Tropical Hodge Theory and the tropical Hodge conjecture

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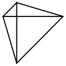
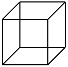
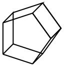
Global picture

Number of faces of simple polytopes

- P polytope of dimension d,
- f_i number of faces of dimension i,
- $F(x) = \sum_i f_i x^i$,
- $H(x) = \sum_i h_i x^i := F(x-1)$.

P	$F(x)$	$H(x)$
	$4 + 6x + 4x^2 + x^3$	$1 + x + x^2 + x^3$
	$8 + 12x + 6x^2 + x^3$	$1 + 3x + 3x^2 + x^3$
	$10 + 15x + 7x^2 + x^3$	$1 + 4x + 4x^2 + x^3$

Theorem (Stanley 80)

If P is a simple polytope of dimension d then

- $h_i \geqslant 0$,
- $h_i = h_{d-i}$,
- $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{\lfloor d/2 \rfloor}$.

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Idea of the proof: The h-vector corresponds to the Betti numbers of some smooth projective complex variety Σ_P associated to P. Classical Hodge theory then implies the above properties.

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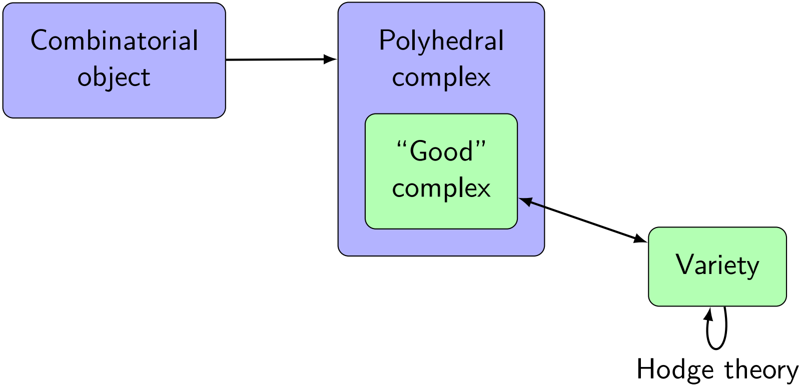
Theorem (Stanley 87, Karu 04)

Extension to non-simple polytopes.

Other similar results

- h-vector of rational non-simple polytopes (Stanley 87)
- h-vector of any polytope (Karu 02)
- g-theorem for simplicial d-spheres (Adiprasito 18)
- coefficients of characteristic (chromatic) polynomials for matroids over $\mathbb{C}$ (Huh 12) $+$ number of independent sets (Lenz 12)
- idem over any field (Huh, Katz 12)
- idem for any matroid (Adiprasito, Huh, Katz 18)
- Kazhdan-Lusztig polynomial of a matroid (Elias, Proudfoot, Wakefield 16)

Global scheme of proof



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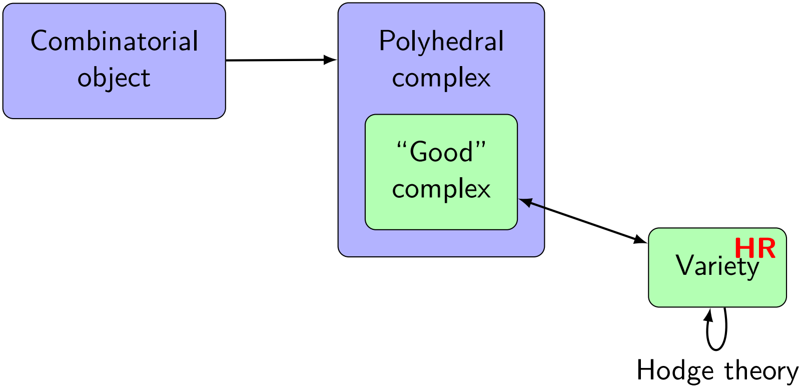
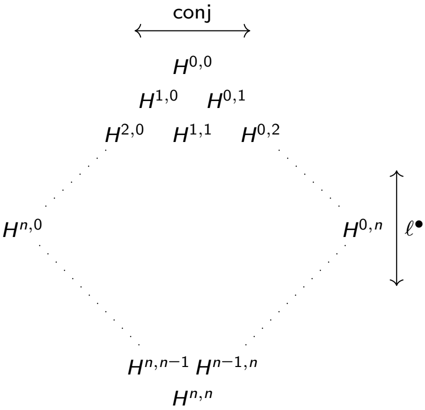


HR



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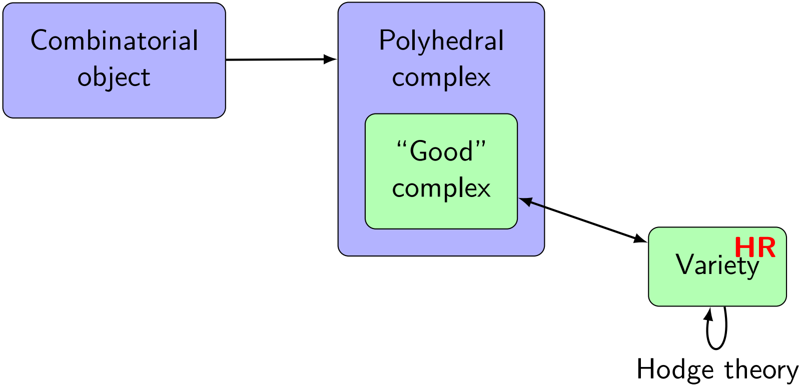

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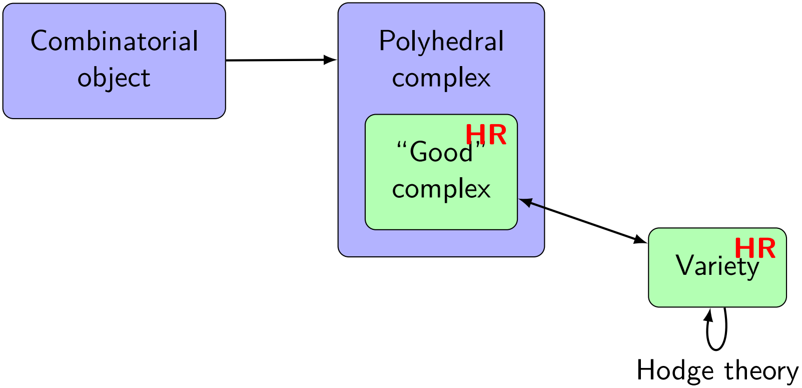

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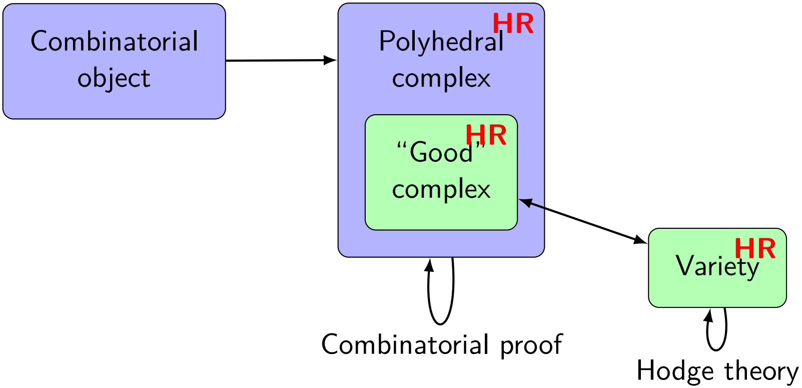

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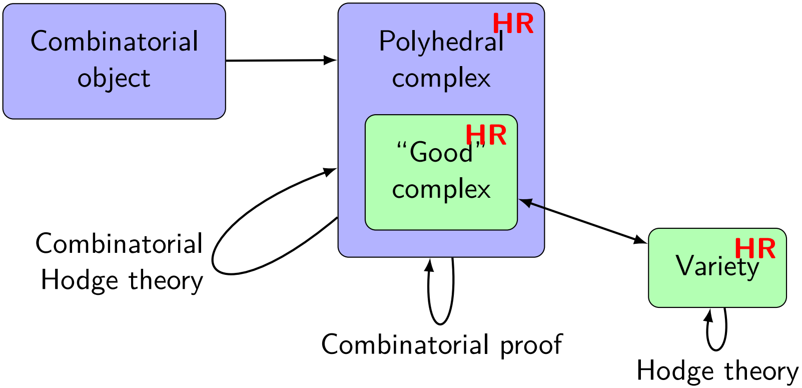

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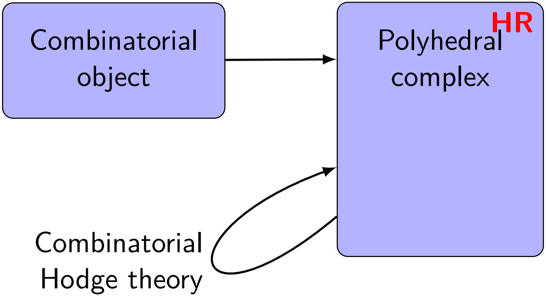


Hints for the existence of a combinatorial Hodge theory

Theorem (Itenberg, Katzarkov, Mikhalkin, Zharkov 19 (08))

Let $X \subset \mathbb{TP}^n$ be the tropical limit of a family of projective complex varieties $\mathfrak{X}$. If X is smooth then

$$h^{\mathrm{trop}}_{p,q}(X) = h^{p,q}(\mathfrak{X}_t)$$

where $\mathfrak{X}_t$ is any generic fiber of the family.

Theorem (Jell, Shaw, Smacka 15)

If X is a smooth compact tropical variety of dimension d, then

$$H^{p,q}_{\mathrm{trop}}(X) \simeq H^{d-p,d-q}_{\mathrm{trop}}(X).$$

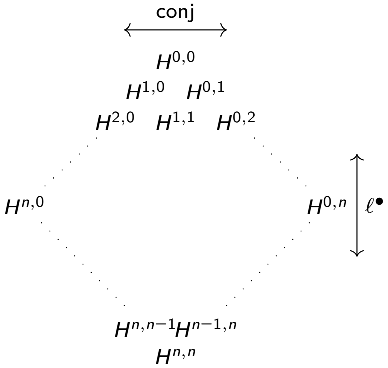
Theorem (Adiprasito, Huh, Katz 18)

*Let Σ be the Bergman fan associated to a matroid. Then the Chow ring of Σ verifies **HR**.*

The main theorem

Theorem (Amini, P. 20)

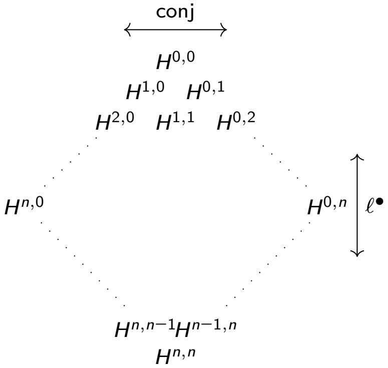
Let X be a smooth projective tropical variety. Then $H^{\bullet,\bullet}_{\mathrm{trop}}(X)$ verifies ***HR****.*



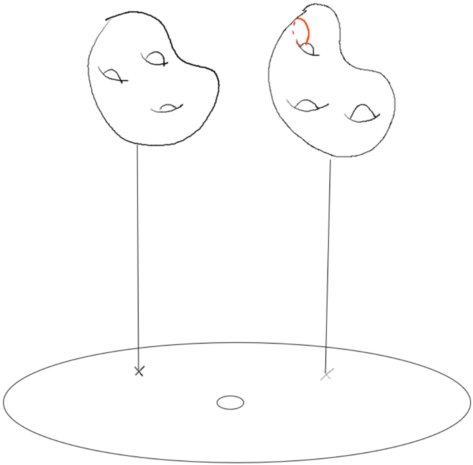
The main theorem

Theorem (Amini, P. 20)

Let X be a smooth projective tropical variety. Then $H^{\bullet,\bullet}_{\mathrm{trop}}(X)$ verifies ***HR****. Moreover, the monodromy operator N induces an isomorphism $H^{p,q}_{\mathrm{trop}}(X) \simeq H^{q,p}_{\mathrm{trop}}(X)$.*



Degenerescence of a family of varieties



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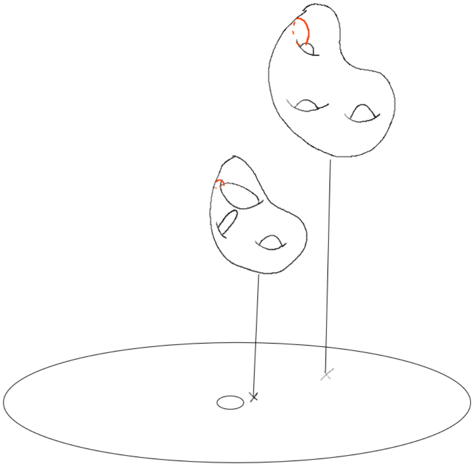

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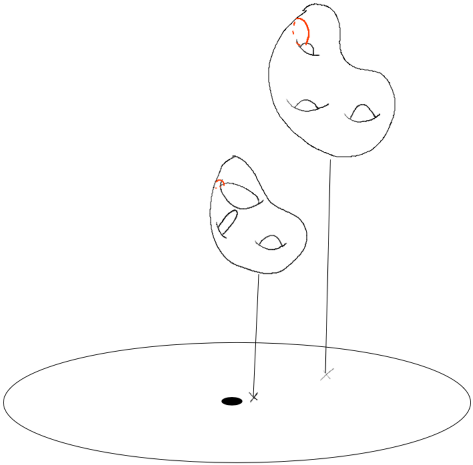

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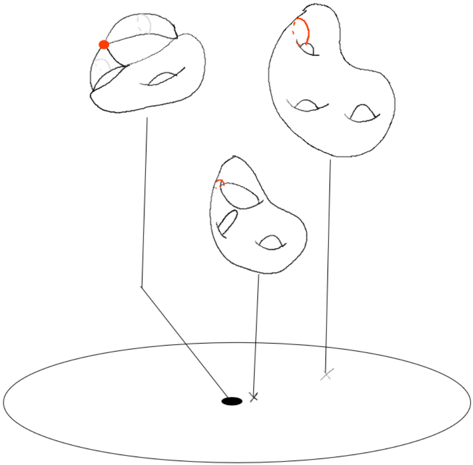

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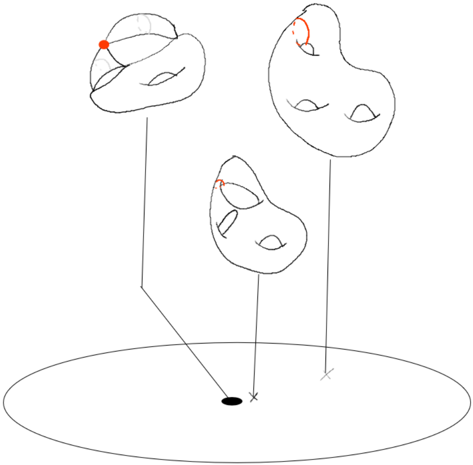

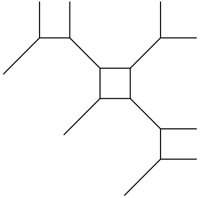

The Hodge conjecture

Conjecture

Let X be a smooth projective complex variety. Then, for any p, any element of $\mathrm{Hdg}^p(X) = H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$ is given by some complex subvarieties with rational coefficients.

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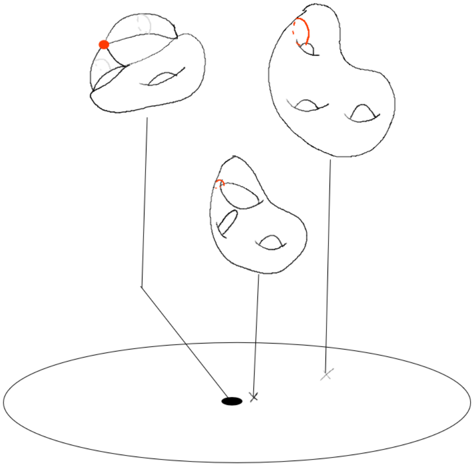
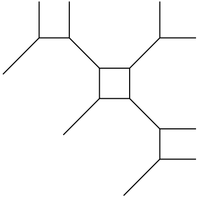
Theorem (Lefschetz)

This is true for $p = 1$.

Tropicalization of the Hodge conjecture





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Theorem (Amini, P. (20?))

This is true for any p if the tropical variety is rationally triangulable.

Classical Hodge theory

Dolbeault cohomology

Let X be a *projective* (compact) *smooth* complex variety of complex dimension d.

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Betti numbers: $h^k = \dim(H^k)$.

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- Let $\Omega^{p,q}$ be the set of (p,q)-forms, i.e., locally

$$f(z)\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\bar{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_q}.$$

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- $\mathrm{d} = \partial + \overline{\partial} \qquad \partial^2 = \overline{\partial}^2 = 0$

$$H^{p,q}(X) = \frac{\ker\left(\overline{\partial} \colon \Omega^{p,q} \to \Omega^{p,q+1}\right)}{\mathrm{Im}\left(\overline{\partial} \colon \Omega^{p,q-1} \to \Omega^{p,q}\right)}.$$

Dolbeault cohomology

$$\wedge\colon H^{p,q} \times H^{p',q'} \to H^{p+p',q+q'}.$$

$$H^{p,q}(X) = \overline{H^{q,p}(X)}.$$

$$H^k(X) = \bigoplus_{p+q=k} H^{p,q}(X).$$

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$$H^{p,q}(X) = \overline{H^{q,p}(X)}.$$
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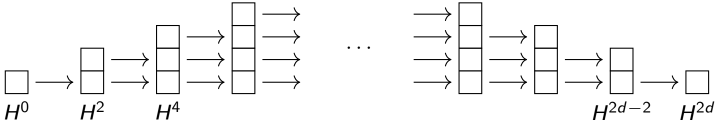
If $\alpha \in \Omega^{p,q}(X)$ and $\beta \in \Omega^{d-q,d-p}(X)$ then
$$\langle \alpha, \beta \rangle = \kappa \int_X \alpha \wedge \overline{\beta}.$$
It induces a perfect pairing $H^{p,q}(X) \times H^{d-q,d-p}(X) \to \mathbb{C}$. Hence,
$$H^{p,q}(X) \simeq (H^{d-q,d-p}(X))^\star.$$
$$H^k(X) \simeq H^{2d-k}(X)^\star.$$

In particular $(h^{2k})_k$ is symmetric.

Kälher form and the hard Lefschetz theorem

There exists a Lefschetz operator $\ell \in H^{1,1}(X)$, i.e., an element such that

$$H^{p,q}(X) \xrightarrow[\ell^{d-q-p}]{\simeq} H^{d-q,d-p}(X).$$



In particular, $(h^{2k})_k$ is unimodal.

Hodge-Riemann bilinear relations

We get a quadratic form

$$\alpha \in H^{p,q} \mapsto \langle \alpha, \ell^{d-p-q}\alpha \rangle.$$

The signature of this form is entirely described by the $(h^{p,q})$.

HR

conj $\longleftrightarrow$

$$H^{0,0}$$

$$H^{1,0} \quad H^{0,1}$$

$$H^{2,0} \quad H^{1,1} \quad H^{0,2}$$

$$H^{n,0} \qquad\qquad\qquad H^{0,n} \quad \updownarrow \ell^\bullet$$

$$H^{n,n-1} \; H^{n-1,n}$$

$$H^{n,n}$$

The cycle class map

If Z is a complex subvariety of X of complex codimension p. We get a map

$$\alpha \in \Omega^{d-p+k,d-p-k} \mapsto \int_Z i^*(\alpha).$$

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$$Z \mapsto \mathrm{cl}(Z) \in H^{p,p}(X) \simeq H^{d-p,d-p}(X)^\star.$$

Moreover, $\mathrm{cl}(Z) \in H^{2p}(X, \mathbb{Z})$.

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Conjecture (Hodge conjecture)

Any element of $\mathrm{Hdg}^p = H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$ *is the class of a combination with rational coefficients of complex subvarieties.*

Tropical Hodge theory

What do we need?

- Tropical variety.
- Compactification.
- Projectivity.
- Smoothness.
- (p, q)-forms and differential.
- Integration.
- Tropical subvariety.
- Complex conjugation?

Tropicalization

$$X \subset \mathbb{C}^d \rightsquigarrow \lim_{t\to 0}\big(\log_t(|z_1|), \dots, \log_t(|z_d|)\big).$$

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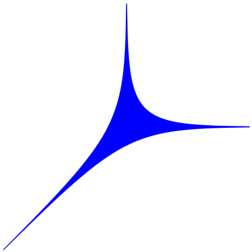
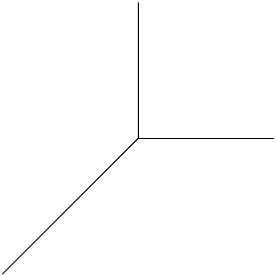
$$z_1 - 2z_2 - 1 = 0$$

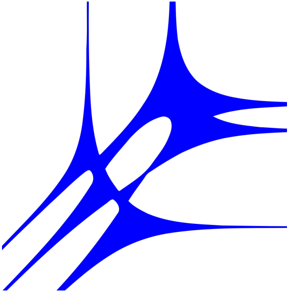




Tropicalization

$$\mathfrak{X} \subset \mathbb{C}^d \times \Delta^* \rightsquigarrow \lim_{t\to 0}\big(\log_t(|z_1|), \dots, \log_t(|z_d|)\big).$$

$$t^{100}z_1^3 + t^{100}z_2^3 + t^3z_1^2z_2 + t^3z_1z_2^2 + t^{10}z_1^2 + t^{10}z_2^2 + z_1z_2 + z_1 + z_2 + 1 = 0$$



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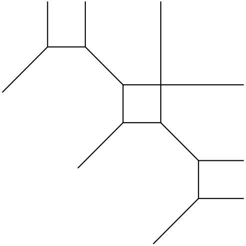

Tropical variety

TROPICAL SURFACES

KRISTIN SHAW

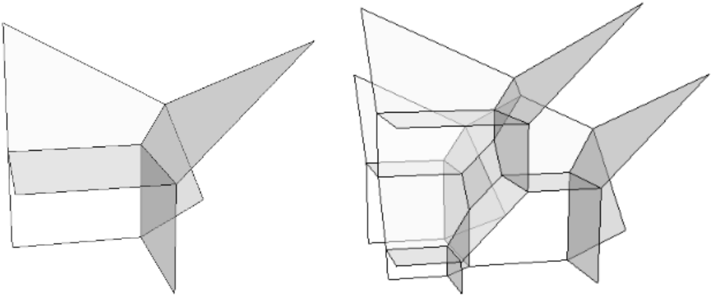

FIGURE 4. A tropical plane in $\mathbb{T}P^3$ on the left and a quadric hypersurface in $\mathbb{T}P^3$ on the right.

Compactification and projectivity

Tropical variety

Let $N \simeq \mathbb{Z}^n$ be a lattice and let $N_\mathbb{R} = N \otimes \mathbb{R} \simeq \mathbb{R}^n$.

Definition

A *tropical variety* of dimension d is (the support of) some polyhedral complex:

- rational,
- of pure dimension d,
- with some weight $w(\sigma)$ on each facet (in this talk $w \equiv 1$),
- verifying the balancing condition,
- one can replace $\sigma = \mathbb{R}_+ u_1 + \cdots + \mathbb{R}_+ u_k$ by $\overline{\sigma} = \mathbb{T}_+ u_1 + \cdots + \mathbb{T}_+ u_k$ where $\mathbb{T}_+ = \mathbb{R}_+ \cup \{+\infty\}$.

Tropical smoothness

A tropical variety is *smooth* if it is locally of the form $\Sigma \times \mathbb{T}^k$ where Σ is a Bergman fan.

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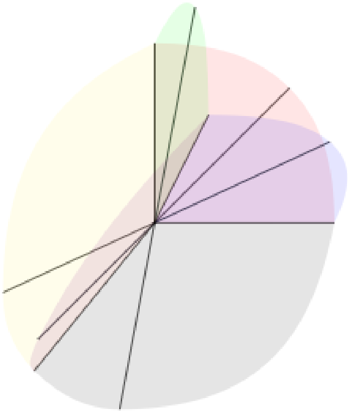

Tropical smoothness

A tropical variety is *smooth* if it is locally of the form $\Sigma \times \mathbb{T}^k$ where Σ is a Bergman fan.

- Σ is (a generalization of) the tropicalization of a complement of hyperplane arrangement,
- can be constructed from the whole space by simple transformations,
- many possible ways to compactify/combine,
- cohomology of the compactifications can be described combinatorially,
- this cohomology is of Tate type ($H^{p,q} = 0$ if $p \neq q$),
- this cohomology verifies **HR** (Adiprasito-Huh-Katz 19).

What do we need?

- Tropical variety. ✓
- Compactification. ✓
- Projectivity. ✓
- Smoothness. ✓
- (p, q)-forms and differential.
- Integration.
- Tropical subvariety.
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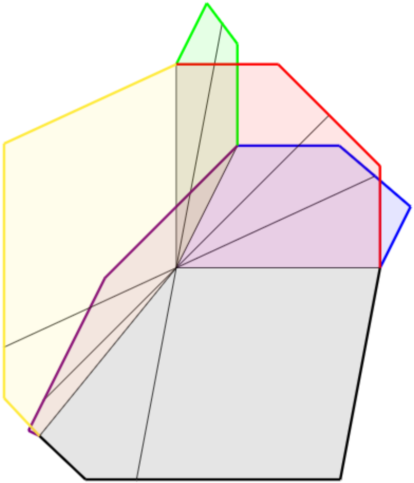
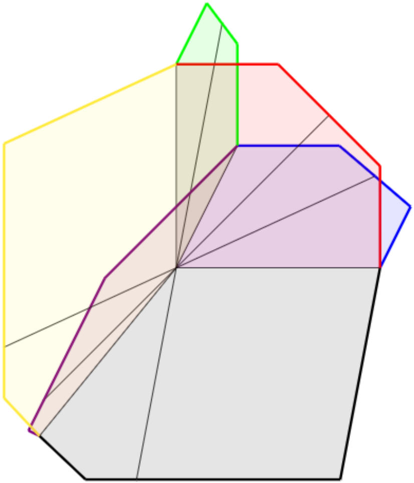
Tropical (p, q)-form



- X tropical variety of dimension d.

Tropical (p, q)-form



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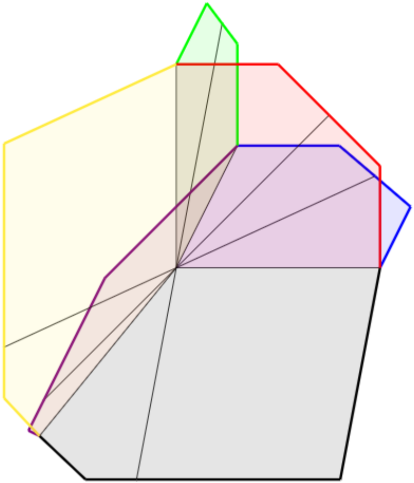
Tropical (p, q)-form



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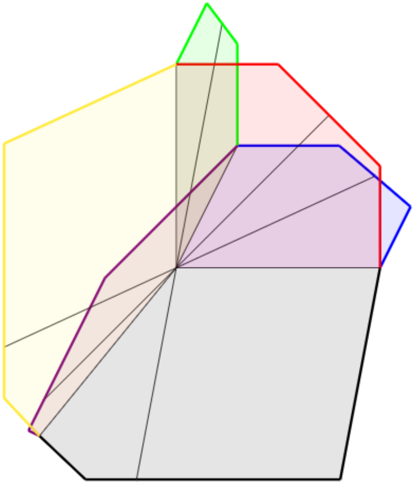
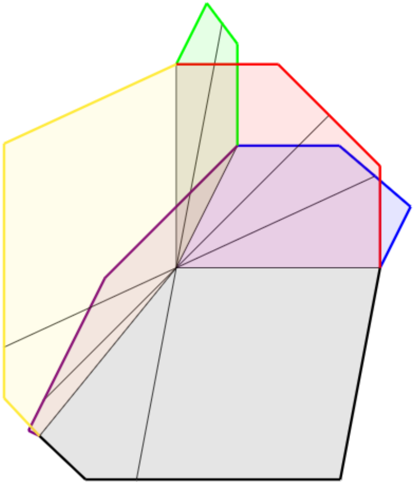
- Dual form $\mathrm{d}x_1, \ldots, \mathrm{d}x_d$.

Tropical (p, q)-form



- X tropical variety of dimension d.
- On a facet $x_1, \ldots, x_d$ coordinates.
- Dual form $\mathrm{d}'x_1, \ldots, \mathrm{d}'x_d$.
  Dual form $\mathrm{d}''x_1, \ldots, \mathrm{d}''x_d$.
  $\mathrm{d}'x_1 \wedge \mathrm{d}''x_1 \neq 0$

Tropical (p, q)-form



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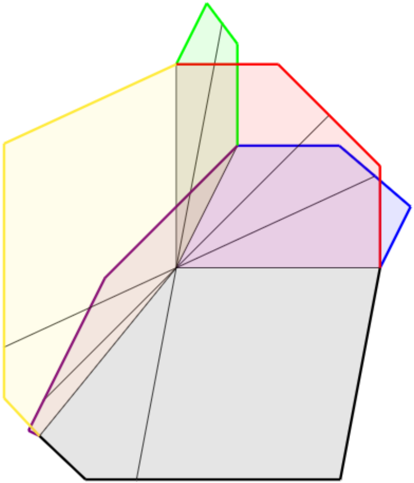
  Dual form $\mathrm{d}''x_1, \ldots, \mathrm{d}''x_d$.
  $\mathrm{d}'x_1 \wedge \mathrm{d}''x_1 \neq 0$
- $\alpha = \pi^*(\alpha_\infty)$ in a neighborhood of infinity strata.
- We can differentiate by d''.

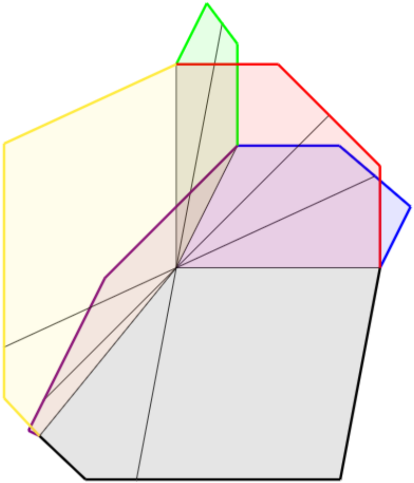
Tropical cohomology

$$H^{p,q}_{\mathrm{trop}}(X) = \frac{\ker\left(\mathrm{d}'' \colon \Omega^{p,q}_{\mathrm{trop}} \to \Omega^{p,q+1}_{\mathrm{trop}}\right)}{\mathrm{Im}\left(\mathrm{d}'' \colon \Omega^{p,q-1}_{\mathrm{trop}} \to \Omega^{p,q}_{\mathrm{trop}}\right)}$$

Integration



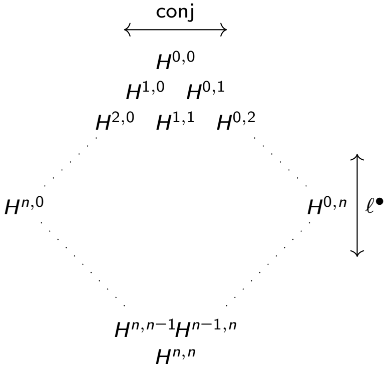
Tropical subvarieties and Minkowski weights



The main results

Theorem (Amini, P. 20)

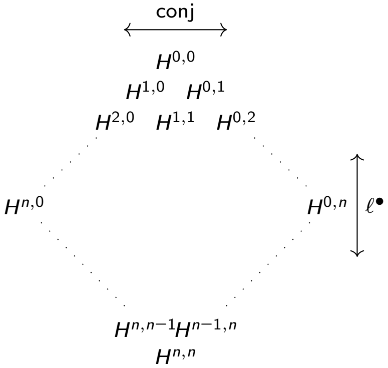
*Let X be a smooth projective tropical variety. Then $H^{\bullet,\bullet}_{\mathrm{trop}}(X)$ verifies **HR**.*



The main results

Theorem (Amini, P. 20)

Let X be a smooth projective tropical variety. Then $H^{\bullet,\bullet}_{\mathrm{trop}}(X)$ verifies ***HR****. Moreover, the monodromy operator N induces an isomorphism $H^{p,q}_{\mathrm{trop}}(X) \simeq H^{q,p}_{\mathrm{trop}}(X)$.*



The main results

Theorem (Amini, P. 20)

Let X be a rationally triangulable smooth projective tropical variety. Then any element in $\ker(N \colon H^{p,p}_{\mathrm{trop}}(X, \mathbb{Q}) \to H^{p-1,p+1}_{\mathrm{trop}}(X, \mathbb{Q}))$ *is a rational combination of elements of the form* $\mathrm{cl}(Z)$ *for some tropical subvarieties Z.*

Asymptotic Hodge theory and tropical limit

Recall

Theorem (Itenberg, Katzarkov, Mikhalkin, Zharkov 19 (08))

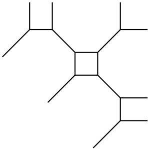
Let $X \subset \mathbb{TP}^n$ be the tropical limit of a family of projective complex varieties $\mathfrak{X}$. If X is smooth then

$$h_{p,q}^{\mathrm{trop}}(X) = h^{p,q}(\mathfrak{X}_t)$$

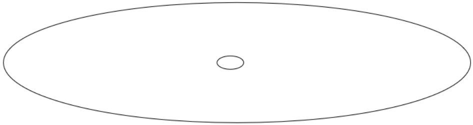
where $\mathfrak{X}_t$ is any generic fiber of the family.

$$\mathfrak{X} \subset \mathbb{C}^d \times \Delta^* \rightsquigarrow \lim_{t \to 0} \left(\log_t(|z_1|), \ldots, \log_t(|z_d|)\right).$$

$$t^{100}x^3 + t^{100}y^3 + t^3x^2y + t^3xy^2 + t^{10}x^2 + t^{10}y^2 + txy + x + y + 1 = 0$$



Degenerescence of a family of varieties



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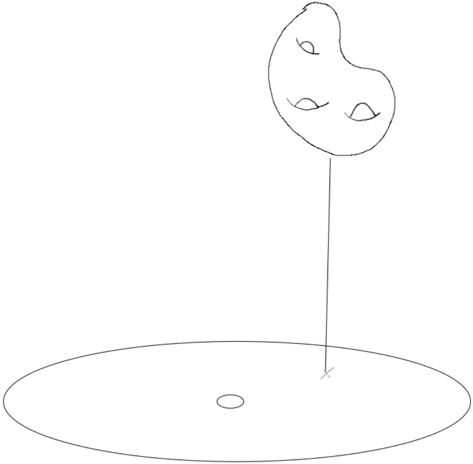

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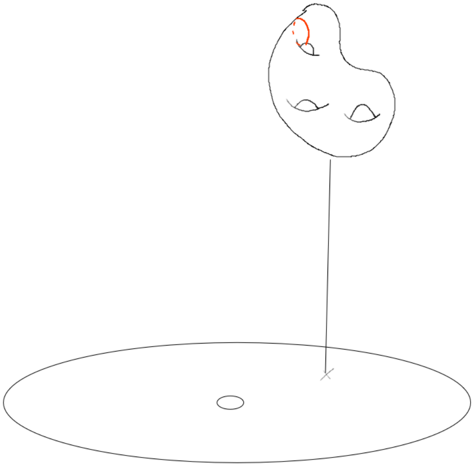

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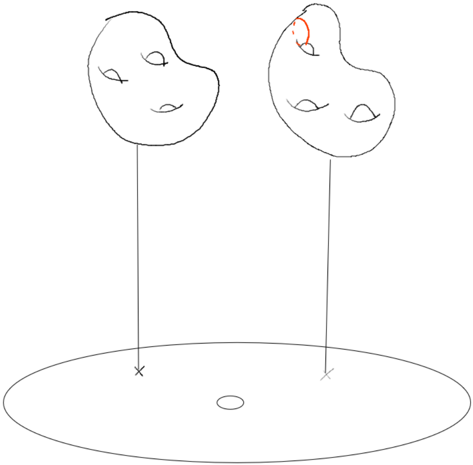

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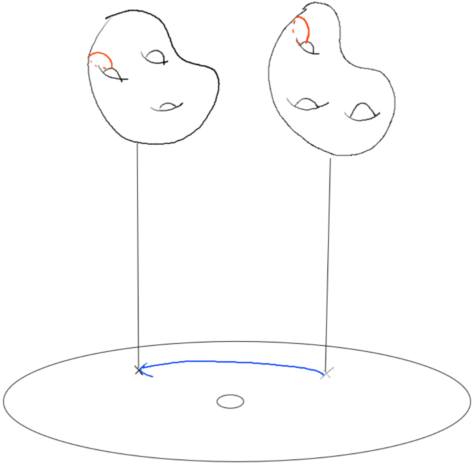

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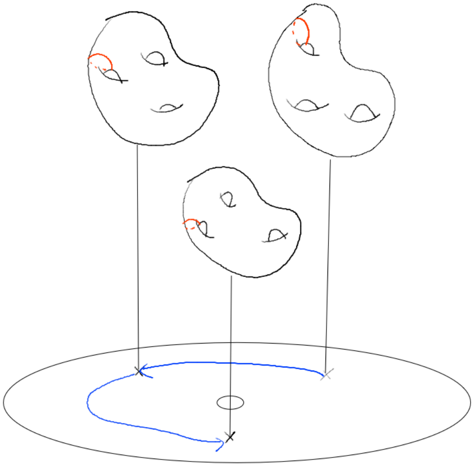

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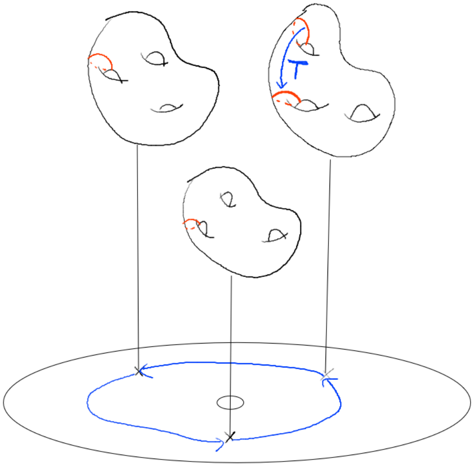

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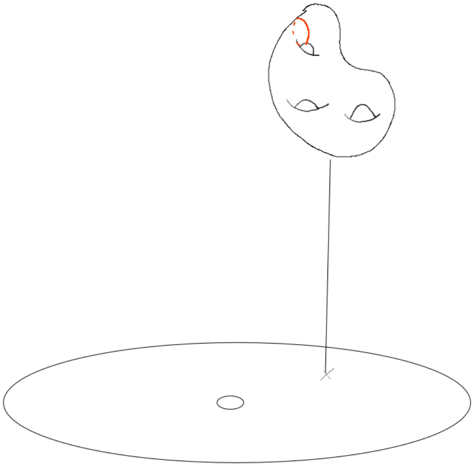

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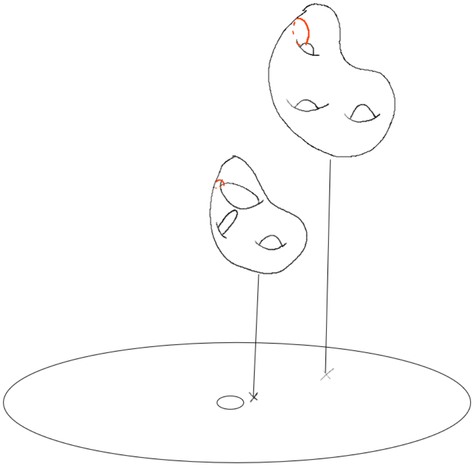

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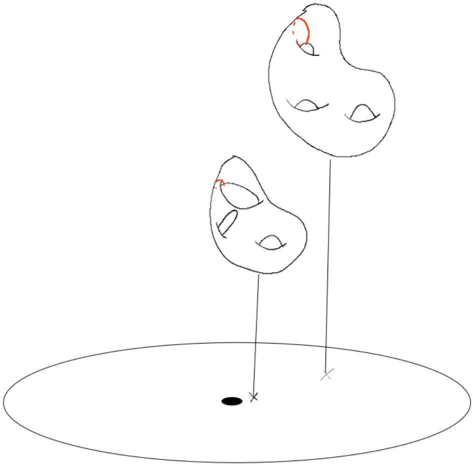
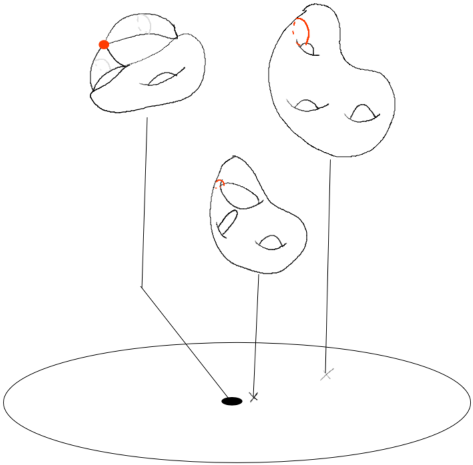
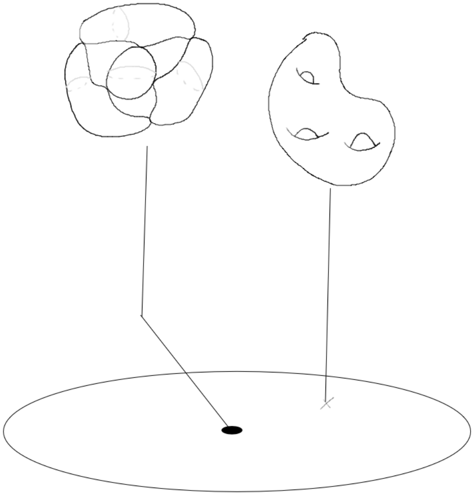


Degenerescence of a family of varieties



Degenerescence of a family of varieties



Clemens-Schmidt exact sequence

Classical Clemens-Schmidt exact sequence:

$$\cdots \to H^k(\mathscr{X}) \to H^k(\widetilde{\mathscr{X}^*}) \xrightarrow{N} H^k(\widetilde{\mathscr{X}^*}) \to H^{k+2}(\mathscr{X},\mathscr{X}^*) \to H^{k+2}(\mathscr{X}) \to \cdots$$

Clemens-Schmidt exact sequence

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Tropical analog:

$$\cdots \to H_s^k(X) \to H^k(X) \xrightarrow{N} H^k(X) \to H_{\mathrm{rel}}^{k+2}(X) \to H_s^{k+2}(X) \to \cdots$$

$H_s^k(X)$ and construction of the cycle

TROPICAL SURFACES

KRISTIN SHAW

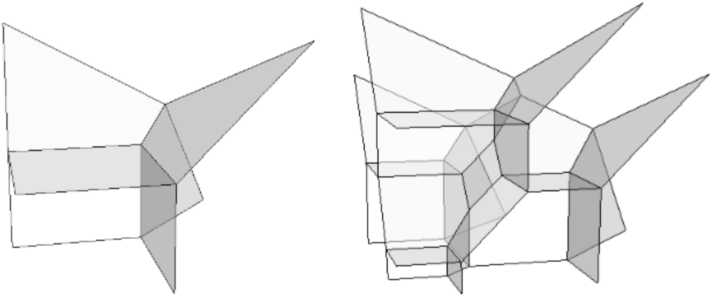

FIGURE 4. A tropical plane in $\mathbb{T}P^3$ on the left and a quadric hypersurface in $\mathbb{T}P^3$ on the right.

Conclusion

- We have a tropical Hodge theory.
- Tropical Hodge conjecture is a way to test (to prove?) classical Hodge conjecture.
- Tropical theory is a very interesting tool in both asymptotic Hodge theory and in combinatorics.