Numerical functions for graded ideals: Application to Coding Theory and Combinatorics.

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Abstract.

Let S be a polynomial ring over the field K and let I be a graded ideal of S. In this talk we introduce and study two functions associated to I: the minimum distance function δ_I and the footprint function fp_I. To define δ_I and fp_I we use the Hilbert function, the degree (multiplicity), and a Gröbner basis for I. We study these functions from a theoretical point of view and examine their asymptotic behavior. These functions can be expressed in terms of the algebraic invariants of I. One of the main results to be presented shows that fp_I is a lower bound for δ_I. We give formulas to compute fp_I and δ_I in the case of certain complete intersections. In the case of complete intersection monomial ideals δ_I is equal to fp_I and give an explicit formula in terms of the degrees of a minimal set of generators of I.

Let $K = \mathbb{F}_q$ be a finite field and let $\mathbb{X} \subset \mathbb{P}^{s-1}$ be a finite subset of points in the projective space $\mathbb{P}^{s-1}$ over the field K. We show a formula to compute the number of zeros that a homogeneous polynomial has in $\mathbb{X}$. We use the minimum distance function of the vanishing ideal associated to $\mathbb{X}$ in order to give an algebraic formulation for the minimum distance in coding theory, in particular for projective Reed–Muller-type codes defined on $\mathbb{X}$, we also compute its dimension and length. Following the footprint method, we present bounds for the number of zeros of polynomials in a projective nested Cartesian set $\mathbb{X}$ and for the minimum distance of the corresponding projective nested Cartesian codes.

To show applications of the footprint method we need to study certain monomial ideals that occur as initial ideals of vanishing ideals over finite fields. This leads us to introduce the edge ideal $I = I(\mathcal{D})$ of a weighted oriented graph $\mathcal{D}$. Using the combinatorial structure of digraphs, we determine the irredundant irreducible decomposition of I. Furthermore, we give a combinatorial characterization for the unmixed property of I, when the digraph is bipartite, a whisker or a cycle. We will also study the Cohen–Macaulay property of I and show that in certain cases I is unmixed if and only if I is Cohen–Macaulay.