Using tropical differential equations

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Abstract

In this survey we explain from scratch the recent formalism behind tropical differential algebraic geometry and how it may be used to extract combinatorial information from the set of power series solutions for systems of differential equations.

1 Introduction

Finding solutions for systems of differential equations is in general a very difficult task. For example, we know from [8] that it does not exist an algorithm for computing formal power series solutions for systems of algebraic partial differential equations with coefficients in the ring of complex formal power series.

In this regard, the recently introduced branch of tropical differential algebraic geometry yields an algebraic framework which gives precise combinatorial information about the set of all the formal power series solutions of systems of differential equations, incarnated in the form of a *tropical fundamental theorem* version for differential algebraic geometry. Both the concept and insight of the method were introduced in [6].

The construction of such correspondences is one of the main goals of the application of tropical methods in commutative algebra, whose basis is the following: one considers a tropicalization scheme (R, S, v) as in [4], consisting of a commutative ring R (possibly endowed with some extra structure), a commutative idempotent semiring S and a map $v : R \longrightarrow S$ that satisfies the usual properties of a valuation. Then one considers a system of algebraic equations Σ living in an R-algebra R' together with its set of algebraic solutions $\mathrm{Sol}(\Sigma)$. If the *tropicalization* $v(\mathrm{Sol}(\Sigma))$ of $\mathrm{Sol}(\Sigma)$ coincides with the set of solutions $\mathrm{Sol}(v(\Sigma))$ of the *tropicalization* $v(\Sigma)$ of the original system, then we have a tropical fundamental theorem for this particular setting.

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The first application of these ideas was a tropical fundamental theorem for very affine varieties over valued fields, in which the tropicalization scheme (R, S, v) corresponds to a valuation $v : K \longrightarrow \mathbb{R} \cup \{-\infty\}$ of rank ≤ 1, the R-algebra R' is $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $\mathrm{Sol}(\Sigma)$ is just the Zariski closed set associated to Σ. Another example is a tropical fundamental theorem for *ordinary differential algebraic geometry* (over rings of formal power series with coefficients in an uncountable, algebraically closed field of characteristic zero), which was originally proved in [1].

A generalization of the previous tropical fundamental theorem which is valid for both the ordinary and the partial case has been proved in [3], and it stands as the main result in the area until now, since it opens the door for the potential application of combinatorial methods to problems in differential algebraic geometry. We would like to use this recent development to present a unified introduction to this framework, with the hope of turning this potential into a reality.

The purpose of this survey is twofold and it is intended to be accessible to a wide audience. The first one is to introduce the formalism behind tropical differential algebraic geometry, which we do in Sections 3 and 4, without assuming any kind of familiarity with the subject. The second one is to explain in Sections 4 and 5 the tropical fundamental theorem for *differential algebraic geometry*, and to explain how it may be used potentially to tackle problems in differential algebraic geometry, both in the archimedean (complex analytic) and in the non-archimedean (p-adic analytic) worlds. As a bonus, we use this note as an opportunity to present in Section 2 an application of the theory of semirings, in the form of a unified theory of differential algebraic geometry over formal power series semirings.

Conventions. With the exception of Section 2.2, every algebraic structure to be considered here will be commutative. We will denote by $\mathbb{N}$ the semiring of the natural numbers (which includes 0) endowed with the usual operations, and we also fix from now on two non-zero natural numbers m and n.

The basis of the free commutative $\mathbb{N}$-module $\mathbb{N}^m$ will be denoted by $\{e_1, \ldots, e_m\}$. If $I = (i_1, \ldots, i_m), J = (j_1, \ldots, j_m) \in \mathbb{N}^m$, we denote by $||I||_\infty := \max\{i_1, \ldots, i_m\} = \max(I)$ and by $I - J = (i_1 - j_1, \ldots, i_m - j_m)$. If $A \subseteq \mathbb{N}^m$, we write $A - I = \{J - I \,:\, J \in A\}$.

2 Differential algebraic geometry over formal power series semirings

We recall that a (not necessarily commutative) semiring with unit is an algebraic structure which satisfies all the axioms of rings, possibly with the exception of the existence of additive inverses. We also have a category of semirings which is a proper enlargement of the category of rings, and it can be described succinctly as the category of $\mathbb{N}$-algebras. The reader might consult [5] to learn more about the general theory of semirings.

In this section we will denote by $S = (S, +, \times, 0, 1)$ a commutative semiring with 1, and its purpose is to introduce the concept of differential polynomial with coefficients in a power series semiring, which is just an adaptation from the case of rings from [10]. Then the (algebraic) differential equations with coefficients in a ring are obtained by equating these objects to zero; unfortunately this definition does not give a satisfactory concept for general semirings, as we will see later for the case of idempotent semirings.

We consider the tuple $T = (t_1, \ldots, t_m)$ and for $I = (i_1, \ldots, i_m) \in \mathbb{N}^m$, we denote by T^I the formal monomial $t_1^{i_1} \cdots t_m^{i_m}$.

Definition: The semiring $S[[T]] := (S[[T]], +, \times, 0, 1)$ of **formal power series** with coefficients in S in the variables $t_1, \ldots, t_m$ consists of the set

$$S[[T]] := \left\{ a = \sum_{I \in A} a_I T^I \ : \ \emptyset \subseteq A \subseteq \mathbb{N}^m, \quad a_I \in S \setminus \{0\} \right\}$$

endowed with the usual operations of term-wise sum and convolution product.

Thus, if $a = \sum_{I \in A} a_I T^I$ and $b = \sum_{I \in B} b_I T^I$, then $a + b = \sum_{I \in A \cup B} (a_I + b_I) T^I$ and $ab = \sum_{I \in A+B} (\sum_{J+K=I} a_J b_K) T^I$.

The next step is to define the differential polynomials with coefficients in $S[[T]]$. A **differential monomial** (in n variables) is a monomial in the variables $\{x_{i,J} \ : \ 1 \leq i \leq n, J \in \mathbb{N}^m\}$. The **order** of $x_{i,J}$ is $||J||_\infty$, so a differential monomial of order less than or equal to $r \in \mathbb{N}$ is an expression of the form

$$E := \prod_{\substack{1 \leq i \leq n \\ ||J||_\infty \leq r}} x_{i,J}^{m_{i,J}} \tag{1}$$

and we can encode (1) by means of the array $M := (m_{i,J}) \in \mathbb{N}^{n \times (r+1)^m}$, in which case we write $E = E_M$.

Definition: A **differential polynomial** with coefficients in $S[[T]]$ is a finite sum of terms $a_M E_M$ consisting of a differential monomial E_M as in (1) and a coefficient $a_M \in S[[T]] \setminus \{0\}$:

$$P = \sum_i a_{M_i} E_{M_i}. \tag{2}$$

Definition: The semiring of differential polynomials with coefficients in $S[[T]]$ is the set $S[[T]]\{x_1, \ldots, x_n\}$ of all such objects endowed with the usual operations of term-wise addition and convolution product of polynomials. The element P in (2) is **ordinary** if $m = 1$ and **partial** otherwise.

We end up with an extension of semirings $S[[T]] \subset S[[T]]\{x_1, \ldots, x_n\}$. The next step is to endow them with a differential structure.

2.1 Differential semirings

Definition: A map $d : S \longrightarrow S$ is a **derivation** if it is additive and satisfies the Leibniz rule for the product. A **differential semiring** is a couple (S, D) consisting of a semiring S together with a finite family D of pairwise commutative derivations defined on it.

The usual definition of partial derivations in a ring of formal power series with coefficients in a ring can be extended to the context of semirings, namely for $a \in S \setminus \{0\}$, $I = (i_1, \ldots, i_m) \in \mathbb{N}^m$ and $j = 1, \ldots, m$, we set

$$\frac{\partial}{\partial t_j}(aT^I) = \begin{cases} i_j a T^{I-e_j}, & \text{if } i_j \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

This is well-defined since $\mathbb{N}$ acts on S. Note that $\frac{\partial}{\partial t_j}(aT^I bT^J) = aT^I \frac{\partial}{\partial t_j}(bT^J) + bT^J \frac{\partial}{\partial t_j}(aT^I)$, thus $\frac{\partial}{\partial t_j}(\sum_I a_I T^I) = \sum_I \frac{\partial}{\partial t_j}(a_I T^I)$ defines a derivation on $S[[T]]$. The pair $S_m := (S[[T]], D = \{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_m}\})$ is the differential semiring of formal power series in the variables $T = (t_1, \ldots, t_m)$ with coefficients in S.

We can extend each derivation $\frac{\partial}{\partial_{t_i}}$ from $S[[T]]$ to $S[[T]]\{x_1, \ldots, x_n\}$ by setting $\frac{\partial}{\partial_{t_i}} x_{k,J} = x_{k,J+e_i}$. We denote $S_{m,n} = (S[[T]]\{x_1, \ldots, x_n\}, D)$ the resulting differential semiring, and we end up with an extension of differential semirings $S_m \subset S_{m,n}$.

The next definition is the key to interpret these differential polynomials as differential equations with coefficients in S.

Definition: For $(j_1, \ldots, j_m) = J \in \mathbb{N}^m$ we denote by $\Theta_{S_m}(J)$ the map $S_m \to S_m$ induced by $\frac{\partial^{\sum_i j_i}}{\partial t_1^{j_1} \cdots \partial t_m^{j_m}}$.

Let E_M be a differential monomial as in (1). Using the operators $\Theta_{S_m}(J)$, we can define an evaluation map $E_M : S[[T]]^n \longrightarrow S[[T]]$:

$$a = (a_1, \ldots, a_n) \mapsto E_M(a) := \prod_{\substack{1 \leq i \leq n \\ ||J||_\infty \leq r}} (\Theta_{S_m}(J) a_i)^{m_{i,J}}.$$

Thus, given $\sum_i a_{M_i} E_{M_i} = P \in S_{m,n}$, we get an evaluation map $P : S[[T]]^n \longrightarrow S[[T]]$ extending the above map by linearity: $P(a) = \sum_i a_{M_i} E_{M_i}(a)$. Now that we have access to these evaluation maps, the next step is to define when $a \in S[[T]]^n$ should be considered a *solution* of P, which must depend on the value $P(a)$. We will see later that the definition of solution depends on the type of semiring under consideration.

Remark 2.1: *We see that the role of the indices $J \in \mathbb{N}^m$ of the variables $x_{i,J}$ of our differential polynomials is to encode an action $\Theta_{S_m} : \mathbb{N}^m \longrightarrow S_m$ under which e_i acts like $\frac{\partial}{\partial t_i}$. In fact, if $\Theta_{S_{m,n}} : \mathbb{N}^m \longrightarrow S_{m,n}$ denotes* *the action of the module $\mathbb{N}^m$ on $S_{m,n}$ under which e_i acts like $\frac{\partial}{\partial t_i}$, then since $S_m \subset S_{m,n}$ is an extension of differential semirings, we have that differentiation of differential polynomials commutes with evaluation, i.e.*

$$(\Theta_{S_{m,n}}(J)P)(a) = \Theta_{S_m}(J)((P(a)), \quad a \in S[[T]]^n.$$

In Section 2.2 we will turn this into a ring action, to show that an approach via differential modules is also possible.

2.2 An approach via the ring of twisted polynomials

We make a pause to explain how one can adapt the construction of differential polynomials using differential modules over differential semirings. At the same time, this formalism is more connected with more modern presentations of differential algebra (see [9] for example).

Modules over a semiring S (again not necessarily commutative) are pairs $(M, \cdot)$ consisting of a commutative monoid $M = (M, +, 0_M)$ and a scalar multiplication $\cdot : S \times M \longrightarrow M$ that satisfies the usual axioms for modules over rings, plus the condition $s \cdot 0_M = 0_M = 0_R \cdot m$. See [5, Chapter 14].

Likewise, the definition of differential modules over a differential semiring is an adaptation of the usual definition for differential rings. We will focus on the case $S_m = (S[[T]], D)$.

Definition: A **differential module** over S_m is a pair (M, D_M) where M is a module over $S[[T]]$ equipped with additive maps $D_M = \{\delta_1, \ldots, \delta_m\}$ satisfying

$$\delta_i(a \cdot m) = a \cdot \delta_i(m) + \tfrac{\partial a}{\partial t_i} \cdot m, \quad a \in S[[T]],\, m \in M.$$

Definition: The **semiring of twisted polynomials** $S[[T]]\{d_1, \ldots, d_m\}$ is the additive semigroup $\bigoplus_{I \in \mathbb{N}^m} S[[T]] \cdot D^I$, where $D^I = d_1^{i_1} \cdots d_m^{i_m}$, and we impose the following rules for the product:

1. $d_i a = a d_i + \frac{\partial a}{\partial t_i}$, if $a \in S[[T]]$, and
2. $d_i d_j = d_j d_i$ for $i, j = 1, \ldots, m$.

Like in the case of differential rings, the semiring of twisted polynomials $S[[T]]\{d_1, \ldots, d_m\}$ is the object which enables us to identify the category of differential modules over S_m with the category of left $S[[T]]\{d_1, \ldots, d_m\}$-modules, since we obtain an action of $S[[T]]\{d_1, \ldots, d_m\}$ on (M, D_M) under which d_i acts like δ_i. In particular, $S[[T]]\{d_1, \ldots, d_m\}$ acts on $S_{m,n}$ and on its differential sub-semiring S_m with d_i acting like $\frac{\partial}{\partial t_i}$. It is clear that this action restricts to the corresponding actions $\Theta_{S_{m,n}}$ and Θ_{S_m} considered in the previous section.

3 The tropical formalism

In this Section we study the case of tropical differential polynomials and we introduce the notion of formal power series solution for them. In practice, this means considering the constructions from the Section 2 for the case $S = \mathbb{B}$, where $\mathbb{B} = (\{0,1\},+,\times)$ is **boolean semiring** in which $\times$ is the usual product and $a+b=1$ whenever a or b are nonzero.

Most of these concepts were introduced in [3], and they are re-introduced here with a different, more algebraic notation using the language of semiring algebra, in order to represent them as closer to their classical counterparts as we can. The reader can refer to the Example in Section 5.1 to see how these concepts work in practice.

Recall that a semiring S is (additively) **idempotent** if $a+a=a$ for all $a \in S$, and that the category of idempotent semirings is just the category of $\mathbb{B}$-algebras. Thus the structure semiring map $\mathbb{N} \longrightarrow S$ factors through the structure idempotent semiring map $\mathbb{B} \longrightarrow S$ via the (unique) homomorphism $\mathbb{N} \longrightarrow \mathbb{B}$.

A formal power series with coefficients in $\mathbb{B}$ is of the form $a = \sum_{I \in A} a_I T^I$ where $\emptyset \subseteq A \subseteq \mathbb{N}^m$ and $a_I = 1$, so the series a gets identified with the set A and the semiring $\mathbb{B}[[T]] := (\mathbb{B}[[T]],+,\times,0,1)$ consists of the set

$$\mathbb{B}[[T]] = \left\{ a = \sum_{I \in A} T^I \ : \ \emptyset \subseteq A \subseteq \mathbb{N}^m \right\}$$

endowed with the operations $a+b = \sum_{I \in A \cup B} T^I$ and $ab = \sum_{I \in A+B} T^I$, where $a = \sum_{I \in A} T^I$ and $b = \sum_{I \in B} t^I$. Recall that $\mathbb{B}_m = (\mathbb{B}[[T]], D)$ where D is the set of partial derivations; in particular note that for $(j_1,\ldots,j_m) = J \in \mathbb{N}^m$ the map $\Theta_{\mathbb{B}_m}(J) : \mathbb{B}_m \to \mathbb{B}_m$ sends $a = \sum_{I \in A} T^I$ to

$$\Theta_{\mathbb{B}_m}(J)a := \{(i_1,\ldots,i_m) \in A - J \mid i_1,\ldots,i_m \geq 0\}.$$

This is because in this case the action of $\mathbb{N}$ used to define (3) factors through the structure map $\mathbb{B} \longrightarrow S$. Note that the algebraic operations and partial derivation on boolean formal power series are simplified versions of the corresponding operations performed on usual formal power series with coefficients in a ring.

The next step is to use the elements of the semiring $\mathbb{B}_{m,n} = (\mathbb{B}[[T]]\{x_1,\ldots,x_n\}, D)$ of differential polynomials with coefficients in $\mathbb{B}[[T]]$ to define tropical differential equations. If we follow the same route as for the case of rings, i.e. considering the equation $P=0$ for $\sum_i a_{M_i} E_{M_i} = P \in \mathbb{B}_{m,n}$ and then trying to find the solutions $a \in \mathbb{B}[[T]]^n$ of this equation by the condition $P(a) = \sum_i a_{M_i} E_{M_i}(a) = 0$, then this happens if and only if $E_{M_i}(a) = 0$ for all i, since the semiring $\mathbb{B}[[T]]$ is idempotent and the equation $a+b=0$ has no nontrivial solutions.

In order to do this, we need to introduce some auxiliary concepts.

Definition: Let $\sum_{I\in A} T^I = a \in \mathbb{B}[[T]]$. The **Newton polygon** $\mathrm{New}(a) \subseteq \mathbb{R}^m_{\geq 0}$ of a is the convex hull of the set $\{I + J \ : \ I \in A, \ J \in \mathbb{N}^m\}$. We say that $a' = \sum_{I\in A'} T^I$ with $A' \subseteq A$ is a **spanning element** if $\mathrm{New}(a') = \mathrm{New}(a)$.

Theorem 3.1 (See [3]). *Any $a \in \mathbb{B}[[T]]$ has a unique minimal spanning element.*

We denote this unique minimal spanning element a' of a as $V(a) = \sum_{I\in A'} T^I$ and we call it the **vertex polynomial** of a. It is a polynomial since A' is finite. Note that in the ordinary case, $V(\sum_{i\in A} t^i)$ is the monomial $t^{\min(A)}$.

Remark 3.2: *The previous result can be considered as an analogue for Newton polygons of Dickson's lemma for monomial ideals : if $B(a)$ is the minimal basis for the monomial ideal generated by a, we have $V(a) \subseteq B(a)$, but the inclusion may be strict in general.*

Now we are ready to give the notion of solution of a tropical differential polynomial. This definition is slightly different from the original presented in [3], but it is equivalent and it has the advantage of being less technical.

Definition: We say that $a \in \mathbb{B}[[T]]^n$ is a **solution** of $\sum_i a_{M_i} E_{M_i} = P \in \mathbb{B}_{m,n}$ if for every monomial T^I appearing in $V(P(a))$, there are at least two distinct terms $a_{M_k} E_{M_k}$ and $a_{M_\ell} E_{M_\ell}$ of P such that T^I appears in both $V(a_{M_k} E_{M_k}(a))$ and $V(a_{M_\ell} E_{M_\ell}(a))$.

We denote by $\mathrm{Sol}(P) \subset \mathbb{B}[[T]]^n$ the set of solutions of P, and if $\Sigma \subset \mathbb{B}_{m,n}$ is a system of tropical differential polynomials, then we denote by $\mathrm{Sol}(\Sigma)$ the set of common solutions of all the elements $P \in \Sigma$.

4 The tropical fundamental theorem for differential algebraic geometry

In this Section we introduce the valuation maps which complete the basic tropicalization scheme $v : R \longrightarrow S$ for the case of differential algebraic geometry. Here R will be $\mathbb{K}_{m,n}$ for $\mathbb{K}$ an uncountable, algebraically closed field of characteristic zero, S will be $\mathbb{B}_{m,n}$, discussed in the previous Section, and v will be a support map. After that, we will formulate the tropical fundamental theorem for this case, and we will discuss how it may help us to get some combinatorial information about the set of solutions of systems of differential equations.

If $\Sigma \subseteq \mathbb{K}_{m,n} = (\mathbb{K}[[T]]\{x_1, \ldots, x_n\}, D)$ is a system of differential polynomials, we are interested in computing the set of n-tuples $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{K}[[T]]^n$ such that $P(\varphi) = 0$ for all $P \in \Sigma$. The set of all such n-tuples form the **set of solutions** of Σ, and we denote it by $\mathrm{Sol}(\Sigma)$.

Remark 4.1: *Perhaps the first examples of such fields that come to one's mind are the complex numbers $\mathbb{C}$ and the algebraic closure of the field of* *p-adic numbers $\mathbb{Q}_p^{alg}$. Moreover, these fields come equipped with natural absolute values, so one could ask when the elements of Sol(Σ) consists of tuples of analytic functions, archimedean for the complex case, or non-archimedean for the p-adic case.*

In principle computing $\mathrm{Sol}(\Sigma)$ may be very hard, but we can ask instead about its **set of supports** $\mathrm{Supp}(\mathrm{Sol}(\Sigma))$: if $\varphi = \sum_{I\in A}\alpha_I T^I \in \mathbb{K}[[T]]$, we denote by $\mathrm{Supp}(\varphi) \in \mathbb{B}[[T]]$ the boolean formal power series $\sum_{I\in A} T^I$. The resulting map $\mathrm{Supp} : \mathbb{K}[[T]] \longrightarrow \mathbb{B}[[T]]$ can be extended to a map $\mathrm{Supp} : \mathbb{K}[[T]]\{x_1,\ldots,x_n\} \longrightarrow \mathbb{B}[[T]]\{x_1,\ldots,x_n\}$ that sends $P = \sum_{i=1}^d \alpha_{M_i}E_{M_i}$ to $\mathrm{Supp}(P) = \sum_{i=1}^d \mathrm{Supp}(\alpha_{M_i})E_{M_i}$.

Finally we define the map $\mathrm{Supp} : \mathbb{K}[[T]]^n \longrightarrow \mathbb{B}[[T]]^n$ as the coordinate-wise application of Supp.

We now introduce a necessary technical concept.

Definition: An ideal of the differential ring $\mathbb{K}_{m,n}$ is **differential** if it is closed under the action of D.

If $[\Sigma] \subset \mathbb{K}_{m,n}$ denotes the differential ideal generated by the system $\Sigma \subseteq \mathbb{K}_{m,n}$, then it follows from Remark 2.1 that both $[\Sigma]$ and Σ share the same set of solutions.

Now we have all the ingredients to state the tropical fundamental theorem for differential algebraic geometry, which represents in a unified form the content of [1] and [3]. Let $G \subset \mathbb{K}_{m,n}$ be a differential ideal. We denote by

$$\mathrm{Supp}(G) = \{p \in \mathbb{B}_{m,n} \ : \ p = \mathrm{Supp}(P) \text{ for some } P \in G\}$$

and by

$$\mathrm{Supp}(\mathrm{Sol}(G)) = \{a = (a_1,\ldots,a_n) \in \mathbb{B}[[T]]^n : a = \mathrm{Supp}(\varphi) \text{ for some } \varphi \in \mathrm{Sol}(G)\}.$$

Then

$$\mathrm{Supp}(\mathrm{Sol}(G)) = \mathrm{Sol}(\mathrm{Supp}(G)). \tag{4}$$

To recapitulate, (4) characterizes the set of all the $\mathbb{B}$-formal power series solutions of the system of tropical differential equations $\mathrm{Supp}(G)$ as the set of supports of all the $\mathbb{K}$-formal power series solutions $\mathrm{Sol}(G)$ of the original system G. Equivalently, it says that any $\mathbb{B}$-formal power series solution of the system of tropical differential equations $\mathrm{Supp}(G)$ can be lifted to a $\mathbb{K}$-formal power series solution of G.

To end this section, we record some important properties of the map $\mathrm{Supp} : \mathbb{K}_{m,n} \longrightarrow \mathbb{B}_{m,n}$, which have to be compared with those of [3, Lemma 4.3]. It is worth noticing that this map has properties which are very close to those of a valuation. In particular, it may be useful for a future study of tropical differential algebra from the perspective of differential semirings.

We now record the definition of valuation from [4]. We also recall that in an (additively) idempotent semiring, the expression $a \leq b$ means $a + b = b$.

Definition: A **valuation** is a map $v : R \longrightarrow S$ where R is a ring, S an idempotent semiring, and $v : R \longrightarrow S$ a map that satisfies

1. $v(0) = 0$ and $v(\pm 1) = 1$,
2. $v(ab) = v(a)v(b)$,
3. $v(a + b) \leq v(a) + v(b)$

The valuation v is **non-degenerate** if $v(a) = 0$ implies $a = 0$, and **sub-multiplicative** if $v(ab) \leq v(a)v(b)$.

Proposition 4.2. *The map Supp* : $\mathbb{K}_{m,n} \longrightarrow \mathbb{B}_{m,n}$ *is a non-degenerate, sub-multiplicative valuation that satisfies* $Supp \circ \Theta_{\mathbb{K}_{m,n}} \leq \Theta_{\mathbb{B}_{m,n}} \circ Supp$.

Proof. The conditions $\mathrm{Supp}(P) = 0$ if and only if $P = 0$, and $\mathrm{Supp}(\pm 1) = 1$ are clear.

We first consider $a, b \in \mathbb{K}[[T]]$ and write $\alpha, \beta \in \mathbb{B}[[T]]$ for their corresponding supports. We define the elements $\mathrm{tad}(a, b), \mathrm{tmd}(a, b) \in \mathbb{B}[[T]]$ as

$$\mathrm{tad}(a,b) = \sum_{\substack{I \in \alpha+\beta \\ a_I + b_I = 0}} T^I, \quad \mathrm{tmd}(a,b) = \sum_{\substack{I \in \alpha\beta \\ \sum_{I=K+L} a_K b_L = 0}} T^I. \tag{5}$$

Then these are the unique elements in $\mathbb{B}[[T]]$ satisfying

1. $\mathrm{Supp}(a + b) \cap \mathrm{tad}(a, b) = \emptyset$ and $\mathrm{Supp}(a + b) + \mathrm{tad}(a, b) = \alpha + \beta$,
2. $\mathrm{Supp}(ab) \cap \mathrm{tmd}(a, b) = \emptyset$ and $\mathrm{Supp}(ab) + \mathrm{tmd}(a, b) = \alpha\beta$

We call $\mathrm{tad}(a, b)$ the **tropical additive difference** and $\mathrm{tmd}(a, b)$ the **tropical multiplicative difference** of a and b, respectively. The associativity of the operations in $\mathbb{B}[[T]]$ guarantees that the definitions from (5) can be extended to any finite amount of factors.

Now we write $P = \sum_{M \in \Lambda(P)} a_M E_M$ and $Q = \sum_{M \in \Lambda(Q)} b_M E_M$. We have $P + Q = \sum_{M \in \Lambda(P) \cup \Lambda(Q)} (a_M + b_M) E_M$, and the condition $\mathrm{Supp}(P + Q) \leq \mathrm{Supp}(P) + \mathrm{Supp}(Q)$ follows from $\mathrm{Supp}(a_M + b_M) + \mathrm{tad}(a_M, b_M) = \mathrm{Supp}(a_M) + \mathrm{Supp}(b_M)$ for every M.

The proof of the condition $\mathrm{Supp}(PQ) \leq \mathrm{Supp}(P)\mathrm{Supp}(Q)$ is similar : we have $PQ = \sum_{O \in \Lambda(P) + \Lambda(Q)} c_O E_O$, where $c_O = \sum_{M+N=O} a_M b_N$, so we need to show that for any $O \in \Lambda(P) + \Lambda(Q)$, there exists $\gamma_O \in \mathbb{B}_m$ such that $\mathrm{Supp}(c_O) + \gamma_O = \sum_{M+N=O} \mathrm{Supp}(a_M)\mathrm{Supp}(b_N)$.

This follows from

1. $\mathrm{Supp}(c_O) + \mathrm{tad}_{M+N=O}(a_M b_N) = \sum_{M+N=O} \mathrm{Supp}(a_M b_N)$, and
2. $\mathrm{Supp}(a_M b_N) + \mathrm{tmd}(a_M, b_N) = \mathrm{Supp}(a_M)\mathrm{Supp}(b_N)$.

The final condition follows from the following observation: if $P = \sum_M \alpha_M E_M$ and $J \in \mathbb{N}^m$, then $\Theta_{\mathbb{K}_{m,n}}(J)P = \sum_M (\Theta_{\mathbb{K}_{m,n}}(J)\alpha_M)E_M + \sum_M \alpha_M(\Theta_{\mathbb{K}_{m,n}}(J)E_M)$, and by the point 2 we have:

$$\mathrm{Supp}(\Theta_{\mathbb{K}_{m,n}}(J)P) \leq \sum_M \mathrm{Supp}(\Theta_{\mathbb{K}_{m,n}}(J)\alpha_M)E_M + \sum_M \mathrm{Supp}(\alpha_M)(\Theta_{\mathbb{K}_{m,n}}(J)E_M).$$

The result follows from $(\Theta_{\mathbb{K}_{m,n}}(J)E_M) = (\Theta_{\mathbb{B}_{m,n}}(J)E_M)$ and $\mathrm{Supp} \circ \Theta_{\mathbb{K}_{m,n}}(J)(a) = \Theta_{\mathbb{B}_{m,n}}(J) \circ \mathrm{Supp}(a)$ for every $a \in \mathbb{K}_m$. ■

5 Computational aspects

In this section we discuss two enhancements which can be made to the previous setting towards effective computations.

The first one is the following. As stated, the fundamental theorem (4) is about differential ideals $G \subset \mathbb{K}_{m,n}$, which are infinite systems of differential polynomials. The Ritt-Raudenbush Basis Theorem (see [2]) says that we can always find $P_1, \ldots, P_s \in I$ such that $\mathrm{Sol}(G) = \mathrm{Sol}(P_1, \ldots, P_s)$. However, it is not true in general that

$$\mathrm{Supp}(\mathrm{Sol}(G)) = \mathrm{Sol}(\mathrm{Supp}(P_1), \ldots, \mathrm{Supp}(P_s)).$$

It would be desirable to have a notion of *tropical basis* for $\mathrm{Supp}(G)$, which would be a (preferably smaller) subsystem $\Phi \subset G$ such that $\mathrm{Sol}(\mathrm{Supp}(\Phi)) = \mathrm{Sol}(\mathrm{Supp}(G))$. In this respect, notions of Groebner basis exist for both the tropical and the differential settings, and the concept of tropical differential Groebner basis for the ordinary case was introduced in [7]. It can be considered as the adaptation of the tropical Groebner bases to the differential setting, and at the same time as the adaptation of the differential Groebner bases to the tropical setting.

The second one is about considering vertex polynomials, which are always finite (in contrast to the case of the $\mathbb{B}$-formal power series). The map $V : \mathbb{B}[[T]] \longrightarrow \mathbb{B}[[T]]$ sending a formal power series a to its vertex polynomial $V(a)$ satisfies $V^2 = \mathrm{Id}$, so we can express the set $V\mathbb{B}[T]$ of **vertex polynomials** as

$$V\mathbb{B}[T] = \{a \in \mathbb{B}[T] \;:\; V(a) = a\}.$$

We can define new operations on $V\mathbb{B}[T]$ as follows : $a \oplus b := V(a+b)$ and $a \odot b = V(ab)$. All the technical framework can be summarized in the following result.

Theorem 5.1 ([3]). *The tuple $V\mathbb{B}[T] = (V\mathbb{B}[T], \oplus, \odot, 1, 0)$ is an idempotent semiring and we have a commutative diagram*

$$\begin{array}{ccc} (\mathbb{K}[[T]], D) & \xrightarrow{Supp} & (\mathbb{B}[[T]], D) \\ & \searrow^{trop} & \downarrow V \\ & & V\mathbb{B}[T] \end{array} \qquad (6)$$

where

1. *the map Supp is a non-degenerate, sub-multiplicative valuation which commutes with D,*
2. *the map $V : \mathbb{B}[[T]] \longrightarrow V\mathbb{B}[T]$ is a homomorphism of semirings.*
3. *the map trop : $\mathbb{K}[[T]] \longrightarrow V\mathbb{B}[T]$ is a non-degenerate, surjetive valuation*

Remark 5.2: *In particular, since $p(a) = \sum_M a_M E_M(a)$, we have*

$$V(p(a)) = V\left(\sum_M a_M E_M(a)\right) = \bigoplus_M V(a_M E_M(a)) = \bigoplus_M V(a_M) \odot V(E_M(a)).$$

Thus the definition of solution of p can be formulated either with respect to the vertex polynomials $V(a_M E_M(a)$ (as we did) or with respect to the vertex polynomials $V(a_M) \odot V(E_M(a))$ (as in the original definition from [3]).

Finally the solutions $\varphi \in \mathbb{K}[[T]]^n$ have been considered as formal power series, but it would be interesting to know whether they represent analytic functions $\Omega \longrightarrow \mathbb{K}$ defined in a common domain $\Omega \subset \mathbb{A}_{\mathbb{K}}^{m,\mathrm{an}}$. We have two main cases:

1. Archimedean. The only possibility is $\mathbb{K} = \mathbb{C}$ with the Euclidean topology, thus $\mathbb{A}_{\mathbb{K}}^{m,\mathrm{an}} = (\mathbb{C}^m, \mathcal{H})$, where $\mathcal{H}$ is the sheaf of holomorphic functions.
2. Non-archimedean. As we mentioned before, one of the most active cases is $\mathbb{K} = \mathbb{Q}_p^{\mathrm{alg}}$, and one could consider the Berkovich space $\mathbb{A}_{\mathbb{C}_p}^{m,\mathrm{an}}$ associated to the completion $\mathbb{C}_p$ of $\mathbb{K}$.

5.1 An example

It is difficult to construct illustrative examples, so we will content ourselves by presenting the way in which tropical differential algebra works concretely by evaluating a tropical differential polynomial $p \in \mathbb{B}[[t,u]]\{x,y\}$ at an element.

We denote an element a of $\mathbb{B}_2 = \mathbb{B}[[t,u]]$ as $a = \sum_{(i,j)\in A} t^i u^j$ for some $A \subseteq \mathbb{N}^2$. A differential monomial of order less than $r = 1$ is of the form

$$E_M = x_{(0,0)}^{m_{1,(0,0)}} x_{(0,1)}^{m_{1,(0,1)}} x_{(1,0)}^{m_{1,(1,0)}} x_{(1,1)}^{m_{1,(1,1)}} y_{(0,0)}^{m_{2,(0,0)}} y_{(0,1)}^{m_{2,(0,1)}} y_{(1,0)}^{m_{2,(1,0)}} y_{(1,1)}^{m_{2,(1,1)}}$$

where $M = (m_{i,J})_{i,J} \in \mathbb{N}^{2\times 4}$. So an example of a polynomial of order at most 1 would be

$$p = (t+u)x_{(0,0)}y_{(1,1)}^3 + (1+t^2u^2)x_{(1,0)}x_{(0,1)} + ty_{(1,0)}^2.$$

The polynomial p has three terms, namely $a_{M_1}E_{M_1} = (t+u)x_{(0,0)}y_{(1,1)}^3$, $a_{M_2}E_{M_2} = (1+t^2u^2)x_{(1,0)}x_{(0,1)}$ and $a_{M_3}E_{M_3} = ty_{(1,0)}^2$. If $a = (a_1, a_2) = (t + t^2 + tu + u^3, 1 + u + t^2u + t^3u^2)$, then

- $a_{M_1}E_{M_1}(a) = (t+u)a_1(\Theta_{\mathbb{B}}(1,1)a_2)^3 = (t+u)(t+t^2+tu+u^3)(t+t^2u)^3$,
- $a_{M_2}E_{M_2}(a) = (1+t^2u^2)(\Theta_{\mathbb{B}}(1,0)a_1)(\Theta_{\mathbb{B}}(0,1)a_1) = (1+t^2u^2)(1+u+t)(t+u^2)$
- $a_{M_3}E_{M_3}(a) = t(\Theta_{\mathbb{B}}(1,0)a_2)^2 = t(tu+t^2u^2)^2 = t^3u^2(1+tu+t^2u^2)$.

The evaluation $p(a)$ is the union of the three formal series $a_{M_i}E_{M_i}(a)$, $i = 1, 2, 3$; they are diplayed in Figure 1.

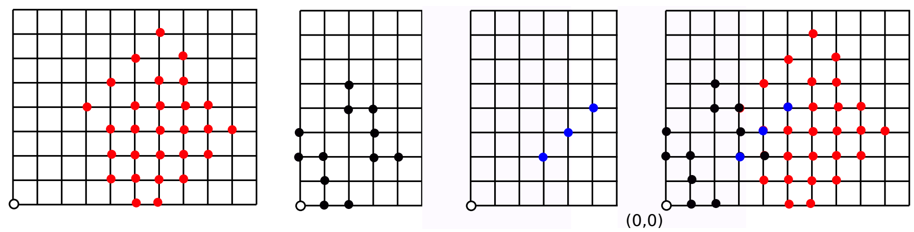

Figure 1: Values of the terms $a_{M_i}E_{M_i}(a)$ for $i = 1, 2, 3$, and $p(a)$.

On the other hand, if we use vertex polynomials, we get

- $V(a_{M_1}) \odot V(E_{M_1}(a)) = V((t+u)(t^4+t^3u^3)) = t^5+t^4u+t^3u^4$,
- $V(a_{M_2}) \odot V(E_{M_2}(a)) = V(1(u^2+t)) = t+u^2$,
- $V(a_{M_3}) \odot V(E_{M_3}(a)) = V(t(t^2u^2)) = t^3u^2$.

Finally, note that $V(p(a)) = V(\sum_M V(a_M) \odot V(E_M(a))) = t+u^2$, and since these monomials only appear in $a_{M_2}E_{M_2}(a)$, we deduce that a is not a solution of p.

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