Special Subgroups of the Cremona Group via Calabi-Yau Pairs

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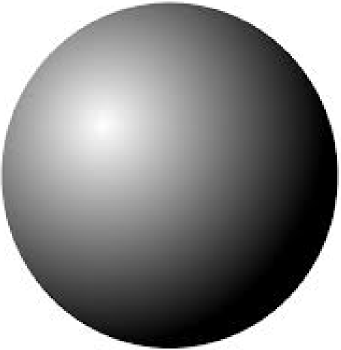
(remotely from Rio de Janeiro)

Joint with Alessio Corti and Alex Massarenti

(We always work over $\mathbb{C}$)

Automorphisms in Algebraic Geometry

$X \subset \mathbb{P}^n$ complex projective variety



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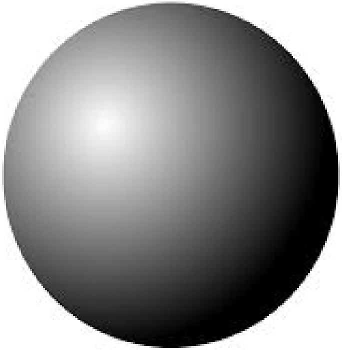

$$\mathrm{Aut}(X) \; = \; \left\{ \, f : X \to X \textit{ automorphism} \, \right\}$$

Automorphisms in Algebraic Geometry

Example ($X = \mathbb{P}^n$)

$\mathrm{Aut}(\mathbb{P}^n) \; = \; \mathrm{PGL}_{n+1}(\mathbb{C})$

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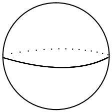
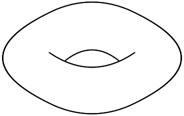
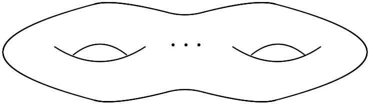
Example (X smooth projective curve of genus g)



$g = 0$



$g = 1$



$g \geq 2$

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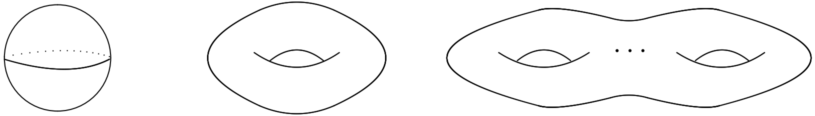

$g = 0$ $g = 1$ $g \geq 2$

- $g = 0$ $\mathrm{Aut}^0(\mathbb{P}^1) = \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$
- $g = 1$ $\mathrm{Aut}^0(X) \cong X$
- $g \geq 2$ $\mathrm{Aut}(X)$ is finite

Birational Geometry

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Equivalently:

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The problem of birational classification :

Given a projective variety X, to find a simplest representative in its birational class - minimal model of X

Birational Geometry

Definition

X is rational if it is birationally equivalent to $\mathbb{P}^n$

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For which d and n, is the generic hypersurface $X_d \subset \mathbb{P}^{n+1}$ rational?

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$\mathrm{Aut}(X)$ is not a birational invariant

$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[d, dashed, "\wr"] & X \arrow[d, dashed, "\wr"] \\ Y \arrow[r, dashed, "\sim"] & Y \end{tikzcd}$$

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Definition (The Birational Group)

$$\mathrm{Bir}(X) := \{\ \varphi : X \overset{\sim}{\dashrightarrow} X \ \text{ birational self-map}\ \}$$

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Definition (The Cremona Group in dimension n)

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Example (The standard quadratic transformation)

$$\begin{array}{cccc} \tau : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ & (x:y:z) & \longmapsto & \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right) = (yz : xz : xy) \end{array}$$

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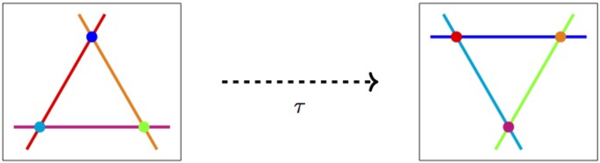

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$$\mathrm{Bir}(\mathbb{P}^2) \ = \ \langle \ \mathrm{Aut}(\mathbb{P}^2) \ , \ \tau \ \rangle$$

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Theorem (Bertini 1877, $\cdots$, Dolgachev-Iskovskikh 2009)

Classification of finite subgroups of $\mathsf{Bir}(\mathbb{P}^2)$.

The Cremona Group in higher dimensions

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For $n \geq 3$, $\mathrm{Bir}(\mathbb{P}^n)$ cannot be generated by elements of bounded degree.

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Problem

To construct interesting subgroups of $\mathrm{Bir}(\mathbb{P}^n)$.

Symplectic birational transformations of $\mathbb{P}^2$

$\frac{dx}{x} \wedge \frac{dy}{y}$ - meromorphic volume form on $\mathbb{P}^2$

$\mathrm{Bir}\left(\mathbb{P}^2, \frac{dx}{x} \wedge \frac{dy}{y}\right) \subset \mathrm{Bir}(\mathbb{P}^2)$

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$$\mathrm{Bir}\left(\mathbb{P}^2, \frac{dx}{x} \wedge \frac{dy}{y}\right) = \left\langle \underbrace{\mathrm{SL}(2,\mathbb{Z}),\ (\mathbb{C}^*)^2}_{\text{preserve the torus } (\mathbb{C}^*)^2}\ , (x,y) \mapsto \left(y, \frac{1+y}{x}\right) \right\rangle$$

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} :\ (x,y) \mapsto (\, x^a y^b\ ,\ x^c y^d\,)$$

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To determine $\mathrm{Bir}\left(\mathbb{P}^n, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}\right)$.

Special subgroups of $\mathrm{Bir}(\mathbb{P}^n)$

ω meromorphic volume form on $\mathbb{P}^n$

$$\mathrm{Bir}\big(\mathbb{P}^n, \omega\big) \ \subset \ \mathrm{Bir}(\mathbb{P}^n)$$

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Remark

$D \subset \mathbb{P}^n$ hypersurface of degree $n+1 \;\rightsquigarrow\; \exists\; \omega_D$ (unique up to scaling)

$$D \;=\; -\,\big[\mathrm{div}(\omega_D)\big]$$

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If $n \geq 3$ and D is very general , then

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(D is smooth and $\mathrm{Pic}(D) = \mathbb{Z} \cdot (H_{|D})$)

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Example

If $D = -(n+1)H$, then

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$^{(*)}$ $(\mathbb{P}^n, D)$ has log canonical singularities

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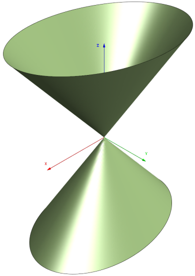
$D \subset \mathbb{P}^3$ general quartic hypersurface with 1 singular point P

$$x_0^2 A_2(x_1, x_2, x_3) + x_0 B_3(x_1, x_2, x_3) + C_4(x_1, x_2, x_3) = 0$$



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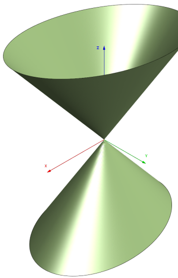

Example

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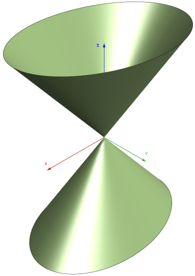

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Theorem B

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$$\mathbb{G} = \left\{ \left[(AG - BF)x_0 - CF : A(Fx_0 + G)x_1 : A(Fx_0 + G)x_2 : A(Fx_0 + G)x_3 \right] \right\}$$

$(F, G \in \mathbb{C}[x_1, x_2, x_3]$ homogeneous with $\deg(G) = \deg(F) + 1)$

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$D \subset \mathbb{P}^3$ general quartic hypersurface with 1 singular point P

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Remark

This is not true for arbitrary D

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The X_i's are Mori fiber spaces

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Definition (Mori Fiber Space)

$f : X \to Y$ fibration such that

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- $\mathbb{F}_m \to \mathbb{P}^1$ $\quad$ ($\mathbb{P}^1$-bundle)

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Type 4

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Calabi-Yau pairs

Definition (Calabi-Yau pair)

(X, D) such that

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Definition

(X, D_X) and (Y, D_Y) Calabi-Yau pairs

$f : X \dashrightarrow Y$ birational map $\rightsquigarrow$ $f_* : \ \Omega^n_{\mathbb{C}(X)/\mathbb{C}} \ \to \ \Omega^n_{\mathbb{C}(Y)/\mathbb{C}}$

If $f_*\omega_{D_X} = \omega_{D_Y}$ (up to scaling) then we say that

$$f : \ (X, D_X) \ \dashrightarrow \ (Y, D_Y) \ \text{ is volume preserving}$$

Volume Preserving Sarkisov Program

Theorem (Corti-Kaloghiros 2016)

A volume preserving birational map between Mori fibered Calabi-Yau pairs is a composition of volume preserving Sarkisov links.

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Example

If $D \subset \mathbb{P}^n$ is a smooth hypersurface of degree $n+1$, and $f : X \to \mathbb{P}^n$ is a volume preserving blowup along a smooth center Z, then

$$Z \subset D \quad \text{and} \quad \mathrm{codim}_{\mathbb{P}^n}(Z) = 2.$$

Volume Preserving Sarkisov Program

Theorem A

If $n \geq 3$ and D is very general hypersurface of degree $n+1$, then

$$\mathrm{Bir}\big(\mathbb{P}^n, D\big) \;=\; \mathrm{Aut}\big(\mathbb{P}^n, D\big).$$

$\Big($ D is smooth and $\mathrm{Pic}(D) = \mathbb{Z} \cdot \big(H_{|D}\big)$ $\Big)$

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X_1 has worst than terminal singularities

Volume Preserving Sarkisov Program

Theorem B

If $D \subset \mathbb{P}^3$ general quartic hypersurface with 1 singular point P, then

$$\mathrm{Bir}(\mathbb{P}^3, D) \ \cong \ \mathbb{G} \rtimes \mathbb{Z}/2\mathbb{Z},$$

where $\mathbb{G}$ is a form of $\mathbb{G}_m$ over $\mathbb{C}(x, y)$.

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Thank you! [1]

[1] Thanks to Santiago Arango and Wikipedia for some nice pictures