Using tropical differential equations

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Abstract

We explain from scratch the formalism behind tropical differential algebraic geometry and how it may be used to extract combinatorial information from the set of power series solutions of systems of classical differential equations.

1 Introduction

Finding solutions for systems of differential equations is in general a very difficult task. The recently introduced branch of tropical differential algebraic geometry yields an algebraic framework which extracts combinatorial information from the set of all the formal power series solutions of these systems. Both the concept and insight of the method was introduced in [4], and it was proved recently in [3] (which is a generalization of the ordinary case proved in [1]).

The purpose of this note is twofold. The first one is to introduce the formalism behind tropical differential algebraic geometry. We do this in Sections 3 and 4, without assuming any kind of familiarity with the subject. The second is to explain how it may be used concretely to tackle problems in classical differential algebraic geometry, which we do in Sections 4 and 5. We also use this note as an opportunity to present a unified theory of differential equations with coefficients in a commutative semiring in Section 2.

Conventions. Every algebraic structure to be considered here will be commutative .We will denote by $\mathbb{N}$ the semiring of natural numbers (which includes 0) and we also fix from now two non-zero natural numbers m and n.

The basis of the free $\mathbb{N}$-module $\mathbb{N}^m$ will be denoted by $\{e_1, \ldots, e_m\}$. If $J = (j_1, \ldots, j_m), I = (i_1, \ldots, i_m) \in \mathbb{N}^m$, we denote by $||J||_\infty := \max\{j_1, \ldots, j_m\} = \max(J)$ and by $I - J = (i_1 - j_1, \ldots, i_m - j_m)$. If $A \subseteq \mathbb{N}^m$, we write $A - J = \{I - J \,:\, I \in A\}$.

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2 Differential equations with coefficients in a semiring

In this Section we introduce the concept of differential equation (or differential polynomial) with coefficients in a commutative semiring with unit $S = (S, +, \times, 0, 1)$. Recall that these objects form a category which is just the category of $\mathbb{N}$-algebras.

We consider the tuple $T = (t_1, \dots, t_m)$ and for $I = (i_1, \dots, i_m) \in \mathbb{N}^m$, we denote by T^I the formal monomial $t_1^{i_1} \cdots t_m^{i_m}$.

Definition: The semiring $S_m := (S[[T]], +, \times, 0, 1)$ of **formal power series** with coefficients in S in the variables $t_1, \dots, t_m$ consists of the set

$$S[[T]] := \left\{ a = \sum_{I \in A} a_I T^I \ : \ \emptyset \subseteq A \subseteq \mathbb{N}^m, \quad a_I \in S \setminus \{0\} \right\}$$

endowed with the operations $a + b = \sum_{I \in A \cup B} c_I T^I$ and $ab = \sum_{I \in A+B} d_I T^I$, where $a = \sum_{I \in A} a_I T^I$, $b = \sum_{I \in B} b_I T^I$, $c_I = a_I + b_I$ and $d_I = \sum_{J+K=I} a_J b_K$.

We now define a set $D = \{\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_m}\}$ of pairwise commutative derivations on the semiring S_m. For $a \in S \setminus \{0\}$, $I = (i_1, \dots, i_m) \in \mathbb{N}^m$ and $j = 1, \dots, m$, set

$$\tfrac{\partial}{\partial t_j}(aT^I) = \begin{cases} i_j a T^{I - e_j}, & \text{if } i_j \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\frac{\partial}{\partial t_j}(aT^I bT^J) = aT^I \frac{\partial}{\partial t_j}(bT^J) + bT^J \frac{\partial}{\partial t_j}(aT^I)$, thus $\frac{\partial}{\partial t_j}(\sum_I a_I T^I) = \sum_I \frac{\partial}{\partial t_j}(a_I T^I)$, defines a derivation on S_m.

The pair (S_m, D) is an example of a differential semiring (a pair of a semiring and a finite family of pairwise commutative derivations defined on it). We call it the (standard) differential semiring of formal power series in the variables $t_1, \dots, t_m$ with coefficients in S.

The next step is to introduce the semiring $S_{m,n}$ of differential polynomials in the variables $x_1, \dots, x_n$ with coefficients in S_m. A **differential monomial** is a monomial in the variables $\{x_{i,J} \ : \ 1 \leq i \leq n, J \in \mathbb{N}^m\}$. The **order** of $x_{i,J}$ is $||J||_\infty$, so a differential monomial of order less than or equal to $r \in \mathbb{N}$ is an expression of the form

$$E_M := \prod_{\substack{1 \leq i \leq n \\ ||J||_\infty \leq r}} x_{i,J}^{m_{i,J}} \tag{1}$$

where $M = (m_{i,J}) \in \mathbb{N}^{n \times (r+1)^m}$.

A **differential polynomial** with coefficients in S_m is a finite sum of terms $a_M E_M$ consisting of a differential monomial E_M as in (1) and a coefficient $a_M \in S_m \setminus \{0\}$:

$$p = \sum_{i=1}^{d} a_{M_i} E_{M_i}. \tag{2}$$

We denote by $S_m\{x_1, \ldots, x_n\}$ the set of all differential polynomials with coefficients in S_m. If we endow this set with the usual operations of termwise addition and convolution product of polynomials, then we get the semiring $S_{m,n} = (S_m\{x_1, \ldots, x_n\}, +, \times, 0, 1)$ of differential polynomials with coefficients in S_m. An element $p \in S_{m,n}$ is called **ordinary** if $m = 1$ and **partial** otherwise.

The next definition is the key to interpret these differential polynomials as differential equations with coefficients in S.

Definition: For $(j_1, \ldots, j_m) = J \in \mathbb{N}^m$ we denote $\Theta_S(J) : S_m \to S_m$ the map that sends a to $\Theta_S(J)a = \frac{\partial^{\sum_i j_i} a}{\partial t_1^{j_1} \cdots \partial t_m^{j_m}}$

Let E_M be a differential monomial as in (1). Using the operators $\Theta_S(J)$, we can define an evaluation map $E_M : S_m^n \longrightarrow S_m$ described as

$$a = (a_1, \ldots, a_n) \mapsto E_M(a) := \prod_{\substack{1 \leq i \leq n \\ ||J||_\infty \leq r}} (\Theta_S(J) a_i)^{m_{i,J}}.$$

Thus, given $\sum_i a_{M_i} E_{M_i} = p \in S_{m,n}$, we get an evaluation map $p : S_m^n \longrightarrow S_m$ extending the above map by linearity: $p(a) = \sum_i a_{M_i} E_{M_i}(a)$. Thus the indices $J = (j_1, \ldots, j_m) \in \mathbb{N}^m$ in the variables $x_{i,J}$ of our polynomials are encoding the derivations $\Theta_S(J) = \frac{\partial^{\sum_i j_i}}{\partial t_1^{j_1} \cdots \partial t_m^{j_m}}$.

The next step is to define when $a \in S_m^n$ is a solution of the differential equation $p \in S_{m,n}$, for which we will use the evaluation maps $a \mapsto p(a)$. However, this definition can not be done in general since it depends on the type of semiring under consideration.

3 The tropical formalism

In this Section we study the case of tropical differential equations and we introduce the notion of formal power series solution for them. In practice, this means setting $S = \mathbb{B}$, where $\mathbb{B} = (\{0, 1\}, +, \times)$ is **boolean semiring** in which $\times$ is the usual product and $a + b = 1$ whenever a or b are nonzero. We make a special effort in representing the objects that make up the tropical setting as closer to their classical counterparts as we can. The reader can refer to the Example in Section 5.1 to see how these concepts work in practice.

Then the semiring $\mathbb{B}_m := (\mathbb{B}[[T]], +, \times, 0, 1)$ of **boolean formal power series** in the variables $t_1, \ldots, t_m$ consists of the set

$$\mathbb{B}[[T]] = \left\{ a = \sum_{I \in A} T^I \ : \ \emptyset \subseteq A \subseteq \mathbb{N}^m \right\}$$

endowed with the operations $a + b = \sum_{I \in A \cup B} T^I$ and $ab = \sum_{I \in A+B} T^I$, where $a = \sum_{I \in A} T^I$ and $b = \sum_{I \in B} t^I$. Also for $(j_1, \ldots, j_m) = J \in \mathbb{N}^m$ the map $\Theta_{\mathbb{B}}(J) : \mathbb{B}_m \to \mathbb{B}_m$ sends $a = \sum_{I \in A} t^I$ to

$$\Theta_{\mathbb{B}}(J)a := \{(i_1, \ldots, i_m) \in (A - J) \mid i_1, \ldots, i_m \geq 0\}.$$

Note that the algebraic operations and partial derivation on boolean formal power series are simplified versions of the corresponding operations performed on usual formal power series with coefficients in a ring.

We now want to express the condition for which $a \in \mathbb{B}_m^n$ is a *solution* of $p \in \mathbb{B}_{m,n}$, based on the evaluation $p(a) = \sum_{I \in S} t^I \in \mathbb{B}_m$. Note that this is not immediate, since the semiring $\mathbb{B}_m$ is idempotent, and not totally ordered (with respect to the inclusion of subsets of $\mathbb{N}^m$).

Definition: Let $\sum_{I \in A} t^I = a \in \mathbb{B}_m$. The **Newton polygon** $\mathrm{New}(a) \subseteq \mathbb{R}_{\geq 0}^m$ of a is the convex hull of the set $a + w$, where $w := \sum_{I \in \mathbb{N}^m} t^I$. We say that $a' = \sum_{I \in A'} t^I$ with $A' \subseteq A$ is a **spanning element** if $\mathrm{New}(a') = \mathrm{New}(a)$.

Theorem 3.1 ([3]). *Any $a \in \mathbb{B}_m$ has a unique minimal spanning element.*

We denote this unique minimal spanning element a' of a as $V(a) = \sum_{I \in A'} t^I$ and we call it the **vertex polynomial** of a. It is a polynomial since A' is finite.

Remark 3.2: *The previous result can be considered as an analogue for Newton polygons of Dickson's lemma for monomial ideals : if $B(a)$ is the minimal basis for the monomial ideal generated by a, we have $V(a) \subseteq B(a)$, but the inclusion may be strict in general.*

Now we are ready to give the notion of solution of a tropical differential polynomial.

Definition: We say that $a \in \mathbb{B}_m^n$ is a **solution** of $\sum_i a_{M_i} E_{M_i} = p \in \mathbb{B}_{m,n}$ if for every monomial t^I in $V(p(a))$, there are at least two distinct terms $a_{M_k} E_{M_k}$ and $a_{M_\ell} E_{M_\ell}$ such that the monomial t^I appears in both $a_{M_k} E_{M_k}(a)$ and $a_{M_\ell} E_{M_\ell}(a)$.

We denote by $\mathrm{Sol}(p) \subset \mathbb{B}_m^n$ the set of solutions of p, and if $T \subset \mathbb{B}_{m,n}$ is a system of tropical differential polynomials, then we denote by $\mathrm{Sol}(T)$ the set of common solutions of all the elements $p \in T$.

4 The fundamental theorem

In this Section we now discuss how the tropical formalism can help us to get some combinatorial information about the set of solutions of systems of classical differential equations. In practice this means setting $S = \mathbb{C}$ (or in our favourite uncountable algebraically closed field of characteristic zero).

Note that we can extend the action of the set of derivations $D = \{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_m}\}$ from $\mathbb{C}_m$ to $\mathbb{C}_{m,n}$ by defining $\frac{\partial}{\partial_{t_i}} \cdot x_{k,J} = \cdot x_{k,J+e_i}$ for $i = 1, \ldots, m$. In this way we get a new differential ring $(S_{m,n}, D)$, and we say that an ideal of $S_{m,n}$ is **differential** if it is closed under the action of D.

Suppose that we have a system $\Sigma \subseteq \mathbb{C}_{m,n}$. We denote by $[\Sigma] \subset \mathbb{C}_{m,n}$ the differential ideal generated by Σ, we are interested in computing the set of n-tuples of formal power series $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{C}_m^n$ such that $P(\varphi) = 0$ for all $P \in [\Sigma]$. The set of all such n-tuples is what we call the **set of solutions** of $[\Sigma]$, and we denote it by $\mathrm{Sol}([\Sigma])$.

In principle computing $\mathrm{Sol}([\Sigma])$ may be very hard, but we can ask instead about its **set of supports** $\mathrm{Supp}(\mathrm{Sol}([\Sigma]))$: if $\varphi = \sum_{I \in A} \alpha_I t^I \in \mathbb{C}_m$, we denote by $\mathrm{Supp}(\varphi) \in \mathbb{B}_m$ the boolean formal power series $\sum_{I \in A} t^I$. Using the resulting map $\mathrm{Supp} : \mathbb{C}_m \longrightarrow \mathbb{B}_m$ we see that $\Theta_{\mathbb{B}}(J)$ is the combinatorial shadow of the usual operator $\Theta_{\mathbb{C}}(J)$ since

$$\Theta_{\mathbb{B}}(J) \circ \mathrm{Supp} = \mathrm{Supp} \circ \Theta_{\mathbb{C}}(J).$$

We use the map $\mathrm{Supp} : \mathbb{C}_m \longrightarrow \mathbb{B}_m$ to define two other maps $\mathrm{Supp} : \mathbb{C}_m^n \longrightarrow \mathbb{B}_m^n$ and $\mathrm{Supp} : R_{m,n} \longrightarrow \mathbb{B}_{m,n}$. The first one is the coordinate-wise application of Supp, and the second one sends $P = \sum_{i=1}^d \alpha_{M_i} E_{M_i}$ to $\mathrm{Supp}(P) = \sum_{i=1}^d \mathrm{Supp}(\alpha_{M_i}) E_{M_i}$.

We now have all the ingredients to state the *Fundamental theorems of tropical differential algebraic geometry*, which represent in a unified form the content of [1] and [3]. Let $I \subset \mathbb{C}_{m,n}$ be a differential ideal. We denote by

$$\mathrm{Supp}(I) = \{p \in \mathbb{B}_{m,n} \ : \ p = \mathrm{Supp}(P) \text{ for some } P \in I\}$$

and by

$$\mathrm{Supp}(\mathrm{Sol}(I)) = \{a = (a_1, \ldots, a_n) \in \mathbb{B}_m^n \ : \ a = \mathrm{Supp}(\varphi) \text{ for some } \varphi \in \mathrm{Sol}(I)\}.$$

Then

$$\mathrm{Supp}(\mathrm{Sol}(I)) = \mathrm{Sol}(\mathrm{Supp}(I)). \tag{3}$$

To recapitulate, (3) says that the set of supports of all the formal power series solutions $\mathrm{Sol}(I)$ of the system of classical differential equations I arises precisely as the set of all the boolean formal power solutions of the system of tropical differential equations $\mathrm{Supp}(I)$.

5 Computational aspects

In this section we discuss two enhancements which can be made to the previous setting towards effective computations.

The first one is the following. As stated, the fundamental theorem (3) is about classical differential ideals $I \subset \mathbb{C}_{m,n}$, which are infinite systems of differential polynomials. The Ritt-Raudenbush Basis Theorem says that we can always find $P_1, \ldots, P_s \in I$ such that $\mathrm{Sol}(I) = \mathrm{Sol}(P_1, \ldots, P_s)$. For more details on this, see [2]. However, it is not true in general that

$$\mathrm{Supp}(\mathrm{Sol}(G)) = \mathrm{Sol}(\mathrm{Supp}(P_1), \ldots, \mathrm{Supp}(P_s)). \tag{4}$$

In this respect, one can use the tropical differential bases and tropical differential Gröbner bases introduced in [5] for the ordinary case.

The second one is about considering vertex polynomials, which are always finite (contrary to formal boolean power series). The map $V : \mathbb{B}_m \longrightarrow \mathbb{B}_m$ sending a formal power series a to its vertex polynomial $V(a)$ satisfies $V^2 = \mathrm{Id}$, so we can express the set $\mathbb{V}_m$ of **vertex polynomials** as

$$\mathbb{V}_m = \{a \in \mathbb{B}_m \ : \ V(a) = a\}.$$

We can define new operations on $\mathbb{V}_m$ as follows : $a \oplus b := V(a+b)$ and $a \odot b = V(ab)$.

Theorem 5.1 ([3]). *The tuple* $\mathbb{V}_m = (\mathbb{V}_m, \oplus, \odot, 1, 0)$ *is a semiring and* $V : \mathbb{B}_m \longrightarrow \mathbb{V}_m$ *is a homomorphism of semirings.*

In particular, since $p(a) = \sum_M a_M E_M(a)$, we have

$$V(p(a)) = V\left(\sum_M a_M E_M(a)\right) = \bigoplus_M V(a_M) \odot V(E_M(a)) \in \mathbb{V}_m.$$

This yields an alternative formulation for the concept of solution of a tropical differential equation stated in terms of boolean polynomials.

Definition: We say that $a \in \mathbb{B}_m^n$ is a **solution** of $\sum_i a_{M_i} E_{M_i} = p \in \mathbb{B}_{m,n}$ if for every monomial t^I in $V(p(a))$, there are at least two distinct monomials $a_{M_k} E_{M_k}$ and $a_{M_\ell} E_{M_\ell}$ such that the monomial t^I appears in both $V(a_{M_k}) \odot V(E_{M_k}(a))$ and $V(a_{M_\ell}) \odot V(E_{M_\ell}(a))$.

5.1 An example

Suppose $n = m = 2$. We denote an element a of $\mathbb{B}_2 = \mathbb{B}[[t,u]]$ as $a = \sum_{(i,j)\in A} t^i u^j$ for some $A \subseteq \mathbb{N}^2$. A differential monomial of order less than $r = 1$ is of the form

$$E_M = x_{(0,0)}^{m_{1,(0,0)}} x_{(0,1)}^{m_{1,(0,1)}} x_{(1,0)}^{m_{1,(1,0)}} x_{(1,1)}^{m_{1,(1,1)}} y_{(0,0)}^{m_{2,(0,0)}} y_{(0,1)}^{m_{2,(0,1)}} y_{(1,0)}^{m_{2,(1,0)}} y_{(1,1)}^{m_{2,(1,1)}}$$

where $M = (m_{i,J})_{i,J} \in \mathbb{N}^{2\times 4}$. So an example of a polynomial of order at most 1 would be

$$p = (t+u) x_{(0,0)} y_{(1,1)}^3 + (1 + t^2u^2) x_{(1,0)} x_{(0,1)} + t y_{(1,0)}^2.$$

The polynomial p has three terms, namely $a_{M_1}E_{M_1} = (t+u)x_{(0,0)}y^3_{(1,1)}$, $a_{M_2}E_{M_2} = (1+t^2u^2)x_{(1,0)}x_{(0,1)}$ and $a_{M_3}E_{M_3} = ty^2_{(1,0)}$. If $a = (a_1, a_2) = (t+t^2+tu+u^3, 1+u+t^2u+t^3u^2)$, then

- $a_{M_1}E_{M_1}(a) = (t+u)a_1(\Theta_{\mathbb{B}}(1,1)a_2)^3 = (t+u)(t+t^2+tu+u^3)(t+t^2u)^3$,
- $a_{M_2}E_{M_2}(a) = (1+t^2u^2)(\Theta_{\mathbb{B}}(1,0)a_1)(\Theta_{\mathbb{B}}(0,1)a_1) = (1+t^2u^2)(1+u+t)(t+u^2)$
- $a_{M_3}E_{M_3}(a) = t(\Theta_{\mathbb{B}}(1,0)a_2)^2 = t(tu+t^2u^2)^2 = t^3u^2(1+tu+t^2u^2)$.

The value $p(a)$ is the union of the three formal series $a_{M_i}E_{M_i}(a)$, $i = 1,2,3$. The four can be seen in Figure 1. We see that $V(p(a)) = t+u^2$, but these monomials only appear in $a_{M_2}E_{M_2}(a)$, so we deduce that a is not a solution of p.

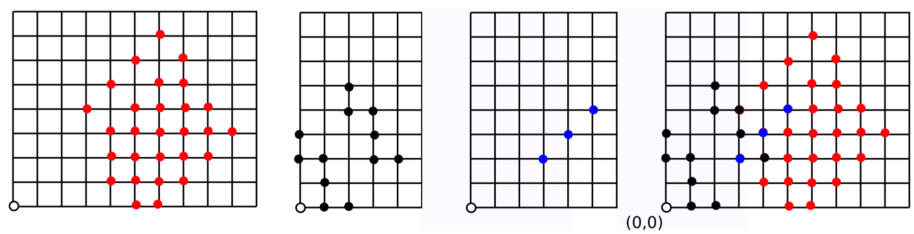

Figure 1: Values of the terms $a_{M_i}E_{M_i}(a)$ for $i = 1,2,3$, and $p(a)$.

On the other hand, if we use vertex polynomials, we get

- $V(a_{M_1}) \odot V(E_{M_1}(a)) = V((t+u)(t^4+t^3u^3)) = t^5+t^4u+t^3u^4$,
- $V(a_{M_2}) \odot V(E_{M_2}(a)) = V(1(u^2+t)) = t+u^2$,
- $V(a_{M_3}) \odot V(E_{M_3}(a)) = V(t(t^2u^2)) = t^3u^2$.

Now again $V(p(a)) = V(\sum_M V(a_M) \odot V(E_M(a))) = t+u^2$.

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