $\mathfrak{h}^*$.

We have already seen how the diagonal matrices act on the space of all traceless matrices: if $E_{i,j}$ is the endomorphism of $\mathbb{C}^n$ carrying e_j to e_i and killing e_k for all $k \neq j$, then we have

$$\operatorname{ad}(a_1 H_1 + a_2 H_2 + \cdots + a_n H_n)(E_{i,j}) = (a_i - a_j) \cdot E_{i,j}; \qquad (15.1)$$

or, in other words, *$E_{i,j}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i - L_j$*; in particular, *the roots of $\mathfrak{sl}_n\mathbb{C}$ are just the pairwise differences of the L_i.*

Before we try to visualize anything taking place in $\mathfrak{h}$ or $\mathfrak{h}^*$, let us take a moment out and describe the Killing form. To this end, note that the automorphism φ of $\mathbb{C}^n$ sending e_i to e_j, e_j to $-e_i$ and fixing e_k for all $k \neq i, j$ induces an automorphism $\operatorname{Ad}(\varphi)$ of the Lie algebra $\mathfrak{sl}_n\mathbb{C}$ (or even $\mathfrak{gl}_n(\mathbb{C})$) that carries $\mathfrak{h}$ to itself, exchanges H_i and H_j, and fixes all the other H_k. Since the Killing form on $\mathfrak{h}$ must be invariant under all these automorphisms, it must satisfy $B(L_i, L_i) = B(L_j, L_j)$ for all i and j and $B(L_i, L_k) = B(L_j, L_k)$ for all i, j and $k \neq i, j$; it follows that on $\mathfrak{h}$ it must be a linear combination of the forms

$$B'(\textstyle\sum a_i H_i, \sum b_i H_i) = \sum a_i b_i$$

and

$$B''(\textstyle\sum a_i H_i, \sum b_i H_i) = \sum_{i \neq j} a_i b_j.$$

On the space $\{\sum a_i H_i : \sum a_i = 0\}$, however, we have $0 = (\sum a_i)(\sum b_j) = \sum a_i b_i + \sum a_i b_j$, so in fact these two forms are dependent; and hence we can write the Killing form simply as a multiple of B'. Similarly, the Killing form on $\mathfrak{h}^*$ must be a linear combination of the forms $B'(\sum a_i L_i, \sum b_i L_i) = \sum a_i b_i$ and $B''(\sum a_i L_i, \sum b_i L_i) = \sum_{j \neq i} a_i b_j$; the condition that $B(\sum a_i L_i, \sum b_i L_i) = 0$ whenever $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$ implies that it must be a multiple of

$$\begin{aligned} B(\textstyle\sum a_i L_i, \sum b_i L_i) &= \left(\dfrac{n-1}{n}\right) \displaystyle\sum_i a_i b_i - \frac{1}{n} \sum_{i \neq j} a_i b_j \\ &= \sum_i a_i b_i - \frac{1}{n} \sum_{i,j} a_i b_j. \end{aligned} \qquad (15.2)$$

We may, of course, also calculate the Killing form directly from the definition. By (14.21), since the roots of $\mathfrak{sl}_n\mathbb{C}$ are $\{L_i - L_j\}_{i \neq j}$, we have

$$\begin{aligned} B(\textstyle\sum a_i H_i, \sum b_i H_i) &= \textstyle\sum_{i \neq j} (a_i - a_j)(b_i - b_j) \\ &= \textstyle\sum_i \sum_{j \neq i} (a_i b_i + a_j b_j - a_i b_j - a_j b_i). \end{aligned}$$

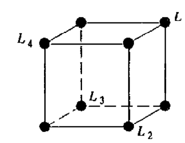
Noting that $\sum_{j \neq i} a_j = -a_i$ and, similarly, $\sum_{j \neq i} b_j = -b_i$, this simplifies to

$$B(\textstyle\sum a_i H_i, \sum b_i H_i) = 2n \sum a_i b_i. \qquad (15.3)$$

It follows with a little calculation that the dual form on $\mathfrak{h}^*$ is

$$B(\textstyle\sum a_i L_i, \sum b_i L_i) = (1/2n)(\sum_i a_i b_i - (1/n) \sum_{i,j} a_i b_j). \qquad (15.4)$$

It is probably simpler just to think of this as the form, unique up to scalars, invariant under the symmetric group $\mathfrak{S}_n$ of permutations of $\{1, 2, \ldots, n\}$. The L_i, therefore, all have the same length, and the angles between all pairs are the same. To picture the roots in $\mathfrak{h}^*$, then, we should think of the points L_i as situated at the vertices of a regular $(n-1)$-simplex Δ, with the origin located at the barycenter of that simplex. This picture is easiest to visualize in the special case $n = 4$, where the L_i will be located at every other vertex of a unit cube centered at the origin:



(15.5)

Now, as we said, the roots of $\mathfrak{sl}_n\mathbb{C}$ are now just the pairwise differences of the L_i. The root lattice Λ_R they generate can thus be described as

$$\Lambda_R = \{\textstyle\sum a_i L_i : a_i \in \mathbb{Z}, \sum a_i = 0\}/(\sum L_i = 0).$$

Both the roots and the root lattice can be drawn in the case of $\mathfrak{sl}_4\mathbb{C}$: if we think of the vectors $L_i \in \mathfrak{h}^*$ as four of the vertices of a cube centered at the origin, the roots will comprise all the midpoints of the edges of a second cube whose linear dimensions are twice the dimensions of the first:

(15.6)

The next step, finding the distinguished subalgebras $\mathfrak{s}_\alpha$, is also very easy. The root space $\mathfrak{g}_{L_i-L_j}$ corresponding to the root $L_i - L_j$ is generated by $E_{i,j}$, so the subalgebra $\mathfrak{s}_{L_i-L_j}$ is generated by

$$E_{i,j},\quad E_{j,i},\quad \text{and}\quad [E_{i,j}, E_{j,i}] = H_i - H_j.$$

The eigenvalue of $H_i - H_j$ acting on $E_{i,j}$ is $(L_i - L_j)(H_i - H_j) = 2$, so that the corresponding distinguished element $H_{L_i-L_j}$ in $\mathfrak{h}$ must be just $H_i - H_j$. The annihilator, of course, is the hyperplane $\Omega_{L_i-L_j} = \{\sum a_i L_i\colon a_i = a_j\}$; note that this is indeed perpendicular to the root $L_i - L_j$ with respect to the Killing form B as described above.

Knowing the H_α we know the weight lattice: in order for a linear functional $\sum a_i L_i \in \mathfrak{h}^*$ to have integral values on all the distinguished elements, it is clearly necessary and sufficient that all the a_i be congruent to one another modulo $\mathbb{Z}$. Since $\sum L_i = 0$ in $\mathfrak{h}^*$, this means that the weight lattice is given as

$$\Lambda_W = \mathbb{Z}\{L_1, \ldots, L_n\}/(\textstyle\sum L_i = 0).$$

In sum, then, *the weight lattice of $\mathfrak{sl}_n\mathbb{C}$ may be realized as the lattice generated by the vertices of a regular $(n-1)$-simplex Δ centered at the origin; and the roots as the pairwise differences of these vertices.*

While we are at it, having determined Λ_R and Λ_W we might as well compute the quotient Λ_W/Λ_R. This is pretty easy: since the lattice Λ_W can be generated by Λ_R together with any of the vertices L_i of our simplex, the quotient Λ_W/Λ_R will be cyclic, generated by any L_i; since, modulo Λ_R,

$$0 = \textstyle\sum_j (L_i - L_j) = nL_i - \sum_j L_j = nL_i.$$

we see that L_i has order dividing n in Λ_W/Λ_R.

Exercise 15.7. Show that L_i has order exactly n in Λ_W/Λ_R, so that $\Lambda_W/\Lambda_R \cong \mathbb{Z}/n\mathbb{Z}$.

From the above we can also say what the Weyl group is: the reflection in the hyperplane perpendicular to the root $L_i - L_j$ will exchange L_i and $L_j \in \mathfrak{h}^*$ and leave the other L_k alone, so that *the Weyl group $\mathfrak{W}$ is just the group $\mathfrak{S}_n$, acting as the symmetric group on the generators L_i of $\mathfrak{h}^*$.* Note that we have already verified that these automorphisms of $\mathfrak{h}^*$ do come from automorphisms of the whole Lie algebra $\mathfrak{sl}_n\mathbb{C}$ preserving $\mathfrak{h}$.

To continue, let us choose a direction, and describe the corresponding Weyl chamber. We can write our linear functional l as

$$l(\textstyle\sum a_i L_i) = \sum c_i a_i$$

with $\sum c_i = 0$; let us suppose that $c_1 > c_2 > \cdots > c_n$. The corresponding ordering of the roots will then be

$$R^+ = \{L_i - L_j\colon i < j\}$$

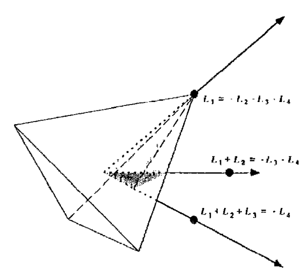
and

$$R^- = \{L_i - L_j\colon j < i\}.$$

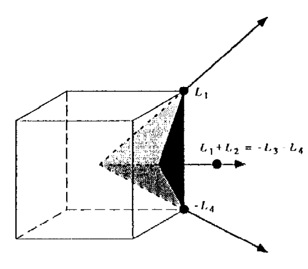
The primitive negative roots for this ordering are simply the roots $L_{i+1} - L_i$. (Note that the ordering of the roots depends only on the relative sizes of the c_i, so that the Weyl group acts simply transitively on the set of orderings.) The (closed) Weyl chamber associated to this ordering will then be the set

$$\mathscr{W} = \{\textstyle\sum a_i L_i\colon a_1 \geq a_2 \geq \cdots \geq a_n\}.$$

One way to describe this geometrically is to say that if we take the barycentric subdivision of the faces of the simplex Δ, the Weyl chamber will be the cone over one $(n-2)$-simplex of the barycentric subdivision: e.g., in the case $n = 4$

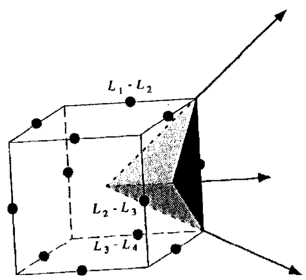


It may be easier to visualize the case $n = 4$ if we introduce the associated cubes: in terms of the cube with vertices at the points $\pm L_i$, we can draw the Weyl chamber as



(15.8)

Alternatively, in terms of the slightly larger cube with vertices at the points $\pm 2L_i$, we can draw $\mathscr{W}$ as



(15.9)

From the first of these pictures we see that the edges of the Weyl chamber are the rays generated by the vectors L_1, $L_1 + L_2$, and $L_1 + L_2 + L_3$; and that the faces of the Weyl chamber are the planes orthogonal to the primitive negative roots $L_2 - L_1$, $L_3 - L_2$, and $L_4 - L_3$. The picture in general is analogous: for $\mathfrak{sl}_n\mathbb{C}$, the Weyl chamber will be the cone over an $(n-2)$-simplex, with edges generated by the vectors

$$L_1, \quad L_1 + L_2, \quad L_1 + L_2 + L_3, \ldots, L_1 + \cdots + L_{n-1} = -L_n.$$

The faces of $\mathscr{W}$ will thus be the hyperplanes

$$\Omega_{L_i - L_{i+1}} = \{\textstyle\sum a_j L_j : a_i = a_{i+1}\}$$

perpendicular to the primitive negative roots $L_{i+1} - L_i$.

Note the important phenomenon: the intersection of the closed Weyl chamber with the lattice Λ_W will be a free semigroup $\mathbb{N}^{n-1}$ generated by the fundamental weights $\omega_i = L_1 + \cdots + L_i$ occurring along the edges of the Weyl chamber. One aspect of its significance that is immediate is that it allows us to index the irreducible representations $\mathfrak{sl}_n\mathbb{C}$ nicely: for an arbitrary $(n-1)$-tuple of natural numbers $(a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1}$ we will denote by $\Gamma_{a_1,\ldots,a_{n-1}}$ the irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $a_1L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1}) = (a_1 + \cdots + a_{n-1})L_1 + (a_2 + \cdots + a_{n-1})L_2 + \cdots + a_{n-1}L_{n-1}$:

$$\Gamma_{a_1,\ldots,a_{n-1}} = \Gamma_{a_1L_1 + a_2(L_1+L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1})}.$$

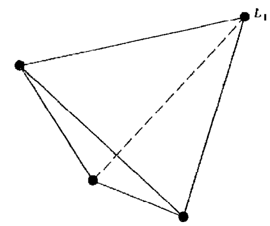
This also has the nice consequence that once we have located the irreducible representations $V^{(i)}$ with highest weight $L_1 + \cdots + L_i$, the general irreducible representation $\Gamma_{a_1,\ldots,a_{n-1}}$ with highest weight $\sum a_i(L_1 + \cdots + L_i)$ will occur inside the tensor product of symmetric powers

$$\mathrm{Sym}^{a_1}V^{(1)} \otimes \mathrm{Sym}^{a_2}V^{(2)} \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}V^{(n-1)}$$

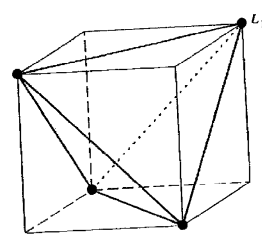
of these representations. Thus, the existence part of the basic Theorem 14.18 is reduced to finding the basic representations $V^{(i)}$; we will do this in due course, though at this point it is probably not too hard an exercise to guess what they are.

§15.2. Representations of $\mathfrak{sl}_4\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}$

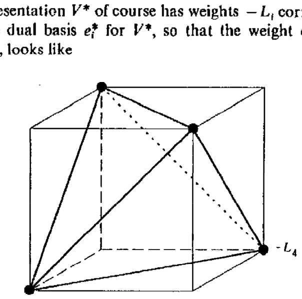
We begin as usual with the standard representation of $\mathfrak{sl}_4\mathbb{C}$ on $V = \mathbb{C}^4$. The standard basis vectors e_i of $\mathbb{C}^4$ are eigenvectors for the action of $\mathfrak{h}$, with eigenvalues L_i, so that the weight diagram looks like



or, with the reference cube drawn as well,

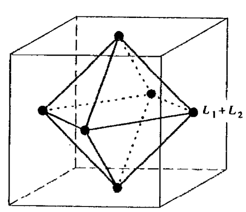


The dual representation V^* of course has weights $-L_i$ corresponding to the vectors of the dual basis e_i^* for V^*, so that the weight diagram, with its reference cube, looks like



Note that the highest weight for this representation is $-L_4$, which lies along the bottom edge of the Weyl chamber, as depicted in Diagram (15.8). Note also that the weights of the representation $\wedge^3 V$—the triple sums $L_1 + L_2 + L_3$, $L_1 + L_2 + L_4$, $L_1 + L_3 + L_4$, and $L_2 + L_3 + L_4$ of distinct weights of V—are the same as those of V^*, reflecting the isomorphism of these two representations.

This suggests that we look next at the second exterior power $\wedge^2 V$. This is a six-dimensional representation, with weights $L_i + L_j$ the pairwise sums of distinct weights of V; its weight diagram, in its reference cube, looks like



The diagram shows clearly that $\wedge^2 V$ is irreducible since it is not the nontrivial union of two configurations invariant under the Weyl group $\mathfrak{S}_4$ (and all weights occur with multiplicity 1). Note also that the weights are symmetric about the origin, reflecting the isomorphism of $\wedge^2 V$ with $(\wedge^2 V)^* = \wedge^2(V^*)$.

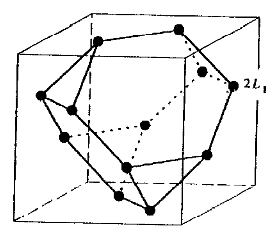
Note that the highest weight $L_1 + L_2$ of the representation $\wedge^2 V$ is the primitive vector along the front edge of the Weyl chamber $\mathscr{W}$ as pictured in Diagram (15.8). Now, we have already seen that the intersection of the closed

Weyl chamber with the weight lattice is a free semigroup generated by the primitive vectors along the three edges of $\mathscr{W}$—that is, every vector in $\mathscr{W} \cap \Lambda_W$ is a non-negative integral linear combination of the three vectors L_1, $L_1 + L_2$, and $L_1 + L_2 + L_3$. As we remarked at the end of the first section of this lecture, it follows that *we have proved the existence half of the general existence and uniqueness theorem* (14.18) *in the case of the Lie algebra* $\mathfrak{sl}_4\mathbb{C}$. Explicitly, since V, $\wedge^2 V$, and $\wedge^3 V = V^*$ have highest weight vectors with weights L_1, $L_1 + L_2$, and $L_1 + L_2 + L_3$, respectively, it follows that *the representation*

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b(\wedge^2 V) \otimes \mathrm{Sym}^c(\wedge^3 V)$$

contains a highest weight vector with weight $aL_1 + b(L_1 + L_2) + c(L_1 + L_2 + L_3)$, *and hence a copy of the irreducible representation* $\Gamma_{a,b,c}$ *with this highest weight.*

Let us continue our examination of representations of $\mathfrak{sl}_4\mathbb{C}$ with a pair of tensor products of the three basic representations: $V \otimes \wedge^2 V$ and $V \otimes \wedge^3 V$. As for the first of these, its weights are easy to find: they consist of the sums $2L_i + L_j$ (which occur once, as the sum of L_i and $L_i + L_j$) and $L_i + L_j + L_k$ (which occur three times). The diagram of these weights looks like



(We have drawn only the vertices of the convex hull of this diagram, thus omitting the weights $L_i + L_j + L_k$; they are located at the centers of the hexagonal faces of this polyhedron.)

Now, the representation $V \otimes \wedge^2 V$ cannot be irreducible, for at least a couple of reasons. First off, just by looking at weights, we see that the irreducible representation $W = \Gamma_{1,1,0}$ with highest weight $2L_1 + L_2$ can have multiplicity at most 2 on the weight $L_1 + L_2 + L_3$: by Observation 14.16, the weight space $W_{L_1+L_2+L_3}$ is generated by the images of the highest weight vector $v \in W_{2L_1+L_2}$ by successive applications of the primitive negative root spaces $\mathfrak{g}_{L_2-L_1}$, $\mathfrak{g}_{L_3-L_2}$, and $\mathfrak{g}_{L_4-L_3}$. But $L_1 + L_2 + L_3$ is uniquely expressible as a sum of $2L_1 + L_2$ and the primitive negative roots:

$$L_1 + L_2 + L_3 = 2L_1 + L_2 + (L_2 - L_1) + (L_3 - L_2);$$

so that $V_{L_1+L_2+L_3}$ is generated by the subspaces $\mathfrak{g}_{L_2-L_1}(\mathfrak{g}_{L_3-L_2}(v))$ and $\mathfrak{g}_{L_3-L_2}(\mathfrak{g}_{L_2-L_1}(v))$. We can in fact check that the representation $\Gamma_{1,1,0}$ takes on the weight $L_1 + L_2 + L_3$ with multiplicity 2 by writing out these generators explicitly and checking that they are independent: for example, we have

$$\begin{aligned}\mathfrak{g}_{L_2-L_1}(\mathfrak{g}_{L_3-L_2}(v)) &= \mathbb{C}\cdot E_{2,1}(E_{3,2}(e_1 \otimes (e_1 \wedge e_2)))\\ &= \mathbb{C}\cdot E_{2,1}(e_1 \otimes (e_1 \wedge e_3))\\ &= \mathbb{C}\cdot (e_2 \otimes (e_1 \wedge e_3) + e_1 \otimes (e_2 \wedge e_3)).\end{aligned}$$

This is in fact what is called for in Exercise 15.10.

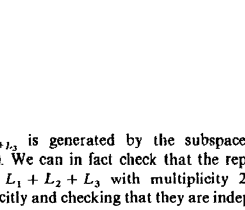
Alternatively, forgetting weights entirely, we can see from standard multilinear algebra that the representation $V \otimes \wedge^2 V$ cannot be irreducible: we have a natural map of representations

$$\varphi\colon V \otimes \wedge^2 V \to \wedge^3 V$$

which is obviously surjective. The kernel of this map is a representation with the same set of weights as $V \otimes \wedge^2 V$ (but taking on the weights $L_i + L_j + L_k$ with multiplicity 2 rather than 3), and so must contain the irreducible representation $\Gamma_{1,1,0}$ with highest weight $2L_1 + L_2$.

Exercise 15.10. Prove that the kernel of φ is indeed the irreducible representation $\Gamma_{1,1,0}$.

Finally, consider the tensor product $V \otimes \wedge^3 V$. This has weights $2L_i + L_k + L_l = L_i - L_j$, each occurring once, and 0, occurring four times. Its weight diagrams thus look like



This we may recognize as simply a direct sum of the adjoint representation with a copy of the trivial; this corresponds to the kernel and image of the obvious contraction (or trace) map

$$V \otimes \wedge^3 V = V \otimes V^* = \mathrm{Hom}(V, V) \to \mathbb{C}.$$

(Note that the adjoint representation is the irreducible representation with highest weight $2L_1 + L_2 + L_3$, or in other words the representation $\Gamma_{1,0,1}$.)

Exercise 15.11. Describe the weights of the representations $\mathrm{Sym}^n V$, and deduce that they are all irreducible.

Exercise 15.12. Describe the weights of the representations $\mathrm{Sym}^n(\wedge^2 V)$, and deduce that they are not irreducible. Describe maps

$$\varphi_n\colon \mathrm{Sym}^n(\wedge^2 V) \to \mathrm{Sym}^{n-2}(\wedge^2 V)$$

and show that the kernel of φ_n is the irreducible representation with highest weight $n(L_1 + L_2)$.

Exercise 15.13. The irreducible representation $\Gamma_{1,1,1}$ with highest weight $3L_1 + 2L_2 + L_3$ occurs as a subrepresentation of the tensor product $V \otimes \wedge^2 V \otimes \wedge^3 V$ lying in the kernel of each of the three maps

$$\begin{aligned}&V \otimes \wedge^2 V \otimes \wedge^3 V \to \wedge^3 V \otimes \wedge^3 V\\ &V \otimes \wedge^2 V \otimes \wedge^3 V \to \wedge^2 V \otimes \wedge^4 V \cong \wedge^2 V\\ &V \otimes \wedge^2 V \otimes \wedge^3 V \cong V \otimes \wedge^2 V^* \otimes V^* \to V \otimes \wedge^3 V^* \cong V \otimes V\end{aligned}$$

obtained by wedging two of the three factors. Is it equal to the intersection of these kernels? To test your graphic abilities, draw a diagram of the weights (ignoring multiplicities) of this representation.

Representations of $\mathfrak{sl}_n\mathbb{C}$

Once the case of $\mathfrak{sl}_4\mathbb{C}$ is digested, the case of the special linear group in general offers no surprises; the main difference in the general case is just the absence of pictures. Of course, the standard representation V of $\mathfrak{sl}_n\mathbb{C}$ has highest weight L_1, and similarly the exterior power $\wedge^k V$ is irreducible with highest weight $L_1 + \cdots + L_k$. It follows that the irreducible representation $\Gamma_{a_1,\ldots,a_{n-1}}$ with highest weight $(a_1 + \cdots + a_{n-1})L_1 + \cdots + a_{n-1}L_{n-1}$ will appear inside the tensor product

$$\mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2}(\wedge^2 V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} V),$$

demonstrating the existence theorem (14.18) for representations of $\mathfrak{sl}_n\mathbb{C}$.

Exercise 15.14. Verify that the exterior powers of the standard representations of $\mathfrak{sl}_n\mathbb{C}$ are indeed irreducible (though this is not necessary for the truth of the last sentence).

§15.3. Weyl's Construction and Tensor Products

At the end of the preceding section, we saw that the irreducible representation $\Gamma_{a_1,\ldots,a_{n-1}}$ of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $(a_1 + \cdots + a_{n-1})L_1 + \cdots + a_{n-1}L_{n-1}$ will appear as a subspace of the tensor product

$$\mathrm{Sym}^{a_1}V \otimes \mathrm{Sym}^{a_2}(\Lambda^2 V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\Lambda^{n-1}V),$$

or equivalently as a subspace of the dth tensor power $V^{\otimes d}$ of the standard representation V. The natural question is, how can we describe this subspace? We have seen the answer in one case already (two cases, if you count the trivial answer $\Gamma_a = \mathrm{Sym}^a V$ in the case $n = 2$): the representation $\Gamma_{a,b}$ of $\mathfrak{sl}_3\mathbb{C}$ can be realized as the kernel of the contraction map

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b(\Lambda^2 V) \to \mathrm{Sym}^{a-1}V \otimes \mathrm{Sym}^{b-1}(\Lambda^2 V).$$

This raises the question of whether the representation Γ_a can in general be described as a subspace of the tensor power $\bigotimes(\mathrm{Sym}^{a_i}(\Lambda^i V))$ by intersecting kernels of such contraction/wedge product maps. Specifically, for i and j with $i + j \le n$ we can define maps

$$\begin{aligned}\mathrm{Sym}^{a_1}V \otimes \mathrm{Sym}^{a_2}(\Lambda^2 V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\Lambda^{n-1}V)\\ \to \Lambda^i V \otimes \Lambda^j V \otimes \mathrm{Sym}^{a_1}V \otimes \cdots \otimes \mathrm{Sym}^{a_i-1}(\Lambda^i V) \otimes \cdots\\ \otimes \mathrm{Sym}^{a_j-1}(\Lambda^j V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\Lambda^{n-1}V)\end{aligned}$$

and we have similar maps for $i < j$ with $i + j \ge n$ and i even with $2i \ge n$; there are likewise analogously defined maps in which we split off three or more factors. The representation $\Gamma_{a_1,\ldots,a_{n-1}}$ is in the kernel of all such maps; and we may ask whether the intersection of all such kernels is equal to Γ_a.

The answer, it turns out, is no. (It is a worthwhile exercise to find an example of a representation Γ_a that cannot be realized in this way.) There is, however, another way of describing Γ_a as a subspace of $V^{\otimes d}$: in fact, we have already met these representations in Lecture 6, under the guise of *Schur functors* or *Weyl modules*. In fact, at the end of this lecture we will see how to describe them explicitly as subspaces of the above spaces $\bigotimes(\mathrm{Sym}^{a_i}(\Lambda^i V))$. Recall that for $V = \mathbb{C}^n$ an n-dimensional vector space, and any partition

$$\lambda\colon \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0,$$

we can apply the Schur functor $\mathbb{S}_\lambda$ to V to obtain a representation $\mathbb{S}_\lambda V = \mathbb{S}_\lambda(\mathbb{C}^n)$ of $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$. If $d = \sum \lambda_i$, this was realized as

$$\mathbb{S}_\lambda V = V^{\otimes d} \cdot c_\lambda = V^{\otimes d} \otimes_{\mathbb{C}\mathfrak{S}_d} V_\lambda,$$

where c_λ is the Young symmetrizer corresponding to λ, and V_λ is the irreducible representation of $\mathfrak{S}_d$ corresponding to λ.

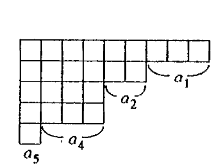
We saw in Lecture 6 that $\mathbb{S}_\lambda V$ is an irreducible representation of $\mathrm{GL}_n\mathbb{C}$. It follows immediately that $\mathbb{S}_\lambda V$ remains irreducible as a representation of $\mathrm{SL}_n\mathbb{C}$, since any element of $\mathrm{GL}_n\mathbb{C}$ is a scalar multiple of an element of $\mathrm{SL}_n\mathbb{C}$. In particular, it determines an irreducible representation of the Lie algebra $\mathfrak{sl}_n\mathbb{C}$.

Proposition 15.15. *The representation $\mathbb{S}_\lambda(\mathbb{C}^n)$ is the irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_n L_n$.*

In particular, $\mathbb{S}_\lambda(\mathbb{C}^n)$ and $\mathbb{S}_\mu(\mathbb{C}^n)$ are isomorphic representations of $\mathfrak{sl}_n\mathbb{C}$ if and only if $\lambda_i - \mu_i$ is constant, independent of i. To relate this to our earlier notation, we may say that the irreducible representation $\Gamma_{a_1,\ldots,a_{n-1}}$ of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1})$ is obtained by applying the Schur functor $\mathbb{S}_\lambda$ to the standard representation V, where

$$\lambda = (a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, \ldots, a_{n-1}, 0).$$

(If we want a unique Schur functor for each representation, we can restrict to those λ with $\lambda_n = 0$.) In terms of the Young diagram for λ, the coefficients $a_i = \lambda_i - \lambda_{i+1}$ are the differences of lengths of rows. For example, if $n = 6$,



is the Young diagram corresponding to $\Gamma_{3,2,0,3,1}$.

Proof of the proposition. In Theorem 6.3 we calculated that the trace of a diagonal matrix with entries $x_1, \ldots, x_n$ on $\mathbb{S}_\lambda(\mathbb{C}^n)$ is the Schur polynomial $S_\lambda(x_1, \ldots, x_n)$. By Equation (A.19), when the Schur polynomial is written out it takes the form

$$S_\lambda(x_1, \ldots, x_n) = M_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} M_\mu, \tag{15.16}$$

where M_μ is the sum of the monomial $X^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdot \ldots \cdot x_n^{\mu_n}$ and all distinct monomials obtained from it by permuting the variables, and the $K_{\lambda\mu}$ are certain non-negative integers called Kostka numbers. When $\mathbb{S}_\lambda(\mathbb{C}^n)$ is diagonalized with respect to the group of diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$, it is also diagonalized with respect to $\mathfrak{h} \subset \mathfrak{sl}_n(\mathbb{C})$. There is one monomial in the displayed equation for each one-dimensional eigenspace. The weights of $\mathbb{S}_\lambda(\mathbb{C}^n)$ as a representation of $\mathfrak{sl}_n(\mathbb{C})$ therefore consist of all

$$\mu_1 L_1 + \mu_2 L_2 + \cdots + \mu_n L_n,$$

each occurring as often as it does in the monomial X^μ in the polynomial

$S_\lambda(x_1, \ldots, x_n)$. Since the sum is over those partitions μ for which the first nonzero $\lambda_i - \mu_i$ is positive, the highest weight that appears is $\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_n L_n$, which concludes the proof. [In fact one can describe an explicit basis of eigenvectors for $\mathbb{S}_\lambda(\mathbb{C}^n)$ which correspond to the monomials that appear in (15.16), cf. Problem 6.15 or Proposition 15.55.] □

In particular, we have (by Theorem 6.3) formulas for the dimension of the representation with given highest weight. Explicitly, one formula says that

$$\dim(\Gamma_{a_1,\ldots,a_{n-1}}) = \prod_{1 \le i < j \le n} \frac{(a_i + \cdots + a_{j-1}) + j - i}{j - i}. \tag{15.17}$$

As we saw in the proof, this proposition also gives the multiplicities of all weight spaces as the integers $K_{\lambda\mu}$ that appear in (15.16), which have a simple combinatorial description (p. 456): *the dimension of the weight space with weight μ in the representation $\mathbb{S}_\lambda(\mathbb{C}^n)$ is the number of ways one can fill the Young diagram of λ with μ_1 1's, μ_2 2's, $\ldots$, μ_n n's, in such a way that the entries in each row are nondecreasing and those in each column are strictly increasing.*

Exercise 15.18. Use the formula in case $n = 4$ to calculate the dimensions of the irreducible representations $\Gamma_{1,1,0}$ and $\Gamma_{1,1,1}$ of $\mathfrak{sl}_4\mathbb{C}$. In the former case, use this to redo Exercise 15.10; in the latter case, to do Exercise 15.13.

Exercise 15.19*. Use this formula to show that the dimension of the irreducible representation $\Gamma_{a,b}$ of $\mathfrak{sl}_3$ with highest weight $aL_1 + b(L_1 + L_2)$ is $(a + b + 1)(a + 1)(b + 1)/2$. This is the same as the dimension of the kernel of the contraction map

$$\iota_{a,b}\colon \operatorname{Sym}^a V \otimes \operatorname{Sym}^b V^* \to \operatorname{Sym}^{a-1} V \otimes \operatorname{Sym}^{b-1} V^*.$$

Use this to give another proof of the assertion made in Claim 13.4 that $\Gamma_{a,b}$ is this kernel.

Exercise 15.20*. As an application of the above formula, show that if V is the standard representation of $\mathfrak{sl}_n\mathbb{C}$, then the kernel of the wedge product map

$$V \otimes \wedge^k V \to \wedge^{k+1} V$$

is the irreducible representation $\Gamma_{1,0,\ldots,0,1,0,\ldots}$ with highest weight $2L_1 + L_2 + \cdots + L_k$; and that the irreducible representation $\Gamma_{k-1,1,0,\ldots}$ with highest weight $k \cdot L_1 + L_2$ is the kernel of the product map

$$V \otimes \operatorname{Sym}^k V \to \operatorname{Sym}^{k+1} V.$$

Exercise 15.21*. Show that the only nontrivial irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ of dimension less than or equal to n are V and V^*.

One important consequence of the fact that the irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ are obtained by applying Schur functors to the standard representation

is that *identities among the Schur–Weyl functors give rise to identities among representations of* GL_n (and hence SL_n and $\mathfrak{sl}_n$), as we saw in Lecture 6. For example, the representation

$$\operatorname{Sym}^{\lambda_1}(V) \otimes \operatorname{Sym}^{\lambda_2}(V) \otimes \cdots \otimes \operatorname{Sym}^{\lambda_n}(V) \tag{15.22}$$

is a direct sum of representations $\mathbb{S}_\lambda(V) \oplus \bigoplus_\mu K_{\mu\lambda}\mathbb{S}_\mu(V)$, where $K_{\mu\lambda}$ is the coefficient described above. The particular application of this principle that we will use most frequently in the sequel, however, is the consequence that *one knows the decomposition of a tensor product of any two irreducible representations of* $\mathfrak{sl}_n\mathbb{C}$: specifically, the tensor power $\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V)$ decomposes into a direct sum of irreducible representations

$$\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V) = \bigoplus_\nu N_{\lambda\mu\nu}\mathbb{S}_\nu(V), \tag{15.23}$$

where the coefficients $N_{\lambda\mu\nu}$ are given by the *Littlewood–Richardson rule*, which is a formula in terms of the number of ways to fill the Young diagram between λ and ν with μ_1 1's, μ_2 2's, $\ldots$, μ_n n's, satisfying a certain combinatorial condition described in (A.8).

Exercise 15.24. Use the Littlewood–Richardson rule to show that the representation $\Gamma_{a_1+b_1,\ldots,a_{n-1}+b_{n-1}}$ occurs exactly once in the tensor product $\Gamma_{a_1,\ldots,a_{n-1}} \otimes \Gamma_{b_1,\ldots,b_{n-1}}$.

A special case of this is the analogue of Pieri's formula, which allows us to decompose the tensor product of an arbitrary irreducible representation with either $\operatorname{Sym}^k V = \Gamma_{k,0,\ldots,0}$ or the fundamental representation $\wedge^k V = \Gamma_{0,\ldots,1,0,\ldots,0}$, (where the 1 occurs in the kth place):

Proposition 15.25. (i) *The tensor product of $\Gamma_{a_1,\ldots,a_{n-1}}$ with $\operatorname{Sym}^k V = \Gamma_{k,0,\ldots,0}$ decomposes into a direct sum:*

$$\Gamma_{a_1,\ldots,a_{n-1}} \otimes \Gamma_{k,\ldots,0} = \bigoplus \Gamma_{b_1,\ldots,b_{n-1}},$$

the sum over all $(b_1, \ldots, b_{n-1})$ for which there are non-negative integers $c_1, \ldots, c_n$ whose sum is k, with $c_{i+1} \le a_i$ for $1 \le i \le n - 1$, and with $b_i = a_i + c_i - c_{i+1}$ for $1 \le i \le n - 1$.

(ii) *The tensor product of $\Gamma_{a_1,\ldots,a_{n-1}}$ with $\wedge^k V = \Gamma_{0,\ldots,0,1,0,\ldots,0}$ decomposes into a direct sum:*

$$\Gamma_{a_1,\ldots,a_{n-1}} \otimes \Gamma_{0,\ldots,0,1,0,\ldots,0} = \bigoplus \Gamma_{b_1,\ldots,b_{n-1}},$$

the sum over all $(b_1, \ldots, b_{n-1})$ for which there is a subset S of $\{1, \ldots, n\}$ of cardinality k, such that if $i \notin S$ and $i + 1 \in S$, then $a_i > 0$, with

$$b_i = \begin{cases} a_i - 1 & \text{if } i \notin S \text{ and } i + 1 \in S \\ a_i + 1 & \text{if } i \in S \text{ and } i + 1 \notin S \\ a_i & \text{otherwise.} \end{cases}$$

Proof. This is simply a matter of translating the prescriptions of (6.8) and (6.9), which describe the decompositions in terms of adding boxes to the Young diagrams. In (i), the c_i are the number of boxes added to the ith row, and in (ii), S is the set of rows to which a box is added. □

Exercise 15.26. Verify the descriptions in Section 2 of this lecture of $V \otimes \Lambda^2 V$ and $V \otimes \Lambda^3 V$, where V is the standard representation of $\mathfrak{sl}_4\mathbb{C}$.

Exercise 15.27. Use Pieri's formula (with $n = 4$) twice to find the decomposition into irreducibles of $V \otimes \Lambda^2 V \otimes \Lambda^3 V$, where V is the standard representation of $\mathfrak{sl}_4\mathbb{C}$. Use this to redo Exercise 15.13.

Exercise 15.28. Use Pieri's formula to prove (13.5). You may also want to look around in Lecture 13 to see which other of the decompositions found there by hand may be deduced from these formulas.

Exercise 15.29. Verify that the statement of Exercise 15.20 follows directly from Pieri's formula.

In the following exercises, $V = \mathbb{C}^n$ is the standard representation of $\mathfrak{sl}_n\mathbb{C}$.

Exercise 15.30. Consider now tensor products of the form $\Lambda^k V \otimes \Lambda^l V$, with, say, $k \geq l$. Show that there is a natural map

$$\Lambda^k V \otimes \Lambda^l V \to \Lambda^{k+1} V \otimes \Lambda^{l-1} V$$

given by contraction with the element "trace" (or "identity") in $V \otimes V^* = \operatorname{End}(V)$. Explicitly, this map may be given by

$$(v_1 \wedge \cdots \wedge v_k) \otimes (w_1 \wedge \cdots \wedge w_l)$$
$$\mapsto \sum_{i=1}^{l} (-1)^i (v_1 \wedge \cdots \wedge v_k \wedge w_i) \otimes (w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_l).$$

What is the image of this map? Show that the kernel is the irreducible representation $\Gamma_{0,\ldots,0,1,0,\ldots,0,1,0,\ldots}$ with highest weight $2L_1 + \cdots + 2L_l + L_{l+1} + \cdots + L_k$.

Exercise 15.31*. Carry out an analysis similar to that of the preceding exercise for the maps

$$\operatorname{Sym}^k V \otimes \operatorname{Sym}^l V \to \operatorname{Sym}^{k+1} V \otimes \operatorname{Sym}^{l-1} V$$

defined analogously.

Exercise 15.32*. As a special case of Pieri's formula, we see that if V is the standard representation of $\mathfrak{sl}_n\mathbb{C}$, the tensor product

$$\Lambda^k V \otimes \Lambda^k V = \bigoplus \mathbb{S}_{(2,\ldots,2,1,\ldots,1,0,\ldots)}(V)$$
$$= \bigoplus \Gamma_{0,\ldots,0,1,0,\ldots,0,1,0,\ldots},$$

where in the ith factor the 1's occur in the $(k - i)$th and $(k + i)$th places. At the same time, of course, we know that

$$\Lambda^k V \otimes \Lambda^k V = \operatorname{Sym}^2(\Lambda^k V) \oplus \Lambda^2(\Lambda^k V).$$

If we denote the ith term on the right-hand side of the first displayed equation for $\Lambda^k V \otimes \Lambda^k V$ by Θ_i, show that

$$\operatorname{Sym}^2(\Lambda^k V) = \bigoplus \Theta_{2i} \quad \text{and} \quad \Lambda^2(\Lambda^k V) = \bigoplus \Theta_{2i+1}.$$

Exercise 15.33*. As another special case of Pieri's formula, we see that the tensor product

$$\operatorname{Sym}^k V \otimes \operatorname{Sym}^k V = \bigoplus \mathbb{S}_{(k+i,k-i)}(V)$$
$$= \bigoplus \Gamma_{2i,k-i,0\ldots0}.$$

At the same time, of course, we know that

$$\operatorname{Sym}^k V \otimes \operatorname{Sym}^k V = \operatorname{Sym}^2(\operatorname{Sym}^k V) \oplus \Lambda^2(\operatorname{Sym}^k V).$$

Which of the factors appearing in the first decomposition lie in $\operatorname{Sym}^2(\operatorname{Sym}^k V)$, and which in $\Lambda^2(\operatorname{Sym}^k V)$?

It follows from the Littlewood–Richardson rule that if λ, μ, and ν all have at most two rows, then the coefficient $N_{\lambda\mu\nu}$ is zero or one (and it is easy to say which occurs). In particular, for the Lie algebras $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$, the decomposition of the tensor product of two irreducible representations is always multiplicity free. Groups whose representations have this property, such as SU(2), SU(3), and SO(3) which are so important in physics, are called "simply reducible," cf. [Mack].

§15.4. Some More Geometry

Let V be an n-dimensional vector space, and $G(k, n) = G(k, V) = \operatorname{Grass}_k V$ the Grassmannian of k-planes in V. $\operatorname{Grass}_k V$ is embedded as a subvariety of the projective space $\mathbb{P}(\Lambda^k V)$ by the *Plücker embedding*:

$$\rho: \operatorname{Grass}_k V \hookrightarrow \mathbb{P}(\Lambda^k V)$$

sending the plane W spanned by vectors $v_1, \ldots, v_k$ to the alternating tensor $v_1 \wedge \cdots \wedge v_k$. Equivalently, noting that if $W \subset V$ is a k-dimensional subspace, then $\Lambda^k W$ is a line in $\Lambda^k V$, we may write this simply as

$$\rho: W \mapsto \Lambda^k W.$$

This embedding is compatible with the action of the general linear group:

$$\mathrm{PSL}_n\mathbb{C} = \mathrm{Aut}(\mathbb{P}(V)) = \{\sigma \in \mathrm{Aut}(\mathbb{P}(\wedge^k V)) : \sigma(G(k, V)) = G(k, V)\}^\circ.$$

This follows from a fact in algebraic geometry ([Ha]): all automorphisms of the Grassmannian are induced by automorphisms of V, unless $n = 2k$, in which case we can choose an arbitrary isomorphism of V with V^* and compose these with the automorphism that takes W to $(\mathbb{C}^n/W)^*$. Here the superscript $\circ$ denotes the connected component of the identity. As in previous lectures, if we want symmetric powers to correspond to homogeneous polynomials on projective space, we should consider the dual situation: $G = \mathrm{Grass}^k V$ is the Grassmannian of k-dimensional *quotient* spaces of V, and the Plücker embedding embeds G in the projective space $\mathbb{P}(\wedge^k V^*)$ of one-dimensional quotients of $\wedge^k V$.

The space of all homogeneous polynomials of degree m on $\mathbb{P}(\wedge^k V^*)$ is naturally the symmetric power $\mathrm{Sym}^m(\wedge^k V)$. Let $I(G)_m$ denote the subspace of those polynomials of degree m on $\mathbb{P}(\wedge^k V^*)$ that vanish on G. Each $I(G)_m$ is a representation of $\mathfrak{sl}_n\mathbb{C}$:

$$0 \to I(G)_m \to \mathrm{Sym}^m(\wedge^k V) \to W_m \to 0,$$

where W_m denotes the restrictions to G of the polynomials of degree m on the ambient space $\mathbb{P}(\wedge^k V^*)$. We shall see later that W_m is the irreducible representation $\Gamma_{0,\ldots,0,m,0,\ldots}$ with highest weight $m(L_1 + \cdots + L_k)$ (the case $m = 2$ will be dealt with below). In the following discussion, we consider the problem of describing the quadratic part $I(G)_2$ of the ideal as a representation of $\mathfrak{sl}_n\mathbb{C}$.

Exercise 15.34. Consider the first case of a Grassmannian that is not a projective space, that is, $k = 2$. The ideal of the Grassmannian $G(2, V)$ of 2-planes in a vector space is easy to describe: a tensor $\varphi \in \wedge^2 V$ is decomposable if and only if $\varphi \wedge \varphi = 0$ (equivalently, if we think of φ as given by a skew-symmetric $n \times n$ matrix, if and only if the Pfaffians of symmetric 4×4 minors all vanish); and indeed the quadratic relations we get in this way generate the ideal of the Grassmannian. We, thus, have an isomorphism

$$I(G)_2 \cong \wedge^4 V$$

and correspondingly a decomposition into irreducibles

$$\mathrm{Sym}^2(\wedge^2 V) \cong \wedge^4 V \oplus \Gamma_{0,2,0,\ldots,0},$$

where $\Gamma_{0,2,0,\ldots,0}$ is, as above, the irreducible representation with highest weight $2(L_1 + L_2)$, cf. Exercise 15.32.

Exercise 15.35. When $k = 2$ and $n = 4$, G is a quadric hypersurface in $\mathbb{P}^5$, so polynomials vanishing on G are simply those divisible the quadratic polynomial that defines G. Deduce an isomorphism.

$$I(G)_m = \mathrm{Sym}^{m-2}(\wedge^2 V).$$

The first case of a Grassmannian that is not a projective space or of the form $G(2, V)$ is, of course, $G(3, 6)$, and this yields an interesting example.

Exercise 15.36. Let V be six dimensional. By examining weights, show that the space $I(G)_2$ of quadratic polynomials vanishing on the Grassmannian $G(3, V) \subset \mathbb{P}(\wedge^3 V)$ is isomorphic to the adjoint representation of $\mathfrak{sl}_6\mathbb{C}$, i.e., that we have a map

$$\varphi\colon \mathrm{Sym}^2(\wedge^3 V) \to V \otimes V^*$$

with image the space of traceless matrices.

Exercise 15.37. Find explicitly the map φ of the preceding exercise.

Exercise 15.38. Again, let V be six dimensional. Show that the representation $\mathrm{Sym}^4(\wedge^3 V)$ has a trivial direct summand, corresponding to the hypersurface in $\mathbb{P}(\wedge^3 V^*)$ dual to the Grassmannian $G = G(3, V) \subset \mathbb{P}(\wedge^3 V)$.

In general, the ideal $I(G) = \bigoplus I(G)_m$ is generated by the famous *Plücker equations*. These are homogeneous polynomials of degree two, and may be written down explicitly, cf. (15.53), [H-P], or [Ha]. In the following exercises, we will give a more intrinsic description of these relations, which will allow us to identify the space $I(G)_2$ they span as a representation on $\mathfrak{sl}_n\mathbb{C}$ (and to see the general pattern of which the above are special cases).

Exercise 15.39. For a given tensor $\Lambda \in \wedge^k V$, we introduce two associated subspaces:

$$W = \{v \in V : v \wedge \Lambda = 0\} \subset V$$

and

$$W^* = \{v^* \in V^* : v^* \wedge \Lambda^* = 0\} \subset V^*,$$

where, abusing notation slightly, Λ^* is the tensor Λ viewed as an element of $\wedge^k V = \wedge^{n-k} V^*$. Show that the dimensions of W and W^* are at most k and $n - k$, respectively, and that Λ is decomposable if and only if W has dimension k or W^* has dimension $n - k$; and deduce that Λ is decomposable if and only if the annihilator W' of W^* is equal to W.

Exercise 15.40. Now let $\Xi \in \wedge^{k+1} V^* = \wedge^{n-k-1} V$. Wedge product gives a map

$$\iota_\Xi\colon \wedge^k V \to \wedge^{n-1} V = V^*.$$

Using the preceding exercise, show that Λ is decomposable if and only if

$$\iota_\Xi(\Lambda) \wedge \Lambda = 0 \in \wedge^{k-1} V$$

for all $\Xi \in \wedge^{k+1} V^*$.

Exercise 15.41. Observe that in the preceding exercise we construct a map

$$\wedge^{k+1}V^* \otimes \mathrm{Sym}^2(\wedge^k V) \to \wedge^{k-1}V,$$

or, by duality, a map

$$\wedge^{k+1}V^* \otimes \wedge^{k-1}V^* \to \mathrm{Sym}^2(\wedge^k V^*) \tag{15.42}$$

whose image is a vector space of quadrics on $\mathbb{P}(\wedge^k V)$ whose common zeros are exactly the locus of decomposable vectors, that is, the Grassmannian $G(k, V)$. Show that this image is exactly the span of the Plücker relations above.

Exercise 15.43. Show that the map (15.42) of the preceding exercise is just the dual of the map constructed in Exercise 15.30, with $k = l$ and restricted to the symmetric product. Combining this with the result of Exercise 15.32 (and assuming the statement that the Plücker relations do indeed span $I(G)_2$), deduce that in terms of the description

$$\mathrm{Sym}^2(\wedge^k V) = \bigoplus \Theta_{2l}$$

of the symmetric square of $\wedge^k V$, we have

$$W_2 = \Theta_0 = \Gamma_{0,\dots,0,2,0,\dots}$$

(the irreducible representation with highest weight $2(L_1 + \cdots + L_k)$), and

$$I(G)_2 = \bigoplus_{l\geq 1} \Theta_{2l}.$$

Hard Exercise 15.44. Show that in the last equation the sub-direct sum

$$I(l) = \bigoplus_{i\geq l} \Theta_{2i}$$

is just the quadratic part of the ideal of the *restricted chordal variety* of the Grassmannian: that is, the union of the chords $\overline{LM}$ joining pairs of points in G corresponding to pairs of planes L and M meeting in a subspace of dimension at least $k - 2l + 1$. (Question: What is the actual zero locus of these quadrics?)

Exercise 15.45. Carry out an analysis similar to the above to relate the ideal of a Veronese variety $\mathbb{P}V^* \subset \mathbb{P}(\mathrm{Sym}^k V^*)$ to the decomposition given in Exercise 15.33 of $\mathrm{Sym}^2(\mathrm{Sym}^k V)$. For which k do the quadratic polynomials vanishing the Veronese give an irreducible representation?

Exercise 15.46. (For algebraic geometers and/or commutative algebraists.) Just as the group $\mathrm{PGL}_n\mathbb{C}$ acts on the ring S of polynomials on projective space $\mathbb{P}^N$, preserving the ideal of the Veronese variety, so it acts on that space of relations on the ideal (that is, inasmuch as the ideal is generated by quadrics, the kernel of the multiplication map $I_X(2) \otimes S \to S$), and likewise on the entire minimal resolution of the ideal of X. Show that this resolution has the form

$$\cdots \to R_2 \otimes S \to R_1 \otimes S \to I_X(2) \otimes S,$$

where all the R_i are finite-dimensional representations of $\mathrm{PGL}_n\mathbb{C}$, and identify the representations R_i in the specific cases of

(i) the rational normal curve in $\mathbb{P}^3$,
(ii) the rational normal curve in $\mathbb{P}^4$, and
(iii) the Veronese surface in $\mathbb{P}^5$.

§15.5. Representations of $\mathrm{GL}_n\mathbb{C}$

We have said that there is little difference between representations of $\mathrm{GL}_n\mathbb{C}$ and those of the subgroup $\mathrm{SL}_n\mathbb{C}$ of matrices of determinant 1. Our object here is to record the difference, which, naturally enough, comes from the determinant: if $V = \mathbb{C}^n$ is the standard representation, $\wedge^n V$ is trivial for $\mathrm{SL}_n\mathbb{C}$ but not for $\mathrm{GL}_n\mathbb{C}$. Similarly, V and $\wedge^{n-1}V^*$ are isomorphic for $\mathrm{SL}_n\mathbb{C}$ but not for $\mathrm{GL}_n\mathbb{C}$.

To relate representations of $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{GL}_n\mathbb{C}$, we first need to define some representations of $\mathrm{GL}_n\mathbb{C}$. To begin with, let D_k denote the one-dimensional representation of $\mathrm{GL}_n\mathbb{C}$ given by the kth power of the determinant. When k is non-negative, $D_k = (\wedge^n V)^{\otimes k}$; D_{-k} is the dual $(D_k)^*$ of D_k. Next, note that the irreducible representations of $\mathrm{SL}_n\mathbb{C}$ may be lifted to representations of $\mathrm{GL}_n\mathbb{C}$ in two ways. First, for any index $\mathbf{a} = (a_1, \dots, a_n)$ of length n we may take $\Phi_{\mathbf{a}}$ to be the subrepresentation of the tensor product

$$\mathrm{Sym}^{a_1}V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1}V) \otimes \mathrm{Sym}^{a_n}(\wedge^n V)$$

spanned by the highest weight vector with weight $a_1L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1})$—that is, the vector

$$v = (e_1)^{a_1} \cdot (e_1 \wedge e_2)^{a_2} \cdot \ldots \cdot (e_1 \wedge \cdots \wedge e_n)^{a_n}.$$

This restricts to $\mathrm{SL}_n\mathbb{C}$ to give the representation $\Gamma_{a'}$, where $a' = (a_1, \dots, a_{n-1})$; taking different values of a_n amounts to tensoring the representation with different factors $\mathrm{Sym}^{a_n}(\wedge^n V) = (\wedge^n V)^{\otimes a_n} = D_{a_n}$. In particular, we have

$$\Phi_{a_1,\dots,a_n+k} = \Phi_{a_1,\dots,a_n} \otimes D_k,$$

which allows us to extend the definition of $\Phi_{\mathbf{a}}$ to indices a with $a_n < 0$: we simply set

$$\Phi_{a_1,\dots,a_n} = \Phi_{a_1,\dots,a_n+k} \otimes D_{-k}$$

for large k.

Alternatively, we may consider the Schur functor $\mathbb{S}_\lambda$ applied to the standard representation V of $\mathrm{GL}_n\mathbb{C}$, where

$$\lambda = (a_1 + \cdots + a_n, a_2 + \cdots + a_n, \dots, a_{n-1} + a_n, a_n).$$

We will denote this representation $\mathbb{S}_\lambda V$ of $\mathrm{GL}_n\mathbb{C}$ by Ψ_λ; note that

$$\Psi_{\lambda_1+k,\dots,\lambda_n+k} = \Psi_{\lambda_1,\dots,\lambda_n} \otimes D_k$$

which likewise allows us to define Ψ_λ for any index λ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, even if some of the λ_i are negative: we simply take

$$\Psi_{\lambda_1,\ldots,\lambda_n} = \Psi_{\lambda_1+k,\ldots,\lambda_n+k} \otimes D_{-k}$$

for any sufficiently large k.

As is not hard to see, *the two representations* $\Phi_{\mathbf{a}}$ *and* Ψ_λ *are isomorphic as representations of* $GL_n\mathbb{C}$: by §15.3 their restrictions to $SL_n\mathbb{C}$ agree, so it suffices to check their restrictions to the center $\mathbb{C}^* \subset GL_n\mathbb{C}$, where each acts by multiplication by $z^{\sum\lambda_i} = z^{\sum ia_i}$. It is even clearer that there are no coincidences among the $\Phi_{\mathbf{a}}$ (i.e., $\Phi_{\mathbf{a}}$ will be isomorphic to $\Phi_{\mathbf{a}'}$ if and only if $\mathbf{a} = \mathbf{a}'$): if $\Phi_{\mathbf{a}} \cong \Phi_{\mathbf{a}'}$, we must have $a_i = a_i'$ for $i = 1, \ldots, n-1$, so the statement follows from the nontriviality of D_k for $k \neq 0$. Thus, to complete our description of the irreducible finite-dimensional representations of $GL_n\mathbb{C}$, we just have to check that we have found them all. We may then express the completed result as

Proposition 15.47. *Every irreducible complex representation of* $GL_n\mathbb{C}$ *is isomorphic to* Ψ_λ *for a unique index* $\lambda = \lambda_1, \ldots, \lambda_n$ *with* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *(equivalently, to* $\Phi_{\mathbf{a}}$ *for a unique index* $\mathbf{a} = a_1, \ldots, a_n$ *with* $a_1, \ldots, a_{n-1} \geq 0$*).*

PROOF. We start by going back to the corresponding Lie algebras. The scalar matrices form a one-dimensional ideal $\mathbb{C}$ in $\mathfrak{gl}_n\mathbb{C}$, and in fact $\mathfrak{gl}_n\mathbb{C}$ is a product of Lie algebras:

$$\mathfrak{gl}_n\mathbb{C} = \mathfrak{sl}_n\mathbb{C} \times \mathbb{C}. \tag{15.48}$$

In particular, $\mathbb{C}$ is the radical of $\mathfrak{gl}_n\mathbb{C}$, and $\mathfrak{sl}_n\mathbb{C}$ is the semisimple part. It follows from Proposition 9.17 that every irreducible representation of $\mathfrak{gl}_n\mathbb{C}$ is a tensor product of an irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ and a one-dimensional representation. More precisely, let $W_\lambda = \mathbb{S}_\lambda(\mathbb{C}^n)$ be the representation of $\mathfrak{sl}_n\mathbb{C}$ determined by the partition λ (extended to $\mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ by making the second factor act trivially). For $w \in \mathbb{C}$, let $L(w)$ be the one-dimensional representation of $\mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ which is zero on the first factor and multiplication by w on the second; the proof of Proposition 9.17 shows that any irreducible representation of $\mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ is isomorphic to a tensor product $W_\lambda \otimes L(w)$. The same is therefore true for the simply connected[1] group $SL_n\mathbb{C} \times \mathbb{C}$ with this Lie algebra.

We write $GL_n\mathbb{C}$ as a quotient modulo a discrete subgroup of the center of $SL_n\mathbb{C} \times \mathbb{C}$:

$$1 \to \mathrm{Ker}(\rho) \to SL_n\mathbb{C} \times \mathbb{C} \xrightarrow{\rho} GL_n\mathbb{C} \to 1, \tag{15.49}$$

where $\rho(g \times z) = e^z \cdot g$, so the kernel of ρ is generated by $e^s \cdot I \times (-s)$, where $s = 2\pi i/n$.

Our task is simply to see which of the representations $W_\lambda \otimes L(w)$ of $SL_n\mathbb{C} \times \mathbb{C}$ are trivial on the kernel of ρ. Now $e^s \cdot I$ acts on $\mathbb{S}_\lambda\mathbb{C}^n$ by multiplication by e^{sd}, where $d = \sum\lambda_i$; indeed, this is true on the entire representation $(\mathbb{C}^n)^{\otimes d}$ which contains $\mathbb{S}_\lambda\mathbb{C}^n$. And $-s$ acts on $L(w)$ by multiplication by e^{-sw}, so $e^s \cdot I \times (-s)$ acts on the tensor product by multiplication by e^{sd-sw}. The tensor product is, therefore, trivial on the kernel of ρ precisely when $sd - sw \in 2\pi i\mathbb{Z}$, i.e., when

$$w = \sum\lambda_i + kn$$

for some integer k.

We claim finally that any representation $W_\lambda \otimes L(w)$ satisfying this condition is the pullback via ρ of a representation Ψ on $GL_n\mathbb{C}$. In fact, it is not hard to see that it is the pullback of the representation $\Psi_{\lambda_1+k,\ldots,\lambda_n+k}$: the two clearly restrict to the same representation on $SL_n\mathbb{C}$, and their restrictions to $\mathbb{C}$ are just multiplication by $e^{wz} = e^{(\sum\lambda_i+nk)z}$. □

[1] For a proof that $SL_n\mathbb{C}$ is simply connected, see §23.1.

Exercise 15.50. Show that the dual of the representation Ψ_λ which is isomorphic to $\mathbb{S}_\lambda(V^*)$ is the representation $\Psi_{(-\lambda_n,\ldots,-\lambda_1)}$.

Exercise 15.51*. Show that if $\rho: GL_n\mathbb{C} \to GL(W)$ is a representation (assumed to be holomorphic), then W decomposes into a direct sum of irreducible representations.

Exercise 15.52*. Show that the Hermite reciprocity isomorphism of Exercise 11.34 is an isomorphism over $GL_2\mathbb{C}$, not just over $SL_2\mathbb{C}$.

More Remarks on Weyl's Construction

We close out this lecture by looking once more at the Weyl construction of these representations of $GL(V)$. This will include a realization "by generators and relations," as well as giving a natural basis for each representation. First, it may be illuminating—and it will be useful later—to look more closely at how $\mathbb{S}_\lambda V$ sits in $V^{\otimes d}$. We want to realize $\mathbb{S}_\lambda V$ as a subspace of the subspace

$$\mathrm{Sym}^{a_k}(\textstyle\bigwedge^k V) \otimes \mathrm{Sym}^{a_{k-1}}(\bigwedge^{k-1} V) \otimes \cdots \otimes \mathrm{Sym}^{a_1}(V) \subset V^{\otimes d},$$

where a_i is the number of columns of the Young diagram of λ of length i (and k is the number of rows). This space is embedded in $V^{\otimes d}$ in the natural way: from left to right, a factor $\mathrm{Sym}^a(\bigwedge^b V)$ is embedded in the corresponding $V^{\otimes ab}$ by mapping a symmetric product of exterior products

$$(v_{1,1} \wedge v_{2,1} \wedge \cdots \wedge v_{b,1}) \cdot (v_{1,2} \wedge v_{2,2} \wedge \cdots \wedge v_{b,2}) \cdot \ldots \cdot (v_{1,a} \wedge v_{2,a} \wedge \cdots \wedge v_{b,a})$$

to

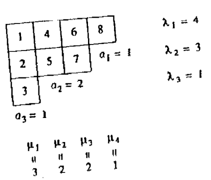
$$\sum \mathrm{sgn}(q)(v_{q_1(1),p(1)} \otimes \cdots \otimes v_{q_1(b),p(1)}) \otimes \cdots \otimes (v_{q_a(1),p(a)} \otimes \cdots \otimes v_{q_a(b),p(a)}),$$

the sum over $p \in \mathfrak{S}_a$ and $q = (q_1, \ldots, q_a) \in \mathfrak{S}_b \times \cdots \times \mathfrak{S}_b$. In other words, one first symmetrizes by permuting columns of the same length, and then performs an alternating symmetrizer on each column.

Letting $\mathbf{a} = (a_1, \ldots, a_k)$, let $A^{\mathbf{a}}(V)$ denote this tensor product of symmetric powers of exterior powers, i.e., set

$$A^{\mathbf{a}}V = \mathrm{Sym}^{a_k}(\Lambda^k V) \otimes \mathrm{Sym}^{a_{k-1}}(\Lambda^{k-1} V) \otimes \cdots \otimes \mathrm{Sym}^{a_1}(V).$$

We want to realize $\mathbb{S}_\lambda V$ as a subspace of $A^{\mathbf{a}}V$. To do this we use the construction of $\mathbb{S}_\lambda V$ as $V^{\otimes d} \cdot c_\lambda$, where c_λ is a Young symmetrizer; to get compatibility with the embedding of $A^{\mathbf{a}}V$ we have just made, we use the tableau which numbers the columns from top to bottom, then left to right.



We take $\mu = \lambda' = (\mu_1 \geq \cdots \geq \mu_l > 0)$ to be the conjugate of λ. The symmetrizer c_λ is a product $a_\lambda \cdot b_\lambda$, where $a_\lambda = \sum e_p$, the sum over all p in the subgroup $P = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}$ of $\mathfrak{S}_d$ preserving the rows, $b_\lambda = \sum \mathrm{sgn}(q)q$, the sum over the subgroup $Q = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_l}$ preserving the columns, as described in Lecture 4. The symmetrizing by rows can be done in two steps as follows. There is a subgroup

$$R = \mathfrak{S}_{a_k} \times \cdots \times \mathfrak{S}_{a_1}$$

of P, which consists of permutations that move all entries of each column to the same position in some column of the same length; in other words, permutations in R are determined by permuting columns which have the same length. (In the illustration, $R = \{1, (46)(57)\}$.) Set

$$a'_\lambda = \sum_{r \in R} e_r \quad \text{in } \mathbb{C}\mathfrak{S}_d.$$

Now if we define a''_λ to be $\sum e_p$, where the sum is over any set of representatives in P for the left cosets P/R, then the row symmetrizer a_λ is the product of a''_λ and a'_λ. So

$$\mathbb{S}_\lambda(V) = (V^{\otimes d} \cdot a''_\lambda) \cdot a'_\lambda \cdot b_\lambda.$$

The point is that, by what we have just seen,

$$V^{\otimes d} \cdot a'_\lambda \cdot b_\lambda = A^{\mathbf{a}}V.$$

Since $V^{\otimes d} \cdot a''_\lambda$ is a subspace of $V^{\otimes d}$, its image $\mathbb{S}_\lambda(V)$ by $a'_\lambda \cdot b_\lambda$ is a subspace of $A^{\mathbf{a}}(V)$, as we claimed.

There is a simple way to construct all the representations $\mathbb{S}_\lambda V$ of $GL(V)$ at once. In fact, the direct sum of all the representations $\mathbb{S}_\lambda V$, over all (non-negative) partitions λ, can be made into a commutative, graded ring, which we denote by $\mathbb{S}^\bullet$ or $\mathbb{S}^\bullet(V)$, with simple generators and relations. This is similar to the fact that the symmetric algebra $\mathrm{Sym}^\bullet V = \bigoplus \mathrm{Sym}^k V$ and the exterior algebra $\Lambda^\bullet V = \bigoplus \Lambda^k V$ are easier to describe than the individual graded pieces, and it has some of the similar advantages for studying all the representations at once. This algebra has appeared and reappeared frequently, cf. [H-P]; the construction we give is essentially that of Towber [Tow1].

To construct $\mathbb{S}^\bullet(V)$, start with the symmetric algebra on the sum of all the positive exterior products of V: set

$$\begin{aligned} A^\bullet(V) &= \mathrm{Sym}^\bullet(V \oplus \Lambda^2 V \oplus \Lambda^3 V \oplus \cdots \oplus \Lambda^n V) \\ &= \bigoplus_{a_1, \ldots, a_n} \mathrm{Sym}^{a_n}(\Lambda^n V) \otimes \cdots \otimes \mathrm{Sym}^{a_2}(\Lambda^2 V) \otimes \mathrm{Sym}^{a_1}(V), \end{aligned}$$

the sum over all n-tuples $a_1, \ldots, a_n$ of non-negative integers. So $A^\bullet(V)$ is the direct sum of the $A^{\mathbf{a}}(V)$ just considered. The ring $\mathbb{S}^\bullet = \mathbb{S}^\bullet(V)$ is defined to be the quotient of this ring $A^\bullet(V)$ modulo the graded, two-sided ideal $I^\bullet$ generated by all elements ("Plücker relations") of the form

$$\begin{aligned} &(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q) \\ &\quad - \sum_{i=1}^{p} (v_1 \wedge \cdots \wedge v_{i-1} \wedge w_1 \wedge v_{i+1} \wedge \cdots \wedge v_p) \cdot (v_i \wedge w_2 \wedge \cdots \wedge w_q) \end{aligned} \tag{15.53}$$

for all $p \geq q \geq 1$ and all $v_1, \ldots, v_p, w_1, \ldots, w_q \in V$. (If $p = q$, this is an element of $\mathrm{Sym}^2(\Lambda^p V)$; if $p > q$, it is in $\Lambda^p V \otimes \Lambda^q V = \mathrm{Sym}^1(\Lambda^p V) \otimes \mathrm{Sym}^1(\Lambda^q V)$. Note that the multiplication in $\mathbb{S}^\bullet(V)$ comes entirely from its being a symmetric algebra and does not involve the wedge products in $\Lambda^\bullet V$.)

Exercise 15.54*. Show that $I^\bullet$ contains all elements of the form

$$\begin{aligned} &(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q) \\ &\quad - \sum (v_1 \wedge \cdots \wedge w_1 \wedge \cdots \wedge w_r \wedge \cdots \wedge v_p) \\ &\qquad \cdot (v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_r} \wedge w_{r+1} \wedge \cdots \wedge w_q) \end{aligned}$$

for all $p \geq q \geq r \geq 1$ and all $v_1, \ldots, v_p, w_1, \ldots, w_q \in V$, where the sum is over all $1 \leq i_1 < i_2 < \cdots < i_r \leq p$, and the elements $w_1, \ldots, w_r$ are inserted at the corresponding places in $v_1 \wedge \cdots \wedge v_p$.

Remark. You can avoid this exercise by simply taking the elements in the exercise as defining generators for the ideal $I^\bullet$. When $p = q = r$, the calculation of Exercise 15.54 shows that the relation $(v_1 \wedge \cdots \wedge v_p)\cdot(w_1 \wedge \cdots \wedge w_p) = (w_1 \wedge \cdots \wedge w_p)\cdot(v_1 \wedge \cdots \wedge v_p)$ follows from the generating equations for $I^\bullet$. In particular, this commutativity shows that one could define $\mathbb{S}^\bullet(V)$ to be the full tensor algebra on $V \oplus \Lambda^2 V \oplus \cdots \oplus \Lambda^n V$ modulo the ideal generated by the same generators.

The algebra $\mathbb{S}^\bullet(V)$ is the direct sum of the images $\mathbb{S}^{\mathbf{a}}(V)$ of the summands $A^{\mathbf{a}}(V)$. Let $e_1, \ldots, e_n$ be a basis for V. We will construct a basis for $\mathbb{S}^{\mathbf{a}}(V)$, with a basis element e_T for every semistandard tableau T on the partition λ which corresponds to $\mathbf{a}$. Recall that a semistandard tableau is a numbering of the boxes of the Young diagram with the integers $1, \ldots, n$, in such a way that the entries in each row are nondecreasing, and the entries in each column are strictly increasing. Let $T(i, j)$ be the entry of T in the ith row and the jth column. Define e_T to be the image in $\mathbb{S}^{\mathbf{a}}(V)$ of the element

$$\prod_{j=1}^{l} e_{T(1,j)} \wedge e_{T(2,j)} \wedge \cdots \wedge e_{T(\mu_j,j)} \in \mathrm{Sym}^{a_n}(\Lambda^n V) \otimes \cdots \otimes \mathrm{Sym}^{a_1}(V),$$

i.e., wedge together the basis elements corresponding to the entries in the columns, and multiply the results in $\mathbb{S}^\bullet(V)$.

Proposition 15.55. (1) *The projection from $A^{\mathbf{a}}(V)$ to $\mathbb{S}^{\mathbf{a}}(V)$ maps the subspace $\mathbb{S}_\lambda(V)$ isomorphically onto $\mathbb{S}^{\mathbf{a}}(V)$.*

(2) *The e_T for T a semistandard tableau on λ form a basis for $\mathbb{S}^{\mathbf{a}}(V)$.*

PROOF. We show first that the elements e_T span $\mathbb{S}^{\mathbf{a}}(V)$. It is clear that the e_T span if we allow all tableaux T that number the boxes of λ with integers between 1 and n with strictly increasing columns, for such elements span before dividing by the ideal $I^\bullet$. We order such tableaux by listing their entries column by column, from left to right and top to bottom, and using the reverse lexicographic order: $T' > T$ if the last entry where they differ has a larger entry for T' than for T. If T is not semistandard, there will be two successive columns of T, say the jth and $(j+1)$st, in which we have $T(r, j) > T(r, j+1)$ for some r. It suffices to show how to use relations in $I^\bullet$ to write e_T as a linear combination of elements $e_{T'}$ with $T' > T$. For this we use the relation in Exercise 15.54, with $v_i = e_{T(i,j)}$ for $1 \le i \le p = \mu_j$, and $w_i = e_{T(i,j+1)}$ for $1 \le i \le q = \mu_{j+1}$, to interchange the first r of the $\{w_i\}$ with subsets of r of the $\{v_i\}$. The terms on the right-hand side of the relation will all correspond to tableaux T' in which the r first entries in the $(j+1)$st column of T are replaced by r of the enties in the jth column, and are not otherwise changed beyond the jth column. All of these are larger than T in the ordering, which proves the assertion.

It is possible to give a direct proof that the e_T corresponding to semistandard tableaux T are linearly independent (see [Tow1]), but we can get by with less. Among the semistandard tableaux on λ there is a smallest one T_0 whose ith row is filled with the integer i. We need to know that e_{T_0} is not zero in $\mathbb{S}^\bullet$. This is easy to see directly. In fact, the relations among the e_T in $I^\bullet \cap A^{\mathbf{a}}(V)$ are spanned by those obtained by substituting r elements from some column of some T to an earlier column, as in the preceding paragraph. Such will never involve the generator e_{T_0} unless the T that is used is T_0, and in this case, the resulting element of $I^\bullet$ is zero. Since e_{T_0} occurs in no nontrivial relation, its image in $\mathbb{S}^\bullet$ cannot vanish.

Since e_{T_0} comes from $\mathbb{S}_\lambda(V)$, it follows that the projection from $\mathbb{S}_\lambda(V)$ to $\mathbb{S}^{\mathbf{a}}(V)$ is not zero. Since this projection is a mapping of representations of $\mathrm{SL}(V)$, it follows that $\mathbb{S}^{\mathbf{a}}(V)$ must contain a copy of the irreducible representation $\mathbb{S}_\lambda(V)$. We know from Theorem 6.3 and Exercise A.31 that the dimension of $\mathbb{S}_\lambda(V)$ is the number of semistandard tableaux on λ. Since we have proved that the dimension of $\mathbb{S}^{\mathbf{a}}(V)$ is at most this number, the projection from $\mathbb{S}_\lambda(V)$ to $\mathbb{S}^{\mathbf{a}}(V)$ must be surjective, and since $\mathbb{S}_\lambda(V)$ is irreducible, it must be injective as well, and the e_T for T a semistandard tableau on λ must form a basis, as asserted. □

Note that this proposition gives another description of the representations $\mathbb{S}_\lambda(V)$, as the quotient of the space $A^{\mathbf{a}}(V)$ by the subspace generated by the "Plücker" relations (15.53).

Exercise 15.56. Show that, if the factor $\Lambda^n V$ is omitted from the construction, the resulting algebra is the direct sum of all irreducible representations of $\mathrm{SL}(V) = \mathrm{SL}_n\mathbb{C}$.

It is remarkable that all the representations $\mathbb{S}_\lambda(\mathbb{C}^n)$ of $\mathrm{GL}_n\mathbb{C}$ were written down by Deruyts (following Clebsch) a century ago, before representation theory was born, as in the following exercise.

Exercise 15.57*. Let $X = (x_{i,j})$ be an $n \times n$ matrix of indeterminants. The group $G = \mathrm{GL}_n\mathbb{C}$ acts on the polynomial ring $\mathbb{C}[x_{i,j}]$ by $g \cdot x_{i,j} = \sum_{k=1}^{n} a_{k,i} x_{k,j}$ for $g = (a_{i,j}) \in \mathrm{GL}_n\mathbb{C}$. For any tableau T on the Young diagram of λ consisting of the integers from 1 to n, strictly increasing in the columns, let e_T be the product of minors constructed from X, one for each column, as follows: if the column of T has length μ_j, form the minor using the first μ_j columns, and use the rows that are numbered by the entries of the column of T. Let D_λ be the subspace of $\mathbb{C}[x_{i,j}]$ spanned by these e_T, where d is the number partitioned by λ. Show that: (i) D_λ is preserved by $\mathrm{GL}_n\mathbb{C}$; (ii) the e_T, where T is semistandard, form a basis for D_λ; (iii) D_λ is isomorphic to $\mathbb{S}_\lambda(\mathbb{C}^n)$.

(№7) Lectures 16, 17

LECTURE 16

Symplectic Lie Algebras

In this lecture we do for the symplectic Lie algebras exactly what we did for the special linear ones in §15.1 and most of §15.2: we will first describe in general the structure of a symplectic Lie algebra (that is, give a Cartan subalgebra, find the roots, describe the Killing form, and so on). We will then work out in some detail the representations of the specific algebra $\mathfrak{sp}_4\mathbb{C}$. As in the case of the corresponding analysis of the special linear Lie algebras, this is completely elementary.

§16.1: The structure of $\mathrm{Sp}_{2n}\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$
§16.2 Representations of $\mathfrak{sp}_4\mathbb{C}$

§16.1. The Structure of $\mathrm{Sp}_{2n}\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$

Let V be a $2n$-dimensional complex vector space, and

$$Q\colon V \times V \to \mathbb{C},$$

a nondegenerate, skew-symmetric bilinear form on V. The symplectic Lie group $\mathrm{Sp}_{2n}\mathbb{C}$ is then defined to be the group of automorphisms A of V preserving Q—that is, such that $Q(Av, Aw) = Q(v, w)$ for all $v, w \in V$—and the symplectic Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ correspondingly consists of endomorphisms $A\colon V \to V$ satisfying

$$Q(Av, w) + Q(v, Aw) = 0$$

for all v and $w \in V$. Clearly, the isomorphism classes of the abstract group and Lie algebra do not depend on the particular choice of Q; but in order to be able to write down elements of both explicitly we will, for the remainder of our discussion, take Q to be the bilinear form given, in terms of a basis $e_1, \ldots,$

e_{2n} for V, by

$$Q(e_i, e_{i+n}) = 1,$$
$$Q(e_{i+n}, e_i) = -1,$$

and

$$Q(e_i, e_j) = 0 \quad \text{if } j \neq i \pm n.$$

The bilinear form Q may be expressed as

$$Q(x, y) = {}^t x \cdot M \cdot y,$$

where M is the $2n \times 2n$ matrix given in block form as

$$M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix};$$

the group $\mathrm{Sp}_{2n}\mathbb{C}$ is thus the group of $2n \times 2n$ matrices A satisfying

$$M = {}^tA \cdot M \cdot A$$

and the Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ correspondingly the space of matrices X satisfying the relation

$$ {}^tX \cdot M + M \cdot X = 0. \tag{16.1}$$

Writing a $2n \times 2n$ matrix X in block form as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we have

$$ {}^tX \cdot M = \begin{pmatrix} -{}^tC & {}^tA \\ -{}^tD & {}^tB \end{pmatrix}$$

and

$$M \cdot X = \begin{pmatrix} C & D \\ -A & -B \end{pmatrix}$$

so that this relation is equivalent to saying that *the off-diagonal blocks B and C of X are symmetric, and the diagonal blocks A and D of X are negative transposes of each other.*

With this said, there is certainly an obvious candidate for Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{sp}_{2n}\mathbb{C}$, namely the subalgebra of matrices diagonal in this representation; in fact, this works, as we shall see shortly. The subalgebra $\mathfrak{h}$ is thus spanned by the n $2n \times 2n$ matrices $H_i = E_{i,i} - E_{n+i,n+i}$ whose action on V is to fix e_i, send e_{n+i} to its negative, and kill all the remaining basis vectors; we will correspondingly take as basis for the dual vector space $\mathfrak{h}^*$ the dual basis L_j, where $\langle L_j, H_i \rangle = \delta_{i,j}$.

We have already seen how the diagonal matrices act on the algebra of all matrices, so that it is easy to describe the action of $\mathfrak{h}$ on $\mathfrak{g}$. For example, for

$1 \leq i, j \leq n$ the matrix $E_{i,j} \in \mathfrak{gl}_{2n}\mathbb{C}$ is carried into itself under the adjoint action of H_i, into minus itself by the action of H_j, and to 0 by all the other H_k; and the same is true of the matrix $E_{n+j,n+i}$. The element

$$X_{i,j} = E_{i,j} - E_{n+j,n+i} \in \mathfrak{sp}_{2n}\mathbb{C}$$

is thus an eigenvector for the action of $\mathfrak{h}$, with eigenvalue $L_i - L_j$. Similarly, for $i \neq j$ we see that the matrices $E_{i,n+j}$ and $E_{j,n+i}$ are carried into themselves by H_i and H_j and killed by all the other H_k; and likewise $E_{n+i,j}$ and $E_{n+j,i}$ are each carried into their negatives by H_i and H_j and killed by the others. Thus, the elements

$$Y_{i,j} = E_{i,n+j} + E_{j,n+i}$$

and

$$Z_{i,j} = E_{n+i,j} + E_{n+j,i}$$

are eigenvectors for the action of $\mathfrak{h}$, with eigenvalues $L_i + L_j$ and $-L_i - L_j$, respectively. Finally, when $i = j$ the same calculation shows that $E_{i,n+i}$ is doubled by H_i and killed by all other H_j; and likewise $E_{n+i,i}$ is sent to minus twice itself by H_i and to 0 by the others. Thus, the elements

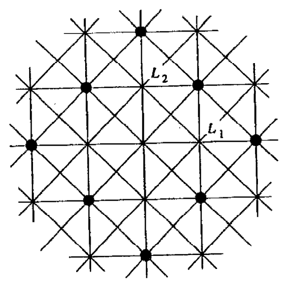
$$U_i = E_{i,n+i}$$

and

$$V_i = E_{n+i,i}$$

are eigenvectors with eigenvalues $2L_i$ and $-2L_i$, respectively. In sum, then, *the roots of the Lie algebra* $\mathfrak{sp}_{2n}\mathbb{C}$ *are the vectors* $\pm L_i \pm L_j \in \mathfrak{h}^*$.

In the first case $n = 1$, of course we just get the root diagram of $\mathfrak{sl}_2\mathbb{C}$, which is the same algebra as $\mathfrak{sp}_2\mathbb{C}$. In case $n = 2$, we have the diagram



(16.2)

As in the case of the special linear Lie algebras, probably the easiest way to determine the Killing form on $\mathfrak{sp}_{2n}\mathbb{C}$ (at least up to scalars) is to use its

invariance under the automorphisms of $\mathfrak{sp}_{2n}\mathbb{C}$ preserving $\mathfrak{h}$. For example, w have the automorphisms of $\mathfrak{sp}_{2n}\mathbb{C}$ induced by permutations of the basi vectors e_i of V: for any permutation σ of $\{1, 2, \ldots, n\}$ we can define a automorphism of V preserving Q by sending e_i to $e_{\sigma(i)}$ and e_{n+i} to $e_{n+\sigma(i)}$, an this induces an automorphism of $\mathfrak{sp}_{2n}\mathbb{C}$ preserving $\mathfrak{h}$ and carrying H_i to $H_{\sigma(i)}$ Also, for any i we can define an involution of V—and thereby of $\mathfrak{sp}_{2n}\mathbb{C}$—b sending e_i to e_{n+i}, e_{n+i} to $-e_i$, and all the other basis vectors to themselves this will have the effect of sending H_i to $-H_i$ and preserving all the other H_j Now, the Killing form on $\mathfrak{h}$ must be invariant under these automorphisms from the first batch it follows that for some pair of constants α and β we mus have

$$B(H_i, H_i) = \alpha$$

and

$$B(H_i, H_j) = \beta \quad \text{for } i \neq j;$$

from the second batch it follows that, in fact, $\beta = 0$. Thus, B is just a multipl of the standard quadratic form $B(H_i, H_j) = \delta_{i,j}$, and the dual form correspond ingly a multiple of $B(L_i, L_j) = \delta_{i,j}$, so that the angles in the diagram above ar correct.

Also as in the case of $\mathfrak{sl}_n\mathbb{C}$, one can also compute the Killing form directl from the definition: $B(H, H') = \sum \alpha(H)\alpha(H')$, the sum over all roots α. Fo $H = \sum a_i H_i$ and $H' = \sum b_i H_i$, this gives $B(H, H')$ as a sum

$$\sum_{i \neq j} (a_i + a_j)(b_i + b_j) + 2\sum_i (2a_i)(2bi) + \sum_{i \neq j} (a_i - a_j)(b_i - b_j)$$

which simplifies to

$$B(H, H') = (4n + 4)(\textstyle\sum a_i b_i). \tag{16.3}$$

Our next job is to locate the distinguished copies $\mathfrak{s}_\alpha$ of $\mathfrak{sl}_2\mathbb{C}$, and the corresponding elements $H_\alpha \in \mathfrak{h}$. This is completely straightforward. We start with the eigenvalues $L_i - L_j$ and $L_j - L_i$ corresponding to the elements $X_{i,j}$ and $X_{j,i}$; we have

$$\begin{aligned}[X_{i,j}, X_{j,i}] &= [E_{i,j} - E_{n+j,n+i}, E_{j,i} - E_{n+i,n+j}] \\ &= [E_{i,j}, E_{j,i}] + [E_{n+j,n+i}, E_{n+i,n+j}] \\ &= E_{i,i} - E_{j,j} + E_{n+j,n+j} - E_{n+i,n+i} \\ &= H_i - H_j.\end{aligned}$$

Thus, the distinguished element $H_{L_i - L_j}$ is a multiple of $H_i - H_j$. To see what multiple, recall that $H_{L_i - L_j}$ should act on $X_{i,j}$ by multiplication by 2 and on $X_{j,i}$ by multiplication by -2; since we have

$$\begin{aligned}\operatorname{ad}(H_i - H_j)(X_{i,j}) &= ((L_i - L_j)(H_i - H_j)) \cdot X_{i,j} \\ &= 2X_{i,j},\end{aligned}$$

we conclude that

$$H_{L_i - L_j} = H_i - H_j.$$

Next consider the pair of opposite eigenvalues $L_i + L_j$ and $-L_i - L_j$, corresponding to the eigenvectors $Y_{i,j}$ and $Z_{i,j}$. We have

$$\begin{aligned}[Y_{i,j}, Z_{i,j}] &= [E_{i,n+j} + E_{j,n+i}, E_{n+i,j} + E_{n+j,i}] \\ &= [E_{i,n+j}, E_{n+j,i}] + [E_{j,n+i}, E_{n+i,j}] \\ &= E_{i,i} - E_{n+j,n+j} + E_{j,j} - E_{n+i,n+i} \\ &= H_i + H_j.\end{aligned}$$

We calculate then

$$\begin{aligned}\mathrm{ad}(H_i + H_j)(Y_{i,j}) &= ((L_i + L_j)(H_i + H_j)) \cdot Y_{i,j} \\ &= 2 \cdot Y_{i,j},\end{aligned}$$

so we have

$$H_{L_i + L_j} = H_i + H_j$$

and similarly

$$H_{-L_i - L_j} = -H_i - H_j.$$

Finally, we look at the pair of eigenvalues $\pm 2L_i$ coming from the eigenvectors U_i and V_i. To complete the span of U_i and V_i to a copy of $\mathfrak{sl}_2\mathbb{C}$ we add

$$\begin{aligned}[U_i, V_i] &= [E_{i,n+i}, E_{n+i,i}] \\ &= E_{i,i} - E_{n+i,n+i} \\ &= H_i.\end{aligned}$$

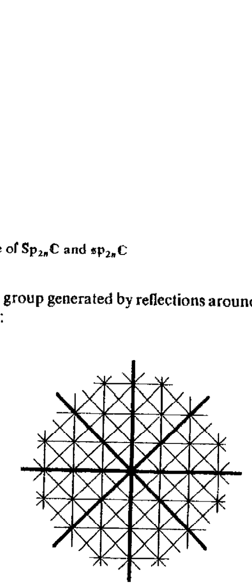
Since

$$\begin{aligned}\mathrm{ad}(H_i)(U_i) &= (2L_i(H_i)) \cdot U_i \\ &= 2 \cdot U_i,\end{aligned}$$

we conclude that the distinguished element H_{2L_i} is H_i, and likewise $H_{-2L_i} = -H_i$. Thus, the distinguished elements $\{H_\alpha\} \subset \mathfrak{h}$ are $\{\pm H_i \pm H_j, \pm H_i\}$; in particular, the weight lattice Λ_W of linear forms on $\mathfrak{h}$ integral on all the H_α is exactly the lattice of integral linear combinations of the L_i. In Diagram (16.2), for example, this is just the lattice of intersections of the horizontal and vertical lines drawn; observe that for all n the index $[\Lambda_W : \Lambda_R]$ of the root lattice in the weight lattice is just 2.

Next we consider the group of symmetries of the weights of an arbitrary representation of $\mathfrak{sp}_{2n}\mathbb{C}$. For each root α we let W_α be the involution in $\mathfrak{h}^*$ fixing the hyperplane Ω_α given by $\langle H_\alpha, L\rangle = 0$ and acting as $-I$ on the line spanned by α; we observe in this case that, as we claimed will be true in general, the line generated by α is perpendicular to the hyperplane Ω_α, so that the involution is just a reflection in this plane. In the case $n = 2$, for example,

we get the dihedral group generated by reflections around the four lines drawn through the origin:



so that the weight diagram of a representation of $\mathfrak{sp}_4\mathbb{C}$ will look like an octagon in general, or (in some cases) a square.

In general, reflection in the plane Ω_{2L_i} given by $\langle H_i, L\rangle = 0$ will simply reverse the sign of L_i while leaving the other L_j fixed; reflection in the plane $\langle H_i - H_j, L\rangle = 0$ will exchange L_i and L_j and leave the remaining L_k alone. The Weyl group $\mathfrak{W}$ acts as the full automorphism group of the lines spanned by the L_i and fits into a sequence

$$1 \to (\mathbb{Z}/2\mathbb{Z})^n \to \mathfrak{W} \to \mathfrak{S}_n \to 1.$$

Note that the sequence splits: $\mathfrak{W}$ is a semidirect product of $\mathfrak{S}_n$ and $(\mathbb{Z}/2\mathbb{Z})^n$. (This is a special case of a *wreath product.*) In particular the order of $\mathfrak{W}$ is $2^n n!$.

We can choose a positive direction as before:

$$l(\textstyle\sum a_i L_i) = c_1 a_1 + \cdots + c_n a_n, \qquad c_1 > c_2 > \cdots > c_n > 0.$$

The positive roots are then

$$R^+ = \{L_i + L_j\}_{i \le j} \cup \{L_i - L_j\}_{i<j}, \tag{16.4}$$

with primitive positive roots $\{L_i - L_{i+1}\}_{i=1,\dots,n-1}$ and $2L_n$. The corresponding (closed) Weyl chamber is

$$\mathscr{W} = \{a_1 L_1 + a_2 L_2 + \cdots + a_n L_n : a_1 \ge a_2 \ge \cdots \ge a_n \ge 0\}; \tag{16.5}$$

note that the walls of this chamber—the cones

$$\{\textstyle\sum a_i L_i : a_1 > \cdots > a_i = a_{i+1} > \cdots > a_n > 0\}$$

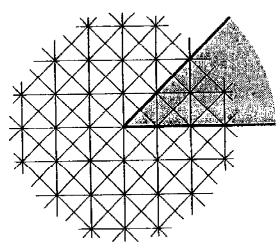
and

$$\{\textstyle\sum a_i L_i : a_1 > a_2 > \cdots > a_n = 0\}$$

lie in the hyperplanes $\Omega_{L_i - L_{i+1}}$ and Ω_{2L_n} perpendicular to the primitive positive or negative roots, as expected.

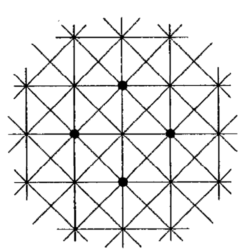
§16.2. Representations of $\mathfrak{sp}_4\mathbb{C}$

Let us consider now the representations of the algebra $\mathfrak{sp}_4\mathbb{C}$ specifically. Recall that, with the choice of Weyl chamber as above, there is a unique irreducible representation Γ_α of $\mathfrak{sp}_4\mathbb{C}$ with highest weight α for any α in the intersection of the closed Weyl chamber $\mathscr{W}$ with the weight lattice: that is, for each lattice vector in the shaded region in the diagram



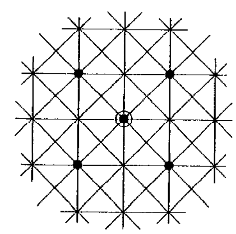
Any such highest weight vector can be written as a non-negative integral linear combination of L_1 and $L_1 + L_2$; for simplicity we will just write $\Gamma_{a,b}$ for the irreducible representation $\Gamma_{aL_1+b(L_1+L_2)}$ with highest weight $aL_1 + b(L_1 + L_2) = (a + b)L_1 + bL_2$.

To begin with, we have the standard representation as the algebra of endomorphisms of the four-dimensional vector space V; the four standard basis vectors e_1, e_2, e_3, and e_4 are eigenvectors with eigenvalues $L_1, L_2, -L_1$, and $-L_2$, respectively, so that the weight diagram of V is



V is just the representation $\Gamma_{1,0}$ in the notation above. Note that the dual of this representation is isomorphic to it, which we can see either from the symmetry of the weight diagram, or directly from the fact that the corresponding group representation preserves a bilinear form $V \times V \to \mathbb{C}$ giving an identification of V with V^*.

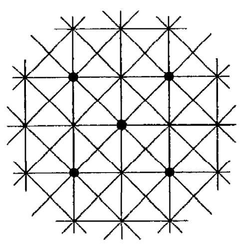
The next representation to consider is the exterior square $\wedge^2 V$. The weights of $\wedge^2 V$, the pairwise sums of distinct weights of V, are just the linear forms $\pm L_i \pm L_j$ (each appearing once) and 0 (appearing twice, as $L_1 - L_1$ and $L_2 - L_2$), so that its weight diagram looks like



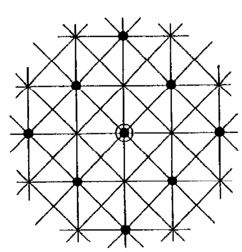
Clearly this representation is not irreducible. We can see this from the weight diagram, using Observation 14.16: there is only one way of getting to the weight space 0 from the highest weight $L_1 + L_2$ by successive applications of the primitive negative root spaces $\mathfrak{g}_{-L_1+L_2}$ (spanned by $X_{2,1} = E_{2,1} - E_{3,4}$) and $\mathfrak{g}_{-2L_2}$ (spanned by $V_2 = E_{4,2}$)—that is, by applying first V_2, which takes you to the weight space of $L_1 - L_2$, and then $X_{2,1}$—and so the dimension of the zero weight space in the irreducible representation $\Gamma_{0,1}$ with highest weight $L_1 + L_2$ must be one. Of course, we know in any event that $\wedge^2 V$ cannot be irreducible: the corresponding group action of $\mathrm{Sp}_4\mathbb{C}$ on V by definition preserves the skew form $Q \in \wedge^2 V^* \cong \wedge^2 V$. Either way, we conclude that we have a direct sum decomposition

$$\wedge^2 V = W \oplus \mathbb{C},$$

where W is the irreducible, five-dimensional representation of $\mathfrak{sp}_4\mathbb{C}$ with highest weight $L_1 + L_2$—in our notation, $\Gamma_{0,1}$—and weight diagram



Let us consider next some degree 2 tensors in V and W. To begin with, we can write down the weight diagram for the representation $\mathrm{Sym}^2 V$; the weights being just the pairwise sums of the weights of V, the diagram is

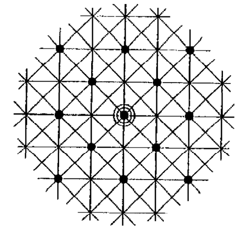


This looks like the weight diagram of the adjoint representation, and indeed that is what it is: in terms of the identification of V and V^* given by the skew form Q, the relation (16.1) defining the symplectic Lie algebra says that the subspace

$$\mathfrak{sp}_4\mathbb{C} \subset \mathrm{Hom}(V, V) = V \otimes V^* = V \otimes V$$

is just the subspace $\mathrm{Sym}^2 V \subset V \otimes V$. In particular, $\mathrm{Sym}^2 V$ is the irreducible representation $\Gamma_{2,0}$ with highest weight $2L_1$.

Next, consider the symmetric square $\mathrm{Sym}^2 W$, which has weight diagram



To see if this is irreducible we first look at the weight diagram: this time there are three ways of getting from the weight space with highest weight $2L_1 + 2L_2$ to the space of weight 0 by successively applying $X_{2,1} = E_{2,1} - E_{3,4}$ and $Y_2 = E_{4,2}$, so if we want to proceed by this method we are forced to do a little calculation, which we leave as Exercise 16.7.

Alternatively, we can see directly that $\mathrm{Sym}^2 W$ decomposes: the natural map given by wedge product

$$\Lambda^2 V \otimes \Lambda^2 V \to \Lambda^4 V = \mathbb{C}$$

is symmetric, and so factors to give a map

$$\mathrm{Sym}^2(\Lambda^2 V)) \to \mathbb{C}.$$

Moreover, since this map is well defined up to scalars—in particular, it does not depend on the choice of skew form Q—it cannot contain the subspace $\mathrm{Sym}^2 W \subset \mathrm{Sym}^2(\Lambda^2 V))$ in its kernel, so that it restricts to give a surjection

$$\varphi\colon \mathrm{Sym}^2 W \to \mathbb{C}.$$

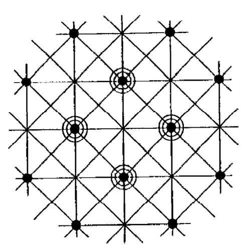
This approach would appear to leave two possibilities open: either the kernel of this map is irreducible, or it is the direct sum of an irreducible representation and a further trivial summand. In fact, however, from the principle that an irreducible representation cannot have two independent invariant bilinear forms, we see that $\mathrm{Sym}^2 W$ can contain at most one trivial summand, and so the former alternative must hold, i.e., we have

$$\mathrm{Sym}^2 W = \Gamma_{0,2} \oplus \mathbb{C}. \qquad (16.6)$$

Exercise 16.7*. Prove (16.6) directly, by showing that if v is a highest weight vector, then the three vectors $X_{2,1} Y_2 X_{2,1} Y_2 v$, $X_{2,1} X_{2,1} Y_2 Y_2 v$, and $Y_2 X_{2,1} X_{2,1} Y_2 v$ span a two-dimensional subspace of the kernel of φ.

Exercise 16.8. Verify that $\Lambda^2 W \cong \mathrm{Sym}^2 V$. The significance of this isomorphism will be developed further in Lecture 18.

Lastly, consider the tensor product $V \otimes W$. First, its weight diagram:



This obviously must contain the irreducible representation $\Gamma_{1,1}$ with highest weight $2L_1 + L_2$; but it cannot be irreducible, for either of two reasons. First, looking at the weight diagram, we see that $\Gamma_{1,1}$ can take on the eigenvalues $\pm L_i$ with multiplicity at most 2, so that $V \otimes W$ must contain at least one copy of the representation V. Alternatively, we have a natural map given by wedge product

$$\wedge : V \otimes \Lambda^2 V \to \Lambda^3 V = V^* = V;$$

and since this map does not depend on the choice of skew form Q, it must restrict to give a nonzero (and hence surjective) map

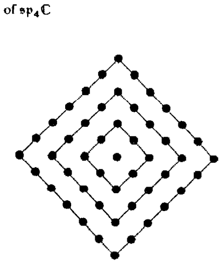
$$\varphi \colon V \otimes W \to V.$$

Exercise 16.9. Show that the kernel of this map is irreducible, and hence that we have

$$V \otimes W = \Gamma_{1,1} \oplus V.$$

What about more general tensors? To begin with, note that we have established the existence half of the standard existence and uniqueness theorem (14.18) in the case of $\mathfrak{sp}_4\mathbb{C}$: the irreducible representation $\Gamma_{a,b}$ may be found somewhere in the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b W$. The question that remains is, where? In other words, we would like to be able to say how these tensor products decompose. This will be, as it was in the case of $\mathfrak{sl}_3\mathbb{C}$, nearly tantamount (modulo the combinatorics needed to count the multiplicity with which the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b W$ assumes each of its eigenvalues) to specifying the multiplicities of the irreducible representations $\Gamma_{a,b}$.

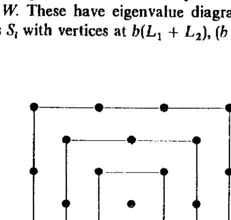
Let us start with the simplest case, namely, the representations $\mathrm{Sym}^a V$. These have weight diagram a sequence of nested diamonds D_i with vertices at aL_1, $(a-2)L_1$, etc.:



Moreover, it is not hard to calculate the multiplicities of $\mathrm{Sym}^a V$: the multiplicity on the outer diamond D_1 is one, of course; and then the multiplicities will increase by one on successive rings, so that the multiplicity along the diamond D_i will be i.

Exercise 16.10. Using the techniques of Lecture 13, show that the representations $\mathrm{Sym}^a V$ are irreducible.

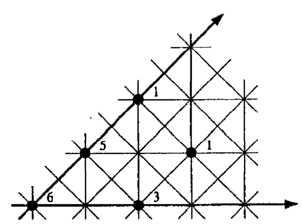
The next simplest representations, naturally enough, are the symmetric powers $\mathrm{Sym}^b W$ of W. These have eigenvalue diagrams in the shape of a sequence of squares S_i with vertices at $b(L_1 + L_2)$, $(b-1)(L_1 + L_2)$, and so on:



Here, however, the multiplicities increase in a rather strange way: they grow quadratically, but only on every other ring. Explicitly, the multiplicity will be one on the outer two rings, then 3 on the next two rings, 6 on the next two; in general, it will be $i(i+1)/2$ on the $(2i-1)$st and $(2i)$th squares S_{2i-1} and S_{2i}.

Exercise 16.11. Show that contraction with the skew form $\varphi \in \mathrm{Sym}^2 W^*$ introduced in the discussion of $\mathrm{Sym}^2 W$ above determines a surjection from $\mathrm{Sym}^b W$ onto $\mathrm{Sym}^{b-2} W$, and that the kernel of this map is the irreducible representation $\Gamma_{0,b}$ with highest weight $b(L_1 + L_2)$. Show that the multiplicities of $\Gamma_{0,b}$ are i on the squares S_{2i-1} and S_{2i} described above.

We will finish by analyzing, naively and in detail, one example of a representation $\Gamma_{a,b}$ with a and b both nonzero, namely, $\Gamma_{2,1}$; one thing we may observe on the basis of this example is that there is not a similarly simple pattern to the multiplicities of the representations $\Gamma_{a,b}$ with general a and b. To carry out our analysis, we start of course with the product $\mathrm{Sym}^2 V \otimes W$. We can readily draw the weight diagram for this representation; drawing only one-eighth of the plane and indicating multiplicities by numbers, it is



We know that the representation $\mathrm{Sym}^2 V \otimes W$ contains a copy of the irreducible representation $\Gamma_{2,1}$ with highest weight $2L_1 + (L_1 + L_2)$; and we can see immediately from the diagram that it cannot equal this: for example, $\Gamma_{2,1}$ can take the weight $2L_1$ with multiplicity at most 2 (if $v \in \Gamma_{2,1}$ is its highest weight vector, the corresponding weight space $(\Gamma_{2,1})_{2L_1} \subset \Gamma_{2,1}$ will be spanned by the two vectors $X_{2,1}(V_2(v))$ and $V_2(X_{2,1}(v))$); since it cannot contain a copy of the representation $\Gamma_{0,2}$ (the multiplicity of the weight $2(L_1 + L_2)$ being just one) it follows that $\mathrm{Sym}^2 V \otimes W$ must contain a copy of the representation $\Gamma_{2,0} = \mathrm{Sym}^2 V$.

We can, in this way, narrow down the list of possibilities a good deal. For example, $\Gamma_{2,1}$ cannot have multiplicity just one at each of the weights $2L_1$ and $L_1 + L_2$: if it did, $\mathrm{Sym}^2 V \otimes W$ would have to contain two copies of $\mathrm{Sym}^2 V$ and a further two copies of W to make up the multiplicity at $L_1 + L_2$; but since 0 must appear as a weight of $\Gamma_{2,1}$, this would give a total multiplicity of at least 7 for the weight 0 in $\mathrm{Sym}^2 V \otimes W$. Similarly, $\Gamma_{2,1}$ cannot have multiplicity 1 at $2L_1$ and 2 at $L_1 + L_2$: we would then have two copies of $\mathrm{Sym}^2 V$ and one of W in $\mathrm{Sym}^2 V \otimes W$; and since the multiplicity of 0 in $\Gamma_{2,1}$ will in this case be at least 2 (being greater than or equal to the multiplicity of $L_1 + L_2$), this would again imply a multiplicity of at least 7 for the weight 0 in $\mathrm{Sym}^2 V \otimes W$. It follows that $\mathrm{Sym}^2 V \otimes W$ must contain exactly one copy of $\mathrm{Sym}^2 V$; and since the multiplicity of $L_1 + L_2$ in $\Gamma_{2,1}$ is at most 3, it follows that $\mathrm{Sym}^2 V \otimes W$ will contain at least one copy of $\Gamma_{0,1} = W$ as well.

Exercise 16.12. Prove, independently of the above analysis, that $\mathrm{Sym}^2 V \otimes W$ must contain a copy of $\mathrm{Sym}^2 V$ and a copy of W by looking at the map

$$\varphi: \mathrm{Sym}^2 V \otimes W \to V \otimes V$$

obtained by sending

$$u \cdot v \otimes (w \wedge z) \mapsto u \otimes \tilde{Q}(v \wedge w \wedge z) + v \otimes \tilde{Q}(u \wedge w \wedge z),$$

where we are identifying $\Lambda^3 V$ with the dual space V^* and denoting by $\tilde{Q}: V^* \to V$ the isomorphism induced by the skew form Q on V. Specifically, show that the image of this map is complementary to the line spanned by the element $Q \in \Lambda^2 V^* = \Lambda^2 V \subset V \otimes V$.

The above leaves us with exactly two possibilities for the weights of $\Gamma_{2,1}$: we know that the multiplicity of $2L_1$ in $\Gamma_{2,1}$ is exactly 2; so either the multiplicities of $L_1 + L_2$ and 0 in $\Gamma_{2,1}$ are both 3 and we have

$$\mathrm{Sym}^2 V \otimes W = \Gamma_{2,1} \oplus \mathrm{Sym}^2 V \oplus W;$$

or the multiplicities of $L_1 + L_2$ and 0 in $\Gamma_{2,1}$ are both 2 and we have

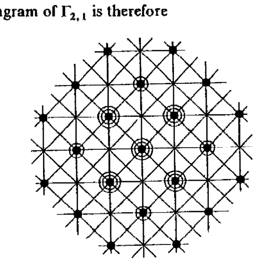
$$\mathrm{Sym}^2 V \otimes W = \Gamma_{2,1} \oplus \mathrm{Sym}^2 V \oplus W^{\oplus 2}.$$

Exercise 16.13. Show that the former of these two possibilities actually occurs, by

(a) Showing that if v is the highest weight vector in $\Gamma_{2,1} \subset \mathrm{Sym}^2 V \otimes W$, then the images $(X_{2,1})^2 V_2(v)$, $X_{2,1} V_2 X_{2,1}(v)$, and $V_2(X_{2,1})^2 v$ are independent; and (redundantly)

(b) Showing that the representation $\mathrm{Sym}^2 V \otimes W$ contains only one highest weight vector of weight $L_1 + L_2$.

The weight diagram of $\Gamma_{2,1}$ is therefore



We see from all this that, in particular, the weights of the irreducible representations of $\mathfrak{sp}_4\mathbb{C}$ are not constant on the rings of their weight diagrams.

Exercise 16.14. Analyze the representation $V \otimes \mathrm{Sym}^2 W$ of $\mathfrak{sp}_4\mathbb{C}$. Find in particular the multiplicities of the representation $\Gamma_{1,2}$.

Exercise 16.15. Analyze the representation $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 W$ of $\mathfrak{sp}_4\mathbb{C}$. Find in particular the multiplicities of the representation $\Gamma_{2,2}$.

LECTURE 17

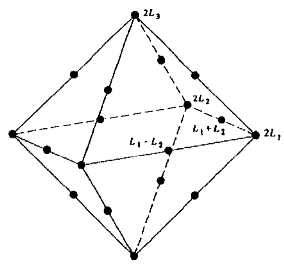
$\mathfrak{sp}_6\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$

In the first two sections of this lecture we complete our classification of the representations of the symplectic Lie algebras: we describe in detail the example of $\mathfrak{sp}_6\mathbb{C}$, then sketch the representation theory of symplectic Lie algebras in general, in particular proving the existence part of Theorem 14.18 for $\mathfrak{sp}_{2n}\mathbb{C}$. In the final section we describe an analog for the symplectic algebras of the construction given in §15.3 of the irreducible representations of the special linear algebras via Weyl's construction, though we postpone giving analogous formulas for the decomposition of tensor products of irreducible representations. Sections 17.1 and 17.2 are completely elementary, given the by now standard multilinear algebra of Appendix B. Section 17.3, like §15.3, requires familiarity with the contents of Lecture 6 and Appendix A; but, like that section, it can be skipped without affecting most of the rest of the book.

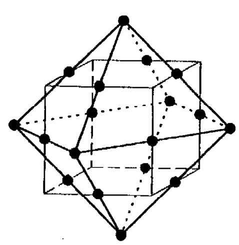
§17.1: Representations of $\mathfrak{sp}_6\mathbb{C}$
§17.2: Representations of the symplectic Lie algebras in general
§17.3: Weyl's construction for symplectic groups

§17.1. Representations of $\mathfrak{sp}_6\mathbb{C}$

As we have seen, the Cartan algebra $\mathfrak{h}$ of $\mathfrak{sp}_6\mathbb{C}$ is three-dimensional, with the linear functionals L_1, L_2, and L_3 forming an orthonormal basis in terms of the Killing form; and the roots of $\mathfrak{sp}_6\mathbb{C}$ are then the 18 vectors $\pm L_i \pm L_j$. We can draw this in terms of a "reference cube" in $\mathfrak{h}^*$ with faces centered at the points $\pm L_i$; the vectors $\pm L_i \pm L_j$ with $i \neq j$ are then the midpoints of edges of this reference cube and the vectors $\pm 2L_i$ the midpoints of the faces of a cube twice as large. Alternatively, we can draw a reference octahedron with vertices at the vectors $\pm 2L_i$; the roots $\pm L_i \pm L_j$ with $i \neq j$ will then be the midpoints of the edges of this octahedron:

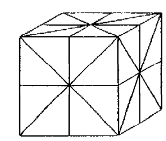


or, if we include the reference cube as well, as

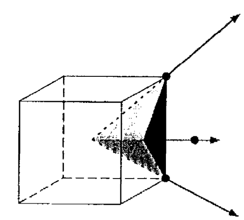


(17.1)

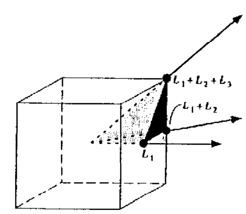
This last diagram, however ineptly drawn, suggests a comparison with the root diagram of $\mathfrak{sl}_4\mathbb{C}$; in fact the 12 roots of $\mathfrak{sp}_6\mathbb{C}$ of the form $\pm L_i \pm L_j$ for $i \neq j$ are congruent to the 12 roots of $\mathfrak{sl}_4\mathbb{C}$. In particular, the Weyl group of $\mathfrak{sp}_6\mathbb{C}$ will be generated by the Weyl group of $\mathfrak{sl}_4\mathbb{C}$, plus any of the additional three reflections in the planes perpendicular to the L_i (i.e., the planes parallel to the faces of the reference cube in the root diagram of either Lie algebra). We can indicate the planes perpendicular to the roots of $\mathfrak{sp}_6\mathbb{C}$ by drawing where they cross the visible part of the reference cube:



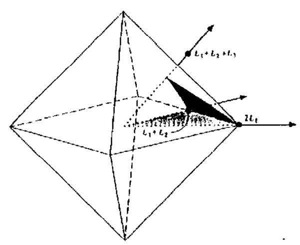
We see from this that the effect of the additional reflections in the Weyl group of $\mathfrak{sp}_6\mathbb{C}$ on the Weyl chamber of $\mathfrak{sl}_4\mathbb{C}$ is simply to cut it in half; whereas the Weyl chamber of $\mathfrak{sl}_4\mathbb{C}$ looked like



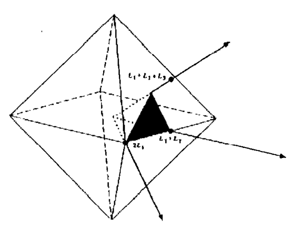
the Weyl chamber of $\mathfrak{sp}_6\mathbb{C}$ will look like just the upper half of this region:



In terms of the reference octahedron, this is the cone over one part of the barycentric subdivision of a face:



or, if we rotate 90° around the vertical axis in an attempt to make the picture clearer,



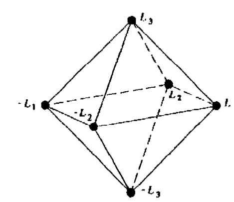
(17.2)

We should remark before proceeding that the comparison between the root systems of the special linear algebra $\mathfrak{sl}_4\mathbb{C}$ and the symplectic algebra $\mathfrak{sp}_6\mathbb{C}$ is peculiar to this case; in general, the root systems of $\mathfrak{sl}_{n+1}\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$ will bear no such similarity.

As we saw in the preceding lecture, the weight lattice of $\mathfrak{sp}_6\mathbb{C}$ consists simply of the integral linear combinations of the weights L_i. In particular, the intersection of the weight lattice with the closed Weyl chamber chosen above will consist exactly of integral linear combinations $a_1L_1 + a_2L_2 + a_3L_3$ with $a_1 \geq a_2 \geq a_3 \geq 0$. By our general existence and uniqueness theorem, then, for every triple (a, b, c) of non-negative integers there will exist a unique irreducible representation of $\mathfrak{sp}_6\mathbb{C}$ with highest weight $aL_1 + b(L_1 + L_2) +$

$c(L_1 + L_2 + L_3) = (a + b + c)L_1 + (b + c)L_2 + cL_3$; we will denote this re resentation by $\Gamma_{a,b,c}$ and will demonstrate its existence in the following.

We start by considering the standard representation of $\mathfrak{sp}_6\mathbb{C}$ on $V = \mathbb{C}$ The eigenvectors of the action of $\mathfrak{h}$ on V are just the standard basis vecto e_i, and these have eigenvalues $\pm L_i$, so that the weight diagram of V looks li the midpoints of the faces of the reference cube (or the vertices of octahedron one-half the size of the reference octahedron):



In particular, V is the representation $\Gamma_{1,0,0}$.

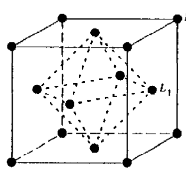
Since we are going to want to find a representation with highest weig $L_1 + L_2$, the natural thing to look at next is the second exterior power $\wedge^2$ of the standard representation. This will have weights the pairwise sum distinct weights of V, or in other words the 12 weights $\pm L_i \pm L_j$ with $i \neq$ and the weight 0 taken three times. This is not irreducible: by definition th action of $\mathfrak{sp}_6\mathbb{C}$ on the standard representation preserves a skew form, so th the representation on $\wedge^2 V$ will have a trivial summand. On the other han the skew form on V preserved by $\mathfrak{sp}_6\mathbb{C}$, and hence that trivial summand $\wedge^2 V$, is unique; and since all the nonzero weights of $\wedge^2 V$ occur with mult plicity 1 and are conjugate under the Weyl group, it follows that the compl ment W of the trivial representation in $\wedge^2 V$ is irreducible. So $W = \Gamma_{0,1,0}$.

As in previous examples, we can also see that $\wedge^2 V$ is not irreducible b using the fact (Observation 14.16) that the irreducible representation $\Gamma_{0,1,}$ with highest weight $L_1 + L_2$ will be generated by applying to a single highe weight vector v the root spaces $\mathfrak{g}_{L_2-L_1}$, $\mathfrak{g}_{L_3-L_2}$, and $\mathfrak{g}_{-2L_3}$ corresponding t primitive negative roots. We can then verify that in the irreducible representa tion W with highest weight $L_1 + L_2$, there are only three ways of going fro the highest weight space to the zero weight space by successive application o these roots spaces: we can go

$$\begin{array}{ccccccc} L_1 + L_2 & \to & L_1 + L_3 & \to & L_1 - L_3 & \to & L_1 - L_2 \\ & \searrow & & \searrow & & \searrow & \\ & & L_2 + L_3 & \to & L_2 - L_3 & \to & 0 \end{array}$$

Exercise 17.3. Verify this, and also verify that the lower two routes to the zero-weight space in $\wedge^2 V$ yield the same nonzero vector, and that the upper route yields an independent element of $\wedge^2 V$, so that 0 does indeed occur with multiplicity 2 as a weight of $\Gamma_{0,1,0}$.

To continue, we look next at the third exterior power $\wedge^3 V$ of the standard representation; we know that this will contain a copy of the irreducible representation $\Gamma_{0,0,1}$ with highest weight $L_1 + L_2 + L_3$. The weights of $\wedge^3 V$ are of two kinds: we have the eight sums $\pm L_1 \pm L_2 \pm L_3$, corresponding to the vertices of the reference cube and each occurring once; and we have the weights $\pm L_i$ each occurring twice (as $\pm L_i + L_j - L_j$ and $\pm L_i + L_k - L_k$). The weight diagram thus looks like the vertices of the reference cube together with the midpoints of its faces:



Now, the weights $\pm L_i$ must occur in the representation $\Gamma_{0,0,1}$ with highest weight $L_1 + L_2 + L_3$, since they are congruent to $L_1 + L_2 + L_3$ modulo the root lattice and lie in the convex hull of the translates of $L_1 + L_2 + L_3$ under the Weyl group (that is, they lie in the closed reference cube). But they cannot occur with multiplicity greater than 1: for example, the only way to get from the point $L_1 + L_2 + L_3$ to the point L_1 by translations by the basic vectors $L_2 - L_1$, $L_3 - L_2$, and $-2L_3$ pictured in Diagram (17.1) above (while staying inside the reference cube) is by translation by $-2L_3$ first, and then by $L_3 - L_2$. it follows that the multiplicities of the weights $\pm L_i$ in $\Gamma_{0,0,1}$ are 1. On the other hand, we have a natural map

$$\wedge^3 V \to V$$

obtained by contracting with the element of $\wedge^2 V^*$ preserved by the action of $\mathfrak{sp}_6\mathbb{C}$, and the kernel of this map, which must contain the representation $\Gamma_{0,0,1}$, will have exactly these weights. The kernel of φ is thus the irreducible representation with highest weight $L_1 + L_2 + L_3$; we will call this representation U for now.

At this point, we have established the existence theorem for representations of $\mathfrak{sp}_6\mathbb{C}$: the irreducible representation $\Gamma_{a,b,c}$ with highest weight $(a + b + c)L_1 + (a + b)L_2 + cL_3$ will occur inside the representation

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b W \otimes \mathrm{Sym}^c U.$$

For example, suppose we want to find the irreducible representation $\Gamma_{1,1,0}$ with highest weight $2L_1 + L_2$. The weights of this representation will be the 24 weights $\pm 2L_i \pm L_j$, each taken with multiplicity 1; the 8 weights $\pm L_1 \pm L_2 \pm L_3$, taken with a multiplicity we do not a priori know (but that the reader can verify must be either 1 or 2), and the weights $\pm L_i$ taken with some other multiplicity. At the same time, the representation $V \otimes W$, which contains $\Gamma_{1,1,0}$, will take on these weights, with multiplicities 1, 3, and 6, respectively. In particular, it follows that $V \otimes W$ will contain a copy of the irreducible representation U with highest weight $L_1 + L_2 + L_3$ as well; alternatively, we can see this directly by observing that the wedge product map

$$V \otimes \wedge^2 V \to \wedge^3 V$$

factors to give a map

$$V \otimes W \to U$$

and that $\Gamma_{1,1,0}$ must lie in the kernel of this map. To say more about the location of $\Gamma_{1,1,0}$ inside $V \otimes W$, and its exact weights, would require either explicit calculation or something like the Weyl character formula. We will see in Lecture 24 how the latter can be used to solve the problem; for the time being we leave this as

Exercise 17.4. Verify by direct calculation that the multiplicities of the weights of $\Gamma_{1,1,0}$ are 1, 2, and 5, and hence that the kernel of the map φ above is exactly the representation $\Gamma_{1,1,0}$.

§17.2. Representations of $\mathfrak{sp}_{2n}\mathbb{C}$ in General

The general picture for representations of the symplectic Lie algebras offers no further surprises. As we have seen, the weight lattice consists simply of integral linear combinations of the L_i. And our typical Weyl chamber is a cone over a simplex in n-space, with edges the rays defined by

$$a_1 = a_2 = \cdots = a_i > a_{i+1} = \cdots = a_n = 0.$$

The primitive lattice element on the ith ray is the weight $\omega_i = L_1 + \cdots + L_i$, and we may observe that, similarly to the case of the special linear Lie algebras, these n fundamental weights generate as a semigroup the intersection of the closed Weyl chamber with the lattice. Thus, our basic existence and uniqueness theorem asserts that for an arbitrary n-tuple of natural numbers $(a_1, \ldots, a_n) \in \mathbb{N}^n$ there will be a unique irreducible representation with highest weight

$$\begin{aligned} a_1\omega_1 + a_2\omega_2 + \cdots + a_n\omega_n \\ = (a_1 + \cdots + a_n)L_1 + (a_2 + \cdots + a_n)L_2 + \cdots + a_nL_n. \end{aligned}$$

As before, we denote this by $\Gamma_{a_1,\ldots,a_n}$:

$$\Gamma_{a_1,\ldots,a_n} = \Gamma_{a_1L_1+a_2(L_1+L_2)+\cdots+a_n(L_1+\cdots+L_n)}.$$

These exhaust all irreducible representations of $\mathfrak{sp}_{2n}\mathbb{C}$.

We can find the irreducible representation $V^{(k)} = \Gamma_{0,\ldots,1,\ldots,0}$ with highest weight $L_1 + \cdots + L_k$ easily enough. Clearly, it will be contained in the kth exterior power $\Lambda^k V$ of the standard representation. Moreover, we have a natural contraction map

$$\varphi_k: \Lambda^k V \to \Lambda^{k-2} V$$

defined by

$$\varphi_k(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} Q(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k$$

(see §B.3 of Appendix B for an intrinsic definition and explanation). Since the representation $\Lambda^{k-2}V$ does not have the weight $L_1 + \cdots + L_k$, the irreducible representation with this highest weight will have to be contained in the kernel of this map. We claim now that conversely

Theorem 17.5. *For $1 \le k \le n$, the kernel of the map φ_k is exactly the irreducible representation $V^{(k)} = \Gamma_{0,\ldots,0,1,0,\ldots,0}$ with highest weight $L_1 + \cdots + L_k$.*

Proof. Clearly, it is enough to show that the kernel of φ_k is an irreducible representation of $\mathfrak{sp}_{2n}\mathbb{C}$. We will do this by restricting to a subalgebra of $\mathfrak{sp}_{2n}\mathbb{C}$ isomorphic to $\mathfrak{sl}_n\mathbb{C}$, and using what we have learned about representations of $\mathfrak{sl}_n\mathbb{C}$.

To describe this copy of $\mathfrak{sl}_n\mathbb{C}$ inside $\mathfrak{sp}_{2n}\mathbb{C}$, consider the subgroup $G \subset \mathrm{Sp}_{2n}\mathbb{C}$ of transformations of the space $V = \mathbb{C}^{2n}$ preserving the skew form Q introduced in Lecture 16 and preserving as well the decomposition $V = \mathbb{C}\{e_1, \ldots, e_n\} \oplus \mathbb{C}\{e_{n+1}, \ldots, e_{2n}\}$. These can act arbitrarily on the first factor, as long as they do the opposite on the second; in coordinates, they are the matrices

$$G = \left\{\begin{pmatrix} X & 0 \\ 0 & {}^tX^{-1} \end{pmatrix}, X \in \mathrm{GL}_n\mathbb{C}\right\}.$$

We have, correspondingly, a subalgebra

$$\mathfrak{s} = \left\{\begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix}, A \in \mathfrak{sl}_n\mathbb{C}\right\} \subset \mathfrak{sp}_{2n}\mathbb{C}$$

isomorphic to $\mathfrak{sl}_n\mathbb{C}$.

Now, denote by W the standard representation of $\mathfrak{sl}_n\mathbb{C}$. The restriction of the representation V of $\mathfrak{sp}_{2n}\mathbb{C}$ to the subalgebra $\mathfrak{s}$ then splits

$$V = W \oplus W^*$$

into a direct sum of W and its dual; and we have, correspondingly,

$$\Lambda^k V = \bigoplus_{a+b=k} (\Lambda^a W \otimes \Lambda^b W^*).$$

How does the tensor product $\Lambda^a W \otimes \Lambda^b W^*$ decompose as a representation of $\mathfrak{sl}_n\mathbb{C}$? We know the answer to this from the discussion in Lecture 15 (see Exercise 15.30): we have contraction maps

$$\Psi_{a,b}: \Lambda^a W \otimes \Lambda^b W^* \to \Lambda^{a-1} W \otimes \Lambda^{b-1} W^*;$$

and the kernel of $\Psi_{a,b}$ is the irreducible representation $W^{(a,b)} = \Gamma_{0,\ldots,0,1,0,\ldots,0,1,0,\ldots}$ with (if, say, $a \le n - b$) highest weight $2L_1 + \cdots + 2L_a + L_{a+1} + \cdots + L_{n-b}$. The restriction of $\Lambda^k V$ to $\mathfrak{s}$ is thus given by

$$\Lambda^k V = \bigoplus_{\substack{a+b \le k \\ a+b \equiv k(2)}} W^{(a,b)}$$

and by the same token,

$$\mathrm{Ker}(\varphi_k) = \bigoplus_{a+b=k} W^{(a,b)}.$$

Note that the actual highest weight factor in the summand $W^{(a,b)} \subset \mathrm{Ker}(\varphi_k) \subset \Lambda^k V$ is the vector

$$\begin{aligned} w^{(a,b)} &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \\ &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-k+a+1} \wedge \cdots \wedge e_{2n}. \end{aligned}$$

Exercise 17.6. Show that more generally the highest weight vector in any summand $W^{(a,b)} \subset \Lambda^k V$ is the vector

$$\begin{aligned} w^{(a,b)} &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-k+a+1} \wedge \cdots \wedge e_{2n} \wedge Q^{(k-a-b)/2} \\ &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-k+a+1} \wedge \cdots \wedge e_{2n} \wedge \left(\sum (e_i \wedge e_{n+i})\right)^{(k-a-b)/2}. \end{aligned}$$

By the above, any subspace of $\mathrm{Ker}(\varphi_k)$ invariant under $\mathfrak{sp}_{2n}\mathbb{C}$ must be a direct sum, over a subset of pairs (a, b) with $a + b = k$, of subspaces $W^{(a,b)}$. But now (supposing for the moment that $k < n$) we observe that the element

$$Z_{a,n-b} = E_{2n-b,a} + E_{n+a,n-b} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries the vector $w^{(a,b)}$ into $w^{(a-1,b+1)}$ and, likewise,

$$Y_{a+1,n-b+1} = E_{a+1,2n-b+1} + E_{n-b+1,n+a+1} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries $w^{(a,b)}$ to $w^{(a+1,b-1)}$. In case $a + b = k = n$, we see similarly that

$$V_a = E_{n+a,a} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries the vector $w^{(a,b)}$ into $w^{(a-1,b+1)}$, and

$$U_{a+1} = E_{a+1,n+a+1} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries $w^{(a,b)}$ to $w^{(a+1,b-1)}$. Thus, any representation of $\mathfrak{sp}_{2n}\mathbb{C}$ contained in $\mathrm{Ker}(\varphi_k)$ and containing any one of the factors $W^{(a,b)}$ will contain them all, and we are done. □

Exercise 17.7. Another way to conclude this proof would be to remark that, inasmuch as all the $w^{(a,b)}$ above are eigenvectors of different weights, any highest weight vector for the action of $\mathfrak{sp}_{2n}\mathbb{C}$ on $\ker(\varphi_k) \subset \wedge^k V$ would have to be (up to scalars) one of the $w^{(a,b)}$. It would thus be sufficient to find, for each (a, b) with $a + b = k$ other than $(a, b) = (k, 0)$, a positive root α such that $\mathfrak{g}_\alpha(w^{(a,b)}) \neq 0$. Do this.

Note that, having found the irreducible representations $V^{(k)} = \Gamma_{0,\dots,1,\dots,0}$ with highest weight $L_1 + \cdots + L_k$, any other representation of $\mathfrak{sp}_{2n}\mathbb{C}$ will occur in a tensor product of these; specifically, the irreducible representation $\Gamma_{a_1,\dots,a_n}$ with highest weight $a_1 L_1 + \cdots + a_n(L_1 + \cdots + L_n)$ will occur in the product $\mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2} V^{(2)} \otimes \cdots \otimes \mathrm{Sym}^{a_n} V^{(n)}$.

One further remark is that there exist geometric interpretations of the action of $\mathfrak{sl}_{2n}\mathbb{C}$ on the fundamental representations $V^{(k)}$. We have said before that the group $\mathrm{PSp}_{2n}\mathbb{C}$ may be characterized as the subgroup of $\mathrm{PGL}_{2n}\mathbb{C}$ carrying isotropic subspaces of V into isotropic subspaces. At the same time, $\mathrm{PGL}_{2n}\mathbb{C}$ acts on the projective space $\mathbb{P}(\wedge^k V)$ as the connected component of the identity in the group of motions of this space carrying the Grassmannian $G = G(k, V) \subset \mathbb{P}(\wedge^k V)$ into itself. Now, the subset $G_L \subset G$ of k-dimensional isotropic subspaces of V is exactly the intersection of the Grassmannian G with the subspace $\mathbb{P}(V^{(k)})$ associated to the kernel of the map φ above; so that $\mathrm{PSp}_{2n}\mathbb{C}$ will act on $\mathbb{P}(V^{(k)})$ carrying G_L into itself and indeed when $1 < k \leq n$ may be characterized as the connected component of the identity in the group of motions of $\mathbb{P}(V^{(k)})$ preserving the variety G_L.

Exercise 17.8. Show that if $k > n$ the contraction φ_k is injective.

§17.3. Weyl's Construction for Symplectic Groups

We have just seen how the basic representations for $\mathfrak{sp}_{2n}\mathbb{C}$ can be obtained by taking certain basic representations of the larger Lie algebra $\mathfrak{sl}_{2n}\mathbb{C}$—in this case, $\wedge^k V$ for $k \leq n$—and intersecting with the kernel of a contraction constructed from the symplectic form. In fact, *all* the representations of the symplectic Lie algebras can be given a similar conrete realization, by intersecting certain of the irreducible representations of $\mathfrak{sl}_{2n}\mathbb{C}$ with the intersections of the kernels of all such contractions.

Recall from Lectures 6 and 15 that the irreducible representations of $\mathfrak{sl}_{2n}\mathbb{C}$ are given by Schur functors $\mathbb{S}_\lambda V$, where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{2n} \geq 0)$ is a partition of some integer $d = \sum \lambda_i$, and $V = \mathbb{C}^{2n}$. This representation is realized as the image of a corresponding Young symmetrizer c_λ acting on the d-fold tensor product space $V^{\otimes d}$. For each pair $I = \{p < q\}$ of integers between 1 and d, the symplectic form Q determines a contraction

$$\Phi_I \colon V^{\otimes d} \to V^{\otimes(d-2)},$$
$$v_1 \otimes \cdots \otimes v_d \mapsto Q(v_p, v_q) v_1 \otimes \cdots \otimes \hat{v}_p \otimes \cdots \otimes \hat{v}_q \otimes \cdots \otimes v_d. \tag{17.9}$$

Let $V^{\langle d \rangle} \subset V^{\otimes d}$ denote the intersection of the kernels of all these contractions. These subspaces is mapped to itself by permutations, so $V^{\langle d \rangle}$ is a subrepresentation of $V^{\otimes d}$ as a representation of the symmetric group $\mathfrak{S}_d$. Now let[1]

$$\mathbb{S}_{\langle\lambda\rangle} V = V^{\langle d \rangle} \cap \mathbb{S}_\lambda V. \tag{17.10}$$

This space is a representation of the symplectic group $\mathrm{Sp}_{2n}\mathbb{C}$ of Q, since $V^{\langle d \rangle}$ and $\mathbb{S}_\lambda(V)$ are subrepresentations of $V^{\otimes d}$ over $\mathrm{Sp}_{2n}\mathbb{C}$.

Theorem 17.11. *The space $\mathbb{S}_{\langle\lambda\rangle}(V)$ is nonzero if and only if the Young diagram of λ has at most n rows, i.e., $\lambda_{n+1} = 0$. In this case, $\mathbb{S}_{\langle\lambda\rangle}(V)$ is the irreducible representation of $\mathfrak{sp}_{2n}\mathbb{C}$ with highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$.*

In other words, for an n-tuple $(a_1, \dots, a_n)$ of non-negative integers

$$\Gamma_{a_1,\dots,a_n} = \mathbb{S}_{\langle\lambda\rangle} V,$$

where λ is the partition $(a_1 + a_2 + \cdots + a_n, a_2 + \cdots + a_n, \dots, a_n)$.

The proof follows the pattern for the general linear group given in §6.2, but we will have to call on a basic result from invariant theory in place of the simple Lemma 6.23. We first show how to find a complement to $V^{\langle d \rangle}$ in $V^{\otimes d}$. For example, if $d = 2$, then

$$V^{\otimes 2} = V^{\langle 2 \rangle} \oplus \mathbb{C} \cdot \psi,$$

where ψ is the element of $V \otimes V$ corresponding to the quadratic form Q. In terms of our canonical basis, $\psi = \sum (e_i \otimes e_{n+i} - e_{n+i} \otimes e_i)$. In general, for any $I = \{p < q\}$ define

$$\Psi_I \colon V^{\otimes(d-2)} \to V^{\otimes d}$$

by inserting ψ in the p, q factors. Note that $\Phi_I \circ \Psi_I$ is multiplication by $2n = \dim V$ on $V^{\otimes(d-2)}$. We claim that

$$V^{\otimes d} = V^{\langle d \rangle} \oplus \textstyle\sum_I \Psi_I(V^{\otimes(d-2)}). \tag{17.12}$$

To prove this, put the standard Hermitian metric (,) on $V = \mathbb{C}^{2n}$, using the given e_i as a basis, so that $(ae_i, be_j) = \delta_{ij}\bar{a}b$. This extends to give a Hermitian metric on each $V^{\otimes d}$. We claim that the displayed equation is a perpendicular direct sum. This follows from the following exercise.

Exercise 17.13. (i) Verify that for $v, w \in V$, $(\psi, v \otimes w) = Q(v, w)$.
(ii) Use (i) to show that $\mathrm{Ker}(\Phi_I) = \mathrm{Im}(\Psi_I)^\perp$ for each I.

Now define $F_r^d \subset V^{\otimes d}$ to be the intersection of the kernels of all r-fold contractions $\Phi_{I_1} \circ \cdots \circ \Phi_{I_r}$, and set

$$V_{d-2r}^{\langle d \rangle} = \sum \Psi_{I_1} \circ \cdots \circ \Psi_{I_r}(V^{\otimes\langle d-2r \rangle}). \tag{17.14}$$

[1] This follows a classical notation of using $\langle\ \rangle$ for the symplectic group and $[\ ]$ for the orthogonal group (although we have omitted the corresponding notation $\{\ \}$ for the general linear group).

Lemma 17.15. *The tensor power $V^{\otimes d}$ decomposes into a direct sum*

$$V^{\otimes d} = V^{\langle d \rangle} \oplus V_{d-2}^{\langle d \rangle} \oplus V_{d-4}^{\langle d \rangle} \oplus \cdots \oplus V_{d-2p}^{\langle d \rangle},$$

with $p = [d/2]$, and, for all $r \geq 1$,

$$F_r^d = V^{\langle d \rangle} \oplus V_{d-2}^{\langle d \rangle} \oplus \cdots \oplus V_{d-2r+2}^{\langle d \rangle}.$$

Exercise 17.16. (i) Show as in the preceding exercise that there is a perpendicular decomposition

$$V^{\otimes d} = F_r^d \oplus \sum \Psi_{I_1} \circ \cdots \circ \Psi_{I_r}(V^{\otimes(d-2r)}).$$

(ii) Verify that $\Psi_I(F_p^{d-2}) \subset F_{p+1}^d$.
(iii) Show by induction that $V^{\otimes d}$ is the sum of the spaces $V_{d-2r}^{\langle d \rangle}$.
(iv) Finish the proof of the lemma, using (i) and (ii) to deduce that both sums are orthogonal splittings. □

All the subspaces in these splittings are invariant by the action of the symplectic group $\mathrm{Sp}_{2n}\mathbb{C}$, as well as the action of the symmetric group $\mathfrak{S}_d$. In particular, we see that

$$\mathbb{S}_{\langle \lambda \rangle} V = V^{\langle d \rangle} \cdot c_\lambda = \mathrm{Im}(c_\lambda \colon V^{\langle d \rangle} \to V^{\langle d \rangle}). \tag{17.17}$$

Exercise 17.18*. (i) Show that if $s > n$, then $\Lambda^s V \otimes V^{\otimes(d-s)}$ is contained in $\sum_I \Psi_I(V^{\otimes(d-2)})$, and deduce that $\mathbb{S}_{\langle \lambda \rangle}(V) = 0$ if λ_{n+1} is not 0.
(ii) Show that $\mathbb{S}_{\langle \lambda \rangle}(V)$ is not zero if $\lambda_{n+1} = 0$.

For any pair of integers I from $\{1, \ldots, d\}$, define

$$\vartheta_I = \Psi_I \circ \Phi_I \colon V^{\otimes d} \to V^{\otimes d}.$$

From what we have seen, $V^{\langle d \rangle}$ is the intersection of the kernels of all these endomorphisms. Note that the endomorphism of $V^{\otimes d}$ determined by any symplectic automorphism of V not only commutes with all permutations of the factors $\mathfrak{S}_d$ but also commutes with the operators ϑ_I. We need a fact which is proved in Appendix F.2:

Invariant Theory Fact 17.19. *Any endomorphism of $V^{\otimes d}$ that commutes with all permutations in $\mathfrak{S}_d$ and all the operators ϑ_I is a finite $\mathbb{C}$-linear combination of operators of the form $A \otimes \cdots \otimes A$, for $A \in \mathrm{Sp}_{2n}\mathbb{C}$.*

Now let B be the algebra of all endomorphisms of the space $V^{\langle d \rangle}$ that are $\mathbb{C}$-linear combinations of operators of the form $A \otimes \cdots \otimes A$, for $A \in \mathrm{Sp}_{2n}\mathbb{C}$.

Proposition 17.20. *The algebra B is precisely the algebra of all endomorphisms of $V^{\langle d \rangle}$ commuting with all permutations in $\mathfrak{S}_d$.*

Proof. If F is an endomorphism of $V^{\langle d \rangle}$ commuting with all permutations of $\mathfrak{S}_d$, then the endomorphism $\tilde{F}$ of $V^{\otimes d}$ that is F on the factor $V^{\langle d \rangle}$ and

zero on the complementary summand $\sum_I \Psi_I(V^{\otimes(d-2)})$ is an endomorphism that commutes with all permutations and all operators ϑ_I. The fact that $\tilde{F}$ is a linear combination of operators from the symplectic group (which we know from Fact 17.19) implies the same for F. □

Corollary 17.21. *The representations $\mathbb{S}_{\langle \lambda \rangle}(V)$ are irreducible representations of $\mathrm{Sp}_{2n}\mathbb{C}$.*

Proof. Since B is the commutator algebra to $A = \mathbb{C}[\mathfrak{S}_d]$ acting on the space $V^{\langle d \rangle}$, Lemma 6.22 implies that $(V^{\langle d \rangle}) \cdot c_\lambda$ is an irreducible B-module. But we have seen that $(V^{\langle d \rangle}) \cdot c_\lambda = \mathbb{S}_{\langle \lambda \rangle} V$, and the proposition shows that being irreducible over B is the same as being irreducible over $\mathrm{Sp}_{2n}\mathbb{C}$. □

Exercise 17.22*. Show that the multiplicity with which $\mathbb{S}_{\langle \lambda \rangle}(V)$ occurs in $V^{\langle d \rangle}$ is the dimension m_λ of the corresponding representation V_λ of $\mathfrak{S}_d$.

As was the case for the Weyl construction over $\mathrm{GL}_n\mathbb{C}$, there are general formulas for decomposing tensor products of these representations, as well as restrictions to subgroups $\mathrm{Sp}_{2n-2}\mathbb{C}$, and for their dimensions and multiplicities of weight spaces. We postpone these questions to Lecture 25, when we will have the Weyl character formula at our disposal.

As we saw in Lecture 15 for $\mathrm{GL}_n\mathbb{C}$, it is possible to make a commutative algebra which we denote by $\mathbb{S}^{\langle \cdot \rangle} = \mathbb{S}^{\langle \cdot \rangle}(V)$ out of the sum of all the irreducible representations of $\mathrm{Sp}_{2n}\mathbb{C}$, where $V = \mathbb{C}^{2n}$ is the standard representation. Probably the simplest way to do this, given what we have proved so far, is to start with the ring

$$\begin{aligned} A^\cdot(V, n) &= \mathrm{Sym}^\cdot(V \oplus \Lambda^2 V \oplus \Lambda^3 V \oplus \cdots \oplus \Lambda^n V) \\ &= \bigoplus_{a_1, \ldots, a_n} \mathrm{Sym}^{a_n}(\Lambda^n V) \otimes \cdots \otimes \mathrm{Sym}^{a_2}(\Lambda^2 V) \otimes \mathrm{Sym}^{a_1}(V), \end{aligned}$$

the sum over all n-tuples $\mathbf{a} = (a_1, \ldots, a_n)$ of non-negative integers. Define a ring $\mathbb{S}^\cdot(V, n)$ to be the quotient of $A^\cdot(V, n)$ by the ideal generated by the same relations as in (15.53). By the argument in §15.5, the ring $\mathbb{S}^\cdot(V, n)$ is the direct sum of all the representations $\mathbb{S}_\lambda(V)$ of $\mathrm{GL}(V)$, as λ varies over all partitions with at most n parts.

The decomposition $V^{\otimes d} = V^{\langle d \rangle} \oplus W^{\langle d \rangle}$ of (17.12) determines a decomposition $V^{\otimes d} \cdot c_\lambda = V^{\langle d \rangle} \cdot c_\lambda \oplus W^{\langle d \rangle} \cdot c_\lambda$, which is a decomposition

$$\mathbb{S}_\lambda(V) = \mathbb{S}_{\langle \lambda \rangle}(V) \oplus J_{\langle \lambda \rangle}(V)$$

of representations of $\mathrm{Sp}_{2n}\mathbb{C}$. We claim that the sum $J^{\langle \cdot \rangle} = \bigoplus_\lambda J_{\langle \lambda \rangle}(V)$ is an ideal in $\mathbb{S}^\cdot(V, n) = \bigoplus_\lambda \mathbb{S}_\lambda(V)$. This is easy to see using weights, since $J_{\langle \lambda \rangle}(V)$ is the sum of all the representations in $\mathbb{S}_\lambda(V)$ whose highest weight is strictly smaller than λ. This implies that the image of $J_{\langle \lambda \rangle}(V) \otimes \mathbb{S}_\mu(V)$ in $\mathbb{S}_{\lambda+\mu}(V)$ is a sum of representations whose highest weights are less than $\lambda + \mu$, so they must be in $J_{\langle \lambda + \mu \rangle}(V)$.

The quotient ring is, therefore, the ring $\mathbb{S}^{\langle \cdot \rangle}(V)$ we were looking for:

$$\mathbb{S}^{\langle\cdot\rangle} = \mathbb{S}^{\cdot}(V, n)/J^{\langle\cdot\rangle} = \bigoplus_{\lambda} \mathbb{S}_{\langle\lambda\rangle}(V).$$

In fact, the ideal $J^{\langle\cdot\rangle}$ is generated by elements of the form $x \wedge \psi$, where $x \in \Lambda^i V$, $i \leq n - 2$, and ψ is the element in $\Lambda^2 V$ corresponding to the skew form Q. An outline of the proof is sketched at the end of Lecture 25. The calculations, as well as other constructions of the ring, can be found in [L-T], where one can also find a discussion of functorial properties of the construction. For bases, see [DC-P], [L-M-S], and [M-S].

LECTURE 18

Orthogonal Lie Algebras

In this and the following two lectures we carry out for the orthogonal Lie algebras what we have already done in the special linear and symplectic cases. As in those cases, we start by working out in general the structure of the orthogonal Lie algebras, describing the roots, root spaces, Weyl group, etc., and then go to work on low-dimensional examples. There is one new phenomenon here: as it turns out, all three of the Lie algebras we deal with in §18.2 are isomorphic to symplectic or special linear Lie algebras we have already analyzed (this will be true of $\mathfrak{so}_6\mathbb{C}$ as well, but of no other orthogonal Lie algebra). As in the previous cases, the analysis of the Lie algebras and their representation theory will be completely elementary. Algebraic geometry does intrude into the discussion, however: we have described the isomorphisms between the orthogonal Lie algebras discussed and special linear and symplectic ones in terms of projective geometry, since that is what seems to us most natural. This should not be a problem; there are many other ways of describing these isomorphisms, and readers who disagree with our choice can substitute their own.

§18.1: $SO_m\mathbb{C}$ and $\mathfrak{so}_m\mathbb{C}$
§18.2: Representations of $\mathfrak{so}_3\mathbb{C}$, $\mathfrak{so}_4\mathbb{C}$, and $\mathfrak{so}_5\mathbb{C}$

§18.1. $SO_m\mathbb{C}$ and $\mathfrak{so}_m\mathbb{C}$

We will take up now the analysis of the Lie algebras of orthogonal groups. Here there is, as we will see very shortly, a very big difference in behavior between the so-called "even" orthogonal Lie algebras $\mathfrak{so}_{2n}\mathbb{C}$ and the "odd" orthogonal Lie algebras $\mathfrak{so}_{2n+1}\mathbb{C}$. Interestingly enough, the latter seem at first glance to be more complicated, especially in terms of notation; but when we analyze their representations we see that in fact they behave more regularly than the even ones. In any event, we will try to carry out the analysis in parallel

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$$\mathbb{S}^{\langle\cdot\rangle} = \mathbb{S}^{\cdot}(V, n)/J^{\langle\cdot\rangle} = \bigoplus_{\lambda} \mathbb{S}_{\langle\lambda\rangle}(V).$$

In fact, the ideal $J^{\langle\cdot\rangle}$ is generated by elements of the form $x \wedge \psi$, where $x \in \bigwedge^i V$, $i \leq n-2$, and ψ is the element in $\bigwedge^2 V$ corresponding to the skew form Q. An outline of the proof is sketched at the end of Lecture 25. The calculations, as well as other constructions of the ring, can be found in [L-T], where one can also find a discussion of functorial properties of the construction. For bases, see [DC-P], [L-M-S], and [M-S].

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§18.1: $\mathrm{SO}_m\mathbb{C}$ and $\mathfrak{so}_m\mathbb{C}$
§18.2: Representations of $\mathfrak{so}_3\mathbb{C}$, $\mathfrak{so}_4\mathbb{C}$, and $\mathfrak{so}_5\mathbb{C}$

§18.1. $\mathrm{SO}_m\mathbb{C}$ and $\mathfrak{so}_m\mathbb{C}$

We will take up now the analysis of the Lie algebras of orthogonal groups. Here there is, as we will see very shortly, a very big difference in behavior between the so-called "even" orthogonal Lie algebras $\mathfrak{so}_{2n}\mathbb{C}$ and the "odd" orthogonal Lie algebras $\mathfrak{so}_{2n+1}\mathbb{C}$. Interestingly enough, the latter seem at first glance to be more complicated, especially in terms of notation; but when we analyze their representations we see that in fact they behave more regularly than the even ones. In any event, we will try to carry out the analysis in parallel fashion for as long as is feasible; when it becomes necessary to split up into cases, we will usually look at the even orthogonal Lie algebras first and then consider the odd.

Let V be a m-dimensional complex vector space, and

$$Q: V \times V \to \mathbb{C}$$

a nondegenerate, symmetric bilinear form on V. The orthogonal group $SO_m\mathbb{C}$ is then defined to be the group of automorphisms A of V of determinant 1 preserving Q—that is, such that $Q(Av, Aw) = Q(v, w)$ for all $v, w \in V$—and the orthogonal Lie algebra $\mathfrak{so}_m\mathbb{C}$ correspondingly consists of endomorphisms $A: V \to V$ satisfying

$$Q(Av, w) + Q(v, Aw) = 0 \tag{18.1}$$

for all v and $w \in V$. As in the case of the symplectic Lie algebras, to carry out our analysis we want to write Q explicitly in terms of a basis for V, and here is where the cases of even and odd m first separate. In case $m = 2n$ is even, we will choose a basis for V in terms of which the quadratic form Q is given by

$$Q(e_i, e_{i+n}) = Q(e_{i+n}, e_i) = 1$$

and

$$Q(e_i, e_j) = 0 \quad \text{if } j \neq i \pm n.$$

The bilinear form Q may be expressed as

$$Q(x, y) = {}^t x \cdot M \cdot y,$$

where M is the $2n \times 2n$ matrix given in block form as

$$M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix};$$

the group $SO_{2n}\mathbb{C}$ is thus the group of $2n \times 2n$ matrices A satisfying

$$M = {}^t A \cdot M \cdot A$$

and the Lie algebra $\mathfrak{so}_{2n}\mathbb{C}$ correspondingly the space of matrices X satisfying the relation

$${}^t X \cdot M + M \cdot X = 0.$$

Writing a $2n \times 2n$ matrix X in block form as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we have

$${}^t X \cdot M = \begin{pmatrix} {}^t C & {}^t A \\ {}^t D & {}^t B \end{pmatrix}$$

and

$$M \cdot X = \begin{pmatrix} C & D \\ A & B \end{pmatrix}$$

so that this relation is equivalent to saying that *the off-diagonal blocks B and C of X are skew-symmetric, and the diagonal blocks A and D of X are negative transposes of each other.*

Exercise 18.2. Show that with this choice of basis,

$$SO_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \cong \mathbb{C}^*,$$

and $\mathfrak{so}_2\mathbb{C} = \mathbb{C}$.

The situation in case the dimension m of V is odd is similar, if a little messier. To begin with, we will take Q to be expressible, in terms of a basis $e_1, \ldots, e_{2n+1}$ for V, by

$$Q(e_i, e_{i+n}) = Q(e_{i+n}, e_i) = 1 \quad \text{for } 1 \leq i \leq n;$$

$$Q(e_{2n+1}, e_{2n+1}) = 1;$$

and

$$Q(e_i, e_j) = 0 \quad \text{for all other pairs } i, j.$$

The bilinear form Q may be expressed as

$$Q(x, y) = {}^t x \cdot M \cdot y,$$

where M is the $(2n + 1) \times (2n + 1)$ matrix

$$M = \left(\begin{array}{c|c|c} 0 & I_n & 0 \\ \hline I_n & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

(the diagonal blocks here having widths n, n, and 1). The Lie algebra $\mathfrak{so}_{2n+1}\mathbb{C}$ is correspondingly the space of matrices X satisfying the relation ${}^t X \cdot M + M \cdot X = 0$; if we write X in block form as

$$X = \left(\begin{array}{c|c|c} A & B & E \\ \hline C & D & F \\ \hline G & H & J \end{array}\right),$$

then this is equivalent to saying that, as in the previous case, *B and C are skew-symmetric and A and D negative transposes of each other; and in addition $E = -{}^t H$, $F = -{}^t G$, and $J = 0$.*

With these choices, we may take as Cartan subalgebra—in both the even and odd cases—the subalgebra of matrices diagonal in this representation.[1]

[1] Note that if we had taken the simpler choice of Q, with M the identity matrix, the Lie algebra would have consisted of skew-symmetric matrices, and there would have been no nonzero diagonal matrices in the Lie algebra.

The subalgebra $\mathfrak{h}$ is thus generated by the n $2n \times 2n$ matrices $H_i = E_{i,i} - E_{n+i,n+i}$ whose action on V is to fix e_i, send e_{n+i} to its negative, and kill all the remaining basis vectors; note that this is the same whether $m = 2n$ or $2n + 1$. We will correspondingly take as basis for the dual vector space $\mathfrak{h}^*$ the dual basis L_j, where $\langle L_j, H_i \rangle = \delta_{i,j}$.

Given that the Cartan subalgebra of $\mathfrak{so}_{2n}\mathbb{C}$ coincides, as a subspace of $\mathfrak{sl}_{2n}\mathbb{C}$, with the Cartan subalgebra of $\mathfrak{sp}_{2n}\mathbb{C}$, we can use much of the description of the roots of $\mathfrak{sp}_{2n}\mathbb{C}$ to help locate the roots and root spaces of $\mathfrak{so}_{2n}\mathbb{C}$. For example, we saw in Lecture 16 that the endomorphism

$$X_{i,j} = E_{i,j} - E_{n+j,n+i} \in \mathfrak{sp}_{2n}\mathbb{C}$$

is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i - L_j$. Since $X_{i,j}$ is also an element of $\mathfrak{so}_{2n}\mathbb{C}$, we see that $L_i - L_j$ is likewise a root of $\mathfrak{so}_{2n}\mathbb{C}$, with root space generated by $X_{i,j}$. Less directly but using the same analysis, we find that the endomorphisms

$$Y_{i,j} = E_{i,n+j} - E_{j,n+i}$$

and

$$Z_{i,j} = E_{n+i,j} - E_{n+j,i}$$

are eigenvectors for the action of $\mathfrak{h}$, with eigenvalues $L_i + L_j$ and $-L_i - L_j$, respectively (note that $Y_{i,j}$ and $Z_{i,j}$ do not coincide with their definitions in Lecture 16). In sum, then, *the roots of the Lie algebra* $\mathfrak{so}_{2n}\mathbb{C}$ *are the vectors* $\{\pm L_i \pm L_j\}_{i \neq j} \subset \mathfrak{h}^*$.

The case of the algebra $\mathfrak{so}_{2n+1}\mathbb{C}$ is similar; indeed, all the eigenvectors for the action of $\mathfrak{h}$ found above in $\mathfrak{so}_{2n}\mathbb{C}$, viewed as endomorphisms of $\mathbb{C}^{2n+1}$, are likewise eigenvectors for the action of $\mathfrak{h}$ on $\mathfrak{so}_{2n+1}\mathbb{C}$. In addition, we have the endomorphisms

$$U_i = E_{i,2n+1} - E_{2n+1,n+i}$$

and

$$V_i = E_{n+i,2n+1} - E_{2n+1,i}$$

which are eigenvectors with eigenvalues $+L_i$ and $-L_i$, respectively. The roots of $\mathfrak{so}_{2n+1}\mathbb{C}$ are thus the roots $\pm L_i \pm L_j$ of $\mathfrak{so}_{2n}\mathbb{C}$, together with additional roots $\pm L_i$.

We note that we could have arrived at these statements without decomposing the Lie algebras $\mathfrak{so}_m\mathbb{C}$: the description (18.1) of the orthogonal Lie algebra may be interpreted as saying that, in terms of the identification of V with V^* given by the form Q, $\mathfrak{so}_m\mathbb{C}$ is just the Lie algebra of skew-symmetric endomorphisms of V (an endomorphism being skew-symmetric if it is equal to minus its transpose). That is, the adjoint representation of $\mathfrak{so}_m\mathbb{C}$ is isomorphic to the wedge product $\wedge^2 V$. In the even case $m = 2n$, since the weights of V are $\pm L_i$ (inasmuch as the subalgebras $\mathfrak{h} \subset \operatorname{End}(V)$ coincide, the weights of V must likewise be the same for $\mathfrak{so}_{2n}\mathbb{C}$ as for $\mathfrak{sp}_{2n}\mathbb{C}$), it follows that the roots of $\mathfrak{so}_{2n}\mathbb{C}$ are just the pairwise distinct sums $\pm L_i \pm L_j$. In the odd case $m = 2n + 1$, we see that $e_{2n+1} \in V$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue 0, so that the weights of the standard representation V are $\{\pm L_i\} \cup \{0\}$ and the weights of the adjoint representation correspondingly $\{\pm L_i \pm L_j\} \cup \{\pm L_i\}$.

Exercise 18.3. Use a similar analysis to find the roots of $\mathfrak{sp}_{2n}\mathbb{C}$ without explicit calculation.

To make a comparison with the Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$, we can say that the root diagram of $\mathfrak{so}_{2n}\mathbb{C}$ looks like that of $\mathfrak{sp}_{2n}\mathbb{C}$ with the roots $\pm 2L_i$ removed, whereas the root diagram of $\mathfrak{so}_{2n+1}\mathbb{C}$ looks like that of $\mathfrak{sp}_{2n}\mathbb{C}$ with the roots $\pm 2L_i$ replaced by $\pm L_i$. Note that this immediately tells us what the Weyl groups are: first, in the case of $\mathfrak{so}_{2n+1}\mathbb{C}$, the Weyl group is the same as that of $\mathfrak{sp}_{2n}\mathbb{C}$:

$$1 \to (\mathbb{Z}/2)^n \to \mathfrak{W}_{\mathfrak{so}_{2n+1}\mathbb{C}} \to \mathfrak{S}_n \to 1.$$

In the case of $\mathfrak{so}_{2n}\mathbb{C}$, the Weyl group is the subgroup of the Weyl group of $\mathfrak{sp}_{2n}\mathbb{C}$ generated by reflection in the hyperplanes perpendicular to the roots $\pm L_i \pm L_j$, without the additional generator given by reflection in the roots $\pm L_i$. This subgroup still acts as the full symmetric group on the set of coordinate axes in $\mathfrak{h}^*$; but the kernel of this action, instead of acting as $\pm I$ on each of the coordinate axes independently, will consist of transformations of determinant 1; i.e., will act as -1 on an even number of axes. (That every such transformation is indeed in the Weyl group is easy to see: for example, reflection in the plane perpendicular to $L_i + L_j$ followed by reflection in the plane perpendicular to $L_i - L_j$ will send L_i to $-L_i$, L_j to $-L_j$, and L_k to L_k for $k \neq i, j$.) Another way to say this is that the Weyl group is the subgroup of the Weyl group of $\mathfrak{sp}_{2n}\mathbb{C}$ consisting of transformations whose determinant agrees with the sign of the induced permutation of the coordinate axes; so that while the Weyl group of $\mathfrak{sp}_{2n}\mathbb{C}$ fits into the exact sequence

$$1 \to (\mathbb{Z}/2)^n \to \mathfrak{W}_{\mathfrak{sp}_{2n}\mathbb{C}} \to \mathfrak{S}_n \to 1,$$

the Weyl group of $\mathfrak{so}_{2n}\mathbb{C}$ has instead the sequence

$$1 \to (\mathbb{Z}/2)^{n-1} \to \mathfrak{W}_{\mathfrak{so}_{2n}\mathbb{C}} \to \mathfrak{S}_n \to 1.$$

We can likewise describe the Weyl chambers of $\mathfrak{so}_{2n}\mathbb{C}$ and $\mathfrak{so}_{2n+1}\mathbb{C}$ by direct comparison with $\mathfrak{sp}_{2n}\mathbb{C}$. To start, to choose an ordering of the roots we take as linear functional on $\mathfrak{h}^*$ a form $l = c_1 H_1 + \cdots + c_n H_n$, where $c_1 > c_2 > \cdots > c_n > 0$. The positive roots in the case of $\mathfrak{so}_{2n+1}\mathbb{C}$ are then

$$R^+ = \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j} \cup \{L_i\}_i,$$

whereas in the case of $\mathfrak{so}_{2n}\mathbb{C}$ we have

$$R^+ = \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j}.$$

The primitive positive roots are

$$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_n \qquad \text{for } \mathfrak{so}_{2n+1}\mathbb{C};$$

$$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_{n-1} + L_n \qquad \text{for } \mathfrak{so}_{2n}\mathbb{C}.$$

In the first case, the Weyl chamber is exactly the same as for $\mathfrak{sp}_{2n}\mathbb{C}$, namely, for $m = 2n + 1$,

$$\mathscr{W} = \{\sum a_i L_i : a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}$$

since the roots are the same except for the factor of 2 on some. In the case of $\mathfrak{so}_{2n}\mathbb{C}$, since there is no root along the line spanned by the L_i, the equality $a_n = 0$ does not describe a face of the Weyl chamber; however, since $L_{n-1} + L_n$ is still a root (and a positive one) we still have the inequality $a_{n-1} + a_n \geq 0$ in $\mathscr{W}$, so that we can write, for $m = 2n$,

$$\mathscr{W} = \{\sum a_i L_i : a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n|\}.$$

(Note that in the case of $\mathfrak{so}_{2n}\mathbb{C}$ we could have chosen our linear functional $l = c_1 H_1 + \cdots + c_n H_n$ with $c_1 > c_2 > \cdots > -c_n > 0$; the ordering of the roots, and consequently the Weyl chamber, would still be the same.)

As for the Killing form, the same considerations as for the symplectic case show that it must be, up to scalars, the standard quadratic form: $B(H_i, H_j) = \delta_{i,j}$. (This was implicit in the above description of the Weyl group.) The explicit calculation is no more difficult, and we leave it as an exercise:

$$B(\sum a_i H_i, \sum b_i H_i) = \begin{cases} (4n-2)\sum a_i b_i & \text{if } m = 2n+1 \\ (4n-4)\sum a_i b_i & \text{if } m = 2n. \end{cases}$$

Next, to describe the representations of the orthogonal Lie algebras we have to determine the weight lattice in $\mathfrak{h}^*$; and to do this we must, as before, locate the copies $\mathfrak{s}_\alpha$ of $\mathfrak{sl}_2\mathbb{C}$ corresponding to the root pairs $\pm\alpha$, and the corresponding distinguished elements H_α of $\mathfrak{h}$. This is so similar to the case of $\mathfrak{sp}_{2n}\mathbb{C}$ that we will leave the actual calculations as an exercise; we will simply state here the results that in $\mathfrak{so}_m\mathbb{C}$ for any m,

(i) the distinguished copy $\mathfrak{s}_{L_i - L_j}$ of $\mathfrak{sl}_2\mathbb{C}$ associated to the root $L_i - L_j$ is the span of the root spaces $\mathfrak{g}_{L_i - L_j} = \mathbb{C}\cdot X_{i,j}$, $\mathfrak{g}_{-L_i - L_j} = \mathbb{C}\cdot X_{j,i}$ and their commutator $[X_{i,j}, X_{j,i}] = E_{i,i} - E_{j,j} + E_{n+j,n+j} - E_{n+i,n+i}$, with distinguished element $H_{L_i - L_j} = H_i - H_j$ (this is exactly as in the case of $\mathfrak{sp}_{2n}\mathbb{C}$);

(ii) the distinguished copy $\mathfrak{s}_{L_i + L_j}$ of $\mathfrak{sl}_2\mathbb{C}$ associated to the root $L_i + L_j$ is the span of the root spaces $\mathfrak{g}_{L_i + L_j} = \mathbb{C}\cdot Y_{i,j}$, $\mathfrak{g}_{-L_i - L_j} = \mathbb{C}\cdot Z_{i,j}$ and their commutator $[Y_{i,j}, Z_{i,j}] = -E_{i,i} + E_{j,j} - E_{n+j,n+j} + E_{n+i,n+i} = -H_i - H_j$, with distinguished element $H_{L_i + L_j} = H_i + H_j$ (so that we have also $H_{-L_i - L_j} = -H_i - H_j$); and in the case of $\mathfrak{so}_{2n+1}\mathbb{C}$,

(iii) the distinguished copy $\mathfrak{s}_{L_i}$ of $\mathfrak{sl}_2\mathbb{C}$ associated to the root L_i is the span of the root spaces $\mathfrak{g}_{L_i} = \mathbb{C}\cdot U_i$, $\mathfrak{g}_{-L_i} = \mathbb{C}\cdot V_i$ and their commutator $[U_i, V_i] = [E_{i,2n+1} - E_{2n+1,n+i}, E_{n+i,2n+1} - E_{2n+1,i}] = -H_i$, with distinguished element $H_{L_i} = 2H_i$ (so that $H_{-L_i} = -2H_i$ as well).

Exercise 18.4. Verify the computations made here.

Again, the configuration of distinguished elements resembles that of $\mathfrak{sp}_{2n}\mathbb{C}$ closely; that of $\mathfrak{so}_{2n+1}\mathbb{C}$ differs from it by the substitution of $\pm 2H_i$ for $\pm H_i$, whereas that of $\mathfrak{so}_{2n}\mathbb{C}$ differs by the removal of the $\pm H_i$. The effect on the weight lattice is the same in either case: *for both even and odd orthogonal Lie algebras, the weight lattice Λ_W is the lattice generated by the L_i together with the element $(L_1 + \cdots + L_n)/2$.*

Exercise 18.5. Show that

$$\Lambda_W/\Lambda_R = \begin{cases} \mathbb{Z}/2 & \text{if } m = 2n+1 \\ \mathbb{Z}/4 & \text{if } m = 2n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } m = 2n \text{ and } n \text{ is even.} \end{cases}$$

§18.2. Representations of $\mathfrak{so}_3\mathbb{C}$, $\mathfrak{so}_4\mathbb{C}$, and $\mathfrak{so}_5\mathbb{C}$

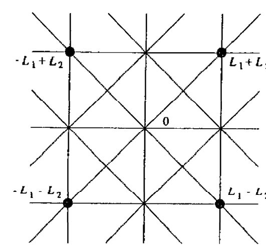
To give some examples, start with the case $n = 1$. Of course, $\mathfrak{so}_2\mathbb{C} \cong \mathbb{C}$ is not semisimple. The root system of $\mathfrak{so}_3\mathbb{C}$, on the other hand, looks like that of $\mathfrak{sl}_2\mathbb{C}$:

This is because, in fact, the two Lie algebras are isomorphic. Indeed, like the symplectic group, the quotient $PSO_m\mathbb{C}$ of the orthogonal group by its center can be realized as the motions of the projective space $\mathbb{P}V$ preserving isotropic subspaces for the quadratic form Q; in particular, this means we can realize $PSO_m\mathbb{C}$ as the group of motions of $\mathbb{P}V = \mathbb{P}^{m-1}$ carrying the quadric hypersurface

$$\bar{Q} = \{[v] : Q(v, v) = 0\}$$

into itself. In the first case of this, we see that the group $PSO_3\mathbb{C}$ is the group of motions of the projective plane $\mathbb{P}^2$ carrying a conic curve $C \subset \mathbb{P}^2$ into itself. But we have seen before that this group is also $PGL_2\mathbb{C}$ (the conic curve is itself isomorphic to $\mathbb{P}^1$, and the group acts as its full group of automorphisms), giving us the isomorphism $\mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$. One thing to note here is that the "standard" representation of $\mathfrak{so}_3\mathbb{C}$ is not the standard representation of $\mathfrak{sl}_2\mathbb{C}$, but rather its symmetric square. In fact, the irreducible representation with highest weight $\frac{1}{2}L_1$ is not contained in tensor powers of the standard representation of $\mathfrak{so}_3\mathbb{C}$. This will turn out to be significant: the standard representation of $\mathfrak{sl}_2\mathbb{C}$, viewed as a representation of $\mathfrak{so}_3\mathbb{C}$, is the first example of a *spin* representation of an orthogonal Lie algebra.

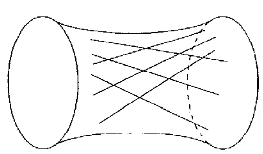
The next examples involve two-dimensional Cartan algebras. First we have $\mathfrak{so}_4\mathbb{C}$, whose root diagram looks like



Note one thing about this diagram: the roots are located on the union of two complementary lines. This says, by Exercise 14.33, that the Lie algebra $\mathfrak{so}_4\mathbb{C}$ is decomposable, and in fact should be the sum of two algebras each of whose root diagrams looks like that of $\mathfrak{sl}_2\mathbb{C}$; explicitly, $\mathfrak{so}_4\mathbb{C}$ is the direct sum of the two algebras $\mathfrak{s}_\alpha$, for $\alpha = L_1 + L_2$ and $\alpha = L_1 - L_2$. In fact, we can see this isomorphism

$$\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}, \tag{18.6}$$

as in the previous example, geometrically. Precisely, we may realize the group $PSO_4\mathbb{C} = SO_4\mathbb{C}/\{\pm I\}$ as the connected component of the identity in the group of motions of projective three-space $\mathbb{P}^3$ carrying a quadric hypersurface $\bar{Q}$ into itself. But a quadric hypersurface in $\mathbb{P}^3$ has two rulings by lines, and these two rulings give an isomorphism of $\bar{Q}$ with a product $\mathbb{P}^1 \times \mathbb{P}^1$



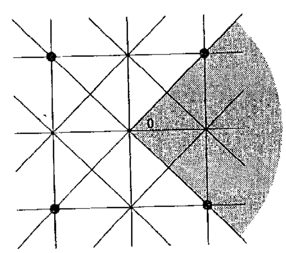
$PSO_4\mathbb{C}$ thus acts on the product $\mathbb{P}^1 \times \mathbb{P}^1$; and since the connected component of the identity in the automorphism group of this variety is just the product $PGL_2\mathbb{C} \times PGL_2\mathbb{C}$, we get an inclusion

$$PSO_4\mathbb{C} \to PGL_2\mathbb{C} \times PGL_2\mathbb{C}.$$

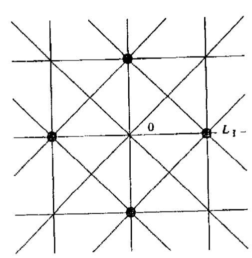
Another way of saying this is to remark that $PSO_4\mathbb{C}$ acts on the variety of isotropic 2-planes for the quadratic form Q on V; and this variety is just the disjoint union of two copies of $\mathbb{P}^1$. To see in this case that the map is an

isomorphism, consider the tensor product $V = U \otimes W$ of the pullbacks to $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$ of the standard representations of the two factors. Clearly the action on $\mathbb{P}(U \otimes W)$ will preserve the points corresponding to decomposable tensors (that is, points of the form $[u \otimes w]$); but the locus of such points is just a quadric hypersurface, giving us the inverse inclusion of $PGL_2\mathbb{C} \times PGL_2\mathbb{C}$ in $PSO_4\mathbb{C}$.

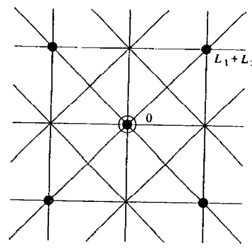
In fact, all of this will fall out of the analysis of the representations of $\mathfrak{so}_4\mathbb{C}$, if we just pursue it as usual. To begin with, the Weyl chamber we have selected looks like



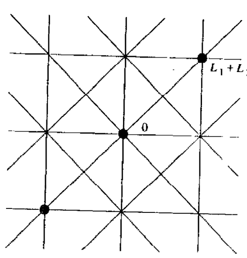
Now, the standard representation has, as noted above, weight diagram



with highest weight L_1 (note that the highest weight of the standard representation lies in this case in the interior of the Weyl chamber, something of an anomaly). Its second exterior power will have weights $\pm L_1 \pm L_2$ and 0 (occurring with multiplicity 2), i.e., diagram



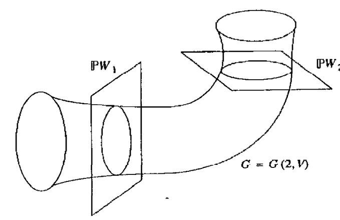
We see one thing about this representation right away, namely, that it cannot be irreducible. Indeed, the images of the highest weight $L_1 + L_2$ under the Weyl group consist just of $\pm(L_1 + L_2)$, so that the diagram of the irreducible representation with this highest weight is



We see from this that the second exterior power $\wedge^2 V$ of the standard representation of $\mathfrak{so}_4\mathbb{C}$ must be the direct sum of the irreducible representations $W_1 = \Gamma_{L_1+L_2}$ and $W_2 = \Gamma_{L_1-L_2}$ with highest weights $L_1 + L_2$ and $L_1 - L_2$. Since $\wedge^2 V$ is at the same time the adjoint representation, this says that $\mathfrak{so}_4\mathbb{C}$ itself must be a product of Lie algebras with adjoint representations $\Gamma_{L_1+L_2}$ and $\Gamma_{L_1-L_2}$.

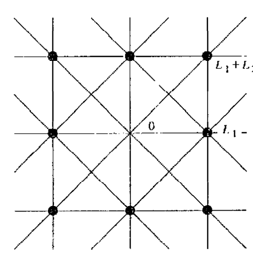
One way to derive the picture of the ruling of the quadric in $\mathbb{P}^3$ from this decomposition is to view $\mathfrak{so}_4\mathbb{C}$ as a subalgebra of $\mathfrak{sl}_4\mathbb{C}$, and the action of $PSO_4\mathbb{C}$ on $\mathbb{P}(\wedge^2 V)$ as a subgroup of the group of motions of $\mathbb{P}^2(\wedge^2 V) = \mathbb{P}^5$ preserving the Grassmannian $G = G(2, V)$ of lines in $\mathbb{P}^3$. In fact, we see from the above that the action of PSO_4 on $\mathbb{P}^5$ will preserve a pair of complementary 2-planes $\mathbb{P}W_1$ and $\mathbb{P}W_2$; it follows that this action must carry into themselves

the intersections of these 2-planes with the Grassmannian. These intersections are conic curves, corresponding to one-parameter families of lines sweeping out a quadric surface (necessarily the same quadric, since the action of $SO_4\mathbb{C}$ on V preserves a unique quadratic form); thus, the two rulings of the quadric.



Note one more aspect of this example: as in the case of $\mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$, the weights of the standard representation of $\mathfrak{so}_4\mathbb{C}$ do not generate the weight lattice, but rather a sublattice $\mathbb{Z}\{L_1, L_2\}$ of index 2 in Λ_W. Thus, there is no way of constructing all the representations of $\mathfrak{so}_4\mathbb{C}$ by applying linear- or multilinear-algebraic constructions to the standard representation; it is only after we are aware of the isomorphism $\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$ that we can construct, for example, the representation $\Gamma_{(L_1+L_2)/2}$ with highest weight $(L_1 + L_2)/2$ (of course, this is just the pullback from the first factor of $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$ of the standard representation of $\mathfrak{sl}_2\mathbb{C}$).

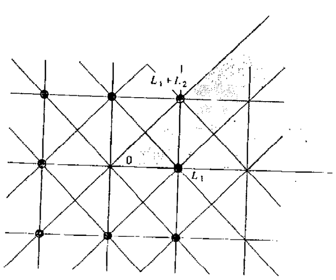
We come now to the case of $\mathfrak{so}_5\mathbb{C}$, which is more interesting. The root diagram in this case looks like



(as in the preceding example, the weight lattice is the lattice of intersections of all the lines drawn). The first thing we should notice about this diagram is that it is isomorphic to the weight diagram of the Lie algebra $\mathfrak{sp}_4\mathbb{C}$; the diagram just appears here rotated through an angle of $\pi/4$. Indeed, this is not accidental; the two Lie algebras $\mathfrak{sp}_4\mathbb{C}$ and $\mathfrak{so}_5\mathbb{C}$ are isomorphic, and it is not hard to construct this isomorphism explicitly. To see the isomorphism geometrically, we simply have to recall the identification, made in Lecture 14, of the group $PSp_4\mathbb{C}$ with a group of motions of $\mathbb{P}^4$. There, we saw that the larger group $PGL_4\mathbb{C}$ could be identified with the automorphisms of the projective space $\mathbb{P}(\wedge^2 V) = \mathbb{P}^5$ preserving the Grassmannian $G = G(2, 4) \subset \mathbb{P}(\wedge^2 V)$. The subgroup $PSp_4\mathbb{C} \subset PGL_4\mathbb{C}$ thus preserves both the Grassmannian G, which is a quadric hypersurface in $\mathbb{P}^5$, and the decomposition of $\wedge^2 V$ into the span $\mathbb{C} \cdot Q$ of the skew form $Q \in \wedge^2 V^* \cong \wedge^2 V$ and its complement W, and so acts on $\mathbb{P}W$ carrying the intersection $G_L = G \cap \mathbb{P}W$ into itself. We thus saw that $PSp_4\mathbb{C}$ was a subgroup of the group of motions of projective space $\mathbb{P}^4$ preserving a quadric hypersurface, and asserted that in fact it was the whole group.

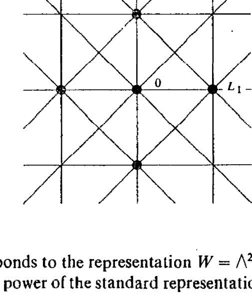
(To see the reverse inclusion directly, we can invoke a little algebraic geometry, which tells us that the locus of isotropic lines for a quadric in $\mathbb{P}^4$ is isomorphic to $\mathbb{P}^3$, so that $PSO_5\mathbb{C}$ acts on $\mathbb{P}^3$. Moreover, this action preserves the subset of pairs of points in $\mathbb{P}^3$ whose corresponding lines in $\mathbb{P}^4$ intersect, which, for a suitably defined skew-symmetric bilinear form $\tilde{Q}$, is exactly the set of pairs $([v], [w])$ such that $\tilde{Q}(v, w) = 0$, so that we have an inclusion of $PSO_5\mathbb{C}$ in $PSp_4\mathbb{C}$.)

Let us proceed to analyze the representations of $\mathfrak{so}_5\mathbb{C}$ as we would ordinarily, bearing in mind the isomorphism with $\mathfrak{sp}_4\mathbb{C}$. To begin with, we draw the Weyl chamber picked out above in $\mathfrak{h}^*$:



As for the representations of $\mathfrak{so}_5\mathbb{C}$, we have to begin with the standard, which has weight diagram

This we see corresponds to the representation $W = \wedge^2 V/\mathbb{C} \cdot Q$ of $\mathfrak{sp}_4\mathbb{C}$. Next, the second exterior power of the standard representation of $\mathfrak{so}_5\mathbb{C}$ has weights



This is of course the adjoint representation of $\mathfrak{so}_5\mathbb{C}$; it is the irreducible representation with highest weight $L_1 + L_2$. Note that it corresponds to the symmetric square $\mathrm{Sym}^2 V$ of the standard representation of $\mathfrak{sp}_4\mathbb{C}$ (see Exercise 16.8).

Exercise 18.7. Show that contraction with the quadratic form $Q \in \mathrm{Sym}^2 V^*$ preserved by the action of $\mathfrak{so}_5\mathbb{C}$ induces maps

$$\varphi: \mathrm{Sym}^a V \to \mathrm{Sym}^{a-2} V.$$

Show that the kernel of this contraction is exactly the irreducible representation with highest weight $a \cdot L_1$. Compare this with the analysis in Exercise 16.11.

Exercise 18.8. Examine the symmetric power $\mathrm{Sym}^a(\wedge^2 V)$ of the representation $\wedge^2 V$. This will contain a copy of the irreducible representation $\Gamma_{a(L_1+L_2)}$; what else will it contain? Interpret these other factors in light of the isomorphism $\mathfrak{so}_5\mathbb{C} \cong \mathfrak{sp}_4\mathbb{C}$.

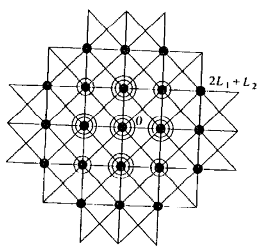
Exercise 18.9. For an example of a "mixed" tensor, consider the irreducible representation $\Gamma_{2L_1+L_2}$. Show that this is contained in the kernels of the wedge product map

$$\varphi \colon V \otimes \wedge^2 V \to \wedge^3 V$$

and the composition

$$\varphi' \colon V \otimes \wedge^2 V \to V^* \otimes \wedge^2 V \to V,$$

where the first map is induced by the isomorphism $\tilde{Q} \colon V \to V^*$ and the second is the contraction $V^* \otimes \wedge^2 V \to V$. Is it equal to the intersection of these kernels? Show that the weight diagram of this representation is



After you are done with this analysis, compare with the analysis given of the corresponding representation in Lecture 16.

Note that, as in the case of the other orthogonal Lie algebras studied so far (and as is the case for all $\mathfrak{so}_m\mathbb{C}$), *the weights of the standard representation do not generate the weight lattice, but only the sublattice of index two generated by the* L_i. Thus, the tensor algebra of the standard representation will contain only one-half of all the irreducible representations of $\mathfrak{so}_5\mathbb{C}$. Now, we do know that there are others, and even something about them—for example, we see in the following exercise that the irreducible representation of $\mathfrak{so}_5\mathbb{C}$ with highest weight $(L_1 + L_2)/2$ is a sort of "symmetric square root" of the adjoint representation:

Exercise 18.10. Show, using only root and weight diagrams for $\mathfrak{so}_5\mathbb{C}$, that the exterior square $\wedge^2 V$ of the standard representation of $\mathfrak{so}_5\mathbb{C}$ is actually the symmetric square of an irreducible representation.

We can also describe this irreducible representation via the isomorphism of $\mathfrak{so}_5\mathbb{C}$ with $\mathfrak{sp}_4\mathbb{C}$: it is just the standard representation of $\mathfrak{sp}_4\mathbb{C}$ on $\mathbb{C}^4$. We do not at this point have, however, a way of constructing this representation without invoking the isomorphism. This representation, the representation of $\mathfrak{so}_3\mathbb{C}$ with highest weight $L_1/2$, and the representation of $\mathfrak{so}_4\mathbb{C}$ with highest weight $(L_1 + L_2)/2$ discussed above are called *spin* representations of the corresponding Lie algebras and will be the subject matter of Lecture 20.

LECTURE 19

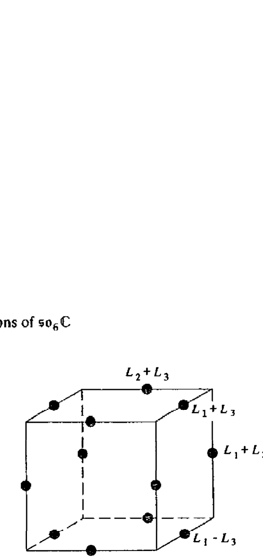
$\mathfrak{so}_6\mathbb{C}$, $\mathfrak{so}_7\mathbb{C}$, and $\mathfrak{so}_m\mathbb{C}$

This lecture is analogous in content (and prerequisites) to Lecture 17: we do some more low-dimensional examples and then describe the general picture of the representations of the orthogonal Lie algebras. One difference is that only half the irreducible representations of $\mathfrak{so}_m\mathbb{C}$ lie in the tensor algebra of the standard; to complete the picture of the representation theory we have to construct the spin representations, which is the subject matter of the following lecture. The first four sections are completely elementary (except possibly for the discussion of the isomorphism $\mathfrak{so}_6\mathbb{C} \cong \mathfrak{sl}_4\mathbb{C}$ in §19.1); the last section assumes a knowledge of Lecture 6 and §15.3, but can be skipped by those who did not read those sections.

§19.1: Representations of $\mathfrak{so}_6\mathbb{C}$
§19.2: Representations of the even orthogonal algebras
§19.3: Representations of $\mathfrak{so}_7\mathbb{C}$
§19.4: Representations of the odd orthogonal algebras
§19.5: Weyl's construction for orthogonal groups

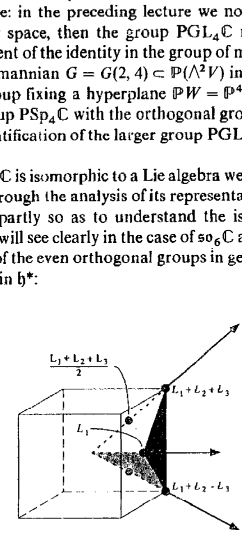
§19.1. Representations of $\mathfrak{so}_6\mathbb{C}$

We continue our discussion of orthogonal Lie algebras with the example of $\mathfrak{so}_6\mathbb{C}$. First, its root diagram:

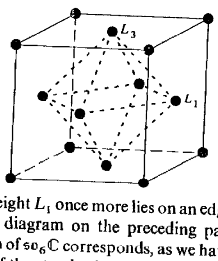


Once more (and for the last time), we notice a coincidence between this and the root diagram of a Lie algebra already studied, namely, $\mathfrak{sl}_4\mathbb{C}$. In fact, the two Lie algebras are isomorphic. The isomorphism is one we have already observed, in a sense: in the preceding lecture we noted that if V is a four-dimensional vector space, then the group $\mathrm{PGL}_4\mathbb{C}$ may be realized as the connected component of the identity in the group of motions of $\mathbb{P}(\wedge^2 V) = \mathbb{P}^5$ carrying the Grassmannian $G = G(2, 4) \subset \mathbb{P}(\wedge^2 V)$ into itself, and $\mathrm{PSp}_4\mathbb{C} \subset \mathrm{PGL}_4\mathbb{C}$ the subgroup fixing a hyperplane $\mathbb{P}W = \mathbb{P}^4 \subset \mathbb{P}^5$. We used this to identify the subgroup $\mathrm{PSp}_4\mathbb{C}$ with the orthogonal group $\mathrm{PSO}_5\mathbb{C}$; at the same time it gives an identification of the larger group $\mathrm{PGL}_4\mathbb{C}$ with the orthogonal group $\mathrm{PSO}_6\mathbb{C}$.

Even though $\mathfrak{so}_6\mathbb{C}$ is isomorphic to a Lie algebra we have already examined, it is worth going through the analysis of its representations for what amounts to a second time, partly so as to understand the isomorphism better, but mainly because we will see clearly in the case of $\mathfrak{so}_6\mathbb{C}$ a number of phenomena that will hold true of the even orthogonal groups in general. To start, we draw the Weyl chamber in $\mathfrak{h}^*$:

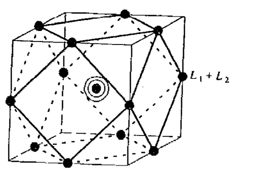


As usual, we begin with the standard representation, which has weights $\pm L_i$, corresponding to the centers of the faces of the cube:



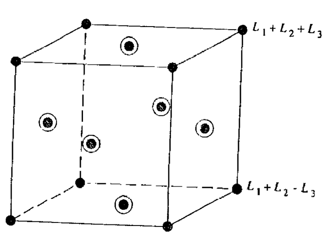
Note that the highest weight L_1 once more lies on an edge of the Weyl chamber (the front edge, in the diagram on the preceding page). Observe that the standard representation of $\mathfrak{so}_6\mathbb{C}$ corresponds, as we have already pointed out, to the exterior square of the standard representation of $\mathfrak{sl}_4\mathbb{C}$.

Next, we look at the exterior square $\wedge^2 V$ of the standard representation of $\mathfrak{so}_6\mathbb{C}$. This will have weights $\pm L_i \pm L_j$ (of course, it is the adjoint representation) and so will have weight diagram

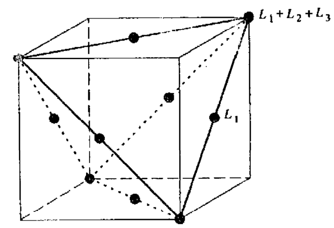


Note that the highest weight vector $L_1 + L_2$ of this representation does not lie on an edge of the Weyl chamber, but rather in the interior of a face (the back face, in the diagram above). In order to generate all the representations, we still need to find the irreducible representations with highest weight along the remaining two edges of the Weyl chamber.

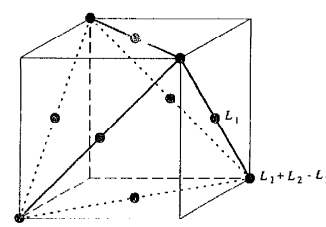
We look next at the exterior cube $\wedge^3 V$ of the standard representation. The weights here are the eight weights $\pm L_1 \pm L_2 \pm L_3$, each taken with multiplicity one, and the six weights $\pm L_i$, each taken with multiplicity 2, as in the diagram



Now, we notice something very interesting: this cannot be an irreducible representation. We can see this in a number of ways: the images of the weight $L_1 + L_2 + L_3$ under the Weyl group, for example, consist of every other vertex of the reference cube; in particular, their convex hull does not contain the remaining four vertices including $L_1 + L_2 - L_3$. Equivalently, there is no way to go from $L_1 + L_2 + L_3$ to $L_1 + L_2 - L_3$ by translation by negative root vectors. The representation $\wedge^3 V$ will thus contain copies of the irreducible representations $\Gamma_{L_1+L_2+L_3}$ and $\Gamma_{L_1+L_2-L_3}$ with highest weights $L_1 + L_2 + L_3$ and $L_1 + L_2 - L_3$, with weight diagrams



and



Since the weight diagram of each of these is a tetrahedron containing the weights $\pm L_i$, we have accounted for all the weights of $\wedge^3 V$ and so must have a direct sum decomposition

$$\wedge^3 V = \Gamma_{L_1+L_2+L_3} \oplus \Gamma_{L_1+L_2-L_3}.$$

We can relate this direct sum decomposition to a geometric feature of a quadric hypersurface in $\mathbb{P}^5$, analogous to the presence of two rulings on a quadric in $\mathbb{P}^3$. We saw before that the locus of lines lying on a quadric surface in $\mathbb{P}^3$ turns out to be disconnected, consisting of two components each isomorphic to $\mathbb{P}^1$ (and embedded, via the Plücker embedding of the Grassmannian $G = G(2, 4)$ of lines in $\mathbb{P}^3$ in $\mathbb{P}(\wedge^2\mathbb{C}) = \mathbb{P}^5$, as a pair of conic curves lying in complementary 2-planes in $\mathbb{P}^5$). In a similar fashion, the variety of 2-planes lying on a quadric hypersurface in $\mathbb{P}^5$ turns out to be disconnected, consisting of two components that, under the Plücker embedding of $G(3, 6)$ in $\mathbb{P}(\wedge^3\mathbb{C}^6) = \mathbb{P}^{19}$, span two complementary 9-planes $\mathbb{P}W_1$ and $\mathbb{P}W_2$; these two planes give the direct sum decomposition of $\wedge^3 V$ as an $\mathfrak{so}_6\mathbb{C}$-module.

In fact, if we think of a quadric hypersurface in $\mathbb{P}^5$ as the Grassmannian $G = G(2, 4)$ of lines in $\mathbb{P}^3$, we can see explicitly what these two families of 2-planes are: for every point $p \in \mathbb{P}^3$ the locus of lines passing through p forms a 2-plane on G, and for every plane $H \subset \mathbb{P}^3$ the locus of lines lying in H is a 2-plane in G. These are the two families; indeed, in this case we can go two steps further. First, we see from this that each of these families is parametrized by $\mathbb{P}^3$, so that the connected component $PSO_6\mathbb{C}$ of the identity in the group of motions of $\mathbb{P}^5$ preserving the Grassmannian acts on $\mathbb{P}^3$, giving us the inverse inclusion $PSO_6\mathbb{C} \subset PGL_4\mathbb{C}$. Second, under the Plücker embedding each of these families is carried into a copy of the quadratic Veronese embedding of $\mathbb{P}^3$ into $\mathbb{P}^9$, giving us the identification of the direct sum factors of the third exterior power of the standard representation of $\mathfrak{so}_6\mathbb{C}$ with the symmetric square of the standard representation of $\mathfrak{sl}_4\mathbb{C}$.

Exercise 19.1. Verify, without using the isomorphism with $\mathfrak{so}_6\mathbb{C}$ and the analysis above, that the standard representation V of $\mathfrak{sl}_4\mathbb{C}$ satisfies

$$\wedge^3(\wedge^2 V) \cong \mathrm{Sym}^2 V \oplus \mathrm{Sym}^2 V^*.$$

Note that we have now identified, in terms of tensor powers of the standard one, irreducible representations of $\mathfrak{so}_6\mathbb{C}$ with highest weight vectors L_1, $L_1 + L_2 + L_3$ and $L_1 + L_2 - L_3$ lying along the edge of the Weyl chamber, as well as one with highest weight $L_1 + L_2$ lying in a face. We can thus find irreducible representations with highest weight γ, if not for every γ in $\Lambda_W \cap \mathscr{W}$, at least for every weight γ in the intersection of $\mathscr{W}$ with a sublattice of index 2 in Λ_W.

§19.2. Representations of the Even Orthogonal Algebras

We will not examine any further representations of $\mathfrak{so}_6\mathbb{C}$ per se, leaving it as an exercise to do so (and to compare the results to the corresponding analysis for $\mathfrak{sl}_4\mathbb{C}$). Instead, we can now describe the general pattern for representations of the even orthogonal Lie algebras $\mathfrak{so}_{2n}\mathbb{C}$. The complete story will have to wait until the following lecture, since at present we cannot construct all the representations of $\mathfrak{so}_{2n}\mathbb{C}$ (as we have pointed out, we have been able to do so in the cases $n = 2$ and 3 studied so far only by virtue of isomorphisms with other Lie algebras; and there are no more such isomorphisms from this point on). We will nonetheless give as much of the picture as we can.

To begin with, recall that the weight lattice of $\mathfrak{so}_{2n}\mathbb{C}$ is generated by $L_1, \ldots, L_n$ together with the further vector $(L_1 + \cdots + L_n)/2$. The Weyl chamber, on the other hand, is the cone

$$\mathscr{W} = \{\textstyle\sum a_i L_i : a_1 \geq a_2 \geq \cdots \geq \pm a_n\}.$$

Note that the Weyl chamber is a simplicial cone, with faces corresponding to the n planes $a_1 = a_2, \ldots, a_{n-1} = a_n$ and $a_{n-1} = -a_n$; the edges of the Weyl chamber are thus the rays generated by the vectors $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-2}$, $L_1 + \cdots + L_n$ and $L_1 + \cdots + L_{n-1} - L_n$ (note that $L_1 + \cdots + L_{n-1}$ is not on an edge of the Weyl chamber). We see from this that, as in every previous case, the intersection of the weight lattice with the closed Weyl cone is a free semigroup generated by fundamental weights, in this case the vectors $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-2}$ and the vectors[1]

$$\alpha = (L_1 + \cdots + L_n)/2 \quad \text{and} \quad \beta = (L_1 + \cdots + L_{n-1} - L_n)/2.$$

As before, the obvious place to start to look for irreducible representations is among the exterior powers of the standard representation. This almost works: we have

Theorem 19.2. (i) *The exterior powers $\wedge^k V$ of the standard representation V of $\mathfrak{so}_{2n}\mathbb{C}$ are irreducible for $k = 1, 2, \ldots, n - 1$; and* (ii) *The exterior power $\wedge^n V$ has exactly two irreducible factors.*

Proof. The proof will follow the same lines as that of the analogous theorem for the symplectic Lie algebras in Lecture 17; in particular, we will start by considering the restriction to the same subalgebra as in the case of $\mathfrak{sp}_{2n}\mathbb{C}$.

Recall that the group $Sp_{2n}\mathbb{C} \subset SL_{2n}\mathbb{C}$ of automorphisms preserving the skew form Q introduced in Lecture 16 contains the subgroup G of automorphisms of the space $V = \mathbb{C}^{2n}$ preserving the decomposition $V = \mathbb{C}\{e_1, \ldots, e_n\} \oplus \mathbb{C}\{e_{n+1}, \ldots, e_{2n}\}$, acting as an arbitrary automorphism on the first factor and as the inverse transpose of that automorphism on the second factor; in matrices

$$G = \left\{\begin{pmatrix} X & 0 \\ 0 & {}^tX^{-1} \end{pmatrix}, X \in GL_n\mathbb{C}\right\}.$$

In fact, the subgroup $SO_{2n}\mathbb{C} \subset SL_{2n}\mathbb{C}$ also contains the same subgroup; we have, correspondingly a subalgebra

[1] To conform to standard conventions, with simple roots $\alpha_i = L_i - L_{i+1}$ for $1 \leq i \leq n - 1$, and $\alpha_n = L_{n-1} + L_n$, to have $\omega_i(H_{\alpha_j}) = \delta_{i,j}$, the fundamental weights ω_i should be put in the order: $\omega_i = L_1 + \cdots + L_i$ for $1 \leq i \leq n - 2$, and

$$\omega_{n-1} = \beta = (L_1 + \cdots + L_{n-1} - L_n)/2, \qquad \omega_n = \alpha = (L_1 + \cdots + L_n)/2.$$

$$\mathfrak{s} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix}, A \in \mathfrak{sl}_n\mathbb{C} \right\} \subset \mathfrak{so}_{2n}\mathbb{C}$$

isomorphic to $\mathfrak{sl}_n\mathbb{C}$.

Denote by W the standard representation of $\mathfrak{sl}_n\mathbb{C}$. As in the previous case, the restriction of the standard representation V of $\mathfrak{so}_{2n}\mathbb{C}$ to the subalgebra $\mathfrak{s}$ then splits

$$V = W \oplus W^*$$

into a direct sum of W and its dual; and we have, correspondingly,

$$\wedge^k V = \bigoplus_{a+b=k} (\wedge^a W \otimes \wedge^b W^*).$$

We also can say how each factor on the right-hand side of this expression decomposes as a representation of $\mathfrak{sl}_n\mathbb{C}$: we have contraction maps

$$\Psi_{a,b}\colon \wedge^a W \otimes \wedge^b W^* \to \wedge^{a-1} W \otimes \wedge^{b-1} W^*;$$

and the kernel of $\Psi_{a,b}$ is the irreducible representation $W^{(a,b)}$ with highest weight $2L_1 + \cdots + 2L_a + L_{a+1} + \cdots + L_{n-b}$. The restriction of $\wedge^k V$ to $\mathfrak{s}$ is thus given by

$$\wedge^k V = \bigoplus_{\substack{a+b \le k \\ a+b \equiv k(2)}} W^{(a,b)},$$

where the actual highest weight factor in the summand $W^{(a,b)} \subset \wedge^k V$ is the vector

$$\begin{aligned} w^{(a,b)} &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge Q^{(k-a-b)/2} \\ &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge \left(\sum (e_i \wedge e_{n+i})\right)^{(k-a-b)/2}. \end{aligned}$$

Now, all the vectors $w^{(a,b)}$ have distinct weights; and it follows, as in Exercise 17.7, that *any highest weight vector for the action of $\mathfrak{so}_{2n}\mathbb{C}$ on $\wedge^k V$ will be a scalar multiple of one of the $w^{(a,b)}$*. It will thus suffice, in order to show that $\wedge^k V$ is irreducible as representation of $\mathfrak{so}_{2n}\mathbb{C}$ for $k < n$, to exhibit for each (a, b) with $a + b \le k < n$ other than $(k, 0)$ a positive root α such that the image $\mathfrak{g}_\alpha(w^{(a,b)}) \ne 0$. This is simplest in the case $a + b = k < n$ (so there is no factor of Q in $w^{(a,b)}$): just as in the case of $\mathfrak{sp}_{2n}\mathbb{C}$ we have

$$\begin{aligned} &Y_{a+1,n-b+1}(w^{(a,b)}) \\ &\quad = (E_{a+1,2n-b+1} - E_{n-b+1,n+a+1})(e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n}) \\ &\quad = w^{(a+1,b-1)} \\ &\quad \ne 0 \end{aligned}$$

and $Y_{i,j}$ is the generator of the positive root space $\mathfrak{g}_{L_i+L_j}$.

In case $a + b < k < n$, we observe first that for any i and j

$$\begin{aligned} Y_{i,j}(Q) &= (E_{i,n+j} - E_{j,n+i})\left(\sum (e_p \wedge e_{n+p})\right) \\ &= 2 \cdot e_j \wedge e_i \\ &\ne 0 \end{aligned}$$

so that whenever $a < i, j \le n - b$,

$$\begin{aligned} &Y_{i,j}(w^{(a,b)}) \\ &\quad = Y_{i,j}(e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge Q^{(k-a-b)/2}) \\ &\quad = e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge Y_{i,j}\left(\left(\sum (e_p \wedge e_{n+p})\right)^{(k-a-b)/2}\right) \\ &\quad = (k - a - b) \cdot (e_1 \wedge \cdots \wedge e_a \wedge e_j \wedge e_i \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge Q^{(k-a-b-2)/2}) \\ &\quad \ne 0. \end{aligned}$$

It is always possible to find a pair (i, j) satisfying the conditions $a < i, j \le n - b$ since we are assuming $a + b < k < n$; this concludes the proof of part (i).

The proof of part (ii) requires only one further step: we have to check the vectors $w^{(a,b)}$ with $a + b = k = n$ to see if any of them might be highest weight vectors for $\mathfrak{so}_{2n}\mathbb{C}$. In fact (as the statement of the theorem implies), two of them are: It is not hard to check that, in fact, $w^{(n,0)}$ and $w^{(n-1,1)}$ are killed by every positive root space $\mathfrak{g}_{L_i+L_j}$. To see that no other vector $w^{(a,n-a)}$ is, look at the action of $Y_{a+1,a+2} \in \mathfrak{g}_{L_{a+1}+L_{a+2}}$: we have

$$\begin{aligned} &Y_{a+1,a+2}(w^{(a,n-a)}) \\ &\quad = (E_{a+1,n+a+2} - E_{a+2,n+a+1})(e_1 \wedge \cdots \wedge e_a \wedge e_{n+a+1} \wedge \cdots \wedge e_{2n}) \\ &\quad = e_1 \wedge \cdots \wedge e_a \wedge e_{a+1} \wedge e_{n+a+1} \wedge e_{n+a+3} \wedge \cdots \wedge e_{2n} \\ &\qquad - e_1 \wedge \cdots \wedge e_a \wedge e_{a+2} \wedge e_{n+a+2} \wedge \cdots \wedge e_{2n} \\ &\quad \ne 0. \end{aligned}$$

□

Remarks. (i) This theorem will be a consequence of the Weyl character formula, which will tell us a priori that the dimension of the irreducible representation of $\mathfrak{so}_{2n}\mathbb{C}$ with highest weight $L_1 + \cdots + L_k$ has dimension $\binom{2n}{k}$ if $k < n$, and half that if $k = n$.

(ii) Note also that by the above, $\wedge^n V$ is the direct sum of the two irreducible representations $\Gamma_{2\alpha}$ and $\Gamma_{2\beta}$ with highest weights $2\alpha = L_1 + \cdots + L_n$ and $2\beta = L_1 + \cdots + L_{n-1} - L_n$. Indeed, the inclusion $\Gamma_{2\alpha} \oplus \Gamma_{2\beta} \subset \wedge^n V$ can be seen just from the weight diagram: $\wedge^n V$ possesses a highest weight vector with highest weight $L_1 + \cdots + L_n$, and so contains a copy of $\Gamma_{2\alpha}$; but this representation does not possess the weight 2β, and so $\wedge^n V$ must contain $\Gamma_{2\beta}$ as well. (Alternatively, we observed in the preceding lecture that in choosing an ordering of the roots we could have chosen our linear functional $l = c_1 H_1 + \cdots + c_n H_n$ with $c_1 > c_2 > \cdots > -c_n > 0$ without altering the positive roots or the Weyl chamber; in this case the weight λ of $\wedge^n V$ with $l(\lambda)$ maximal would be 2β, showing that $\Gamma_{2\beta} \subset \wedge^n V$.)

(iii) If we want to avoid weight diagrams altogether, we can still see that $\wedge^n V$ must be reducible, because the action of $\mathfrak{so}_{2n}\mathbb{C}$ preserves two bilinear forms: first, we have the bilinear form induced on $\wedge^n V$ by the form Q on V; and second we have the wedge product

$$\varphi \colon \wedge^n V \times \wedge^n V \to \wedge^{2n} V = \mathbb{C},$$

the last map taking $e_1 \wedge \cdots \wedge e_{2n}$ to 1. It follows that $\wedge^n V$ is reducible; indeed, if we want to see the direct sum decomposition asserted in the statement of the theorem we can look at the composition

$$\tau \colon \wedge^n V \to \wedge^n V^* \to \wedge^n V,$$

where the first map is the isomorphism given by Q and the second is the isomorphism given by φ. The square of this map is the identity, and decomposing $\wedge^n V$ into $+1$ and -1 eigenspaces for this map gives two subrepresentations.

Exercise 19.3*. Part (i) of Theorem 19.2 can also be proved by showing that for any nonzero vector $w \in \wedge^k V$, the linear span of the vectors $X(w)$, for $X \in \mathfrak{so}_m\mathbb{C}$, is all of $\wedge^k V$. For these purposes take, instead of the basis we have been using, an orthonormal basis $v_1, \ldots, v_m$ for $V = \mathbb{C}^m$, $m = 2n$, so $Q(v_i, v_j) = \delta_{i,j}$. The vectors $v_I = v_{i_1} \wedge \cdots \wedge v_{i_k}$, $I = \{i_1 < \cdots < i_k\}$, form a basis for $\wedge^k V$, and $\mathfrak{so}_m\mathbb{C}$ has a basis consisting of endomorphisms $V_{p,q}$, $p < q$, which takes v_q to v_p, v_p to $-v_q$, and takes the other v_i to zero. Compute the images $V_{p,q}(v_I)$, and prove the claim, first, when $w = v_I$ for some I, and then by induction on the number of nonzero coefficients in the expression $w = \sum a_I v_I$. For (ii) a similar argument shows that $\wedge^n V$ is an irreducible representation of the group $O_n\mathbb{C}$, and the ideas of §5.1 (cf. §19.5) can be used to see how it decomposes over the subgroup $SO_n\mathbb{C}$ of index two.

We return now to our analysis of the representations of $\mathfrak{so}_{2n}\mathbb{C}$. By the theorem, the exterior powers $V, \wedge^2 V, \ldots, \wedge^{n-2} V$ provide us with the irreducible representations with highest weight the fundamental weight along the first $n-2$ edges of the Weyl chamber (of course, the exterior power $\wedge^{n-1} V$ is irreducible as well, but as we have observed, $L_1 + \cdots + L_{n-1}$ is not on an edge of the Weyl chamber, and so $\wedge^{n-1} V$ is not as useful for our purposes). For the remaining two edges, we have found irreducible representations with highest weights located there, namely the two direct sum factors of $\wedge^n V$; but the highest weights of these two representations are not primitive ones; they are divisible by 2. Thus, given the theorem above, we see that we have constructed exactly one-half the irreducible representations of $\mathfrak{so}_{2n}\mathbb{C}$, namely, those whose highest weight lies in the sublattice $\mathbb{Z}\{L_1, \ldots, L_n\} \subset \Lambda_W$. Explicitly, any weight γ in the closed Weyl chamber can be expressed (uniquely) in the form

$$\gamma = a_1 L_1 + \cdots + a_{n-2}(L_1 + \cdots + L_{n-2})$$
$$+ a_{n-1}(L_1 + \cdots + L_{n-1} - L_n)/2 + a_n(L_1 + \cdots + L_n)/2$$

with $a_i \in \mathbb{N}$. If $a_{n-1} + a_n$ is even, with $a_{n-1} \geq a_n$ we see that the representation

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-2}}(\wedge^{n-2} V) \otimes \mathrm{Sym}^{a_n}(\wedge^{n-1} V) \otimes \mathrm{Sym}^{(a_{n-1}-a_n)/2}(\Gamma_{2\beta})$$

will contain an irreducible representation Γ_γ with highest weight γ; whereas if $a_n \geq a_{n-1}$, we will find Γ_γ inside

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-2}}(\wedge^{n-2} V) \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} V) \otimes \mathrm{Sym}^{(a_n-a_{n-1})/2}(\Gamma_{2\alpha}).$$

There remains the problem of constructing irreducible representations Γ_γ whose highest weight γ involves an odd number of α's and β's. To do this, we clearly have to exhibit irreducible representations Γ_α and Γ_β with highest weights α and β. These exist, and are called the *spin representations* of $\mathfrak{so}_{2n}\mathbb{C}$; we will study them in detail in the following lecture. We see from the above that once we exhibit the two representations Γ_α and Γ_β, we will have constructed all the representations of $\mathfrak{so}_{2n}\mathbb{C}$. The representation Γ_γ with highest weight γ written above will be found in the tensor product

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-2}}(\wedge^{n-2} V) \otimes \mathrm{Sym}^{a_{n-1}}(\Gamma_\beta) \otimes \mathrm{Sym}^{a_n}(\Gamma_\alpha).$$

For the time being, we will assume the existence of the spin representations of $\mathfrak{so}_{2n}\mathbb{C}$; there is a good deal we can say about these representations just on the basis of their weight diagrams.

Exercise 19.4*. Find the weights (with multiplicities) of the representations $\wedge^k V$, and also of $\Gamma_{2\alpha}$, $\Gamma_{2\beta}$, Γ_α, and Γ_β.

Exercise 19.5. Using the above, show that Γ_α and Γ_β are dual to one another when n is odd, and that they are self-dual when n is even.

Exercise 19.6. Give the complete decomposition into irreducible representations of $\mathrm{Sym}^2 \Gamma_\alpha$ and $\wedge^2 \Gamma_\alpha$. Show that

$$\Gamma_\alpha \otimes \Gamma_\alpha = \Gamma_{2\alpha} \oplus \wedge^{n-2} V \oplus \wedge^{n-4} V \oplus \wedge^{n-6} V \otimes \cdots.$$

Exercise 19.7. Show that

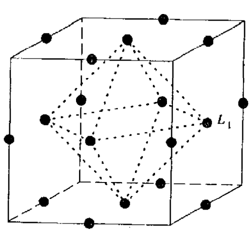
$$\Gamma_\alpha \otimes \Gamma_\beta = \wedge^{n-1} V \oplus \wedge^{n-3} V \oplus \wedge^{n-5} V \oplus \cdots.$$

Exercise 19.8. Verify directly the above statements in the case of $\mathfrak{so}_6\mathbb{C}$, using the isomorphism with $\mathfrak{sl}_4\mathbb{C}$.

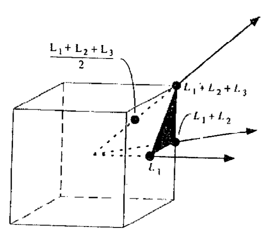
Exercise 19.9. Show that the automorphism of $\mathbb{C}^{2n}$ that interchanges e_n and e_{2n}, leaving the other e_i fixed, determines an automorphism of $\mathfrak{so}_{2n}\mathbb{C}$ that preserves the $n-2$ roots $L_1 - L_2, \ldots, L_{n-2} - L_{n-1}$ and interchanges $L_{n-1} - L_n$ and $L_{n-1} + L_n$. This automorphism takes the representation V to itself, but interchanges Γ_α and Γ_β.

§19.3. Representations of $\mathfrak{so}_7\mathbb{C}$

While we might reasonably be apprehensive about the prospect of a family of Lie algebras even more strangely behaved than the even orthogonal algebras, there is some good news: even though the roots systems of the odd Lie algebras appear more complicated than those of the even, the representation theory of the odd algebras is somewhat tamer. We will describe these representations, starting with the example of $\mathfrak{so}_7\mathbb{C}$; we begin, as always, with a picture of the root diagram:

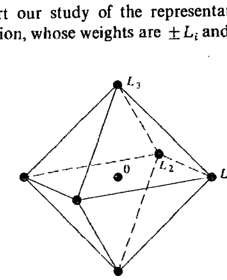


As we said, this looks like the root diagram for $\mathfrak{sp}_6\mathbb{C}$, except that the roots $\pm 2L_i$ have been shortened to $\pm L_i$. Unlike the case of $\mathfrak{so}_5\mathbb{C}$, however, where the long and short roots could be confused and the root diagram was correspondingly congruent to that of $\mathfrak{sp}_4\mathbb{C}$, in the present circumstance the root diagram is not similar to any other; the Lie algebra $\mathfrak{so}_7\mathbb{C}$, in fact, is *not* isomorphic to any of the others we have studied. Next, the Weyl chamber:



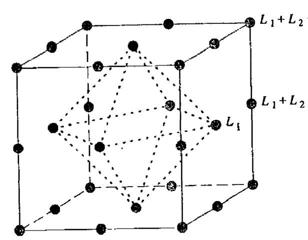
Again, the Weyl chamber itself looks just like that of $\mathfrak{sp}_6\mathbb{C}$; the difference in this picture is in the weight lattice, which contains the additional vector $(L_1 + L_2 + L_3)/2$.

As usual, we start our study of the representations of $\mathfrak{so}_7\mathbb{C}$ with the standard representation, whose weights are $\pm L_i$ and 0:



Note that the highest weight L_1 of this representation lies along the front edge of the Weyl chamber. Next, the weights of the exterior square $\wedge^2 V$ are $\pm L_i \pm L_j$, $\pm L_i$, and 0 (taken three times); this, of course, is just the adjoint representation. Note that the highest weight $L_1 + L_2$ of this representation is the same as that of the exterior square of the standard representation for $\mathfrak{so}_6\mathbb{C}$, but because of the smaller Weyl chamber this weight does indeed lie on an edge of the chamber.

Next, consider the third exterior power $\wedge^3 V$ of the standard. This has weights $\pm L_1 \pm L_2 \pm L_3$, $\pm L_i \pm L_j$, $\pm L_i$ (with multiplicity 2) and 0 (with multiplicity 3), i.e., at the midpoints of all the vertices, edges, and faces of the cube:



It is not obvious, from the weight diagram alone, that this is an irreducible representation; it could be that $\wedge^3 V$ contains a copy of the standard representation V and that the irreducible representation $\Gamma_{L_1+L_2+L_3}$ thus has multiplicity 1 on the weights $\pm L_i$ and multiplicity 2 (or 1) at 0. We can rule out this possibility by direct calculation: for example, if this were the case, then $\wedge^3 V$ would contain a highest weight vector with weight L_1. The weight space with eigenvalue L_1 in $\Lambda^3 V$ is spanned by the tensors $e_1 \wedge e_2 \wedge e_5$ and $e_1 \wedge e_3 \wedge e_6$, however, and if we apply to these the generators $X_{1,2} = E_{1,2} - E_{5,4}$, $X_{2,3} = E_{2,3} - E_{6,5}$, and $U_3 = E_{3,7} - E_{7,6}$ of the root spaces corresponding to the positive roots $L_1 - L_2$, $L_2 - L_3$, and L_3, we see that

$$X_{2,3}(e_1 \wedge e_3 \wedge e_6) = e_1 \wedge e_2 \wedge e_6,$$

$$U_3(e_1 \wedge e_3 \wedge e_6) = e_1 \wedge e_3 \wedge e_7 \neq 0;$$

$$X_{2,3}(e_1 \wedge e_2 \wedge e_5) = e_1 \wedge e_2 \wedge e_6,$$

$$U_3(e_1 \wedge e_2 \wedge e_5) = 0.$$

There is thus no linear combination of $e_1 \wedge e_2 \wedge e_5$ and $e_1 \wedge e_3 \wedge e_6$ killed by both U_3 and $X_{2,3}$, showing that $\Lambda^3 V$ has no highest weight vector of weight L_1.

Exercise 19.10. Verify that $\Lambda^3 V$ does not contain the trivial representation.

We have thus found irreducible representations of $\mathfrak{so}_7\mathbb{C}$ with highest weight vectors along the three edges of the Weyl chamber, and as in the case of $\mathfrak{so}_6\mathbb{C}$ we have thereby established the existence of the irreducible representations of $\mathfrak{so}_7\mathbb{C}$ with highest weight in the sublattice $\mathbb{Z}\{L_1, L_2, L_3\}$. To complete the description, we need to know that the representation Γ_α with highest weight $\alpha = (L_1 + L_2 + L_3)/2$ exists, and what it looks like, and this time there is no isomorphism to provide this; we will have to wait until the following lecture. In the meantime, we can still have fun playing around both with the representations we do know exist, and also with those whose existence is simply asserted.

Exercise 19.11. Find the decomposition into irreducible representations of the tensor product $V \otimes \Lambda^2 V$; in particular find the multiplicities of the irreducible representation $\Gamma_{2L_1+L_2}$ with highest weight $2L_1 + L_2$.

Exercise 19.12. Show that the symmetric square of the representation Γ_α decomposes into a copy of $\Lambda^3 V$ and a trivial one-dimensional representation.

Exercise 19.13. Find the decomposition into irreducible representations of $\Lambda^2\Gamma_\alpha$.

§19.4. Representations of the Odd Orthogonal Algebras

We will now describe as much as we can of the general pattern for representations of the odd orthogonal Lie algebras $\mathfrak{so}_{2n+1}\mathbb{C}$. As in the case of the even orthogonal Lie algebras, the proof of the existence part of the basic theorem (14.18) (that is, the construction of the irreducible representation with given

highest weight) will not be complete until the following lecture, but we can work around this pretty well.

To begin with, recall that the weight lattice of $\mathfrak{so}_{2n+1}\mathbb{C}$ is, like that of $\mathfrak{so}_{2n}\mathbb{C}$, generated by $L_1, \ldots, L_n$ together with the further vector $(L_1 + \cdots + L_n)/2$. The Weyl chamber, on the other hand, is the cone

$$\mathscr{W} = \{\textstyle\sum a_i L_i : a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}.$$

The Weyl chamber is as we have pointed out the same as for $\mathfrak{sp}_{2n}\mathbb{C}$, that is, it is a simplicial cone with faces corresponding to the n planes $a_1 = a_2, \ldots, a_{n-1} = a_n$ and $a_n = 0$. The edges of the Weyl chamber are thus the rays generated by the vectors $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-1}$ and $L_1 + \cdots + L_n$ (note that $L_1 + \cdots + L_{n-1}$ *is* on an edge of the Weyl chamber). Again, the intersection of the weight lattice with the closed Weyl cone is a free semigroup, in this case generated by the fundamental weights $\omega_1 = L_1$, $\omega_2 = L_1 + L_2$, $\ldots$, $\omega_{n-1} = L_1 + \cdots + L_{n-1}$ and the weight $\omega_n = \alpha = (L_1 + \cdots + L_n)/2$. Moreover, as we saw in the cases of $\mathfrak{so}_5\mathbb{C}$ and $\mathfrak{so}_7\mathbb{C}$, the exterior powers of the standard representation do serve to generate all the irreducible representations whose highest weights are in the sublattice $\mathbb{Z}\{L_1, \ldots, L_n\}$: in general we have the following theorem.

Theorem 19.14. *For $k = 1, \ldots, n$, the exterior power $\Lambda^k V$ of the standard representation V of $\mathfrak{so}_{2n+1}\mathbb{C}$ is the irreducible representation with highest weight $L_1 + \cdots + L_k$.*

PROOF. We will leave this as an exercise; the proof is essentially the same as in the case of $\mathfrak{so}_{2n}\mathbb{C}$, with enough of a difference to make it interesting. □

We have thus constructed one-half of the irreducible representations of $\mathfrak{so}_{2n+1}\mathbb{C}$: any weight γ in the closed Weyl chamber can be written

$$\gamma = a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1}) + a_n(L_1 + \cdots + L_n)/2$$

with $a_i \in \mathbb{N}$; and if a_n is even, the representation

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\Lambda^{n-1} V) \otimes \mathrm{Sym}^{a_n/2}(\Lambda^n V)$$

will contain an irreducible representation Γ_γ with highest weight γ. We are still missing, however, any representation whose weights involve odd multiples of α; to construct these, we clearly have to exhibit an irreducible representation Γ_α with highest weight α. This exists and is called (as in the case of the even orthogonal Lie algebras) the *spin representation* of $\mathfrak{so}_{2n+1}\mathbb{C}$. We see from the above that once we exhibit the spin representation Γ_α, we will have constructed all the representations of $\mathfrak{so}_{2n+1}\mathbb{C}$; for any γ as above the tensor

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\Lambda^{n-1} V) \otimes \mathrm{Sym}^{a_n}(\Gamma_\alpha)$$

will contain a copy of Γ_γ.

As in the case of the spin representation Γ_α of the even orthogonal Lie algebras, we can say some things about Γ_α even in advance of its explicit construction; for example, we can do the following exercises.

Exercise 19.15. Find the weights (with multiplicities) of the representations $\wedge^k V$, and also of Γ_α.

Exercise 19.16. Give the complete decomposition into irreducible representations of $\mathrm{Sym}^2\Gamma_\alpha$ and $\wedge^2\Gamma_\alpha$. Show that

$$\Gamma_\alpha \otimes \Gamma_\alpha = \wedge^n V \oplus \wedge^{n-1} V \oplus \wedge^{n-2} V \oplus \cdots \oplus \wedge^1 V \oplus \wedge^0 V.$$

Exercise 19.17. Verify directly the above statements in the case of $\mathfrak{so}_5\mathbb{C}$, using the isomorphism with $\mathfrak{sp}_4\mathbb{C}$.

§19.5. Weyl's Construction for Orthogonal Groups

The same procedure we saw in the symplectic case can be used to construct representations of the orthogonal groups, this time generalizing what we saw directly for $\wedge^k V$ in §§19.2 and 19.4. For the symmetric form Q on $V = \mathbb{C}^m$, the same formula (17.9) determines contractions from $V^{\otimes d}$ to $V^{\otimes(d-2)}$. Denote the intersection of the kernels of all these contractions by $V^{[d]}$. For any partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m \geq 0)$ of d, let

$$\mathbb{S}_{[\lambda]} V = V^{[d]} \cap \mathbb{S}_\lambda V. \tag{19.18}$$

As before, this is a representation of the orthogonal group $\mathrm{O}_m\mathbb{C}$ of Q.

Theorem 19.19. *The space $\mathbb{S}_{[\lambda]}V$ is an irreducible representation of $\mathrm{O}_m\mathbb{C}$; $\mathbb{S}_{[\lambda]}V$ nonzero if and only if the sum of the lengths of the first two columns of the Young diagram of λ is at most m.*

The tensor power $V^{\otimes d}$ decomposes exactly as in Lemma 17.15, with everything the same but replacing the symbol $\langle d \rangle$ by $[d]$. In particular,

$$\mathbb{S}_{[\lambda]} V = V^{[d]} \cdot c_\lambda = \mathrm{Im}(c_\lambda : V^{[d]} \to V^{[d]}).$$

Exercise 19.20. Verify that $\mathbb{S}_{[\lambda]}V$ is zero when the sum of the lengths of the first two columns is greater than m by showing that $\wedge^a V \otimes \wedge^b V \otimes V^{\otimes(d-a-b)}$ is contained in $\sum_I \Psi_I(V^{\otimes(d-2)})$ when $a + b > m$. Show that $\mathbb{S}_{[\lambda]}V$ is not zero when the sum of the lengths of the first two columns is at most m.

Exercise 19.21*. (i) Show that the kernel of the contraction from $\mathrm{Sym}^d V$ to $\mathrm{Sym}^{d-2} V$ is the irreducible representation $\mathbb{S}_{[d]}V$ of $\mathfrak{so}_m\mathbb{C}$ with highest weight dL_1.

(ii) Show that

$$\mathrm{Sym}^d V = \mathbb{S}_{[d]}V \oplus \mathbb{S}_{[d-2]}V \oplus \cdots \oplus \mathbb{S}_{[d-2p]}V,$$

where p is the largest integer $\leq d/2$.

The proof of the theorem proceeds exactly as in §17.3. The fundamental fact from invariant theory is the same statement as (17.19), with, of course, the operators $\vartheta_I = \Psi_I \circ \Phi_I$ defined using the given symmetric form, and the group $\mathrm{Sp}_{2n}\mathbb{C}$ replaced by $\mathrm{O}_m\mathbb{C}$ (and the same reference to Appendix F.2 for the proof). The theorem then follows from Lemma 6.22 in exactly the same way as for the symplectic group.

To find the irreducible representations over $\mathrm{SO}_m\mathbb{C}$ one can proceed as in §5.1. Weyl calls two partitions (each with the sum of the first two column lengths at most m) *associated* if the sum of the lengths of their first columns is m and the other columns of their Young diagrams have the same lengths. Representations of associated partitions restrict to isomorphic representations of $\mathrm{SO}_m\mathbb{C}$. Note that at least one of each pair of associated partitions will have a Young diagram with at most $\frac{1}{2}m$ rows. If $m = 2n + 1$ is odd, no λ is associated to itself, but if $m = 2n$ is even, any λ with a Young diagram with n nonzero rows will be associated to itself, and its restriction will be the sum of two conjugate representations of $\mathrm{SO}_m\mathbb{C}$ of the same dimension. The final result is:

Theorem 19.22. (i) *If $m = 2n + 1$, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, then $\mathbb{S}_{[\lambda]}V$ is the irreducible representation of $\mathfrak{so}_m\mathbb{C}$ with highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$.*

(ii) *If $m = 2n$, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0)$, then $\mathbb{S}_{[\lambda]}V$ is the irreducible representation of $\mathfrak{so}_m\mathbb{C}$ with highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$.*

(iii) *If $m = 2n$, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n > 0)$, then $\mathbb{S}_{[\lambda]}V$ is the sum of two irreducible representations of $\mathfrak{so}_m\mathbb{C}$ with highest weights $\lambda_1 L_1 + \cdots + \lambda_n L_n$ and $\lambda_1 L_1 + \cdots + \lambda_{n-1} L_{n-1} - \lambda_n L_n$.*

Exercise 19.23. When m is odd, show that $\mathrm{O}_m\mathbb{C} = \mathrm{SO}_m\mathbb{C} \times \{\pm I\}$. Show that if λ and μ are associated, then $\mu = \lambda \otimes \varepsilon$, where ε is the sign of the determinant.

We postpone to Lecture 25 all discussion of multiplicities of weight spaces, or decomposing tensor products or restrictions to subgroups.

As we saw in Lecture 15 for $\mathrm{GL}_n\mathbb{C}$ and in Lecture 17 for $\mathrm{Sp}_{2n}\mathbb{C}$, it is possible to make a commutative algebra $\mathbb{S}^{[\cdot]} = \mathbb{S}^{[\cdot]}(V)$ out of the sum of all the irreducible representations of $\mathrm{SO}_m\mathbb{C}$, where $V = \mathbb{C}^m$ is the standard representation. First suppose $m = 2n + 1$ is odd. Define the ring $\mathbb{S}^\cdot(V, n)$ as in §15.5, which is a sum of all the representations $\mathbb{S}_\lambda(V)$ of $\mathrm{GL}(V)$ where λ runs over all partitions with at most n parts. As in the symplectic case, there is a canonical decomposition

$$\mathbb{S}_\lambda(V) = \mathbb{S}_{[\lambda]}(V) \oplus J_{[\lambda]}(V),$$

and the direct sum $J^{[\cdot]} = \bigoplus_\lambda J_{[\lambda]}(V)$ is an ideal in $\mathbb{S}^\cdot(V, n)$. The quotient ring

$$\mathbb{S}^{[\cdot]}(V) = A^\cdot(V, n)/J^{[\cdot]} = \bigoplus_\lambda \mathbb{S}_{[\lambda]}(V)$$

is a commutative graded ring which contains each irreducible representation of $\mathrm{SO}_{2n+1}\mathbb{C}$ once.

If $m = 2n$ is even, the above quotient will contain each representation $\mathbb{S}_{[\lambda]}(V)$ twice if λ has n rows. To cut it down so there is only one of each, one can add to $J^{[\cdot]}$ relations of the form $x - \tau(x)$, for $x \in \wedge^n V$, where $\tau\colon \wedge^n V \to \wedge^n V$ is the isomorphism described in the remark (iii) after the proof of Theorem 19.2. For a detailed discussion, with explicit generators for the ideas, see [L-T].

LECTURE 20

Spin Representations of $\mathfrak{so}_m\mathbb{C}$

In this lecture we complete the picture of the representations of the orthogonal Lie algebras by constructing the spin representations $S^{\pm}$ of $\mathfrak{so}_m\mathbb{C}$; this also yields a description of the spin groups $\mathrm{Spin}_m\mathbb{C}$. Since the representation-theoretic analysis of the spaces $S^{\pm}$ was carried out in the preceding lecture, we are concerned here primarily with the algebra involved in their construction. Thus, §20.1 and §20.2, while elementary, involve some fairly serious algebra. Section 20.3, where we briefly sketch the notion of triality, may seem mysterious to the reader (this is at least in part because it is so to the authors); if so, it may be skipped. Finally, we should say that the subject of the spin representations of $\mathfrak{so}_m\mathbb{C}$ is a very rich one, and one that accommodates many different points of view; the reader who is interested is encouraged to try some of the other approaches that may be found in the literature.

§20.1: Clifford algebras and spin representations of $\mathfrak{so}_m\mathbb{C}$
§20.2: The spin groups $\mathrm{Spin}_m\mathbb{C}$ and $\mathrm{Spin}_m\mathbb{R}$
§20.3: $\mathrm{Spin}_8\mathbb{C}$ and triality

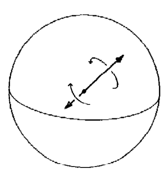
§20.1. Clifford Algebras and Spin Representations of $\mathfrak{so}_m\mathbb{C}$

We begin this section by trying to motivate the definition of Clifford algebras. We may begin by asking, why were we able to find all the representations of $\mathrm{SL}_n\mathbb{C}$ or $\mathrm{Sp}_{2n}\mathbb{C}$ inside tensor powers of the standard representation, but only half the representations of $\mathrm{SO}_m\mathbb{C}$ arise this way? One difference that points in this direction lies in the topology of these groups: $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C}$ are simply connected, while $\mathrm{SO}_m\mathbb{C}$ has fundamental group $\mathbb{Z}/2$ for $m > 2$ (for proofs see §23.1). Therefore $\mathrm{SO}_m\mathbb{C}$ has a double covering, the *spin group* $\mathrm{Spin}_m\mathbb{C}$. (For $m \leq 6$, these coverings could also be extracted from our identifications of the adjoint group $PSO_m\mathbb{C}$ with the adjoint group of other simply connected groups; e.g. the double cover of $SO_3\mathbb{C}$ is $SL_2\mathbb{C}$.) We will see that the missing representations are those representations of $Spin_m\mathbb{C}$ that do not come from representations of $SO_m\mathbb{C}$.

This double covering may be most readily visible, and probably familiar, for the case of the real subgroup $SO_3\mathbb{R}$ of rotations: a rotation is specified by an axis to rotate about, given by a unit vector u, and an angle of rotation about u; the two choices $\pm u$ of unit vector give a two-sheeted covering. In other words, if D^3 is the unit ball in $\mathbb{R}^3$, there is a double covering

$$S^3 = D^3/\partial D^3 \to SO_3\mathbb{R},$$

which sends a vector v in D^3 to rotation by the angle $2\pi\|v\|$ about the unit vector $v/\|v\|$ (the origin and the unit sphere ∂D^3 are sent to the identity transformation).



This covering is even easier to see for the entire orthogonal group $O_3\mathbb{R}$, which is generated by reflections R_v in unit vectors v (with $\pm v$ determining the same reflection): we can describe the double cover of $O_3\mathbb{R}$ as the group generated by unit vectors v, with relations

$$v_1 \cdot \ldots \cdot v_n = w_1 \cdot \ldots \cdot w_m$$

whenever the compositions of the corresponding reflections are equal, i.e., whenever

$$R_{v_1} \circ \cdots \circ R_{v_n} = R_{w_1} \circ \cdots \circ R_{w_m};$$

and also relations

$$(-v)\cdot(-w) = v \cdot w$$

for all pairs of unit vectors v and w. (Note that if we restricted ourselves to products of even numbers of the generators $v \in \partial D^3$ we would get back the double cover of the special orthogonal group $SO_3\mathbb{C}$.)

How should we generalize this? The answer is not obvious. For one thing, for various reasons we will not try to construct directly a group that covers the orthogonal group in general. Instead, given a vector space V (real or complex) and a quadratic form Q on V, we will first construct an algebra $\mathrm{Cliff}(V, Q)$, called the *Clifford algebra*. The algebra $\mathrm{Cliff}(V, Q)$ will then turn

out to contain in its multiplicative group a subgroup which is a double cover of the orthogonal group $O(V, Q)$ of automorphisms of V preserving Q.

By analogy with the construction of the double cover of $SO_3\mathbb{R}$, the Clifford algebra $\mathrm{Cliff}(V, Q)$ associated to the pair (V, Q) is an associative algebra containing and generated by V. (When we want to describe the spin group inside $\mathrm{Cliff}(V, Q)$ we will restrict ourselves to products of even numbers of elements of V having a fixed norm $Q(v, v)$; if odd products are allowed as well, we get a group called "Pin" which is a double covering of the whole orthogonal group.) To motivate the definition, we would like $\mathrm{Cliff}(V, Q)$ to be the algebra generated by V subject to relations analogous to those above for the double cover of the orthogonal group. In particular, for any vector v with $Q(v, v) = 1$, since the reflection R_v in the hyperplane perpendicular to v is an involution, we want

$$v \cdot v = 1$$

in $\mathrm{Cliff}(V, Q)$. By polarization, this is the same as imposing the relation

$$v \cdot w + w \cdot v = 2Q(v, w)$$

for all v and w in V. In particular, $w \cdot v = -v \cdot w$ if v and w are perpendicular. In fact, the Clifford algebra[1] will be defined below to be the associative algebra generated by V and subject to the equation $v \cdot v = Q(v, v)$.

Looking ahead, we will see later in this section that each complex Clifford algebra contains an orthogonal Lie algebra as a subalgebra. The key theorem is then that $\mathrm{Cliff}(V, Q)$ *is isomorphic either to a matrix algebra or to a sum of two matrix algebras.* This in turn determines either one or two representations of the orthogonal Lie algebras, which turn out to be the representations which were needed to complete the story in the last lecture. Just as in the special linear and symplectic cases, the corresponding Lie groups are not really needed to construct the representations; they can be written down directly from the Lie algebra. In this section we do this, using the Clifford algebras to construct these representations of $\mathfrak{so}_m\mathbb{C}$ directly, and verify that they give the missing spin representations. In the second section of this lecture we will show how the spin groups sit as subgroups in their multiplicative groups.

Clifford Algebras

Given a symmetric bilinear form Q on a vector space V, the *Clifford algebra* $C = C(Q) = \mathrm{Cliff}(V, Q)$ is an associative algebra with unit 1, which contains and is generated by V, with $v \cdot v = Q(v, v) \cdot 1$ for all $v \in V$. Equivalently, we have the equation

$$v \cdot w + w \cdot v = 2Q(v, w), \tag{20.1}$$

[1] The mathematical world seems to be about evenly divided about the choice of signs here, and one must translate from Q to $-Q$ to go from one side to the other.

for all v and w in V. The Clifford algebra can be defined to be the universal algebra with this property: if E is any associative algebra with unit, and a linear mapping $j\colon V \to E$ is given such that $j(v)^2 = Q(v, v)\cdot 1$ for all $v \in V$, or equivalently

$$j(v)\cdot j(w) + j(w)\cdot j(v) = 2Q(v, w)\cdot 1 \tag{20.2}$$

for all $v, w \in V$, then there should be a unique homomorphism of algebras from $C(Q)$ to E extending j. The Clifford algebra can be constructed quickly by taking the tensor algebra

$$T^{\cdot}(V) = \bigoplus_{n\geq 0} V^{\otimes n} = \mathbb{C} \oplus V \oplus (V\otimes V) \oplus (V\otimes V\otimes V) \oplus \cdots,$$

and setting $C(Q) = T^{\cdot}(V)/I(Q)$, where $I(Q)$ is the two-sided ideal generated by all elements of the form $v\otimes v - Q(v, v)\cdot 1$. It is automatic that this $C(Q)$ satisfies the required universal property.

The facts that the dimension of C is 2^m, where $m = \dim(V)$, and that the canonical mapping from V to C is an embedding, are part of the following lemma:

Lemma 20.3. *If $e_1, \ldots, e_m$ form a basis for V, then the products $e_I = e_{i_1}\cdot e_{i_2}\cdot\ldots\cdot e_{i_k}$, for $I = \{i_1 < i_2 < \cdots < i_k\}$, and with $e_\emptyset = 1$, form a basis for $C(Q) = \mathrm{Cliff}(V, Q)$.*

Proof. From the equations $e_i\cdot e_j + e_j\cdot e_i = 2Q(e_i, e_j)$ it follows immediately that the elements e_I generate $C(Q)$. Their independence is not hard to verify directly; it also follows by seeing that the images in the matrix algebras under the mappings constructed below are independent. For another proof, note that when $Q \equiv 0$, the Clifford algebra is just the exterior algebra $\bigwedge^{\cdot} V$. In general, the Clifford algebra can be filtered by subspaces F_k, consisting of those elements which can be written as sums of at most k products of elements in V; one checks that the associated graded space F_k/F_{k+1} is $\bigwedge^k V$. For a third proof, one can verify that the Clifford algebra of the direct sum of two orthogonal spaces is the skew commutative tensor product of the Clifford algebras of the two spaces (cf. Exercise B.9), which reduces one to the trivial case where $\dim V = 1$. □

Since the ideal $I(Q) \subset T^{\cdot}(V)$ is generated by elements of even degree, the Clifford algebra inherits a $\mathbb{Z}/2\mathbb{Z}$ grading:

$$C = C^{\text{even}} \oplus C^{\text{odd}} = C^+ \oplus C^-,$$

with $C^+\cdot C^+ \subset C^+$, $C^+\cdot C^- \subset C^-$, $C^-\cdot C^+ \subset C^-$, $C^-\cdot C^- \subset C^+$; C^+ is spanned by products of an even number of elements in V and C^- is spanned by products of an odd number. In particular, C^{even} is a subalgebra of dimension 2^{m-1}.

Since $C(Q)$ is an associative algebra, it determines a Lie algebra, with bracket $[a, b] = a\cdot b - b\cdot a$. From now on we assume Q is nondegenerate. The new representations of $\mathfrak{so}_m\mathbb{C}$ will be found in two steps:

(i) embedding the Lie algebra $\mathfrak{so}(Q) = \mathfrak{so}_m\mathbb{C}$ inside the Lie algebra of the even part of the Clifford algebra $C(Q)$;
(ii) identifying the Clifford algebras with one or two copies of matrix algebras.

To carry out the first step we make explicit the isomorphism of $\bigwedge^2 V$ with $\mathfrak{so}(Q)$ that we have discussed before. Recall that

$$\mathfrak{so}(Q) = \{X \in \mathrm{End}(V)\colon Q(Xv, w) + Q(v, Xw) = 0 \text{ for all } v, w \text{ in } V\}.$$

The isomorphism is given by

$$\textstyle\bigwedge^2 V \xrightarrow{\cong} \mathfrak{so}(Q) \subset \mathrm{End}(V), \qquad a\wedge b \mapsto \varphi_{a\wedge b},$$

for a and b in V, where $\varphi_{a\wedge b}$ is defined by

$$\varphi_{a\wedge b}(v) = 2(Q(b, v)a - Q(a, v)b). \tag{20.4}$$

It is a simple verification that $\varphi_{a\wedge b}$ is in $\mathfrak{so}(Q)$. One sees that the natural bases correspond up to scalars, e.g., $e_i \wedge e_{n+j}$ maps to $2(E_{i,j} - E_{n+j,n+i})$, so the map is an isomorphism. (The choice of scalar factor is unimportant here; it was chosen to simplify later formulas.) One calculates what the bracket on $\bigwedge^2 V$ must be to make this an isomorphism of Lie algebras:

$$\begin{aligned}
[\varphi_{a\wedge b}, \varphi_{c\wedge d}](v) &= \varphi_{a\wedge b}\circ\varphi_{c\wedge d}(v) - \varphi_{c\wedge d}\circ\varphi_{a\wedge b}(v)\\
&= 2\varphi_{a\wedge b}(Q(d, v)c - Q(c, v)d) - 2\varphi_{c\wedge d}(Q(b, v)a - Q(a, v)b)\\
&= 4Q(d, v)(Q(b, c)a - Q(a, c)b)\\
&\quad - 4Q(c, v)(Q(b, d)a - Q(a, d)b)\\
&\quad - 4Q(b, v)(Q(d, a)c - Q(c, a)d)\\
&\quad + 4Q(a, v)(Q(d, b)c - Q(c, b)d)\\
&= 2Q(b, c)\varphi_{a\wedge d}(v) - 2Q(b, d)\varphi_{a\wedge c}(v)\\
&\quad - 2Q(a, d)\varphi_{c\wedge b}(v) + 2Q(a, c)\varphi_{d\wedge b}(v).
\end{aligned}$$

This gives an explicit formula for the bracket on $\bigwedge^2 V$:

$$\begin{aligned}
[a\wedge b, c\wedge d] &= 2Q(b, c)a\wedge d - 2Q(b, d)a\wedge c\\
&\quad - 2Q(a, d)c\wedge b + 2Q(a, c)d\wedge b.
\end{aligned} \tag{20.5}$$

On the other hand, the bracket in the Clifford algebra satisfies

$$\begin{aligned}
[a\cdot b, c\cdot d] &= a\cdot b\cdot c\cdot d - c\cdot d\cdot a\cdot b\\
&= (2Q(b, c)a\cdot d - a\cdot c\cdot b\cdot d) - (2Q(a, d)c\cdot b - c\cdot a\cdot d\cdot b)\\
&= 2Q(b, c)a\cdot d - (2Q(b, d)a\cdot c - a\cdot c\cdot d\cdot b)\\
&\quad - 2Q(a, d)c\cdot b + (2Q(a, c)\cdot d\cdot b - a\cdot c\cdot d\cdot b)\\
&= 2Q(b, c)a\cdot d - 2Q(b, d)a\cdot c - 2Q(a, d)c\cdot b + 2Q(a, c)\cdot d\cdot b.
\end{aligned}$$

It follows that the map $\psi\colon \Lambda^2 V \to \mathrm{Cliff}(V, Q)$ defined by

$$\psi(a \wedge b) = \tfrac{1}{2}(a\cdot b - b\cdot a) = a\cdot b - Q(a, b) \tag{20.6}$$

is a map[2] of Lie algebras, and by looking at basis elements again one sees that it is an embedding. This proves:

Lemma 20.7. *The mapping $\psi \circ \varphi^{-1}\colon \mathfrak{so}(Q) \to C(Q)^{\text{even}}$ embeds $\mathfrak{so}(Q)$ as a Lie subalgebra of $C(Q)^{\text{even}}$.*

Exercise 20.8. Show that the image of ψ is

$$F_2 \cap C(Q)^{\text{even}} \cap \mathrm{Ker}(\text{trace}),$$

where F_2 is the subspace of $C(Q)$ spanned by products of at most two elements of V, and the trace of an element of $C(Q)$ is the trace of left multiplication by that element on $C(Q)$.

We consider first the *even* case: write $V = W \oplus W'$, where W and W' are n-dimensional isotropic spaces for Q. (Recall that a space is isotropic when Q restricts to the zero form on it.) With our choice of standard Q on $V = \mathbb{C}^{2n}$, W can be taken to be the space spanned by the first n basis vectors, W' by the last n.

Lemma 20.9. *The decomposition $V = W \oplus W'$ determines an isomorphism of algebras*

$$C(Q) \cong \mathrm{End}(\Lambda^{\bullet} W),$$

where $\Lambda^{\bullet} W = \Lambda^0 W \oplus \cdots \oplus \Lambda^n W$.

Proof. Mapping $C(Q)$ to the algebra $E = \mathrm{End}(\Lambda^{\bullet} W)$ is the same as defining a linear mapping from V to E, satisfying (20.2). We must construct maps $l\colon W \to E$ and $l'\colon W' \to E$ such that

$$l(w)^2 = 0, \qquad l'(w')^2 = 0, \tag{20.10}$$

and

$$l(w) \circ l'(w') + l'(w') \circ l(w) = 2Q(w, w')I$$

for any $w \in W$, $w' \in W'$. For each $w \in W$, let $L_w \in E$ be left multiplication by w on the exterior algebra $\Lambda^{\bullet} W$:

$$L_w(\xi) = w \wedge \xi, \quad \xi \in \Lambda^{\bullet} W.$$

For $\vartheta \in W^*$, let $D_\vartheta \in E$ be the derivation of $\Lambda^{\bullet} W$ such that $D_\vartheta(1) = 0$, $D_\vartheta(w) = \vartheta(w) \in \Lambda^0 W = \mathbb{C}$ for $w \in W = \Lambda^1 W$, and

[2] Note that the bilinear form ψ given by (20.6) is alternating since $\psi(a \wedge a) = 0$, so it defines a linear map on $\Lambda^2 V$.

$$D_\vartheta(\zeta \wedge \xi) = D_\vartheta(\zeta) \wedge \xi + (-1)^{\deg(\zeta)}\zeta \wedge D_\vartheta(\zeta).$$

Explicitly, $D_\vartheta(w_1 \wedge \cdots \wedge w_r) = \sum (-1)^{i-1}\vartheta(w_i)(w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_r)$. Now set

$$l(w) = L_w, \qquad l'(w') = D_\vartheta, \tag{20.11}$$

where $\vartheta \in W^*$ is defined by the identity $\vartheta(w) = 2Q(w, w')$ for all $w \in W$. The required equations (20.10) are straightforward verifications: one checks directly on elements in $W = \Lambda^1 W$, and then that, if they hold on ζ and ξ, they hold on $\zeta \wedge \xi$. Finally, one may see that the resulting map is an isomorphism by looking at what happens to a basis. □

Exercise 20.12. The left $C(Q)$-module $\Lambda^{\bullet} W$ is isomorphic to a left ideal in $C(Q)$. Show that if f is a generator for $\Lambda^n W'$, then $C(Q)\cdot f = \Lambda^{\bullet} W \cdot f$, and the map $\zeta \mapsto \zeta \cdot f$ gives an isomorphism

$$\Lambda^{\bullet} W \to \Lambda^{\bullet} W \cdot f = C(Q)\cdot f$$

of left $C(Q)$-modules.

Now we have a decomposition $\Lambda^{\bullet} W = \Lambda^{\text{even}} W \oplus \Lambda^{\text{odd}} W$ into the sum of even and odd exterior powers, and $C(W)^{\text{even}}$ respects this splitting. We deduce from Lemma 20.9 an isomorphism

$$C(Q)^{\text{even}} \cong \mathrm{End}(\Lambda^{\text{even}} W) \oplus \mathrm{End}(\Lambda^{\text{odd}} W). \tag{20.13}$$

Combining with Lemma 20.7, we now have an embedding of Lie algebras:

$$\mathfrak{so}(Q) \subset C(Q)^{\text{even}} \cong \mathfrak{gl}(\Lambda^{\text{even}} W) \oplus \mathfrak{gl}(\Lambda^{\text{odd}} W), \tag{20.14}$$

and hence we have two representations of $\mathfrak{so}(Q) = \mathfrak{so}_{2n}\mathbb{C}$, which we denote by

$$S^+ = \Lambda^{\text{even}} W \quad \text{and} \quad S^- = \Lambda^{\text{odd}} W.$$

Proposition 20.15. *The representations $S^\pm$ are the irreducible representations of $\mathfrak{so}_{2n}\mathbb{C}$ with highest weights $\alpha = \frac{1}{2}(L_1 + \cdots + L_n)$ and $\beta = \frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n)$. More precisely,*

$$S^+ = \Gamma_\alpha \quad \text{and} \quad S^- = \Gamma_\beta \quad \text{if } n \text{ is even};$$

$$S^+ = \Gamma_\beta \quad \text{and} \quad S^- = \Gamma_\alpha \quad \text{if } n \text{ is odd}.$$

Proof. We show that the natural basis vectors $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $\Lambda^{\bullet} W$ are weight vectors. Tracing through the isomorphisms established above, we see that $H_i = E_{i,i} - E_{n+i,n+i}$ in $\mathfrak{h} \subset \mathfrak{so}_{2n}\mathbb{C}$ corresponds to $\frac{1}{2}(e_i \wedge e_{n+i})$ in $\Lambda^2 V$, which corresponds to $\frac{1}{2}(e_i \cdot e_{n+i} - 1)$ in $C(Q)$, which maps to

$$\tfrac{1}{2}(L_{e_i} \circ D_{2e_i^*} - I) = L_{e_i} \circ D_{e_i^*} - \tfrac{1}{2}I \in \mathrm{End}(\Lambda^{\bullet} W).$$

A simple calculation shows that