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BISET FUNCTORS AND THE HOMOTOPY TYPE OF CLASSIFYING SPECTRA OF SATURATED FUSION SYSTEMS

T E S I S

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NOTATION

c_g	$g(-)g^{-1}$
${}^gA$	gAg^{-1}
B^g	$g^{-1}Bg$
$\sum_{L \leq_G K}$	a sum running over representatives L of G-conjugacy classes of subgroups of K
$\sum_{P \leq_{\mathcal{F}} Q}$	a sum running over representatives P of $\mathcal{F}$-conjugacy classes of subgroups of Q
$[L]_G$	the G-conjugacy class of L
$[Q]_{\mathcal{F}}$	the $\mathcal{F}$-conjugacy class of Q
$\mathbb{S}$	the sphere spectrum
$F(X, Y)$	the function of spectrum of the spectra X and Y
$O^p(G)$	the largest normal subgroup of G with index a power of p
$\mathrm{Hom}_G(P, Q)$	the set of conjugation homomorphisms $\varphi : P \to Q$ induced by elements of G (i.e. $\varphi = c_g$ for some $g \in G$)
$\mathrm{Out}(G)$	$\mathrm{Aut}(G)/\mathrm{Inn}(G)$
$\mathrm{Out}_G(P)$	$\mathrm{Aut}_G(P)/\mathrm{Inn}(P)$

ABSTRACT

We give a biset-functor theoretic interpretation of the stable homotopy of p-completed classifying spaces of finite groups. In particular, thanks to previous work of Ragnarsson on the subject, we are able to improve some classical results by Martino-Priddy and Webb regarding the complete stable splitting of $BG^{\wedge}_{p}$. We use these improvements to approach the stable Martino-Priddy conjecture and get a better understanding of the problem. On the way, we give a partial answer to a question by O'Hare in his PhD thesis. Subsequently, we aim to generalize our results to saturated fusion systems. This is achieved by introducing the concept of biset functors for saturated fusion systems and exploiting the concept of the characteristic idempotent of a saturated fusion system. More precisely, we give a formula for the evaluation of global Mackey functors for saturated fusion systems, inspired by Webb. Then, this formula is used to show our main result, which involves the stable homotopy type of classifying spaces of saturated fusion systems.

INTRODUCTION

1 LOCALIZATION IN ALGEBRA AND TOPOLOGY

Homotopy theory and representation theory have a long-standing history of fruitful interactions. For instance, group cohomology, which has proven to be a powerful tool in the progress of group theory (e.g. Schur-Zassenhaus theorem), class field theory (e.g. Tate theorem), among other areas; can be approached in algebraic topology via the classifying space of a group: asking a question on $H^*(G; M)$ is the same as asking a question on the twisted cohomology $H^*(BG; M)$ (see Section 1.2).

In fact, several results in group cohomology (and broadly speaking, in group representation theory) were first proven homotopically (e.g. Glauberman's Z^* theorem for odd primes, Mislin's theorem on control of p-fusion). Some of them, only have homotopical proofs so far (e.g. the Carlsson-Thévenaz conjecture).

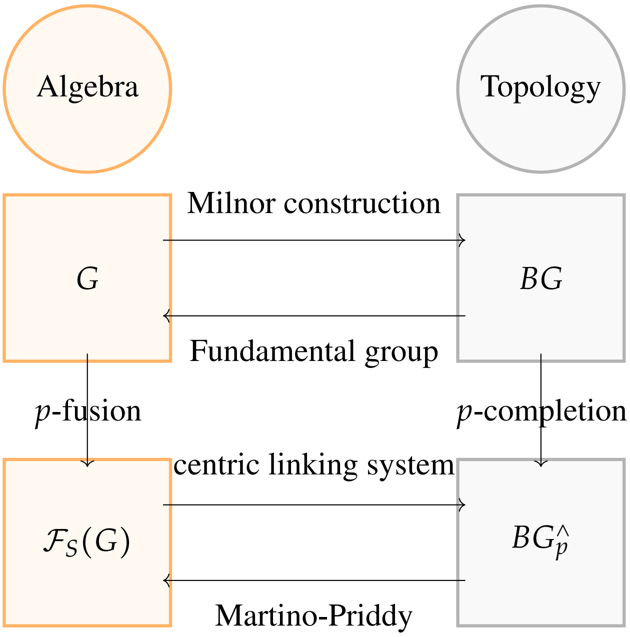
Now, in homotopy theory, the global structure of a space (or a spectrum) X can be hard to deal with, hence we can instead study X locally via its rationalization $X_{\mathbb{Q}}$ and its p-completions $X^\wedge_p$, with the hope that we can recover X from these "pieces". In particular, if $X = BG$, the classifying space of a finite group G, its rationalization does not offer any information, since $BG_{\mathbb{Q}} \simeq *$. Thus, it suffices to study BG "one prime at a time".

Actually, not all primes p are relevant to approach BG, but only those that divide the order of G, since $BG^\wedge_p \simeq *$ otherwise (see comments below Proposition 1.5.5). Now, fixing a prime p, the p-completion is intended to isolate the homotopy p-local data of a space. At this point, given the parallelisms between G and BG outlined above, it is natural to expect that the homotopy data of $BG^\wedge_p$ corresponds with certain p-local structure of G. This is the goal of the Martino-Priddy conjecture.

2 THE MARTINO-PRIDDY CONJECTURE AND ITS GENERALIZATION

The Martino-Priddy conjecture (Theorem 1.7.7) says that given a finite group G, we can recover $BG^\wedge_p$ up to unstable homotopy equivalence, from the p-fusion of G, up to fusion-preserving isomorphism, and conversely. In more categorical words (see Section 4.1), there exists a completely algebraic object (i.e. the fusion system $\mathcal{F}_S(G)$) that determines and is essentially determined, by the unstable homotopy type of $BG^\wedge_p$. This was stated as a theorem by Martino-Priddy in [35], but their proof contained an error. An actual proof was achieved by Oliver in [48, 47]. For the proof, he used the fact that $\mathcal{F}_S(G)$ suffices to construct a category $\mathcal{L}^c_p(G)$, the centric linking system of $\mathcal{F}_S(G)$, such that its geometric realization, once p-completed, is homotopy equivalent to $BG^\wedge_p$ [17].

To summarize, the discussion above can be illustrated in the following figure:



The Martino-Priddy conjecture was later generalized for p-local finite groups [18]. A p-local finite group is a pair $\mathcal{G} = (\mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system over a finite p-group and $\mathcal{L}$, the centric linking system of $\mathcal{F}$, is a category constructed from the $\mathcal{F}$-centric subgroups of S. The space $|\mathcal{L}|_p^\wedge$ is known as the classifying space of $\mathcal{G}$ (see the introduction of Section 4.1). Later, Chermak [21] proved that $\mathcal{L}$ is determined by $\mathcal{F}$, so we can simply speak of the classifying space of $\mathcal{F}$, denoted by $B\mathcal{F}$.

The problem that motivates this thesis is an analogue of the Martino-Priddy conjecture in stable homotopy theory, whose origin dates back to Martino-Priddy too [34], and its generalization to saturated fusion systems. In this context, our interest lies in the homotopy type of the p-completed classifying spectrum $\mathbb{B}G_p^\wedge := \Sigma^\infty BG_p^\wedge$, and more generally, of the classifying spectrum $\mathbb{B}\mathcal{F} := \Sigma^\infty B\mathcal{F}$ (see Section 4.5). The following two sections are intended to outline a preamble for such a problem.

3 THE SEGAL CONJECTURE AND THE STABLE HOMOTOPY OF $BG_p^\wedge$

The Segal conjecture, proved by Carlsson [19], led to several consequences. One of them is that there is an isomorphism

$$\mathbb{Z}_p^\wedge \widetilde{B}^{\triangleleft}(S,S) \cong [\mathbb{B}S, \mathbb{B}S], \tag{3.1}$$

where S is a finite p-group, $[\mathbb{B}S, \mathbb{B}S]$ is the ring of homotopy classes of self-maps of $\mathbb{B}S := \Sigma^\infty BS$, and $\mathbb{Z}_p^\wedge \widetilde{B}^{\triangleleft}(S,S)$ is the reduced double Burnside ring of S, with coefficients in $\mathbb{Z}_p^\wedge$ (see comments below Theorem 3.2.3).

By isomorphism (3.1), the full subcategory $\mathcal{S}_p$ of the homotopy category of spectra, with objects the stable summands of some $\mathbb{B}S$, where S is a finite p-group, is a Krull-Schmidt category. Therefore, any stable summand X of $\mathbb{B}S$ can uniquely be decomposed as a finite wedge sum of indecomposable summands, i.e.

$$X \simeq \bigvee X_i^{n_i}, \tag{3.2}$$

for some $n_i \geq 0$, where X_i runs over representatives of all homotopy types of indecomposable stable summands of $\mathbb{B}S$. The splitting above is called complete. In particular, $\mathbb{B}G^{\wedge}_p$ is a stable summand of $\mathbb{B}S$, hence it has a complete stable splitting:

$$\mathbb{B}G^{\wedge}_p \simeq \bigvee X_i^{n_i(G)} \tag{3.3}$$

Historically, a first progress to understanding the homotopy type of $\mathbb{B}G^{\wedge}_p$ was given by Nishida [44]: given two finite groups G, H, a necessary condition for them to have homotopy equivalent p-completed classifying spectra is to have isomorphic Sylow p -subgroups. Without loss of generality, we can assume that G, H contain a common Sylow p-subgroup S. By splitting (3.3), this amounts to having

$$n_i(G) = n_i(H), \text{ for all } i.$$

Priddy [51, 52], and later Benson-Feshbach [5] and Martino-Priddy [33, 34] worked on determining $n_i(G)$ in terms of a certain p-local structure of G. Their work partly motivated the notion of biset functors.

4 THE POINT OF VIEW OF BISET FUNCTORS

The indecomposable stable summands of $\mathbb{B}S$ are parametrized by pairs (Q, V), where Q is a subgroup of S and V is a simple $\mathbb{F}_p\mathrm{Out}(Q)$-module. This was first observed by Nishida [44]. In particular, we can write

$$X_i = X_{Q,V} \text{ and } n_i(G) = n_{Q,V}(G),$$

for such a pair (Q, V).

Biset functors (e.g. inflation functors, global Mackey functors) were formally introduced by Bouc [7, 8] in a successful attempt to unify previous work of Webb, Symonds, Miller, etc. They are linear functors from an admissible Burnside category to the category of R-modules, for some commutative ring with identity R (see Section 2.3). This notion is closely related to those of Mackey functors for a (fixed) finite group G, a fundamental tool in equivariant stable homotopy theory. In particular, inflation functors provided a suitable framework to study the computation of $n_{Q,V}(G)$ in purely algebraic terms, Webb [70] observed this and reproved the main results in [5] and [33], under this approach.

Biset functors have up to five operations: restriction, induction, isogation, inflation and deflation, depending on the admissible Burnside category chosen (see Definition 2.2.15). Note that the first two operations are also present in the structure of Mackey functors for a finite group. In essence, biset functors are intended to associate an abelian group, module, ring, etc. to any finite group, in a compatible way. Two of the most representative examples are the group cohomology functor (see Example 2.3.6) and the representation ring functor (see Example 2.3.7).

Each category of biset functors has simple objects and the isomorphism classes of such objects are parametrized by pairs (Q, V), which in this context are called seeds (see Definition 2.1.9), in a similar way to the indecomposable stable summands above. Finding $n_{Q,V}(G)$ above amounts to finding the dimension of $\mathbb{S}^{\triangleright}_{Q,V}(G)$, the evaluation of $\mathbb{S}^{\triangleright}_{Q,V}$ at G, where

$\mathbb{S}^{\triangleright}_{Q,V}$ is the simple inflation functor associated to (Q,V) (see Convention 2.3.3 and Section 2.4). This problem has been treated for different kinds of biset functors and has been subject of deep interest until now, see for instance [10, 12, 63].

5 MAIN RESULTS

This thesis concerns first a progress in the understanding of $\mathbb{B}G^{\wedge}_p$ through the eyes of biset functors, much in the spirit of Webb [70]. Then we introduce generalizations for saturated fusion systems. In particular, by exploiting a version of the Segal conjecture due to Ragnarsson [57], we are able to interpret the problem of characterizing the homotopy type of p-complete classifying spectra as an isomorphism problem in algebra, in the following sense:

$$\mathbb{B}G^{\wedge}_p \simeq \mathbb{B}H^{\wedge}_p \text{ if and only if } G \cong H \text{ in } \mathbb{F}_p B^{\triangleright p},$$

where $\mathbb{F}_p B^{\triangleright p}$ is the p-local right-free Burnside category with coefficients in $\mathbb{F}_p$ (see Remark 3.4.8). From this point of view, we refine an implication in the stable Martino-Priddy conjecture [34, Theorem 1] (see Theorem 3.3.5) by showing the following result, where we instead use the the p-local bifree Burnside category with coefficients in $\mathbb{F}_p$, denoted by $\mathbb{F}_p B^{\Delta_p}$ (see Definition 3.4.2).

Proposition (Proposition 3.4.9). *Let G, H be finite groups. Then $G \cong H$ in $\mathbb{F}_p B^{\Delta_p}$ if $\mathbb{F}_p \operatorname{Cen}(Q,G) \cong \mathbb{F}_p \operatorname{Cen}(Q,H)$ as $\mathbb{F}_p\operatorname{Out}(Q)$-modules, for every finite p-group Q, where $\operatorname{Cen}(Q,G) = \{[f] \in \operatorname{Rep}(Q,G) \mid f$ is injective with p-centric image$\}$.*

In fact, the converse should hold by [34, Theorem 1], as discussed in Section 3.6. However, we have an observation on the proof of [34, Theorem 1]. To be more precise, a subtle argument is not clear, see Section 3.5 for a detailed account. In consequence, we find it convenient to give an independent proof of the equivalence in Theorem 3.5.5.

Along the way of proving the above proposition, we introduce in Section 3.4 the "p-localized" representable functors

$$\mathbb{F}_p B^{\Delta_p}(G,-) \text{ and } \mathbb{F}_p B^{\triangleright p}(G,-),$$

and then study their properties.

In particular, we use the characterization of cohomological simple biset functors with values in $\operatorname{Mod}_{\mathbb{F}_p}$ (see Proposition 2.4.25) to give a proof of the following proposition.

Proposition (Proposition 3.4.14). *Let G be a finite group with S a Sylow p-subgroup. Then*

$$\mathbb{F}_p B^{\triangleright p}(G,-) \cong \bigoplus_{(Q,V)} (\mathbb{P}^{\triangleright}_{Q,V})^{n_{Q,V}(G)},$$

where (Q,V) runs over the isomorphism classes of seeds with Q isomorphic to a subgroup of S, $\mathbb{P}_{Q,V}$ is the projective cover of $\mathbb{S}_{Q,V}$ and

$$n_{Q,V}(G) = \frac{\dim_{\mathbb{F}_p} \mathbb{S}^{\triangleright}_{Q,V}(G)}{\dim_{\mathbb{F}_p} \operatorname{End}_{\mathbb{F}_p\operatorname{Out}(Q)}(V)}.$$

An analogous result holds for $\mathbb{F}_p B^{\Delta_p}(G,-)$. The following corollaries would be immediate consequences of the discussed above, if [34, Proposition 4.5] holds.

Corollary (Corollary 3.5.9)**.** *Let $R = \mathbb{Z}_p^\wedge$ or $\mathbb{F}_p$. Then $RB^{\triangleright p}$ has the same isomorphism classes as RB^{Δ_p}.*

This would answer affirmatively a (reformulated) question by O'Hare [46] in his Ph.D. thesis. The other corollary relates the evaluations of simple inflation functors $\mathbb{S}^{\triangleright}_{Q,V}(G)$ with those of simple global Mackey functors $\mathbb{S}^{\Delta}_{Q,V}(G)$.

Corollary (Corollary 3.6.1)**.** *Let G, H be finite groups. The following are equivalent:*

(1) $\mathbb{S}^{\triangleright}_{Q,V}(G) \cong \mathbb{S}^{\triangleright}_{Q,V}(H)$ *for every finite p-group Q.*

(2) $\mathbb{S}^{\Delta}_{Q,V}(G) \cong \mathbb{S}^{\Delta}_{Q,V}(H)$ *for every finite p-group Q.*

The intricacy of inflation functors with respect to global Mackey functors makes this corollary a bit surprising. Indeed, the inflation operation is present in inflation functors, while global Mackey functors lack this operation, thus becoming easier to handle (see comments below Convention 2.3.3).

Chapter 4 starts with a review of the theory of fusion systems, then we exploit the properties of $\omega_{\mathcal{F}}$, the characteristic idempotent of a saturated fusion system $\mathcal{F}$ (see Section 4.4), in order to naturally adapt classical results for finite groups and our previous results, in this new context. Indeed, we have the following result, inspired by [70, Proposition 5.1].

Proposition (Proposition 4.6.7)**.** *The inflation functor $RB^{\triangleleft}(-,\mathcal{F})$ satisfies the following properties:*

(1) *$RB^{\triangleleft}(-,\mathcal{F})$ is projective.*

(2) $\mathrm{Hom}_{\mathrm{Fun}(RB^{\triangleleft},\mathrm{Mod}_R)}(RB^{\triangleleft}(-,\mathcal{F}), M) \cong M(\mathcal{F})$.

(3) *$RB^{\triangleleft}(-,\mathcal{F})$ is generated by its value at S.*

From this, we infer the following proposition adapting our above proposition.

Proposition (Proposition 4.6.8)**.** *Let R be $\mathbb{Z}_p^\wedge$ or $\mathbb{F}_p$. Then*

$$RB^{\triangleleft}(-,\mathcal{F}) \cong \bigoplus_{(Q,V)} \mathbb{P}_{Q,V}^{n_{Q,V}(\mathcal{F})}$$

with $Q \leq S$, where

$$n_{Q,V}(\mathcal{F}) = \frac{\dim \mathbb{S}_{Q,V}(\mathcal{F})}{\dim \mathrm{End}_{R\mathrm{Out}(Q)}(V)},$$

and dimensions are taken over $\mathbb{F}_p$.

This can be interpreted in terms of a complete stable splitting of $\mathbb{B}\mathcal{F}$, in the sense of Benson-Feshbach [5] and Martino-Priddy [33]. In fact, this result can be viewed as a generalization of Proposition 3.4.14 displayed above, since

$$\mathbb{F}_p B^{\triangleleft}(-,\mathcal{F}_S(G)) \cong \mathbb{F}_p B^{\triangleleft p}(-,G) \text{ and } \mathbb{S}_{Q,V}(\mathcal{F}_S(G)) \cong \mathbb{S}_{Q,V}(G) \text{ for all } (Q,V).$$

However, $\mathbb{F}_pB^{\lhd}(-,\mathcal{F}_S(G))$ and $\mathbb{S}_{Q,V}(\mathcal{F}_S(G))$ are quite less explicit than $\mathbb{F}_pB^{\lhd\, p}(-,G)$ and $\mathbb{S}_{Q,V}(G)$, respectively. After all, the tools employed in each proof differ reasonably.

In Section 4.7, we introduce the notion of biset functors for fusion systems, where we have again that the simple biset functors can be parametrized by seeds (Q,V).

Inspired by [70, Theorem 2.6], we achieve a friendly formula for global Mackey functors evaluated at a fusion system $\mathcal{F}$.

Theorem (Theorem 4.7.4)**.** $\mathbb{S}^{\Delta}_{Q,V}(\mathcal{F}) \cong \bigoplus_L W_L({}^L V)$, *where the direct sum runs over $\mathcal{F}$-fully normalized $L \leq S$ isomorphic to Q, taken up to $\mathcal{F}$-conjugation, and* $W_L = k\left(\sum_{\sigma\in \mathrm{Out}_{\mathcal{F}}(L)} \sigma\right)$ *for some* $k \in \mathbb{F}_p$.

Finally, the above theorem gives us the recipe to show the following result.

Theorem (Theorem 4.7.15)**.** *Let $\mathcal{F}_1, \mathcal{F}_2$ be saturated fusion systems. The following are equivalent.*

(1) $\mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_1) \cong \mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_2)$ *as* $\mathbb{F}_p\mathrm{Out}(Q)$*-modules, for every finite p-group Q*

(2) $\mathcal{F}_1 \cong \mathcal{F}_2$ *in* $\mathbb{F}_pB^{\Delta}$.

This brings an immediate application: the following result provides a sufficient algebraic condition for two classifying spectra of fusion systems to be equivalent.

Corollary (Corollary 4.7.17)**.** *If* $\mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_1) \cong \mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_2)$ *as* $\mathbb{F}_p\mathrm{Out}(Q)$*-modules, for every finite p-group Q, then* $\mathbb{B}\mathcal{F}_1 \simeq \mathbb{B}\mathcal{F}_2$.

6 FURTHER DIRECTIONS

We would like to completely reprove [34, Theorem 1], which we call the stable Martino-Priddy conjecture. A good progress towards this goal was achieved in this thesis. This could help us to understand more generally a likely classification of the homotopy type of $\mathbb{B}\mathcal{F}$ in purely algebraic terms. This is displayed in the following conjecture.

Conjecture. *Let $\mathcal{F}_1, \mathcal{F}_2$ be saturated fusion systems. The following are equivalent:*

(1) $\mathbb{F}_p\mathrm{Inj}(Q,\mathcal{F}_1) \cong \mathbb{F}_p\mathrm{Inj}(Q,\mathcal{F}_2)$ *as* $\mathbb{F}_p\mathrm{Out}(Q)$*-modules, for every finite p-group Q.*

(2) $\mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_1) \cong \mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_2)$ *as* $\mathbb{F}_p\mathrm{Out}(Q)$*-modules, for every finite p-group Q.*

(3) $\mathbb{B}\mathcal{F}_1 \simeq \mathbb{B}\mathcal{F}_2$.

The equivalence between conditions (1) and (3) would be, in practice, our best tool towards understanding how much less information $\mathbb{B}\mathcal{F}$ has with respect to $B\mathcal{F}$. Indeed, $\mathbb{B}\mathcal{F}$ does not suffice to essentialy determine $\mathcal{F}$, unlike $B\mathcal{F}$ (see Example 4.5.3). Even without going beyond finite groups, an interesting question arises: Given a fixed p-group S, can we classify, up to p-fusion, all pairs of finite groups G, H containing S as a Sylow p-subgroup, such that

$$BG^\wedge_p \not\simeq BH^\wedge_p \text{ but } \mathbb{B}G^\wedge_p \simeq \mathbb{B}H^\wedge_p?$$

The work of Yagita [73] and Bartel-Spencer [2] makes us believe that this situation, although it occurs (by Example 4.5.3), it rarely does.

CONTENTS

1

PRELIMINARIES

In this chapter, we give a short introduction to the homotopical properties of p-completed classifying spaces of finite groups.

1.1 THE CLASSIFYING SPACE OF A GROUP

In this section we briefly recall the concept of classifying spaces. Although the motivation of the author comes from homotopy theory, the main results of this thesis may be of independent interest for representation theorists, so we find it convenient to give a short review of this subject for the interested reader who is not familiar with these spaces.

Let G be a topological group, its *classifying space* BG is the base space of a universal G-principal bundle [41]

$$p : EG \to BG.$$

By definition, EG is a weakly contractible free G-space. Moreover, BG can be taken as the orbit space EG/G and p as the quotient map

$$\pi : EG \to EG/G = BG.$$

The total space EG is unique up to G-weak equivalence, therefore BG is unique up to weak equivalence. On the other hand, the fiber sequence $G \hookrightarrow EG \to BG$ induces a long exact sequence

$$\cdots \to \pi_{i+1}(BG) \to \pi_i(G) \to \pi_i(EG) \to \pi_i(BG) \to \cdots$$

thus $\pi_i(G) \cong \pi_{i+1}(BG)$, since EG is weakly contractible. In fact, G is weakly equivalent to ΩBG, see for instance [41, Theorem 11.1]. Moreover, if we take a model for EG which is a G-CW-complex, we can construct a diagram of fibrations

$$\begin{array}{ccc} G & \longrightarrow & \Omega BG \\ \downarrow & & \downarrow \\ EG & \longrightarrow & PBG \\ \downarrow & & \downarrow \\ BG & \xrightarrow{1} & BG \end{array}$$

and obtain that $G \to \Omega BG$ is a weak homotopy equivalence.

Example 1.1.1. Classifying spaces cover a wide variety of well-known spaces. We give just a short list: $B\mathbb{Z} \simeq S^1, BS^1 \simeq \mathbb{C}P^\infty, B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty, B\mathbb{Z}^2 \simeq S^1 \times S^1, B\Sigma_n \simeq \mathrm{UConf}_n(\mathbb{R}^\infty)$.

Remark 1.1.2. Hereafter, we will restrict to the case of discrete groups.

Let G be a group. Then we have $\pi_0(G) \cong G$ and $\pi_i(G)$ vanishes otherwise. Hence $\pi_1(BG) \cong G$ and $\pi_i(G)$ vanishes for $i > 1$, i.e. BG is an Eilenberg-MacLane space $K(G, 1)$.

Proposition 1.1.3. *Let G, H be groups. Then the map*

$$[BG, BH]_* \to \mathrm{Hom}(G, H) \tag{1.1.4}$$

sending the pointed homotopy class of $f : BG \to BH$ to $\pi_1(f)$, is a natural bijection.

Consequently, BG is pointed homotopy equivalent to BH if and only if G is isomorphic to H. Now, let

$$\mathrm{Rep}(G, H) = \mathrm{Hom}(G, H)/H,$$

with H acting on $\mathrm{Hom}(G, H)$ by conjugation, i.e. $h \cdot \alpha := c_h \circ \alpha$. On the other hand, we recall [28, Proposition 4A.2] that $[BG, BH] \cong [BG, BH]_*/\pi_1(BH) = [BG, BH]_*/H$, therefore passing to the quotients we obtain:

Proposition 1.1.5. *Let G, H be groups. Then the map*

$$\mathrm{Rep}(G, H) \to [BG, BH]$$

sending $[\alpha]$ to $[B\alpha]$, is bijective with inverse given by sending the homotopy class of f : $BG \to BH$ to the class of $\pi_1(f)$.

We have seen that the homotopy type of BG determines G up to isomorphism. What about its stable homotopy type? Does it suffice to determine G up to isomorphism? The answer is no:

Example 1.1.6 (Minami)**.** Let a and b be different odd integers. Let D_{2a} and Q_{4b} be the dihedral group of order $2a$ and the generalized quaternion group of order $4b$, respectively. Then $B(D_{2a} \times Q_{4b})$ is stably homotopy equivalent to $B(D_{2b} \times Q_{4a})$, but $D_{2a} \times Q_{4b}$ and $D_{2b} \times Q_{4a}$ are not isomorphic. More details can be found in [39, Example 4].

1.2 GROUP (CO)HOMOLOGY

Let G be a group, R a commutative ring with identity, and M a RG-module. We define the *nth cohomology group of G with coefficients in M*

$$H^n(G; M) := \mathrm{Ext}^n_{RG}(R, M),$$

where R is viewed as a RG-module with the trivial G-action. Dually, define the *nth homology group of G with coefficients in M*

$$H_n(G; M) := \mathrm{Tor}^{RG}_n(M, R).$$

Informally speaking, the cohomology groups $H^*(G; -)$ measure the inexactness of taking invariants $(-)^G$. Dually, the homology groups $H^*(G; -)$ measure the inexactness of taking coinvariants $(-)_G$.

Alternatively, we can study these (co)homology groups of G as the singular (co)homology groups of BG with local coefficients, thanks to the following proposition.

Proposition 1.2.1. *With G and M as above, there exist natural isomorphisms*

$$H^n(BG; M) \cong H^n(G; M),$$

$$H_n(BG; M) \cong H_n(G; M).$$

In fact, the first successful definitions of $H^*(G; M)$ as the cohomology groups of a certain cochan complex were accomplished by exploiting the properties of BG or a particular model for BG, see [72] for a more detailed historical account. At last, this allows the direct computation of the (co)homology groups of G, when those of BG are known.

Now, we briefly recall some basic operations in group cohomology, more details can be found in [4, Chapter 3].

- **Restriction**: Let H be a subgroup of G. The *restriction map* $\mathrm{res}^{\mathrm{G}}_{\mathrm{H}} : H^n(G;M) \to H^n(H;M)$ is the homomorphism ι^* induced by the inclusion $\iota : H \hookrightarrow G$.
- **Transfer**: Let G, H as above. The *transfer map* $\mathrm{tr}^{\mathrm{G}}_{\mathrm{H}} : H^n(H;M) \to H^n(G;M)$ is a homomorphism such that $\mathrm{tr}^{\mathrm{G}}_{\mathrm{H}} \circ \mathrm{res}^{\mathrm{G}}_{\mathrm{H}} : H^n(G;M) \to H^n(G;M)$ is multiplication by $[G : H]$.
- **Inflation**: Let N be a normal subgroup of G. The *inflation map* is the composite $\mathrm{inf}^G_{G/N} : H^n(G/N;M^N) \xrightarrow{\pi^*} H^n(G;M^N) \to H^n(G;M)$, where $\pi : G \to G/N$ is the quotient homomorphism.

In particular, for $M = R$, these operations, together with functoriality of $H^n(-;R)$, will define an *inflation functor*, which we will study in Chapter 2.

Remark 1.2.2. Although transfer is the preferred name in the literature for the operation outlined above, other authors prefer to call it *induction*, this little detail will be meaningful in Chapter 2.

1.3 $\mathbb{F}_p$-COHOMOLOGY AND p-FUSION OF G

Let G be a finite group and S a Sylow p-subgroup of G. We now restrict to the case $R = \mathbb{F}_p$, where the composite $\mathrm{tr}^G_S \circ \mathrm{res}^G_S$ is an isomorphism, since $[G : S]$ is invertible in $\mathbb{F}_p$.

$$\begin{array}{ccc} H^*(G;\mathbb{F}_p) & \xrightarrow{\quad [G:S] \quad} & H^*(G;\mathbb{F}_p) \\ & {\scriptstyle \mathrm{res}^G_S} \searrow \quad \nearrow {\scriptstyle \mathrm{tr}^G_S} & \\ & H^*(S;\mathbb{F}_p) & \end{array}$$

Therefore res^G_S is injective, and $H^*(G;\mathbb{F}_p)$ can be identified with a subring of $H^*(S;\mathbb{F}_p)$. This subring can be described in terms of the *p-fusion* of G, i.e. p-subgroups of G and conjugation homomorphisms by elements of G between them. For that, we will need the notion of *stable elements*.

Definition 1.3.1. Let G be a finite group, S a Sylow p-subgroup of G. We say that $a \in H^*(S;\mathbb{F}_p)$ is *stable* if, for every $g \in G$,

$$\operatorname{res}^{gSg^{-1}}_{S\cap gSg^{-1}} \circ c_g^*(a) = \operatorname{res}^{S}_{gSg^{-1}\cap S}(a).$$

The following is known as the *double coset formula* [20, Chapter XII, Section 8].

Proposition 1.3.2 (Cartan-Eilenberg)**.** *For every $H, K \leq G$, we have*

$$\operatorname{res}^G_H \circ \operatorname{tr}^G_K = \sum_{g\in[H\backslash G/K]} \operatorname{tr}^H_{H\cap gKg^{-1}} \circ \operatorname{res}^{gKg^{-1}}_{H\cap gKg^{-1}} \circ c_g^*,$$

where $[H\backslash G/K]$ denotes a set of representatives of the (H,K)-double cosets.

The following result is classical, we find it opportune to give a proof since it works more generally for cohomological functors (see Definition 2.4.24).

Theorem 1.3.3 (Cartan-Eilenberg)**.** *Let G be a finite group, and S a Sylow p-subgroup of G. The image of res^G_S is the subring of stable elements of $H^*(S;\mathbb{F}_p)$.*

Proof. Let a belong to the image of res^G_S. There exists $b \in H^*(G;\mathbb{F}_p)$ such that $a = \operatorname{res}^G_S(b)$. For every $g \in G$, c_g^* is the identity on $H^*(G;\mathbb{F}_p)$ and $c_g^* \circ \operatorname{res}^G_H = \operatorname{res}^G_{gHg^{-1}} \circ c_g^*$ for every $H \leq G$ [20, Chapter XII, Section 8], hence

$$\begin{aligned} c_g^*(a) &= c_g^* \circ \operatorname{res}^G_S(b) \\ &= \operatorname{res}^G_{gSg^{-1}} \circ c_g^*(b) \\ &= \operatorname{res}^G_{gSg^{-1}}(b). \end{aligned}$$

and in particular,

$$\begin{aligned} \operatorname{res}^{gSg^{-1}}_{S\cap gSg^{-1}} \circ c_g^*(a) &= \operatorname{res}^{gSg^{-1}}_{S\cap gSg^{-1}} \circ \operatorname{res}^G_{gSg^{-1}}(b) \\ &= \operatorname{res}^G_{S\cap gSg^{-1}}(b) \\ &= \operatorname{res}^S_{S\cap gSg^{-1}} \circ \operatorname{res}^G_S(b) \\ &= \operatorname{res}^S_{S\cap gSg^{-1}}(a). \end{aligned}$$

This shows that a is stable.

Conversely, let $a \in H^*(S;\mathbb{F}_p)$ be a stable element, we have by Proposition 1.3.2

$$\begin{aligned}\mathrm{res}_S^G \circ \mathrm{tr}_S^G(a) &= \sum_{g\in[S\backslash G/S]} \mathrm{tr}^S_{S\cap gSg^{-1}} \circ \mathrm{res}^{gSg^{-1}}_{S\cap gSg^{-1}} \circ c_g^*(a)\\ &= \sum_{g\in[S\backslash G/S]} \mathrm{tr}^S_{S\cap gSg^{-1}} \circ \mathrm{res}^{S}_{S\cap gSg^{-1}}(a)\\ &= \sum_{g\in[S\backslash G/S]} \left[S : S\cap gSg^{-1}\right] a\\ &= [G:S]a,\end{aligned}$$

where we used that a is stable in the third equality, the fourth equality is a property of double cosets. Then, since $[G:S]$ is invertible in $\mathbb{F}_p$

$$\mathrm{res}_S^G \circ \mathrm{tr}_S^G\left(\frac{a}{[G:S]}\right) = a$$

Thus, $a \in \mathrm{Im}\left(\mathrm{res}_S^G\right)$. □

Consequently, $H^*(G;\mathbb{F}_p)$ is isomorphic to the subring of stable elements of $H^*(S;\mathbb{F}_p)$. Furthermore, let us consider the action of the Weyl group $W_G(S) := N_G(S)/S$ on $H^*(S;\mathbb{F}_p)$ defined by $\overline{g}.x := (c_g)^*(x)$, with $x \in H^*(S;\mathbb{F}_p)$ and $(c_g)^* : H^*(S;\mathbb{F}_p) \to H^*(S;\mathbb{F}_p)$ since, by assumption, $gSg^{-1} = S$. Then we have that

$$\mathrm{Im}(\mathrm{res}_S^G) \subset H^*(S;\mathbb{F}_p)^{W_G(S)}.$$

Indeed, given $b \in H^*(BG;\mathbb{F}_p)$ and $\overline{g} \in W_G(S)$, $c_g^* \circ \mathrm{res}_S^G(b) = \mathrm{res}_S^G(c_g^*(b)) = \mathrm{res}_S^G(b)$.

In particular, it is not hard to check that if S is abelian, this inclusion is actually an equality.

Corollary 1.3.4. *Let G be a finite group and S and Sylow p-subgroup of G that is abelian, then*

$$H^*(G;\mathbb{F}_p) \cong H^*(S;\mathbb{F}_p)^{W_G(S)}.$$

Remark 1.3.5. Alternatively, it is easy to see that $a \in H^*(S;\mathbb{F}_p)$ is stable if and only if

$$\varphi^*(a) = \mathrm{res}_P^S(a)$$

for all $P \leq S$ and each $\varphi = c_g : P \to S$ with $g \in G$.

Remark 1.3.6. Throughout this section, we could have used any ring R such that $[G : S]$ is invertible in R, instead of $\mathbb{F}_p$.

In summary, the Cartan-Eilenberg Theorem says that we can recover the $\mathbb{F}_p$-cohomology of any finite group from that of finite p-groups and then find the stable elements.

We have seen that mod-p cohomology is an invariant that only sees p-fusion, i.e. two groups with isomorphic p-fusion are indistinguishable under the eyes of $H^*(-;\mathbb{F}_p)$.

1.4 COMPLETION OF ABELIAN GROUPS

In this section, we recall the algebraic p-completion and its derived variants, as a motivation to define homotopical p-completion. With this, we will be able to see that $B(-)^\wedge_p$ resembles $H(-;\mathbb{F}_p)$ in the sense explained above (see Theorem 1.7.7).

Let M be an R-module and $I \subset R$ an ideal. The I-adic completion of M is the limit

$$M^\wedge_I = \lim_n M/I^n M.$$

There is always a natural R-module homomorphism $c_M : M \to M^\wedge_I$. We say that M is *I-complete* if c_M is an isomorphism. The nomenclature comes from the fact that the sequence $\{I^n M\}$ forms a local topological basis for $0 \in M$, hence we can speak of Cauchy sequences in the I-adic topology. Thus, equivalently, M is complete if every Cauchy sequence in M converges to an element in M.

If I is finitely generated, then $M^\wedge_I$ is I-complete. On the other hand, if M is finitely generated, then $M^\wedge_I \cong M \otimes_R R^\wedge_I$ and c_M can be identified with the homomorphism $M \to M \otimes_R R^\wedge_I$ that sends m to $m \otimes 1_{R^\wedge_I}$.

Remark 1.4.1. Considered as a functor, the I-adic completion $(-)^\wedge_I$ is neither left nor right exact in general, unlike localization functors, that are exact.

Hereafter, we will concentrate on the case $R = \mathbb{Z}$ and $I = (p)$. In this case, the I-adic completion of an abelian group A is called the *p-completion* of A and is denoted $A^\wedge_p$.

Example 1.4.2. The ring of p-adic integers $\mathbb{Z}_p^\wedge$ is the p-completion of $\mathbb{Z}$. An explicit description of $\mathbb{Z}_p^\wedge$ is the ring of formal series with integer coefficients

$$\left\{ \sum_{i=o}^{\infty} a_i p^i \middle| 0 \leq a_i < p \right\}.$$

Example 1.4.3. $(\mathbb{Z}/p^k)_p^\wedge \cong \mathbb{Z}/p^k$ and $(\mathbb{Z}/q)_p^\wedge = 0$ if q is coprime to p.

More generally, by the classification of finitely generated abelian groups, given a group A of such a kind, we have a decomposition

$$A \cong \mathbb{Z}^r \oplus \bigoplus_{q \text{ prime}} A_q,$$

for a finite sequence of primes q, where $A_q \cong \bigoplus_{k=1}^{s} \mathbb{Z}/q^{m_k}$ for some $m_k \geq 1$. Thus

$$A_p^\wedge \cong (\mathbb{Z}_p^\wedge)^r \oplus A_p.$$

Remark 1.4.4. As discussed above, if A is a finitely generated abelian group, then

$$A_p^\wedge \cong A \otimes_{\mathbb{Z}} \mathbb{Z}_p^\wedge.$$

Even though $(-)_p^\wedge$ is not right exact in general, it still makes sense to speak of its left derived functors. Indeed, given an abelian group, choose a free resolution

$$0 \to F' \to F \to A \to 0.$$

The induced sequence

$$(F')_p^\wedge \to F_p^\wedge \to A_p^\wedge$$

is not exact anymore, but we can define $L_0(A)$ and $L_1(A)$ as the zeroth and first homology group, respectively, of the chain complex

$$\cdots \to 0 \to (F')_p^\wedge \to F_p^\wedge \to 0.$$

In other words, $L_0(A) = \mathrm{Coker}((F')^\wedge_p \to F^\wedge_p)$ and $L_1(A) = \mathrm{Ker}((F')^\wedge_p \to F^\wedge_p)$. The higher left derived functors are zero. Of course, if $(-)^\wedge_p$ were right exact, we would have $L_0(A) \cong A^\wedge_p$.

Thus, despite the lack of exactness of p-completion, we have the following exact sequence

$$0 \to L_1(A) \to (F')^\wedge_p \to F^\wedge_p \to L_0(A) \to 0$$

Functoriality of $(-)^\wedge_p$ allows to induce a natural map $\phi : A \to L_0(A)$ in such a way that we have a map of exact sequences

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & A & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_1(A) & \longrightarrow & (F')^\wedge_p & \longrightarrow & F^\wedge_p & \longrightarrow & L_0(A) & \longrightarrow & 0 \end{array}$$

Fortunately, there are classes of abelian groups where p-completion behaves well:

Lemma 1.4.5 ([38, Lemma 10.1.4])**.** *Let A be either a finitely generated abelian group or a free abelian group. Then $L_0(A) = A^\wedge_p$, $L_1(A) = 0$ and $\phi : A \to L_0(A)$ coincides with p-completion.*

Definition 1.4.6. We say that an abelian group A is *completable* if $L_1(A) = 0$. Under this assumption, $\phi : A \to L_0(A)$ is called the *derived p-completion* of A. On the other hand, we say that A is *derived p-complete* if $\phi : A \to L_0(A)$ is an isomorphism.

There is no danger of confusion, since a derived p-complete group A is completable [38, Proposition 10.1.18]. We can give explicit descriptions for $L_0(A)$ and $L_1(A)$ in general. Indeed [38, Proposition 10.1.17] shows

$$L_0(A) \cong \mathrm{Ext}(\mathbb{Z}/p^\infty, A) \text{ and } L_1(A) \cong \mathrm{Hom}(\mathbb{Z}/p^\infty, A).$$

Example 1.4.7. Using this description, we immediately see that $\mathbb{Z}/p^\infty$ is not completable. Indeed,

$$L_1(\mathbb{Z}/p^\infty) \cong \mathrm{Hom}(\bigcup_n \mathbb{Z}/p^n, \mathbb{Z}/p^\infty) \cong \lim_n \mathrm{Hom}(\mathbb{Z}/p^n, \mathbb{Z}/p^\infty) \cong \lim_n \mathbb{Z}/p^n \cong \mathbb{Z}^\wedge_p.$$

The following proposition allows us to measure the deviation of derived p-completion with respect to p-completion:

Proposition 1.4.8 ([38, Proposition 10.1.11])**.** *Let A be an abelian group, there is a short exact sequence*

$$0 \to \lim_i{}^1 \operatorname{Hom}(\mathbb{Z}/p^i, A) \to L_0 A \to A^\wedge_p \to 0.$$

On the other hand, the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}/p^\infty \to 0$$

induces a long exact sequence of $\operatorname{Ext}^i(-, A)$ groups. From this, the following lemma follows easily.

Lemma 1.4.9 ([38, Remark 10.1.21])**.** *An abelian group A is derived p-complete if and only if*

$$\operatorname{Ext}(\mathbb{Z}[1/p], A) = 0 = \operatorname{Hom}(\mathbb{Z}[1/p], A).$$

1.5 BOUSFIELD LOCALIZATION AND COMPLETION

In classical homotopy theory, we work with homotopy equivalences as isomorphisms. However, there are other weaker notions of equivalences/isomorphisms that will be convenient to better understand what the local study of spaces or spectra means. In particular, this will allow us to find a topological analogue of the algebraic p-completion.

Given a generalized homology theory E_*, represented by a spectrum E, we would like to think of E_*-equivalences (i.e. maps that induce an isomorphism in E_*-homology) as isomorphisms (of spaces or spectra), instead of homotopy equivalences. This is accomplished via the *Bousfield localization* [14, 13]: Let $\mathcal{C}$ be the homotopy category of spaces or spectra, an object $Z \in \mathcal{C}$ is called *E_*-local* if every E_*-equivalence $f : X \to Y$ induces a bijection

$$[Y, Z] \xrightarrow{f^*} [X, Z].$$

Definition 1.5.1 (Bousfield)**.** An *E_*-localization* is a pair (L_E, η), where $L_E : \mathcal{C} \to \mathcal{C}$ is a functor and $\eta : 1_{\mathcal{C}} \to L_E$ a natural transformation such that, for every object X in $\mathcal{C}$, we have

1. $L_E X$ is E_*-local and

2. $\eta_X : X \to L_E X$ is an E_*-equivalence.

It is not hard to check that, for every object X in $\mathcal{C}$,

- η_X is initial among morphisms $f : X \to Y$ with Y E_*-local.
- η_X is terminal among E_*-equivalences $f : X \to Y$.

Moreover, any E_*-equivalence $f : X \to Y$ between E_*-local objects is in fact a homotopy equivalence.

The subcategory $\mathcal{C}_{E_*}$ of E_*-local objects is a model for the ordinary localization $\mathcal{C}[E_* - \text{equiv}^{-1}]$, i.e. there exists an equivalence

$$\mathcal{C}[E_* - \text{equiv}^{-1}] \to \mathcal{C}_{E_*}.$$

Bousfield localization is then a more sophisticated alternative to the ordinary localization, which is rather much more abstract. In summary, we have a tool to detect E_*-equivalences $f : X \to Y$ through its induced map $L_E(f) : L_E X \to L_E Y$. Informally, E_*-localization allows us to concentrate on what E_* sees. In particular, when $E_* = H_*(-;R)$, we can say that $H_*(-;R)$-localization is intended to isolate the R-local information of spaces or spectra, although there is another way to do this:

For every space X, the *Bousfield-Kan R-completion* of X or simply the R-completion of X [16, Chapter I], denoted by $X^\wedge_R$, is another space constructed functorially in such a way that it satisfies the following properties:

- A map $f : X \to Y$ induces an isomorphism $f_* : H_*(X;R) \to H_*(Y;R)$ if and only if the induced map $f^\wedge_R : X^\wedge_R \to Y^\wedge_R$ is a homotopy equivalence.
- For every space X, there exists a natural map

$$\zeta_X : X \to X^\wedge_R.$$

At this point, we note evident similarities between R-completion and $H_*(-;R)$-localization, which may lead us to think that they are the same. It is not the case: unlike η_X, we do not have in general that ζ_X induces an isomorphism in $H_*(-;R)$.

Definition 1.5.2. Let X be a space, then

1. X is called R-*good* if $(\zeta_X)_* : H_*(X;R) \to H_*(X^\wedge_R;R)$ is an isomorphism.

2. X is called R-*complete* if ζ_X is a homotopy equivalence.

Of course, it follows that if X is R-good, then $X^\wedge_R$ is R-complete. In particular, when X is R-good, $X^\wedge_R$ and $L_{HR}X$ are homotopy equivalent. If X is not R-good, we say that X is R-*bad*. When $R = \mathbb{F}_p$, the class of $\mathbb{F}_p$-good spaces is quite large (e.g. all nilpotent spaces are $\mathbb{F}_p$-good). For simplicity, we will say that a space is p-good (resp. p-complete) instead of $\mathbb{F}_p$-good (resp. $\mathbb{F}_p$-complete).

Example 1.5.3. We say that a ring R is *solid* if the map $R \otimes_{\mathbb{Z}} R \to R$ is an isomorphism. In particular, $\mathbb{F}_p$ is solid. The bouquet $S^1 \vee S^1$ is R-bad for any solid ring R [66, Theorem 2.7].

We now concentrate on the cases $R = \mathbb{F}_p$, where $\mathbb{F}_p$-completion is now called p-completion, and $R = \mathbb{Z}$. The following proposition can be found in [15, Chapter VII, Proposition 4.3, Proposition 5.1].

Proposition 1.5.4. *Let X be a space.*

1. *If $\pi_1(X)$ is finite, then X is p-good for every prime p.*

2. *If $\pi_i(X)$ is finite for every i, then X is $\mathbb{Z}$-good.*

Proposition 1.5.5 ([15, Chapter VII, Corollary 4.2])**.** *Let X be a space such that $\pi_i(X)$ is finite for every i. Then*

$$X^\wedge_{\mathbb{Z}} \simeq \prod_p X^\wedge_p.$$

For instance, the classifying space of a finite group BG satisfies these hypotheses, hence BG is both p-good and $\mathbb{Z}$-good. Therefore

$$BG^\wedge_{\mathbb{Z}} \simeq \prod_p BG^\wedge_p. \tag{1.5.6}$$

However, we can just take the product running over the primes p dividing $|G|$. Indeed, let $q \nmid |G|$. It is known that $|G|$ annihilates $H_n(G;\mathbb{F}_q)$, for every $n > 0$, but $|G|$ is invertible in $\mathbb{F}_q$, hence $H_n(G;\mathbb{F}_q) = 0$. Consequently,

$$H_n(BG^\wedge_q;\mathbb{F}_q) \cong H_n(BG;\mathbb{F}_q) \cong H_n(G;\mathbb{F}_q) = 0.$$

By definition of p-completion, we have $BG^\wedge_q \simeq *$.

We recall that the idea behind p-completion and $H_*(-;\mathbb{F}_p)$-localization is to study spaces "one prime at a time", but p-completion has the advantage that its construction is more explicit in general, thereby allowing important tools for computations (e.g. the Bousfield-Kan spectral sequence).

On the other hand, we recall [15, Chapter VI, Section 6] that, when restricting to nilpotent spaces (simply connected spaces, abelian spaces, etc.), p-completion and $H_*(-;\mathbb{F}_p)$ are the same, thus we can use them both indistinctly.

The following proposition illustrates the relationship between topological and algebraic p-completion.

Proposition 1.5.7. *Let X be a simply connected space. Then X is p-complete if and only if $\pi_n(X)$ is derived p-complete for every $n \geq 1$.*

Let us note that the condition for X of being simply connected is to avoid the fundamental group, since the higher homotopy groups are abelian. Nevertheless, we can define (derived) p-completion more generally for nilpotent groups as follows: let G be a nilpotent group, we define $L_0G := \pi_1(BG^\wedge_p)$, $L_1G := \pi_2(BG^\wedge_p)$ and say that G is derived p-complete if the homomorphism $G \to L_0G$ induced by $BG \to BG^\wedge_p$ is an isomorphism. In this way, we can more generally state the following.

Proposition 1.5.8 ([15, Chapter IX, Theorem 3.1])**.** *Let X be a nilpotent space. Then X is p-complete if and only if $\pi_n(X)$ is derived p-complete for every $n \geq 1$. Moreover, there is a short exact sequence*

$$0 \to L_0\pi_nX \to \pi_n(X^\wedge_p) \to L_1\pi_{n-1}X \to 0, \tag{1.5.9}$$

for every $n \geq 1$.

In particular, if A is an abelian group, the p-completed Eilenberg-MacLane space $K(A,n)^\wedge_p$ has $\pi_n(K(A,n)^\wedge_p) \cong L_0A$ and $\pi_{n+1}(K(A,n)^\wedge_p) \cong L_1A$ as its only non-trivial homotopy groups. For instance, by Lemma 1.4.5, if A is finitely generated, then $\pi_n(K(A,n)^\wedge_p) \cong A^\wedge_p$ and its other homotopy groups vanish. More generally, we have the following result.

Corollary 1.5.10. *Let X be an abelian space with finitely generated homotopy groups, then*

$$\pi_n(X^\wedge_p) \cong \pi_n(X)^\wedge_p \cong \pi_n(X) \otimes \mathbb{Z}^\wedge_p,$$

for every $n \geq 1$.

Remark 1.5.11. We cannot expect p-completion to commute with homotopy groups for more general spaces. For instance, $K(\mathbb{Z}/p^\infty, n)^\wedge_p \simeq K(\mathbb{Z}^\wedge_p, n+1)$.

1.6 BOUSFIELD p-COMPLETION FOR SPECTRA

In this section, we define the notion of p-completion for spectra as a special case of Bousfield localization. For that, we recall the definition and properties of the *Moore spectrum*.

Definition 1.6.1. Given an abelian group A, the *Moore spectrum MA* of A is characterized by

(1) $\pi_r MA = 0$, for $r < 0$.

(2) $\pi_0 MA \cong A$.

(3) $H_r(MA;\mathbb{Z}) = 0$, for $r > 0$.

Lemma 1.6.2 ([14, Section 2])**.** *Given a spectrum X and an abelian group A, we have the following universal coefficient short exact sequences*

$$0 \to A \otimes \pi_* X \to \pi_*(MA \wedge X) \to \mathrm{Tor}(A, \pi_{*-1} X) \tag{1.6.3}$$

and

$$0 \to \mathrm{Ext}(A, \pi_{*+1} X) \to [MA, X]_* \to \mathrm{Hom}(A, \pi_* X) \to 0. \tag{1.6.4}$$

These sequences are natural in X.

Let us turn to the case $A = \mathbb{Z}/p$. We say that a spectrum X has uniquely p-divisible homotopy groups if for every $a \in \pi_* X$ there exists $b \in \pi_* X$ with $pb = a$. Then both Ext-term and the Hom-term in the exact sequence (1.6.4) are trivial.

Definition 1.6.5. Let E_* be a generalized homology theory. A spectrum A is called E_*-*acyclic* if $E_* A = 0$.

Remark 1.6.6. Throughout the rest of this section, we reserve the letter A to denote an acyclic spectrum.

We note that a spectrum X is E_*-local if and only if $[A, X]_* = 0$ for every E_*-acyclic spectrum A. When $E = M\mathbb{Z}/p$, we denote $L_{M\mathbb{Z}/p}X$ (recall Definition 1.5.1) simply by $X_p^\wedge$, and call it *the p-completion of X*.

Remark 1.6.7. If $\pi_* X$ are uniquely p-divisible, then X is $M\mathbb{Z}/p$-acyclic by (1.6.4).

Our goal for the next section is to show that the infinite suspension and p-completion commute in the case of the classifying space of a finite group. Previously, we will see some properties of the stable p-completion.

Proposition 1.6.8. *Let X be a spectrum and p a prime. Then*

$$X_p^\wedge \simeq F(\Sigma^{-1}M(\mathbb{Z}/p^\infty), X)$$

Proof. The cofiber sequence

$$\Sigma^{-1}M(\mathbb{Z}/p^\infty) \to \mathbb{S} \to M(\mathbb{Z}[1/p]) \to M(\mathbb{Z}/p^\infty)$$

induces a fiber sequence

$$\begin{array}{ccccc} F(M(\mathbb{Z}[1/p]), X) & \longrightarrow & F(\mathbb{S}, X) \cong X & \longrightarrow & F(\Sigma^{-1}M(\mathbb{Z}/p^\infty), X) \\ & & & & \downarrow \\ & & & & \Sigma F(M(\mathbb{Z}[1/p]), X) \end{array}$$

Let A be an $M\mathbb{Z}/p$-acyclic spectrum. By induction, $A \wedge M\mathbb{Z}/p \simeq *$ implies $A \wedge M(\mathbb{Z}/p^n) \simeq *$, thus $A \wedge M(\mathbb{Z}/p^\infty) \simeq *$.

The spectrum $F(\Sigma^{-1}M(\mathbb{Z}/p^\infty), X)$ is $M\mathbb{Z}/p$-local since, for A as above, by adjunction,

$$[A, F(\Sigma^{-1}M(\mathbb{Z}/p^\infty), X)]_* = [A \wedge \Sigma^{-1}M(\mathbb{Z}/p^\infty), X]_* \cong [*, X]_* = 0.$$

The homotopy groups of $F(M(\mathbb{Z}[1/p]), X)$ are uniquely p-divisible, hence it is $M\mathbb{Z}/p$-acyclic. Therefore $X \to F(\Sigma^{-1}M(\mathbb{Z}/p^\infty), X)$ is a $M\mathbb{Z}/p$-localization. □

The following result is the stable analogue of Corollary 1.5.10. We find it convenient to give a proof based on the properties of $M\mathbb{Z}/p$ and the nice description of $X_p^\wedge$ above.

Proposition 1.6.9. *Let X be a spectrum and p a prime. If $\pi_n(X)$ is finitely generated for all $n \in \mathbb{Z}$, then*

$$\pi_n(X_p^\wedge) \cong \pi_n(X) \otimes \mathbb{Z}_p^\wedge$$

for all $n \in \mathbb{Z}$.

Proof. We use the short exact sequence (1.6.4) for $\mathbb{Z}/p^\infty$

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_* X) \to [M(\mathbb{Z}/p^\infty), X]_{*-1} \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{*-1} X) \to 0. \tag{1.6.10}$$

But

$$[M(\mathbb{Z}/p^\infty), X]_{*-1} \cong [\Sigma^{-1} M(\mathbb{Z}/p^\infty), X]_* \cong [\mathbb{S}, F(\Sigma^{-1} M(\mathbb{Z}/p^\infty), X)]_* \cong \pi_*(X^\wedge_p).$$

As $\pi_{*-1}X$ is finitely generated and abelian, the Hom-term vanishes. Thus, $\pi_*(X^\wedge_p) \cong \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_* X)$. On the other hand, $\lim_n{}^1 \mathrm{Hom}(\mathbb{Z}/p^n, \pi_* X) = 0$, hence

$$\mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_* X) \cong \lim_n \mathrm{Ext}(\mathbb{Z}/p^n, \pi_* X) \cong \lim_n \pi_* X/(p^n) \cong \pi_* X \otimes \mathbb{Z}^\wedge_p,$$

so we conclude. □

Remark 1.6.11. Implicitly in the previous proof, we obtained the stable analogue of Proposition 1.5.8. Indeed, the short exact sequence (1.6.10) corresponds essentially to (1.5.9) and it is not hard to check that a spectrum X is p-complete if and only if $\pi_*(X)$ is derived p-complete.

1.7 HOMOTOPY THEORY OF p-COMPLETED CLASSIFYING SPACES OF FINITE GROUPS

Throughout this section, we let G, H denote finite groups and S, S' denote their respective Sylow p-subgroups.

As we said previously, the p-completion of a space X is aimed at retaining only the p-local data of X. In particular, if $X = BG$, we recall that $H^*(BG^\wedge_p; \mathbb{F}_p)$ is isomorphic to the subring of stable elements of $H^*(S; \mathbb{F}_p)$, which is expressed in terms of the p-fusion of G. Thus, we can naturally expect to find connections between the homotopy theory of $BG^\wedge_p$ and the p-fusion of G. This section is devoted to outlining that philosophy.

Proposition 1.7.1 ([1, Proposition III.1.10])**.** *Let P be a finite p-group. Then BP is p-complete.*

Proposition 1.7.2. *The fundamental group of $BG^\wedge_p$ is isomorphic to $G/O^p(G)$.*

More generally, $\pi_1(X^\wedge_p) \cong \pi_1(X)/O^p(\pi_1(X))$ for every space X with finite fundamental group [1, Proposition III.1.11].

On the other hand, the natural map $\zeta_{BG} : BG \to BG^\wedge_p$ induces a map

$$[BP, BG] \to [BP, BG^\wedge_p] \tag{1.7.3}$$

natural in P and compatible with the $\mathrm{Out}(P)$-actions, but more is true:

Theorem 1.7.4 ([40])**.** *The map $(\zeta_{BG})_* : [BP, BG] \to [BP, BG^\wedge_p]$ is a bijection.*

But we recall from Proposition 1.1.5 that $\mathrm{Rep}(P, G) \cong [BP, BG]$ as $\mathrm{Out}(P)$-sets. Therefore we have the following corollary.

Corollary 1.7.5. *The map above induces a bijection $\mathrm{Rep}(P, G) \cong [BP, BG^\wedge_p]$ as $\mathrm{Out}(P)$-sets, naturally in P.*

Now, let us suppose that G, H have isomorphic Sylow p-subgroups S, S', we say that an isomorphism $\gamma : S \to S'$ *preserves p-fusion* if for every $P, Q \leq S$, we have that $\varphi \in \mathrm{Hom}_G(P, Q)$ if and only if $\gamma \circ \varphi \circ \gamma^{-1} \in \mathrm{Hom}_H(\gamma(P), \gamma(Q))$. In this situation, we say that G and H have *isomorphic p-fusion*, and write $G \cong_p H$.

Lemma 1.7.6 ([1, Proposition III.1.15])**.** *Let G, H be finite groups. Then $G \cong_p H$ if and only if $\mathrm{Rep}(P, G) \cong \mathrm{Rep}(P, H)$ as $\mathrm{Out}(P)$-sets, naturally in P*

In this way, Theorem 1.7.4 and Lemma 1.7.6 tell us that the homotopy type of $BG^\wedge_p$ determines the p-fusion of G. At this point, one naturally would ask if the converse holds. Fortunately, the answer is yes, as proved by Oliver.

Theorem 1.7.7 (Martino-Priddy conjecture)**.** *Let G, H be finite groups. Then $BG^\wedge_p$ is homotopy equivalent to $BH^\wedge_p$ if and only if $G \cong_p H$.*

We are interested now in comparing the unstable and stable p-completion, particularly for BG. We find it convenient to give a proof of the following comparison result, an extensive account for more general spaces can be found in [3].

Proposition 1.7.8. *Let G be a finite group. Then*

$$\Sigma^\infty(BG^\wedge_p) \simeq (\Sigma^\infty BG)^\wedge_p.$$

Proof. Since every $\pi_*(BG)$ is finite, $\pi_*(BG_p^\wedge)$ are all finite p-groups [15, Chapter VII, Proposition 4.3]. We now show that the reduced homology groups $\widetilde{H}_*(BG_p^\wedge;\mathbb{Z})$ are all finite p-groups. Let $\widetilde{BG_p^\wedge}$ denote the universal cover of $BG_p^\wedge$. We use the fibration

$$\widetilde{BG_p^\wedge} \to BG_p^\wedge \to B(\pi_1(BG_p^\wedge))$$

and the fact that the reduced homology groups of $\widetilde{BG_p^\wedge}$ and $B(\pi_1(BG_p^\wedge))$ are finite p-groups. The claim follows from a Serre class argument. The next step is to prove that $\pi_*^s(BG_p^\wedge)$ are all finite p-groups: We use the Atiyah-Hirzebruch spectral sequence

$$E^2_{m,n} \cong H_m(BG_p^\wedge;\pi_n^s(S^0)) \Rightarrow \pi^s_{m+n}(BG_p^\wedge).$$

The second page $E^2_{m,n}$ fits into the short exact sequence

$$0 \to H_m(BG_p^\wedge)\otimes \pi_n^s(S^0) \to H_m(BG_p^\wedge;\pi_n^s(S^0)) \to \mathrm{Tor}(H_{m-1}(BG_p^\wedge),\pi_n^s(S^0)) \to 0$$

Here we remember that $\pi_n^s(S^0)$ is a finite abelian group for any $n>0$, it can be expressed as a direct sum of $\mathbb{Z}/q^l$, with q running over some prime numbers. From this it follows that $\mathrm{Tor}(H_{m-1}(BG_p^\wedge),\pi_n^s(S^0)) \in \mathcal{C}$ and $H_m(BG_p^\wedge)\otimes\pi_n^s(S^0) \in \mathcal{C}$, the Serre class of finite abelian p-groups, then $H_m(BG_p^\wedge;\pi_n^s(S^0)) \in \mathcal{C}$ too. Therefore, $E^\infty_{m,n} \in \mathcal{C}$ and thus $\pi^s_{m+n}(BG_p^\wedge) \in \mathcal{C}$. By Lemma 1.4.9 and Remark 1.6.11, it follows that $\Sigma^\infty(BG_p^\wedge)$ is p-complete. □

Informally, Proposition 1.7.8 says that p-completion and infinite suspension “commute" for BG. Thus, we can denote both $\Sigma^\infty(BG_p^\wedge)$ and $(\Sigma^\infty BG)_p^\wedge$ by $\mathbb{B}G_p^\wedge$, without danger of confusion. Similarly, we use $(\mathbb{B}_+G)_p^\wedge$ to denote $(\Sigma^\infty BG_+)_p^\wedge$. Note that $(\mathbb{B}_+G)_p^\wedge \simeq \mathbb{B}G_p^\wedge \vee \mathbb{S}_p^\wedge$.

1.8 THE STABLE TRANSFER

Now, let $p: X \to B$ be a finite covering of degree n and E^* a generalized cohomology theory. A *transfer map* $\mathrm{Tr}_p : E^*(X) \to E^*(B)$ is thought as a "wrong way" map serving as counterpart of the induced map $p^* : E^*(B) \to E^*(X)$ in the sense that the composite

$$E^*(B) \xrightarrow{p^*} E^*(X) \xrightarrow{\mathrm{Tr}_p} E^*(B) \tag{1.8.1}$$

is multiplication by n.

This transfer is inspired by the transfer in group cohomology discussed in Section 1.2. Indeed, let us suppose that $H \le G$, the map $Bi : BH \to BG$ induced by the inclusion $i : H \to G$ is a covering map with fiber G/H and the transfer map Tr_H^G defined in Section 1.2 is simply Tr_{Bi} for $E^* = H^*(-;\mathbb{F}_p)$.

On the other hand, taking $E = \Sigma^\infty BH$, the *stable transfer map* $\mathrm{tr}_H^G : \Sigma^\infty BG \to \Sigma^\infty BH$ is the map induced by $1 \in [\Sigma^\infty BH, \Sigma^\infty BH]$ via

$$\mathrm{Tr}_{Bi} : [\Sigma^\infty BH, \Sigma^\infty BH] \to [\Sigma^\infty BG, \Sigma^\infty BH].$$

The good news is that we can also recover Tr_{Bi} from tr_H^G, since $\mathrm{Tr}_{Bi} = E^*(\mathrm{tr}_H^G)$. Actually, this is true more generally for any finite covering p. The construction of Tr_{Bi} induces that of tr_H^G and conversely.

Conveniently, we outline the construction of tr_H^G so that the reader familiar with the cohomological transfer tr_H^G can find similarities between them. Let us choose a decomposition $G = \coprod_i \tau_i H$. Given an element τ_i, left multiplication by $g \in G$ sends it to $\tau_{\sigma(g)(i)} h_{i,g}$. This gives us a permutation representation $\sigma : G \to \Sigma_n$ and a homomorphism

$$\begin{aligned} G &\to H^n \rtimes \Sigma_n =: \Sigma_n \wr H \\ g &\mapsto \big(h_{1,g}, \dots, h_{n,g}, \sigma(g)\big). \end{aligned}$$

The stable transfer map $\Sigma^\infty BG \to \Sigma^\infty BH$ is the adjoint to the composite

$$\begin{array}{ccccccc} BG & \longrightarrow & B(\Sigma_n \wr H) & \xrightarrow{\simeq} & BH^n \times_{\Sigma_n} E\Sigma_n & \longrightarrow & (QBH)^n \times_{\Sigma_n} E\Sigma_n \\ & \searrow & & & & & \downarrow \Theta \\ & & & & & & QBH \end{array}$$

where Θ is the Dyer-Lashof map of $QBH := \Omega^\infty \Sigma^\infty BH$.

In fact, we will mainly work with the stable map $\mathbb{B}G_p^\wedge \xrightarrow{(\mathrm{tr}_H^G)_p^\wedge} \mathbb{B}H_p^\wedge$ induced by the tr_H^G. For this reason, we denote it by tr_H^G as well. Similarly, $(\mathbb{B}_+G)_p^\wedge \xrightarrow{(\mathrm{tr}_H^G)_{+p}^\wedge} (\mathbb{B}_+H)_p^\wedge$ is denoted by tr_H^G when it is clear from the context.

We end this chapter by giving another connection between algebraic and homotopical p-completion.

Proposition 1.8.2 ([62, Proposition 2.12]). *The p-completion functor on spectra induces an isomorphism*

$$[\mathbb{B}G, \mathbb{B}H]^{\wedge}_p \to [\mathbb{B}G^{\wedge}_p, \mathbb{B}H^{\wedge}_p].$$

Similarly,

$$[\mathbb{B}G_+, \mathbb{B}H_+]^{\wedge}_p \cong [(\mathbb{B}G_+)^{\wedge}_p, (\mathbb{B}H_+)^{\wedge}_p].$$

We will see in Section 3.4 that all the homotopy classes of stable maps $\mathbb{B}G^{\wedge}_p \to \mathbb{B}H^{\wedge}_p$ are generated by transfer maps and maps induced by homomorphisms $\varphi : K \to H$, with $K \leq G$, and similarly for stable maps $(\mathbb{B}_+G)^{\wedge}_p \to (\mathbb{B}_+H)^{\wedge}_p$.

2

BISET FUNCTORS FOR FINITE GROUPS

In Section 1.2, we remarked that $H^*(-;R)$ is enriched by some crucial operations. For instance, the double coset formula, involving restriction and transfer, is essential to show the Cartan-Eilenberg theorem. In turn, inflation plays a central role in proving the Tate Theorem in class field theory [43, Theorem 3.1.4]. On the other hand, we can define similar operations in group homology: restriction and induction (see Remark 1.2.2) are also well defined, but instead of inflation, there is a deflation operation. Other important functors (e.g. the complex representation ring) have analogous operations.

These kinds of structures can be approached in their full extent thanks to the notion of biset functors, originally introduced by Bouc in [7], which are the subject of this chapter.

2.1 LINEAR FUNCTORS

Let $\mathcal{C}$ be an additive category and R a commutative ring. The induced R-linear category $R\mathcal{C}$, has the same objects as $\mathcal{C}$ and morphisms $R\mathcal{C}(X,Y) = R \otimes_{\mathbb{Z}} \mathcal{C}(X,Y)$. The category of either covariant or contravariant R-linear functors from $R\mathcal{C}$ to Mod_R, denoted $\mathrm{Fun}(R\mathcal{C}, \mathrm{Mod}_R)$, is abelian, because Mod_R is abelian.

Remark 2.1.1. All functors studied in this section are assumed to be covariant. However, all results here can be dually stated for contravariant functors and thus they also work in that context.

Given an object X of $R\mathcal{C}$, the ring $\mathrm{End}_{R\mathcal{C}}(X)$ is an R-algebra. This object induces a functor E_X from $\mathrm{Fun}(R\mathcal{C}, \mathrm{Mod}_R)$ to $\mathrm{Mod}_{\mathrm{End}_{R\mathcal{C}}(X)}$, called *the evaluation functor at X*, such that $E_X(M) := M(X)$, where $M(X)$ is endowed with the $\mathrm{End}_{R\mathcal{C}}(X)$-module structure given by

$$\varphi \cdot m = M(\varphi)(m),$$

with $\varphi \in \mathrm{End}_{R\mathcal{C}}(X)$ and $m \in M(X)$. Conversely, given an $\mathrm{End}_{R\mathcal{C}}(X)$-module V, there exists $L_{X,V}$ in $\mathrm{Fun}(R\mathcal{C}, \mathrm{Mod}_R)$ defined by

$$L_{X,V}(Y) = R\mathcal{C}(X,Y) \otimes_{\mathrm{End}_{R\mathcal{C}}(X)} V,$$
$$L_{X,V}(\phi)(\psi \otimes v) = (\phi\psi \otimes v),$$

with $\phi \in R\mathcal{C}(Y,Z)$. We will see in the following proposition that $L_{X,-}$ is a left adjoint of E_X. Similarly, we will see that E_X has a right adjoint $R_{X,-}$ defined by

$$R_{X,V}(Y) = \mathrm{Hom}_{\mathrm{End}_{R\mathcal{C}}(X)}(R\mathcal{C}(Y,X), V),$$
$$R_{X,V}(\phi)(f)(\alpha) = f(\alpha\phi),$$

with $f \in \mathrm{Hom}_{R\mathcal{C}}(R\mathcal{C}(Y,X), V)$ and $\alpha \in R\mathcal{C}(Z,X)$.

Proposition 2.1.2. *The functor $L_{X,-}$ (resp. $R_{X,-}$) is a left (resp. right) adjoint of E_X.*

Proof. Let $M \in \mathrm{Fun}(R\mathcal{C}, \mathrm{Mod}_R)$, and $V \in \mathrm{Mod}_{\mathrm{End}_{R\mathcal{C}}(X)}$. We have a natural isomorphism of R-modules

$$\mathrm{Hom}_{\mathrm{Fun}(R\mathcal{C},\mathrm{Mod}_R)}(L_{X,V}, M) \cong \mathrm{Hom}_{\mathrm{End}_{R\mathcal{C}}(X)}(V, M(X))$$

by sending $\eta \colon L_{X,V} \to M$ to $V \cong L_{X,V}(X) \xrightarrow{\eta_X} M(X)$ and $f : V \to M(X)$ to $\eta : L_{X,V} \to M$, where

$$\eta_Y(\alpha \otimes v) = M(\alpha)(f(v)), \text{ with } \alpha \in \mathrm{Hom}_{R\mathcal{C}}(X,Y), v \in V.$$

On the other hand, we have a natural isomorphism of R-modules

$$\mathrm{Hom}_{\mathrm{Fun}(R\mathcal{C},\mathrm{Mod}_R)}(M, R_{X,V}) \cong \mathrm{Hom}_{\mathrm{End}_{R\mathcal{C}}(X)}(M(X), V)$$

by sending $\eta : M \to R_{X,V}$ to $M(X) \xrightarrow{\eta_X} R_{X,V}(X) \cong V$ and $f : M(X) \to V$ to $\eta : M \to R_{X,V}$, where

$$\eta_Y(m)(\alpha) = f(M(\alpha)(m)), \text{ with } m \in M(Y) \text{ and } \alpha \in \mathrm{Hom}_{R\mathcal{C}}(Y,X).$$

This concludes the proof. □

Lemma 2.1.3. *If V is a simple $\mathrm{End}_{R\mathcal{C}}(X)$-module, then $L_{X,V}$ has a unique maximal subfunctor $J_{X,V}$ such that*

$$J_{X,V}(Y) = \left\{ \sum_i \varphi_i \otimes v_i \;\middle|\; \sum_i (\psi\varphi_i)v_i = 0, \text{ for all } \psi \in R\mathcal{C}(Y,X) \right\}.$$

Proof. Let M be a subfunctor of $L_{X,V}$. Then $M(X)$ is a submodule of $L_{X,V}(X) = V$. Therefore $M(X) = V$ or $M(X) = 0$, since V is simple. In the first case, given an object Y in $R\mathcal{C}$ and a morphism $\varphi : X \to Y$, we have $M(\varphi)(1_X \otimes v) = \varphi \otimes v$ for any $v \in V$. As $\varphi \otimes v \in M(Y)$ is a generator of $L_{X,V}(Y)$, we have $M(Y) = L_{X,V}(Y)$.

In the second case, we take a morphism $\psi : Y \to X$, and compute

$$M(\psi)\left(\sum_i \varphi_i \otimes v_i\right) = \sum_i (\psi\varphi_i) \otimes v_i = \sum_i 1_X \otimes (\psi\varphi_i)v_i = 1_X \otimes \sum_i (\psi\varphi_i)v_i.$$

This element belongs to $M(X) = 0$ for any $\sum_i \varphi_i \otimes v_i$, so it vanishes. In this way, we have $M(Y) \subset J_{X,V}(Y)$. The natural structure of $J_{X,V}$ as a subfunctor of $L_{X,V}$ is easy to see, then $J_{X,V}$ is maximal, as claimed. □

Quotients of R-linear functors are defined pointwise. In particular, the quotient $L_{X,V}/J_{X,V}$ is denoted $S_{X,V}$. By Lemma 2.1.3, the functor $S_{X,V}$ is simple, i.e. $S_{X,V}$ has no proper non-zero subfunctor, and $S_{X,V}(X) = V$.

Proposition 2.1.4. *If P is a projective (resp. indecomposable projective) $\mathrm{End}_{R\mathcal{C}}(X)$-module, then $L_{X,P}$ is a projective (resp. indecomposable projective) functor.*

Proof. As P is projective, the functor $V \mapsto \mathrm{Hom}_{\mathrm{End}_{R\mathcal{C}}(X)}(P,V)$ is exact. The evaluation functor E_X is exact by definition, hence their composite $F \mapsto \mathrm{Hom}_{\mathrm{End}_{R\mathcal{C}}(X)}(P,F(X))$ is exact as well. By adjunction, the functor $F \mapsto \mathrm{Hom}_{\mathrm{Fun}(R\mathcal{C},\mathrm{Mod}_R)}(L_{X,P},F)$ is exact, i.e., $L_{X,P}$ is projective. Indecomposability is easy to check. □

Let us suppose that V admits a projective cover $P \to V$. By adjunction, this map induces a morphism $p : L_{X,P} \to S_{X,V}$.

Lemma 2.1.5 ([7, Lemma 2])**.** *If P is a projective cover of V, then $L_{X,P}$ is a projective cover of $S_{X,V}$.*

Consequently, if V is simple, the projective cover $L_{X,P}$ of $S_{X,V}$ is indecomposable.

Remark 2.1.6. We note that both $L_{X,V}$ and $J_{X,V}$ make sense even if V is not simple. Therefore we can still define $S_{X,V}$ as $L_{X,V}/J_{X,V}$, even though now $J_{X,V}$ may not be a maximal subfunctor of $L_{X,V}$, thus $S_{X,V}$ may not be simple.

Now we will see that any simple linear functor from $R\mathcal{C}$ to Mod_R is of the form $S_{X,V}$.

Proposition 2.1.7. *Let $S\colon R\mathcal{C} \to \mathrm{Mod}_R$ be a simple linear functor. Then $S \cong S_{X,V}$ for some pair (X, V) with $S(X) = V$, where V is a simple $\mathrm{End}_{R\mathcal{C}}(X)$-module.*

Proof. As S is non-zero, there must exist X such that $S(X) \neq 0$. We show first that $V := S(X)$ is simple as an $\mathrm{End}_{R\mathcal{C}}(X)$-module. Let M be a non-zero submodule of V and $i\colon M \to V$ the inclusion. By adjunction this corresponds to a non-zero morphism $L_{X,M} \to S$, which must be surjective since S is simple. In particular, its evaluation $M \mapsto V$ at X is surjective, but this is the inclusion above, thus $M = V$, as claimed. Secondly, surjectivity of the non-zero morphism $L_{X,V} \to S$ above (with $M = V$) implies that S is a simple quotient of $L_{X,V}$, which implies that $S \cong S_{X,V}$ by Lemma 2.1.3. □

Alternatively, we can describe $S_{X,V}$ from $R_{X,V}$ as follows:

Proposition 2.1.8. *Let V be a simple $\mathrm{End}_{R\mathcal{C}}(X)$-module. Then $R_{X,V}$ has a unique minimal subfunctor $T_{X,V} = \sum_{\varphi \in R\mathcal{C}(X,-)} \mathrm{Im}(R_{X,V}(\varphi))$ and $T_{X,V} \cong S_{X,V}$.*

Proof. By definition $T_{X,V}$ is the subfunctor of $R_{X,V}$ generated by $R_{X,V}(X) = V$. Let M be a subfunctor of $R_{X,V}$. By adjunction, the inclusion $M \to R_{X,V}$ corresponds to the inclusion $M(X) \to R_{X,V}(X) = V$. The simplicity of V implies that $M(X) = 0$ or $M(X) = V$. In the first case, M must be the zero functor by adjunction again. In the second case, $T_{X,V}$ is a subfunctor of M, therefore $T_{X,V}$ is minimal. In particular $T_{X,V}$ is simple (its only subfunctor is the trivial one). Moreover, $T_{X,V}(X) = V$, hence $T_{X,V} \cong S_{X,V}$. □

We end this section with the following definition.

Definition 2.1.9. A *seed* is a pair (X, V) consisting of an object X of $R\mathcal{C}$ and a simple $\mathrm{End}_{R\mathcal{C}}(X)$-module V. We say that two seeds $(X, V), (X', V')$ are isomorphic if there is an isomorphism $\alpha : X \to X'$ and an R-linear isomorphism $f : V \to V'$ such that $f([\alpha^{-1}\varphi\alpha].v) = \varphi.f(v)$ for all $v \in V, \varphi \in R\mathcal{C}(X', X')$.

It is easy to see that if (X,V) and (X',V') are isomorphic seeds, then $S_{X,V} \cong S_{X',V'}$. In the next section, we will study a particular case of linear functors where this parametrization by seeds is refined.

2.2 BURNSIDE MODULES

The next few pages contain a short background on the theory of biset functors, more details can be found in [9, 70]. We must warn the reader that our notation is not standard, but it is inspired by [46, 58].

Definition 2.2.1. Let G,H be finite groups. A (G,H)-*biset* is a set X equipped with a left G-action and a right H-action that commute, i.e., $(g.x).h = g.(x.h)$. A (G,H)-*biset morphism* $f : X \to Y$ is a (G,H)-equivariant map, i.e., $f(g.x.h) = g.f(x).h$.

We note that a (G,H)-biset structure on X induces a left $(G \times H)$-set structure on X, i.e., $(g,h).x = g.x.h^{-1}$, and conversely. Two (G,H)-bisets X,Y are isomorphic if and only if they are isomorphic as $(H \times G)$-sets. A (G,H)-biset is said to be *transitive* if it is transitive as an $(H \times G)$-set.

The *opposite biset* X° of a (G,H)-biset X, is the set X seen as an (H,G)-biset by inverting the actions, i.e. $h.x.g := g^{-1}.x.h^{-1}$. Given a subgroup $L \le G \times H$, the opposite subgroup $L^{\circ} = \{(h,g) \in H \times G \mid (g,h) \in L\}$ of $H \times G$ satisfies

$$(G \times H/L)^{\circ} \cong (H \times G)/L^{\circ}$$

as (H,G)-bisets. Given a subgroup L of $G \times H$, we define

$$\begin{aligned} k_1(L) &:= \{g \in G \mid (g,1) \in L\}, \\ k_2(L) &:= \{H \in H \mid (1,h) \in L\}, \\ q(L) &:= L/(k_1(L) \times k_2(L)). \end{aligned}$$

We note that $k_i(L) \trianglelefteq p_i(L)$, where p_i is the projection of $G \times H$ to its i-th component for $i = 1,2$. The composition $L \xrightarrow{p_i} p_i(L) \xrightarrow{\pi_i} p_i(L)/k_i(L)$ has kernel $k_1(L) \times k_2(L)$, hence we can induce an isomorphism $\overline{p}_i : q(L) \to p_i(L)/k_i(L)$. In this way, we obtain an isomorphism $\eta = \overline{p}_1 \circ \overline{p}_2^{-1} : p_2(L)/k_2(L) \to p_1(L)/k_1(L)$. Thus, L determines a quintuple $(p_1(L), k_1(L), \eta, p_2(L), k_2(L))$. This motivates the following proposition.

Proposition 2.2.2. *The subgroups L of $G \times H$ correspond bijectively to the quintuples (B, A, η, D, C), with B, A (resp. D, C) subgroups of G (resp. H) such that $A \trianglelefteq B$ (resp. $C \trianglelefteq D$).*

By Proposition 2.2.2, we may expect to decompose $(G \times H)/L$ in terms of the quintuple $(p_1(L), k_1(L), \eta, p_2(L), k_2(L))$, up to isomorphism of (G, H)-bisets. This will be indeed the case, and to show that, we need the following definitions.

Definition 2.2.3. Given a subgroup K of G and a homomorphism $\varphi: K \to H$, the subgroup

$$\Delta(\varphi, K) := \{(k, \varphi(k)) \in G \times H \mid k \in K\}$$

of $G \times H$ is called the *twisted diagonal* of φ. Analogously, given a subgroup J of H and a homomorphism $\alpha : J \to G$, the subgroup

$$\Delta(J, \alpha) := \{(\alpha(j), j) \in G \times H \mid j \in J\}$$

is called the twisted diagonal of α. We note that

$$\Delta(J, \alpha)^{\circ} = \Delta(\alpha, J).$$

We are now ready to define the basic bisets that will allow us to define the homonymous operations that characterize all biset functors (see Remark 2.2.12 and Definition 2.3.1).

Definition 2.2.4. Let G and H be finite groups. We define the following basic bisets:

1. Restriction: Suppose $H \leq G$, the biset $\mathrm{Res}_H^G := (H \times G)/\Delta(H)$ is called *restriction from G to H*.

2. Induction: Suppose $H \leq G$, the biset $\mathrm{Ind}_H^G := (G \times H)/\Delta(H)$ is called *induction from G to H*.

3. Inflation: Given a normal subgroup N of G, the biset $\mathrm{Inf}_{G/N}^G := (G \times (G/N))/\Delta(\pi, G/N)$, where $\pi : G \to G/N$ is the canonical projection, is called *inflation from G/N to G*.

4. Deflation: With N as above, the biset $\mathrm{Def}^G_{G/N} := ((G/N) \times G)/\Delta(G/N, \pi)$ is called *deflation from G to G/N*.

5. Isogation: Suppose there is an isomorphism $\sigma : H \to G$, the biset $\mathrm{Iso}(\sigma) := (G \times H)/\Delta(H, \sigma)$ is called *isogation of* σ.

These bisets are basic in the sense that they generate all finite bisets. Indeed, we have

$$(G \times H)/L \cong \mathrm{Ind}^G_{p_1(L)} \times_{p_1(L)} \mathrm{Inf}^{p_1(L)}_{\frac{p_1(L)}{k_1(L)}} \times_{\frac{p_1(L)}{k_1(L)}} \mathrm{Iso}(\sigma) \times_{\frac{p_2(L)}{k_2(L)}} \mathrm{Def}^{p_2(L)}_{\frac{p_2(L)}{k_2(L)}} \times_{p_2(L)} \mathrm{Res}^H_{p_2(L)} \quad (2.2.5)$$

as (G, H)-bisets.

Definition 2.2.6. We say that a pair (B, A) is a *section* of G if A and B are subgroups of G with $A \trianglelefteq B$. We say that a group K is a *subquotient* of G and denote this by $K \sqsubseteq G$ if $K \cong B/A$, where (B, A) is a section of G. On the other hand, K is a *proper subquotient* of G if $K \sqsubseteq G$ and $K \ncong G$, this is written $K \sqsubset G$.

In particular, with G and H as above, $(p_1(L), k_1(L))$ is a section of G and $(p_2(L), k_2(L))$ is a section of H.

Given a section (A, B) of G and $g \in G$, the pair ${}^g(A, B) := ({}^gA, {}^gB)$ is a section of G too. Therefore, G acts on the set of its sections by conjugation.

A general formula for fiber products $X \times_H Y$ is desirable. Of course, it suffices to have one for transitive bisets. For this purpose, we first define the following product.

Definition 2.2.7. Let G, H and K be finite groups, and $L \leq G \times H$ and $M \leq H \times K$, the *star product of* L *and* M is defined by

$$L * M = \{(g, k) \in G \times K \mid (g, h) \in L \text{ and } (h, k) \in M, \text{ for some } h \in H\}.$$

The following is known as the *double coset formula*:

Proposition 2.2.8 ([9, Lemma 2.3.24])**.** *Let G, H and K be finite groups, and $L \leq G \times H$ and $M \leq H \times K$, there is an isomorphism of finite (G, K)-bisets:*

$$(G \times H)/L \times_H (H \times K)/M \cong \bigsqcup_{h \in [p_2(L) \backslash H / p_1(M)]} \frac{G \times K}{L * {}^{(h,1)}M},$$

where $[p_2(L) \backslash H / p_1(M)]$ is a set of representatives of double cosets.

It is not hard to check that fiber products are associative, up to isomorphism.

Definition 2.2.9. Let G, H be finite groups. The *Burnside module of G and H*, denoted $B(G, H)$, is the Grothendieck group of the abelian monoid of isomorphism classes of finite (H, G)-bisets with disjoint union as addition.

The Burnside module $B(G, H)$ is the free abelian group generated by the isomorphism classes of transitive (H, G)-bisets, that is, classes of the form $[H \times G/L]$, with $L \leq H \times G$ chosen up to $(H \times G)$-conjugation. An element of $B(G, H)$ is called a *virtual (H, G)-biset*, or simply a virtual biset if G, H are clear from the context.

Remark 2.2.10. Our definition of $B(G, H)$ is commonly denoted $B(H, G)$ in the literature, but we found it convenient for the following definition.

Definition 2.2.11. Let R be a commutative ring with identity, we define the *R-Burnside category* RB, with objects the finite groups and morphisms $RB(G, H) := R \otimes B(G, H)$. The associative composition $\circ : B(H, K) \times B(G, H) \to B(G, K)$ is defined by

$$[Y] \circ [X] := [Y \times_H X]$$

for isomorphism classes of transitive bisets, and extended bilinearly. We note that RB is a particular case of a category $R\mathcal{C}$ as in Section 2.1.

Remark 2.2.12. By abuse of notation, we omit the square brackets for the isomorphism classes of $\mathrm{Res}^G_H, \mathrm{Ind}^G_H, \mathrm{Inf}^G_{G/N}, \mathrm{Def}^G_{G/N}$ and $\mathrm{Iso}(\sigma)$. In this way, the isomorphism (2.2.5) now becomes the equality

$$[(G \times H)/L] = \mathrm{Ind}^G_{p_1(L)} \circ \mathrm{Inf}^{p_1(L)}_{\frac{p_1(L)}{k_1(L)}} \circ \mathrm{Iso}(\sigma) \circ \mathrm{Def}^{p_2(L)}_{\frac{p_2(L)}{k_2(L)}} \circ \mathrm{Res}^H_{p_2(L)} \tag{2.2.13}$$

Clearly, the right-hand side of equality (2.2.13) is a composite of morphisms in RB. This feature still makes sense for some subcategories of RB, which we define below.

Definition 2.2.14. Let $\mathcal{D}$ be a class of finite groups and $\mathcal{S}$ a class of sets of subgroups $\mathcal{S}_{G,H}$ of $G \times H$ for each pair $G, H \in \mathcal{D}$. We say that $(\mathcal{D}, \mathcal{S})$ is a *preadmissible pair* if $\mathcal{D}$ is closed under taking subquotients and, for each pair $G, H \in \mathcal{D}$, the set $\mathcal{S}_{G,H}$ is closed under $(G \times H)$-conjugation and, if $L \in \mathcal{S}_{G,H}$ and $M \in \mathcal{S}_{H,K}$, then $L * M \in \mathcal{S}_{G,K}$.

Definition 2.2.15. Given a preadmissible pair $(\mathcal{D},\mathcal{S})$, let $RB^{(\mathcal{D},\mathcal{S})}$ be the subcategory of RB such that its objects are those of $\mathcal{D}$ and its morphism sets are $RB^{(\mathcal{D},\mathcal{S})}(G,H)$, that is, the submodule of $RB(G,H)$ generated by the classes $[H\times G/L]$, with $L\in\mathcal{S}_{H,G}$. We say that $(\mathcal{D},\mathcal{S})$ and $RB^{(\mathcal{D},\mathcal{S})}$ are *admissible* if the virtual bisets $\mathrm{Ind}_{p_1(L)}^{H}\circ\mathrm{Inf}_{p_1(L)/k_1(L)}^{p_1(L)}$, $\mathrm{Iso}(\eta)$ and $\mathrm{Def}_{p_2(L)/k_2(L)}^{p_2(L)}\circ\mathrm{Res}_{p_2(L)}^{G}$ are morphisms in $RB^{(\mathcal{D},\mathcal{S})}$.

Observation 2.2.16. We note that the definition of an admissible pair actually does not depend on the ring R.

If a preadmissible pair $(\mathcal{D},\mathcal{S})$ induces an admissible subcategory $RB^{(\mathcal{D},\mathcal{S})}$, we say that $(\mathcal{D},\mathcal{S})$ is *admissible*. In the context of admissible pairs, if $\mathcal{D}$ is the class of all finite groups, we denote $RB^{(\mathcal{D},\mathcal{S})}$ simply by $RB^{\mathcal{S}}$ and say that $\mathcal{S}$ is an *admissible class*.

Example 2.2.17. Let $\mathcal{S}=\triangleright$ be the class of sets $\triangleright_{H,G}:=\{\Delta(\varphi,K)\mid K\le H,\varphi:K\to G\}$. We call $RB^{\triangleright}(G,H)$ the *right-free Burnside module of G and H*, its name comes from the fact that the G-action for the(H,G)-biset $(H\times G)/\Delta(\varphi,K)$ is free. Conversely, it is not hard to check that any transitive (H,G)-biset with free G-action is of that form. This defines the *right-free Burnside category* $RB^{\triangleright}$.

Analogously, we can define the *left-free Burnside category* $RB^{\triangleleft}$ by taking $\triangleleft_{G,H}:=\{\Delta(J,\alpha)\}$ and the *bifree Burnside category* RB^{Δ} with

$$\Delta_{G,H}:=\{\Delta(J,\alpha)\mid \alpha \text{ is a monomorphism}\}$$

or, equivalently,

$$\Delta_{G,H}:=\{\Delta(\varphi,K)\mid \varphi \text{ is a monomorphism}\},$$

Indeed, they give rise to the same admissible subcategory, where each $RB^{\Delta}(G,H)$ is generated by the transitive (H,G)-bisets with both G and H acting freely.

Observation 2.2.18. We note that $\triangleright$ and $\triangleleft$ are mutually opposite categories. Moreover, Δ is isomorphic to its opposite category.

Convention 2.2.19. From now on, for $J\le G$ and $\alpha:J\to H$, we will use the notation

$$[J,\alpha]_G^H:=(H\times G)/\Delta(J,\alpha).$$

Similarly, for $K \leq H$ and $\varphi : K \to G$,

$$[\varphi, K]_G^H := (H \times G)/\Delta(\varphi, K).$$

Example 2.2.20. Using the description $\mathbb{Z}/2 = \{1, \sigma\}$, we obtain

$$B^{\triangleleft}(\mathbb{Z}/2, \Sigma_3) = \mathbb{Z}[\mathbb{Z}/2, j] \oplus \mathbb{Z}[\mathbb{Z}/2, \mathrm{ct}] \oplus \mathbb{Z}[\{1\}, \mathrm{ct}],$$

where $j : \mathbb{Z}/2 \to \Sigma_3$ is the inclusion that sends σ to $(1\ 2)$. In fact, there are three inclusions from $\mathbb{Z}/2$ to Σ_3, but their twisted diagonal are all conjugate, hence they induce the same class in $B^{\triangleleft}(\mathbb{Z}/2, \Sigma_3)$. On the other hand, $B^{\vartriangle}(\mathbb{Z}/2, \Sigma_3) = \mathbb{Z}[\mathbb{Z}/2, j] \oplus \mathbb{Z}[\{1\}, \mathrm{ct}]$.

To end this section, we briefly recall the notion of reduced Burnside modules. For a definition, we will restrict the admissible class $\triangleleft$, but the definition for $\triangleright$ and Δ will follow naturally.

Definition 2.2.21. Let G, H be finite groups, the *reduced left-free Burnside module of G and H*, denoted by $\widetilde{B}^{\triangleleft}(G, H)$, is the quotient of $B(G, H)$ over the submodule N generated by the basis elements of the form $[K, \mathrm{ct}]_G^H$, where $\mathrm{ct} : K \to H$ is the trivial homomorphism.

The equivalence class $[L, \varphi] + N$ is denoted by $\widetilde{[L, \varphi]}$. It is not hard to see that we can speak of the *reduced left-free Burnside category* $R\widetilde{B}^{\triangleleft}$, with composition of morphisms induced from those of $RB^{\triangleleft}$.

2.3 BISET FUNCTORS

We are now ready to define the core notion of this chapter.

Definition 2.3.1. Let $(\mathcal{D}, \mathcal{S})$ be a preadmissible pair inducing an admissible Burnside subcategory $RB^{(\mathcal{D},\mathcal{S})}$. A *$(\mathcal{D}, \mathcal{S})$-biset functor over R* is an either covariant or contravariant R-linear functor $F : RB^{(\mathcal{D},\mathcal{S})} \to \mathrm{Mod}_R$. In particular, when $\mathcal{D}$ is the class of all finite groups, we simply say that F is an *$\mathcal{S}$-biset functor over R*.

If $(\mathcal{D}, \mathcal{S})$ is clear from the context, we refer to F simply as a biset functor. Of course, these are particular cases of the linear functors studied in Section 2.1, so we can speak of projective and simple biset functors, which we will study in the next section.

Remark 2.3.2. Unless otherwise stated, all biset functors are assumed to be covariant.

Convention 2.3.3. In particular, when $\mathcal{S} = \triangleright$, we refer to $\triangleright$-biset functors as *inflation functors*. Similarly, when $\mathcal{S} = \Delta$, we refer to Δ-biset functors as *global Mackey functors*.

We observe that global Mackey functors are biset functors with neither inflation nor deflation, in the sense that any $L \in \Delta_{H,G}$ satisfies that $k_1(L)$ and $k_2(L)$ are trivial subgroups, then $\mathrm{Inf}^{q_1(L)}_{q_1(L)/k_1(L)} = \mathrm{Inf}^{q_1(L)}_{q_1(L)}$ and $\mathrm{Def}^{q_2(L)}_{q_2(L)/k_2(L)} = \mathrm{Def}^{q_2(L)}_{q_2(L)}$ are identities. In contrast, inflation functors do have inflation but no deflation, that is the origin of their nomenclature.

Remark 2.3.4. In a dual manner to the case of inflation functors, we can define the admissible Burnside subcategory given by $S := \triangleleft$, where $\triangleleft_{G,H} = \{\Delta(J,\alpha)\}$. The $\triangleleft$-biset functors do not seem to have a standard name in the literature, we will call them *deflation functors*. Again, the name comes from the fact that they have no inflation, in the sense explained above.

The double coset formula in $RB^{\triangleleft}(G,H)$ is simpler and goes as follows:

$$[J,\alpha]^F_H \circ [K,\varphi]^H_G = \sum_{h\in[J\backslash H/\varphi(K)]} [K \cap \varphi^{-1}(J^h), \alpha \circ c_h \circ \varphi]^F_G.$$

Indeed, $p_2(\Delta(J,\alpha)) = J$, and $p_1(\Delta(K,\varphi)) = \varphi(K)$. And we have $(f,g) \in \Delta(J,\alpha) *^{(h,1)} \Delta(K,\varphi)$ if and only if there exists $l \in H$ such that $\alpha(l) = f$ and $c_h(\varphi(g)) = l$, $g \in K \cap \varphi^{-1}(J^h)$. In turn, the latter holds if and only if $\alpha \circ c_h \circ \varphi(g) = f$ and $g \in K \cap \varphi^{-1}(J^h)$.

Analogously, the double coset formula in $R^{\triangleright}(G,H)$ can be expressed as:

$$\begin{aligned}[][\alpha,J]^F_H \circ [\varphi,K]^H_G &= \sum_{h\in[\alpha(J)\backslash H/K]} [\varphi \circ c_{h^{-1}} \circ \alpha, J \cap \alpha^{-1}({}^hK)]^F_G \\ &= \sum_{l\in[K\backslash H/\alpha(J)]} [\varphi \circ c_l \circ \alpha, J \cap \alpha^{-1}(K^l)]^F_G\end{aligned}$$

Observation 2.3.5. A covariant inflation functor can be viewed as a contravariant deflation functor and vice versa.

Example 2.3.6. Group cohomology $M = H^n(-;R)$ is an example of (covariant) inflation functor. Indeed, we recall from Section 1.2 the restriction $M(\mathrm{Res}^G_H) := \mathrm{res}^G_H : H^n(G;R) \to H^n(H;R)$ and inflation $M(\mathrm{Inf}^G_{G/N}) := \mathrm{inf}^G_{G/N} : H^n(G/N;R) \to H^n(G;R)$ ($R^N = R$ since R has trivial G-action) operations. On the other hand,

$$M(\mathrm{Ind}^G_H) := \mathrm{tr}^G_H : H^n(H;R) \to H^n(G;R).$$

Isogation is obvious by functoriality of $H^n(-;R)$ in the classical sense.

It may be confusing to consider group cohomology as a covariant functor, since it is rather commonly referred as a contravariant one. We can fix this by viewing $E = H^n(-;R)$ as a (contravariant) deflation functor: we set $E(\mathrm{Res}^G_H) := \mathrm{tr}^G_H$, $E(\mathrm{Ind}^G_H) := \mathrm{res}^G_H$, $E(\mathrm{Def}^G_{G/N}) := \mathrm{inf}^G_{G/N}$.

Example 2.3.7. Let $\mathbb{F}$ be a field. Then the group representation functor $R_{\mathbb{F}}$ is an inflation functor. If $M \in R_{\mathbb{F}}(G)$, then $\mathrm{Res}^G_H(M)$ is M viewed as a virtual $\mathbb{F}H$-module via a change of the base ring along $\iota\colon \mathbb{F}H \to \mathbb{F}G$. Similarly, if $M \in R_{\mathbb{F}}(G/N)$, then $\mathrm{Inf}^G_{G/N}(M)$ is M viewed as a virtual $\mathbb{F}G$-module via a change of the base ring along $\pi : \mathbb{F}G \to \mathbb{F}[G/N]$. Isogation follows the same idea. Now, let us suppose that $M \in R_{\mathbb{F}}(H)$, then $\mathrm{Ind}^G_H(M) := \mathbb{F}G \otimes_{\mathbb{F}H} M$, viewing $\mathbb{F}G$ as a $(\mathbb{F}G, \mathbb{F}H)$-bimodule. At last, defining deflation is plausible if we further assume that char $\mathbb{F} = 0$. We do not outline the argument here, but it can be found in [9, Section 1.1]. In other words, $R_{\mathbb{F}}$ has all the basic operations that define a biset functor.

Remark 2.3.8. This point is suitable to start unveiling connections with Chapter 1. Indeed, we can define the homomorphism of abelian groups

$$\alpha : B^{\triangleleft}(G, H) \to [\mathbb{B}_+G, \mathbb{B}_+H]$$

that sends the class $[K, \varphi]^H_G$ to the composite

$$\mathbb{B}_+G \xrightarrow{\mathrm{tr}^G_K} \mathbb{B}_+K \xrightarrow{B\varphi} \mathbb{B}_+H.$$

More importantly, the map α induces an isomorphism after completing over a certain ideal. We do not make the precise statement here, since it appears in this thesis as Theorem 3.2.3.

For convenience and in accordance with the literature [9], from now on we denote the composition $\mathrm{Def}^S_{S/T} \circ \mathrm{Res}^G_S$ simply by $\mathrm{Defres}^G_{S/T}$. Similarly, the composition $\mathrm{Ind}^G_S \circ \mathrm{Inf}^S_{S/T}$ is denoted $\mathrm{Indinf}^G_{S/T}$.

2.4 SIMPLE BISET FUNCTORS

In this section, we study particular properties of simple biset functors that make them of special interest among simple linear functors. We write $\mathbb{S}_{H,V}$ to denote the simple biset functor parametrized by the seed (H,V) (recall Proposition 2.1.7 and Definition 2.1.9).

From now non, we only consider admissible pairs $(\mathcal{D},\mathcal{S})$ with $\mathcal{D}$ the class of all finite groups, hence it will be implicit throughout the rest of this thesis. The simple $\mathcal{S}$-biset functors are properly denoted $\mathbb{S}^{\mathcal{S}}_{H,V}$, but for simplicity we omit the class $\mathcal{S}$ in the notation and simply write $\mathbb{S}_{H,V}$ when there is no need to specify it. In this situation and for consistency, we say that $\mathbb{S}_{H,V}$ is a simple biset functor, instead of a simple $\mathcal{S}$-biset functor.

We mentioned in Section 2.1 that the parametrization of simple functors can be refined for the case of biset functors. To do this, we will use the notion of minimal objects, in the sense outlined below:

Definition 2.4.1. Let M be a biset functor. A *minimal group* for M is a group H such that $M(H) \neq 0$, but $M(K) = 0$ for any group K with $|K| < |H|$.

Let I_H be the R-submodule of $RB^{\mathcal{S}}(H,H)$ generated by morphisms that factor through a finite group K with $|K| < |H|$. It can be shown that I_H is a two-sided ideal of $RB^{\mathcal{S}}(H,H)$ and the quotient $RB^{\mathcal{S}}(H,H)/I_H$ is isomorphic as an R-algebra to $R\mathrm{Out}(H)$. In fact, more is true:

Proposition 2.4.2 ([9, Proposition 4.3.2])**.** *The exact sequence* $0 \to I_H \to RB(H,H) \to R\mathrm{Out}(H) \to 0$ *splits.*

We observe that another more explicit description of I_H for the case $\mathcal{S} = \lhd$ is the R-submodule generated by classes $[K,\varphi]^H_H$, with $|\varphi(K)| < |H|$. Even simpler, for $\mathcal{S} = \Delta$, the ideal I_H is generated by the classes $[K,\varphi]$ with K a proper subgroup of H. If M is a biset functor and H is a minimal group for M, the evaluation $M(H)$ is in principle an $RB(H,H)$-module, but since I_H annihilates $M(H)$, the induced $R\mathrm{Out}(H)$-module structure on $M(H)$ retains all the previous information.

Minimal groups are helpful to better understand non-vanishing evaluations of biset functors, as displayed in the following result.

Proposition 2.4.3. *Let $S := \mathbb{S}_{H,V}$ be a simple biset functor, with H a minimal group for S. If $\mathbb{S}_{H,V}(G) \neq 0$, then H is a subquotient of G.*

Proof sketch. The proof for a general admissible subcategory can be found in [9, Lemma 4.3.9]. We will concentrate on the case $\mathcal{S} = \lhd$ or Δ. As $L_{H,V}(G) \neq J_{H,V}(G)$, there exists an element $\overline{\varphi} = \sum_i r_i[K_i, \varphi_i]_H^G \otimes v_i$ of $L_{H,V}(G)$, which is not in $J_{H,V}(G)$. Then there exists $\overline{\alpha} = \sum_j s_j[L_j, \alpha_j]_G^H$ such that $\sum_i (r_i\alpha[K_i, \varphi_i]).v_i \neq 0$. It follows that $[L_{j_0}, \alpha_{j_0}] \circ [K_{i_0}, \varphi_{i_0}] \notin I_H$ for some i_0, j_0. Setting $L = L_{i_0}, \alpha = \alpha_{i_0}$ and $K = K_{i_0}, \varphi = \varphi_{i_0}$, by the double coset formula, we have:

$$[L, \alpha] \circ [K, \varphi] = \sum_y [K \cap \varphi^{-1}(L^y), \alpha \circ c_y \circ \varphi],$$

but $K \cap \varphi^{-1}(L^y)$ must equal H for some y. Then $K = H$ and $\varphi(H) = L^y$. As the composite

$$H \xrightarrow{\varphi} L^y \xrightarrow{c_y} L \xrightarrow{\alpha} H$$

is surjective, $L \xrightarrow{\alpha} H$ is also surjective, hence $H \cong L/\mathrm{Ker}\,\alpha$, as desired. In the particular case of $\mathcal{S} = \Delta$, we have $\mathrm{Ker}\,\alpha = 0$, therefore $H \cong L$. □

Remark 2.4.4. We note that in the particular case of $\mathcal{S} = \Delta$, the condition $\mathbb{S}_{H,V}(G) \neq 0$ implies that H is isomorphic to a subgroup of G.

We are ready to exploit the advantage of having minimal objects. The simple biset functors over R are parametrized as follows:

Proposition 2.4.5. *The simple biset functors $\mathbb{S}_{H,V}$ and $\mathbb{S}_{H',V'}$, with H, H' minimal groups, respectively, are isomorphic if and only if $(H, V) \cong (H', V')$.*

Proof. It only remains to prove that $\mathbb{S}_{H,V} \cong \mathbb{S}_{H',V'}$ implies $(H, V) \cong (H', V')$ (see comments below Definition 2.1.9). By Proposition 2.4.3, as $S_{H,V}(H') \neq 0$, then $H' \sqsubseteq H$. By symmetry, $H \sqsubseteq H'$, hence there exists an isomorphism $\alpha : H \cong H'$. Now, the seeds (H, V) and $(H', {}^{\alpha}V)$ are isomorphic, so it follows that $\mathbb{S}_{H',{}^{\alpha}V} \cong \mathbb{S}_{H,V} \cong \mathbb{S}_{H',V'}$. By evaluation at H', we have ${}^{\alpha}V \cong V'$, as desired. □

In the same spirit, we can more generally define

$$RI(H, G) = \sum_{K \sqsubset H} B(K, G)B(H, K),$$

where $B(K, G)B(H, K)$ is the image of the associative composition $\circ : B(K, G) \times B(H, K) \to B(H, G)$. This allows us to define $R\overline{B}(G, H) = RB(H, G)/RI(H, G)$. The following proposition follows easily from the definition of $RI(H,G)$ and the fact that the bisets $\mathrm{Defres}^{H}_{S'/T'}$, with $S'/T' \sqsubseteq H$, live in $RI(H,G)$.

Proposition 2.4.6 ([12, Proposition 2.3])**.** *The free R-module $R\overline{B}(H,G)$ is generated by the classes of the form $\overline{\mathrm{Indinf}}^{G}_{T/S}\overline{\mathrm{Iso}}(\sigma)$, where (S,T) runs over the set of G-conjugacy classes of sections of G with $T/S \cong H$ and $\sigma : H \to T/S$ runs over all isomorphisms up to left composition by conjugation homomorphisms by elements of $G_{(S,T)}$.*

In particular, when $\mathcal{S} = \lhd$ or Δ, the basis consists of classes of the form $\overline{\mathrm{Ind}}^{G}_{\sigma(H)}\overline{\mathrm{Iso}}(\sigma) = \overline{[H,\sigma]}$, with $\sigma : H \to G$ a monomorphism. This happens because these submodules have no inflation, unlike in the case of $S = \rhd$, for instance.

It can be shown that $RI(H,-)$ is a subfunctor of $RB(H,-)$, hence $R\overline{B}(H,-)$ is a quotient functor of $RB(H,-)$. For any simple $R\mathrm{Out}(H)$-module V, we have the functor

$$\overline{L}_{H,V}(G) = R\overline{B}(H,G) \otimes_{R\mathrm{Out}(H)} V.$$

Analogously, we define $\overline{J}_{H,V}(G) \cong \bigcap_{\varphi \in RB(G,H)} \mathrm{Ker}(\overline{L}_{H,V}(\varphi))$.

Proposition 2.4.7 ([11, Proposition 4.4])**.** *Let (H,V) be a seed. Then $\mathbb{S}_{H,V} \cong \overline{L}_{H,V}/\overline{J}_{H,V}$.*

Dually, we define

$$R\underline{B}(G,H) = RB(G,H) \Big/ \sum_{K \sqsubset H} B(K,H)B(G,K).$$

In analogy to Proposition 2.4.6, we have the following result.

Proposition 2.4.8. *The free R-module $R\overline{B}(G,H)$ is generated by the classes of the form $\underline{\mathrm{Iso}}(\sigma)\underline{\mathrm{Defres}}^{G}_{T/S}(\sigma)$, where (S,T) runs over the set of G-conjugacy classes of sections of G with $T/S \cong H$ and $\sigma : T/S \to H$ runs over all isomorphisms up to right composition by conjugation homomorphisms by elements of $G_{(S,T)}$.*

In particular, for $\mathcal{S} = \rhd$, the basis of $R\overline{B}^{\rhd}(G,H)$ consists of classes of the form $\underline{[\sigma,H]}$, with $\sigma : H \to G$ a monomorphism, since $\rhd$ has no deflation. $R\underline{B}$ is dual to $R\overline{B}$ in the sense that we can now define

$$\underline{R}_{H,V}(G) = \mathrm{Hom}_{R\mathrm{Out}(H)}(R\underline{B}(G,H),V).$$

Now, analogously to Proposition 2.1.8, we have the following result.

Proposition 2.4.9. *The functor $\underline{R}_{H,V}$ has a unique minimal subfunctor $\underline{T}_{H,V} = \sum_{\varphi \in RB(H,-)} \mathrm{Im}(\underline{R}_{H,V}(\varphi))$ and $\underline{T}_{H,V} \cong \mathbb{S}_{H,V}$.*

In [70, Section 2], Webb defines $\mathbb{S}^{\mathcal{S}}_{H,V}$, for the admissible classes $\mathcal{S} = \triangleright$ or $\mathcal{S} = \Delta$, as a subfunctor of a functor that he denotes $J_{H,V}$. To avoid ambiguity with our notation, we write $W_{H,V}$ for Webb's functor $J_{H,V}$, in such a way that

$$W_{H,V}(G) \cong \bigoplus_{\substack{\alpha : H \to L \\ L \leq_G G}} ({}^{\alpha}V)^{N_G(L)}, \tag{2.4.10}$$

where the direct sum runs over representatives L of G-conjugacy classes of subgroups of G which are isomorphic to H, with a fixed isomorphism $\alpha : H \to L$, and ${}^{\alpha}V$ is the set V viewed as an $R\mathrm{Out}(L)$-module via the isomorphism α. Webb [70, Theorem 2.6] shows isomorphism (2.4.10), we will see that a similar situation holds for $\underline{R}_{H,V}$.

Lemma 2.4.11. *If $\mathcal{S} = \triangleright$ or Δ, we have*

$$\underline{R}^{\mathcal{S}}_{H,V}(G) \cong \overline{W}_{H,V}(G) := \bigoplus_{\substack{\alpha : H \to L \\ L \leq_G G}} ({}^{\alpha}V)^{N_G(L)}, \tag{2.4.12}$$

Proof. We note that $R\underline{B}(G,H)$ is a permutation $R\mathrm{Out}(H)$-module. Indeed, the R-basis $X = \{\underline{[\sigma,H]}^H_G\}$ is a left $\mathrm{Out}(H)$-set with the action

$$[\tau] \cdot \underline{[\sigma,H]}^H_G := \underline{[\sigma \circ \tau^{-1}, H]}^H_G$$

and

$$X \cong \coprod_{\substack{\alpha : H \to L \\ L \leq_G G}} \frac{\mathrm{Out}(H)}{\alpha^{-1}\mathrm{Out}_G(L)\alpha}.$$

since the $\mathrm{Out}(H)$-orbits $\mathrm{Out}(H)\underline{[\alpha,H]}$ are in bijective correspondence with the G-conjugacy classes of subgroups L of G which are isomorphic to H via a chosen isomorphism $\alpha \colon H \to L$ and the stabilizers $\mathrm{Out}(H)_{\underline{[\alpha,H]}}$ are precisely $\alpha^{-1}\mathrm{Out}_G(L)\alpha$.

Now,

$$
\begin{aligned}
\underline{R}_{H,V}(G) &= \mathrm{Hom}_{R\mathrm{Out}(H)}(R\underline{B}(G,H),V)\\
&= \bigoplus_{\substack{\alpha:H\to L\\ L\leq_G G}} \mathrm{Hom}_{R\mathrm{Out}(H)}\left(R\left[\frac{\mathrm{Out}(H)}{\alpha^{-1}\mathrm{Out}_G(L)\alpha}\right],V\right)\\
&\cong \bigoplus_{\substack{\alpha:H\to L\\ L\leq_G G}} \mathrm{Hom}_{R\mathrm{Out}(H)}(R\uparrow^{\mathrm{Out}(H)}_{\alpha^{-1}\mathrm{Out}_G(L)\alpha},V)\\
&\cong \bigoplus_{\substack{\alpha:H\to L\\ L\leq_G G}} \mathrm{Hom}_{\alpha^{-1}\mathrm{Out}_G(L)\alpha}(R,V\downarrow^{\mathrm{Out}(H)}_{\alpha^{-1}\mathrm{Out}_G(L)\alpha})\\
&\cong \bigoplus_{\substack{\alpha:H\to L\\ L\leq_G G}} V^{\alpha^{-1}\mathrm{Out}_G(L)\alpha}\\
&= \bigoplus_{\substack{\alpha:H\to L\\ L\leq_G G}} ({}^{\alpha}V)^{N_G(L)},
\end{aligned}
$$

and the composition sends $f\in\mathrm{Hom}_{R\mathrm{Out}(H)}(R\underline{B}(G,H),V)$ to $(f(\underline{[\alpha,H]}))_{[\alpha(H)]_G}$. □

Moreover, isomorphism (2.4.12) can be exploited to give a functorial structure to $\overline{W}_{H,V}$, in such a way that isomorphism (2.4.12) becomes natural. The inverse $\bigoplus_{\substack{\alpha:H\to L\\ L\leq_G G}} ({}^{\alpha}V)^{N_G(L)}\to$ $\mathrm{Hom}_{R\mathrm{Out}(H)}(R\underline{B}(G,H),V)$ of the isomorphism constructed in the previous lemma sends $v_\alpha\in({}^{\alpha}V)^{N_G(L)}$ to the map f_{v_α} defined by

$$
f_{v_\alpha}(\underline{[\alpha',H]}) = \begin{cases} [\sigma].v_\alpha, & \text{if } [\alpha',H]=[\sigma].[\alpha,H],\\ 0, & \text{otherwise.}\end{cases}
$$

Now, given $[\varphi,M]^{G'}_{G}$, we can use it to define $\hat{f}_{v_\alpha}\in\mathrm{Hom}_{R\mathrm{Out}(H)}(R\underline{B}(G',H),V)$ such that

$$
\begin{aligned}
\hat{f}_{v_\alpha}(\underline{[\beta,H]^{H}_{G'}}) &:= \underline{R}_{H,V}([\varphi,M])(f_{v_\alpha})(\underline{[\beta,H]})\\
&= f_{v_\alpha}(\underline{[\beta,H]\circ[\varphi,M]})\\
&= \sum_{\substack{g\in[\beta(H)\backslash G'/M]\\ \beta(H)^g\leq M}} f_{v_\alpha}(\underline{[\varphi\circ c_{g^{-1}}\circ\beta,H]}),
\end{aligned}
\tag{2.4.13}
$$

where by definition $f_{v_\alpha}(\underline{[\varphi\circ c_{g^{-1}}\circ\beta,H]})=[\sigma].v_\alpha$ if

$$
\underline{[\varphi\circ c_{g^{-1}}\circ\beta,H]}=[\sigma].\underline{[\alpha,H]}=\underline{[H,\alpha\circ\sigma^{-1}]}
\tag{2.4.14}
$$

and zero otherwise. Let us denote $\beta(H) = L'$. A necessary condition for equation (2.4.14) to hold is that $\varphi({}^{g'}L')$ is G-conjugate to $L = \alpha(H)$, for some $g' \in G$. Indeed, $\underline{[\varphi \circ c_{g^{-1}} \circ \beta, H]} = \underline{[\alpha \circ \sigma^{-1}, H]}$ if and only if

$$c_{g''} \circ \alpha \circ \sigma^{-1} = \varphi \circ c_{g^{-1}} \circ \beta \circ c_h = \varphi \circ c_{g^{-1}\beta(h)} \circ \beta \tag{2.4.15}$$

for some $g'' \in G, h \in H$. Then $\varphi({}^{g'}L') = {}^{g''}L$, with $g' = g^{-1}\beta(h)$.

On the other hand, we have that the map σ^{-1} can be seen as the composite

$$H \xrightarrow{\beta} L' \xrightarrow{c_{g'}} {}^{g'}L' \xrightarrow{\varphi} {}^{g''}L \xrightarrow{c_{g''^{-1}}} L \xrightarrow{\alpha^{-1}} H \tag{2.4.16}$$

Consequently, the morphism

$$\mathrm{Hom}_{R\mathrm{Out}(H)}(R\underline{B}(G', H), V) \to \bigoplus_{\substack{\beta: H \to L' \\ L' \leq_{G'} G'}} ({}^{\beta}V)^{N_{G'}(L')}$$

sends $\hat{f}_{v_\alpha}$ to $\left(\sum_g [\sigma].v_\alpha\right)_{\beta(H)}$, with g running over some representatives of $\varphi(H)\backslash G'/M$ satisfying the conditions imposed above. Finally, we obtain a homomorphism

$$\bigoplus_{\substack{\alpha: H \to L \\ L \leq_G G}} ({}^{\alpha}V)^{N_G(L)} \to \bigoplus_{\substack{\beta: H \to L' \\ L \leq_{G'} G'}} ({}^{\beta}V)^{N'_G(L')}$$

sending v_α to $\left(\sum_g [\sigma].v_\alpha\right)_{\beta(H)}$, thus giving the desired functoriality for $\overline{W}_{H,V}$, which we refer to as Webb's model of $\underline{R}_{H,V}$. Now, we want to find a more explicit description for $\underline{T}_{H,V} = \sum_{\varphi \in R\mathcal{C}(H,-)} \mathrm{Im}(\underline{R}_{H,V}(\varphi))$ via the isomorphism given in Lemma 2.4.9.

Given a class $[\pi, K]_H^G$, we have that $\underline{R}_{H,V}([\pi, K]) : V \to \bigoplus_{\substack{\alpha: H \to L \\ L \leq_G G}} ({}^{\alpha}V)^{N_G(L)}$ sends $v = v_{1_H}$ to $(\sum_g [\sigma].v)_{\alpha(H)}$. In particular, we can assume that the expression for σ^{-1} given in (2.4.16) now takes the form

$$H \xrightarrow{\quad\alpha\quad} L \xrightarrow{\quad c_g\quad} {}^{g}L \hookrightarrow K \xrightarrow{\quad\pi\quad} H, \tag{2.4.17}$$

since the actual σ and the inverse of composite (2.4.17) may differ only by composition with c_h, for some $h \in H$, thus they induce the same class in $\mathrm{Out}(H)$. It then follows that $\pi : K \to H$ is a split epimomorphism. As σ^{-1} depends only on g, we better denote it by θ_g. Using equations (2.4.13), (2.4.15) and the fact that $\pi : K \to H$ splits, we obtain that g can be chosen to run over representatives of $N_G(L,K)/K$ such that $B \cap {}^gL = 1$, where $B = \mathrm{Ker}\pi$.

In other words, we obtain that $\mathbb{S}_{H,V}(G)$ consists of the elements of $\bigoplus_{\substack{\alpha:H\to L\\ L\leq_G G}} ({}^{\alpha}V)^{N_G(L)}$ whose α-components are the sums

$$\left(\sum_{\substack{g\in[N_G(L,K)/K]\\ B\cap {}^gL=1}} [\theta_g]^{-1}.v\right), \text{ with } v \in V, \tag{2.4.18}$$

as we find in [70, Theorem 2.6].

For global Mackey functors, we have an even more explicit description of $\mathbb{S}^{\triangle}_{H,V}(G)$.

Proposition 2.4.19.

$$\mathbb{S}^{\Delta}_{H,V}(G) \cong \bigoplus_{\substack{\alpha:Q\to L\\ L\leq_G G}} \mathrm{tr}_L^{N_G(L)}({}^LV)$$

Proof. With $[\pi, K]^G_H$ as above, we now have that $\pi : K \to H$ is also a monomorphism, thus an isomorphism and ${}^gL = K$ in (2.4.17) for every g running in Equation (2.4.18). We can improve this by noting that $[\pi, K] = [\pi \circ c_g, L]$, and then we can assume that $K = L$. In this way, θ_g takes the form

$$H \xrightarrow{\alpha} L \xrightarrow{c_g} L \xrightarrow{\pi} H.$$

As α can be chosen conveniently, we can assume $\alpha = \pi^{-1}$, viewing now θ_g as

$$H \xrightarrow{\alpha} L \xrightarrow{c_g} L \xrightarrow{\alpha^{-1}} H,$$

with g running over representatives of $N_G(L)/L$. Therefore, in this case, Equation (2.4.18) becomes $\sum_{h\in[N_G(L)/L]} [c_h].v.$ □

By definition of trace maps, we have $\mathrm{tr}_L^{N_G(L)}({}^LV) = W_L({}^LV)$, where $W_L = \sum_{x\in N_G(L)/L} c_x$.

Remark 2.4.20. The formula for $\mathbb{S}^{\triangleright}_{H,V}(G)$ is simplified considerably when there is no subretract of G isomorphic to H. Indeed, in that case, the only chance for $[\pi, K]$ is to equal $[1, L]$, therefore

$$\mathbb{S}^{\triangleright}_{H,V}(G) \cong \bigoplus_{\substack{\alpha: Q \to L \\ L \leq_G G}} \mathrm{tr}_L^{N_G(L)}({}^L V),$$

just like a global Mackey functor.

Example 2.4.21. Let $R = \mathbb{F}_p$, $G = \mathbb{Z}/p$ and $H = \mathbb{Z}/p$. We have

$$\mathrm{Out}(\mathbb{Z}/p) = \mathrm{Aut}(\mathbb{Z}/p) \cong \{1, \xi \ldots, \xi^{p-1}\} \cong \mathbb{Z}/(p-1),$$

where ξ is a multiplicative generator of $(\mathbb{F}_p)^\times$. The simple $\mathbb{F}_p[\mathbb{Z}/(p-1)]$-modules are given by the representations $(\det)^k : \mathbb{Z}/(p-1) \to \mathbb{F}_p$, with $0 \leq k \leq p-2$, that send ξ to ξ^k. In particular, $\dim_{\mathbb{F}_p} \mathbb{S}^{\mathcal{S}}_{\mathbb{Z}/p,(\det)^k}(\mathbb{Z}/p) = \dim_{\mathbb{F}_p}(\det)^k = 1$, for any admissible class $\mathcal{S}$.

On the other hand, if $H = 1$, the only simple $\mathbb{F}_p[\mathrm{Out}(H)]$-module is $\mathbb{F}_p$. It is easy to see that $\mathbb{S}^{\Delta}_{1,\mathbb{F}_p}(\mathbb{Z}/p) = 0$ using Proposition 2.4.19 and the fact that $\mathbb{F}_p$ has characteristic p.

Computations of $\mathbb{S}^{\triangleright}_{Q,V}(G)$ using only algebraic tools are scarce in the literature. In fact, computing $\dim_{\mathbb{F}_p} \mathbb{S}^{\triangleright}_{Q,V}(G)$, with $R = \mathbb{F}_p$ and Q nontrivial, amounts to computing the multiplicity of a certain indecomposable summand of $\mathbb{B}G^\wedge_p$ (see Section 3.2), but the latter was a subject of study prior to Webb's definition of inflation functors, which is why several computations of $\dim_{\mathbb{F}_p} \mathbb{S}^{\triangleright}_{Q,V}(G)$ are known partly via homotopy theory.

Example 2.4.22. Let $R = \mathbb{F}_p$ and $A := \mathbb{Z}/p \times \mathbb{Z}/p$. We have $\mathrm{Out}(\mathbb{Z}/p \times \mathbb{Z}/p) = \mathrm{Aut}(\mathbb{Z}/p \times \mathbb{Z}/p) \cong \mathrm{GL}_2(p)$. The simple $\mathbb{F}_p[\mathrm{GL}_2(p)]$-modules are given by

$$M_{q,k} = S(A)^q \otimes (\det)^k, \text{ with } q = 0, \ldots, q-1, \text{ and } k = 0, \ldots, q-2,$$

where $S(A)^q$ is defined as follows: Let us consider the graded polynomial algebra $\mathbb{F}_p[x, y]$, with $|x| = |y| = 2$, then $S(A)^q$ is the component of $\mathbb{F}_p[x, y]$ of homogeneous polynomials of degree $2q$ with the $\mathrm{GL}_2(p)$-action defined by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot x^i y^j := (ax + by)^i (cx + dy)^j.$$

In this case, $\dim_{\mathbb{F}_p} M_{q,k} = q + 1$.

A p-group E is called *extraspecial* if $Z(E)$ has order p and $E/Z(E)$ is elementary abelian. In particular, let us consider the extraspecial group E of order p^3 and exponent p^2, with p odd, which has a presentation

$$E = \langle s, t \mid s^{p^2} = 1, t^p = 1, tst^{-1} = t^{p+1} \rangle$$

The subgroup Q of E generated by s^p and t is the unique subgroup of E isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Moreover, $[N_Q(E)/Q] = \{1, s, \ldots, s^{p-1}\}$ and

$$s.x^i y^j = \left(\begin{smallmatrix} 1 & p-1 \\ 0 & 1 \end{smallmatrix}\right).x^i y^j.$$

The subgroup Q is not a subretract of E, hence $\mathbb{S}^{\triangleright}_{Q.M_{q,k}}(E) = W_Q M_{q,k}$. In fact, it is not hard to check that the evaluation is independent of k, i.e.,

$$W_Q M_{q,0} \cong W_Q M_{q,1} \cong \cdots \cong W_Q M_{q,p-2}.$$

Thus, it suffices to find $\dim_{\mathbb{F}_p} W_Q M_{q,0}$, and this is achieved in [23, Proposition 4.6]. Indeed, we have

$$\dim_{\mathbb{F}_p} W_Q M_{q,0} = \begin{cases} 0, & \text{if } q < p-1, \\ 1, & \text{if } q = p-1. \end{cases}$$

The importance of computing the evaluations of simple biset functors $\mathbb{S}_{H,V}(G)$ can also be appreciated in the algebraic context. For instance, the following result supports this claim.

Theorem 2.4.23 ([70, Theorem 5.6])**.** *Let k be $\mathbb{Z}_p^\wedge$ or $\mathbb{F}_p$. Then*

$$kB^{\triangleright}(-, G) \cong \bigoplus_{(H,V)} (\mathbb{P}^{\triangleright}_{H,V})^{n_{H,V}(G)},$$

where

$$n_{H,V}(G) = \frac{\dim \mathbb{S}^{\triangleright}_{H,V}(G)}{\dim \operatorname{End}_{k\mathrm{Out}(H)}(V)},$$

and dimensions are taken over the residue field of k.

Definition 2.4.24. Let $\mathcal{S} = \triangleright$ or Δ. We say that an $\mathcal{S}$-biset functor F is *cohomological* if $F([\iota_K, K]_G^G) = [G : K]\mathrm{id}_{F(G)}$, for every $K \leq G$, where ι_K is the inclusion.

The denomination "cohomological" is due to the fact that the group cohomology functor $H^n(-, R)$, seen as an inflation/global Mackey functor, satisfies this property.

The following result is [50, Proposition 6.4], although Park restricts the statement to global Mackey functors, the proof is essentially the same.

Proposition 2.4.25. *Let R be a field of characteristic p and $\mathcal{S} = \triangleright$ or Δ. Then $\mathbb{S}^{\mathcal{S}}_{Q,V}$ is cohomological if and only if Q is a p-group.*

Proof. Let us suppose that Q is a p-group. It will suffice to show that $\underline{R}^{\mathcal{S}}_{Q,V}$ is cohomological, since $\mathbb{S}^{\mathcal{S}}_{Q,V}$ is a subfunctor. For that, it will suffice to show that

$$\underline{[\varphi, Q]^Q_G \circ [\iota_K, K]^G_G} = [G : K]\underline{[\varphi, Q]}.$$

Now, since $[\varphi, Q]^Q_G = [\varphi, Q]^Q_{\varphi(Q)} \circ [\iota, \varphi(Q)]^{\varphi(Q)}_G$, we can assume that φ is the inclusion homomorphism $Q \to G$. Hence,

$$\begin{aligned} \underline{[\iota, Q] \circ [\iota, K]} &= \sum_{x \in [K\backslash G/Q]} \underline{[\iota, K^x \cap Q]} \\ &= \sum_{x \in [K\backslash N_G(Q,K)/Q]} \underline{[\iota, Q]} \\ &= |N_G(Q, K) : K|\, \underline{[\iota, Q]} \\ &= [G : K]\underline{[\iota, Q]}. \end{aligned}$$

The first two equalities are clear. For the third equality, note that $K\backslash N_G(Q, K)/Q = K\backslash N_G(Q, K)$. The las equality follows from [22, Lemma 3.5].

Conversely, suppose that $S := \mathbb{S}^{\mathcal{S}}_{Q,V}$ is cohomological. Let R be a Sylow p subgroup of Q. Then

$$S(Q) \xrightarrow{\mathrm{res}^Q_R} S(R) \xrightarrow{\mathrm{tr}^Q_R} S(Q)$$

is multiplication by $|Q : R|$ and hence an isomorphism. Since $S(Q) \cong V \neq 0$, we have $S(R) \neq 0$. Since Q is a minimal group for S, it follows that $Q = R$, so Q is a p-group. □

Let F be a cohomological inflation functor. Inspired by the Cartan-Eilenberg theorem (Theorem 1.3.3), for the following result we use the notation $\varphi^* := M([\varphi, P]^H_P)$, for $\varphi : P \to H$. Its proof goes essentially as in Theorem 1.3.3.

Proposition 2.4.26. *Let R be a field of characteristic p and let F be a cohomological inflation functor over R, and G a finite group with S as a Sylow p-subgroup. Then*

$$M(G) \cong \{x \in M(S) \mid \varphi^*(x) = \iota^*(x) \textit{ for all } P \leq S \textit{ and each } \varphi = c_g : P \to S \textit{ with } g \in G\},$$

as R-modules. In particular, $\dim_R F(G) \leq \dim_R F(S)$.

Proposition 2.4.26 can be used to give a description of $\mathbb{S}^{\triangleright}_{Q,V}(G)$, with Q a p-group, in terms of the elements of $\mathbb{S}^{\triangleright}_{Q,V}(S)$ and the latter's formula by Webb (see comments above Proposition 2.4.19). As a result of this, [33, Theorem 0.1] follows (see Remark 3.2.7).

3

THE SEGAL CONJECTURE AND STABLE HOMOTOPY CLASSIFICATION OF $BG_p^\wedge$ VIA BISET FUNCTORS

This chapter is devoted to, on one side, reviewing classical applications of biset functors to the homotopy theory of $\mathbb{B}G_p^\wedge$ and, on the other side, exploiting Ragnarsson's variants of the Segal conjecture (see Theorem 3.4.3 and Corollary 3.4.4) to give new consequences on this subject (see Section 3.6 for an overview).

3.1 THE SINGLE AND DOUBLE BURNSIDE RING

Let G, H be finite groups. We have a bilinear map $\cdot : B(G,1) \times B(G,H) \to B(G,H)$ by sending $([Y],[X])$ to $[Y \times X]$, with G acting diagonally on $Y \times X$ and H acting only on the second component. In particular, if $H = 1$, this gives $B(G,1)$ the structure of a commutative ring (since $X \times Y \cong Y \times X$ as G-sets), called the *(single) Burnside ring of* G. For simplicity, we use the notation $B(G) := B(G,1)$.

Equivalently, the Burnside ring $B(G)$ can be defined as the Grothendieck construction of the semiring of isomorphism classes of finite G-sets, with addition induced by disjoint union and multiplication induced by Cartesian product. This is in fact the usual definition of the Burnside ring in the literature [8].

As a $\mathbb{Z}$-module, $B(G)$ is freely generated by a set of representatives $[H]$ of G-conjugacy classes of subgroups of G. Indeed, the only transitive $(1,G)$-bisets are of the form $1 \times G/1 \times H$, with $H \leq G$. Using the bilinear map $\cdot$ we can give to $B(G,H)$ a structure of a $B(G)$-module. In particular, the product $\cdot : B(G) \times B(G) \to B(G)$ gives a modified double coset formula:

$$[H] \cdot [K] = \sum_{x \in [H \backslash G / K]} [H \cap {}^x K]$$

The Burnside ring $B(G)$ can be embedded in a ring that is convenient for computations and proofs. We will devote the following lines to define that ring and the embedding of $B(G)$.

Definition 3.1.1. Let H be a subgroup of G and $\Phi_H : B(G) \to \mathbb{Z}$ be the ring homomorphism that sends the class of a finite G-set X to $|X^H|$. The product homomorphism

$$\Phi^G : B(G) \xrightarrow{\prod_{H \le_G G} \Phi_H} \prod_{H \le_G G} \mathbb{Z}$$

is known as the *mark homomorphism for B(G)* and $\Omega(G) := \prod_{H \le_G G} \mathbb{Z}$ is called the *ghost ring of B(G)*.

The following list of results regarding $B(G)$ and $B(G, H)$ are classical and can be found in [59, Section 1.1], for example.

Proposition 3.1.2. *Given $H, K \le G$, we have*

$$\Phi_K([H]) = \frac{|N_G(K, H)|}{|H|}.$$

In particular, $\Phi_K([H]) \neq 0$ if and only if $K \le_G H$.

Proposition 3.1.3. *The map Φ^G is a ring monomorphism. Moreover, the cokernel of Φ is isomorphic as a $\mathbb{Z}$-module to*

$$\prod_{[H] \in C(G)} (\mathbb{Z}/|W_G H|\mathbb{Z}),$$

where $C(G)$ is a set of representatives of G-conjugacy classes of subgroups of G.

This gives us a method to calculate $\mathbb{Q}B(G)$ via its ghost ring, indeed, we have a short exact sequence of abelian groups:

$$0 \to B(G) \to \Omega(G) \to \prod_{[H] \in C(G)} (\mathbb{Z}/|W_G H|\mathbb{Z}) \to 0.$$

The term on the right vanishes after tensoring with $\mathbb{Q}$, so it follows that:

$$\Phi : \mathbb{Q}B(G) \xrightarrow{\cong} \mathbb{Q}\Omega(G) := \mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G),$$

with inverse [26, Section 3] given by

$$\Phi^{-1}(e_H) = \frac{1}{|N_G(H)|} \sum_{K \leq H} \mu(K,H)|K|[G/K], \tag{3.1.4}$$

where $e_H = (0,\ldots,1,\ldots,0)$, with 1 in the H-th position and μ is the Mobius function of the poset of subgroups of G.

Remark 3.1.5. Even though our interest regarding $B(G)$ in this thesis is limited to its applications in the stable homotopy theory of $\mathbb{B}G$ or $\mathbb{B}G^{\wedge}_p$, we must remark that its importance goes beyond this subject. Indeed, the study of Burnside rings gave rise to several induction theorems in representation theory [4, Section 5.6], a characterization of solvable groups [8, Corollary 3.3.9], among others. Regarding further applications to homotopy theory, the Quillen conjecture [54] and its variants have been approached through the theory of Burnside rings, see [8, Section 4] for a detailed account.

Now, given two finite groups G, H, the abelian groups $B(G,H)$ and $B(H \times G)$ are canonically isomorphic. We can use this point of view to speak of the marks of a virtual (H,G)-biset. As an easy-to-check consequence of Proposition 3.1.2, we have the following corollary.

Corollary 3.1.6. *Let $\Delta(K,\varphi) \leq H \times G$ and $[L,\varphi] \in B(G,H)$. Then*

$$\Phi_{\Delta(K,\varphi)}([L,\psi]) = \frac{|N_{\varphi,\psi}|}{|L|}|C_H(K)|,$$

with $N_{\varphi,\psi} = \{x \in N_G(K,L) \mid c_y\varphi = \psi c_x,$ for some $y \in H\}$.

Now, when $H = G$, the associative composition $\circ : B(G,G) \times B(G,G) \to B(G,G)$ gives $B(G,G)$ the structure of a ring. We call $B(G,G)$ the *double Burnside ring of G*. This ring has a richer structure than $B(G)$, in fact, we can see the latter as a subring of the former.

Proposition 3.1.7. *The map $B(G) \to B(G,G)$ that sends $[G/H]$ to $[G \times G/\Delta(H,\iota)]$ is a ring monomorphism.*

However, as we should expect, $B(G,G)$ is more complicated in general. For instance, $B(G,G)$ is not commutative in general, and the mark homomorphism described earlier

$$\Psi : B(G,G) \to \Omega(G \times G),$$

viewing $B(G,G)$ as $B(G \times G)$, is not a ring homomorphism anymore, but only a homomorphism of $\mathbb{Z}$-modules.

3.2 THE SEGAL CONJECTURE

In this section we relate the theory of biset functors to the homotopy theory of classifying spectra of finite groups. This is achieved thanks to the celebrated Segal conjecture, which in fact comes in several forms.

Definition 3.2.1. Let $\epsilon : B(G) \to \mathbb{Z}$ be the ring homomomorphism that sends the class $[X]$ of a finite G-set X to its cardinality $|X|$, extended linearly. We call ϵ the *augmentation homomorphism* and $I(G) := \mathrm{Ker}(\epsilon)$ the *augmentation ideal of B(G).*

We observe that ϵ can be identified with $\mathrm{Res}_1^G : B(G) \to B(1) \cong \mathbb{Z}$, viewing the Burnside ring as a Mackey functor for G [29, Definition 17.3.23]. Thus $I(G) = \mathrm{Ker}(\mathrm{Res}_1^G)$.

Given a G-spectrum X, its collection of H-equivariant homotopy groups $\pi_n^H(X)$, with n fixed and $H \leq G$, form a Mackey functor for G [65, Chapter 3] and each $\pi_n^H(X) = X^{-n}(\mathbb{S}_H)$, where $\mathbb{S}_G$ denotes the *G-equivariant sphere* [65, Example 2.10], has a $B(G)$-module structure. In particular, for $X = \mathbb{S}_G$, we have

$$B(G) \cong \pi_0^G(\mathbb{S}_G) = [\mathbb{S}_G, \mathbb{S}_G]^G$$

as rings [29, Corollary 17.3.36]. Now, let $X = E_G$ be a G-equivariant spectrum and E its underlying non-equivariant spectrum. The projection $\pi : EG_+ \to \mathbb{S}_G$ induces

$$\pi^* : E_G^*(\mathbb{S}_G) \to E_G^*(EG_+).$$

If E_G is a G-split spectrum [27, Definition 3.12], we have $E_G^*(EG_+) \cong E^*(EG_+/G) = E^*(BG_+)$. In this way we obtain

$$\pi^* : E_G^*(\mathbb{S}_G) \to E^*(BG_+).$$

With E_G as above, we say that the *completion theorem holds for* E_G if π^* an $I(G)$-completion map, i.e. π^* induces an isomorphism

$$E^*_G(\mathbb{S}_G)^\wedge_{I(G)} \cong E^*(BG_+).$$

Historically, the first completion theorem is the Atiyah-Segal completion theorem for complex equivariant K-theory K^*_G. For $K = K^0$, this completion theorem takes the form

$$R(G)^\wedge_{I(G)} \cong K(BG),$$

where $R(G)$ is the complex representation ring of G. Now, for our purposes, the case $E_G = \mathbb{S}_G$, proved by Carlsson [19], is of remarkable importance. Originally conjectured by Segal for $\mathbb{S}^0_G$, it takes the form

$$B(G)^\wedge_{I(G)} \cong \pi^0(\mathbb{B}_+G) = [\mathbb{B}_+G, \mathbb{S}]. \tag{3.2.2}$$

Before a proof for the Segal conjecture was finally achieved, it was known that isomorphism 3.2.2 implies a more general version (recall Remark 2.3.8).

Theorem 3.2.3 ([32])**.** *Let G, H be finite groups. The homomorphism of abelian groups*

$$B^\triangleleft(G,H) \xrightarrow{\alpha} [\mathbb{B}_+G, \mathbb{B}_+H],$$

which sends the class $[K, \varphi]$ to the composite $\mathbb{B}G_+ \xrightarrow{\mathrm{tr}^G_K} \mathbb{B}K_+ \xrightarrow{B\varphi} \mathbb{B}H_+$ induces an isomorphism

$$B^\triangleleft(G,H)^\wedge_{I(G)} \cong [\mathbb{B}_+G, \mathbb{B}_+H]$$

This gives us a completely algebraic description of $[\mathbb{B}G_+, \mathbb{B}H_+]$, although $I(G)$-adic completion is most of the times hard to compute. We can simplify this description when $G = S$ is a finite p-group, using the reduced Burnside module. Indeed, α induces a homomorphism

$$\widetilde{B}^\triangleleft(S,H) \xrightarrow{\widetilde{\alpha}} [\mathbb{B}_+S, \mathbb{B}_+H]/[\mathbb{B}_+S, \mathbb{S}] \cong [\mathbb{B}S, \mathbb{B}H]. \tag{3.2.4}$$

Now, $\widetilde{B}^{\triangleleft}(S,H)$ is also a $B(S)$-module, and $\widetilde{\alpha}$ is an $I(S)$-completion map too, but $\widetilde{B}^{\triangleleft}(S,H)$ has the advantage that its $I(S)$-adic topology coincides with its p-adic topology [37]. In other words, α induces an isomorphism

$$\mathbb{Z}_p^{\wedge}\widetilde{B}^{\triangleleft}(S,H) \cong [\mathbb{B}S, \mathbb{B}H]. \tag{3.2.5}$$

Indeed,

$$\mathbb{Z}_p^{\wedge}\widetilde{B}^{\triangleleft}(S,H) = \mathbb{Z}_p^{\wedge} \otimes \widetilde{B}^{\triangleleft}(S,H) \cong \widetilde{B}^{\triangleleft}(S,H)_p^{\wedge},$$

since $\widetilde{B}^{\triangleleft}(S,H)$ is finitely generated. Furthermore, if $H = S$, we have an isomorphism

$$\mathbb{Z}_p^{\wedge}\widetilde{B}^{\triangleleft}(S,S) \cong [\mathbb{B}S, \mathbb{B}S] \tag{3.2.6}$$

as rings, since the homomorphism α (and therefore $\widetilde{\alpha}$) is compatible with the compositions.

Martino-Priddy [33] and Benson-Feshbach [5] used the ring isomorphism in Equation (3.2.6) to classify, up to homotopy equivalence, all the indecomposable stable summands of BS. Indeed, each indecomposable stable summand X of $\mathbb{B}S$ corresponds, up to homotopy equivalence, to a unique primitive idempotent e_X of $\mathbb{Z}_p^{\wedge}\widetilde{B}^{\triangleleft}(S,S)$, up to conjugation. In turn, e_X corresponds to a unique simple $\mathbb{Z}_p^{\wedge}\widetilde{B}^{\triangleleft}(S,S)$-module M_X. Finally, reducing mod p the coefficients of $\mathbb{Z}_p^{\wedge}\widetilde{B}^{\triangleleft}(S,S)$, we obtain that M_X corresponds to a unique simple $\mathbb{F}_p\widetilde{B}^{\triangleleft}(S,S)$-module S_X, up to isomorphism. Now, using the quotient

$$\mathbb{F}_pB^{\triangleleft}(S,S) \to \mathbb{F}_p\widetilde{B}^{\triangleleft}(S,S),$$

which is a homomorphism of $\mathbb{F}_p$-algebras, we can view S_X as an $\mathbb{F}_pB^{\triangleleft}(S,S)$-simple module.

Now, by Section 2.1, to any simple $\mathbb{F}_pB^{\triangleleft}(S,S)$-module we can associate a unique, up to isomorphism, seed (Q,V), where Q is isomorphic to a subgroup of S and V is a simple $\mathbb{F}_p\mathrm{Out}(Q)$-module. However, we must keep in mind that such simple $\mathbb{F}_pB^{\triangleleft}(S,S)$-modules coming from simple $\mathbb{F}_p\widetilde{B}^{\triangleleft}(S,S)$-modules are associated to all seeds (Q,V), except the trivial one $(1,\mathbb{F}_p)$.

All these correspondences are bijective, hence we can associate to any representative seed (Q, V) with $Q \neq 1$, a unique indecomposable stable summand $X_{Q,V}$ of $\mathbb{B}S$. Moreover, by the above arguments, a complete stable splitting

$$\mathbb{B}S \simeq \bigvee_{(Q,V)} X_{Q,V}^{n_{Q,V}}$$

corresponds necessarily to a direct sum decomposition

$$\mathbb{F}_p B^{\triangleright}(S,-) \cong \bigoplus_{(Q,V)} (\mathbb{P}_{Q,V}^{\triangleright})^{n_{Q,V}},$$

therefore

$$n_{Q,V} = \frac{\dim_{\mathbb{F}_p} \mathbb{S}_{Q,V}^{\triangleright}(S)}{\dim_{\mathbb{F}_p} \mathrm{End}_{\mathbb{F}_p \mathrm{Out}(Q)}(V)} = n_{Q,V}(S).$$

Similarly, given a finite group G with a Sylow p-subgroup S, we have that $\mathbb{B}G_p^\wedge$, as a stable summand of $\mathbb{B}S$, can also be decomposed in terms of the summands $X_{Q,V}$ above, although their multiplicities may change. Indeed, the multiplicity of $X_{Q,V}$ as a summand of $\mathbb{B}G_p^\wedge$ is $n_{Q,V}(G)$ (see Theorem 2.4.23 and Proposition 3.4.14). In fact, this is another reason why $n_{Q,V}(G) \leq n_{Q,V}(S)$ (see Proposition 2.4.26).

Remark 3.2.7. We have just seen that $\mathbb{B}G_p^\wedge \simeq \bigvee_{(Q,V)} X_{Q,V}^{n_{Q,V}(G)}$, with $X_{Q,V}$ an indecomposable summand of $\mathbb{B}S$, in particular each Q is a p-group. This is essentially [70, Theorem 6.2], but we will also deduce it in Section 3.4 (see Proposition 3.4.14).

Historically, shortly after Martino-Priddy [33] and Benson-Feshbach [5], Webb [70] defined the notion of an inflation functor and reproved [70, Theorem 6.2] in a more sophisticated manner the main theorems of both papers.

Example 3.2.8. Using Example 2.4.21 for $\mathcal{S} = \triangleright$, and the fact that $\mathbb{F}_p$ is a splitting field for $\mathbb{Z}/(p-1)$, we have

$$\mathbb{B}\mathbb{Z}/p \simeq X_0 \vee \cdots \vee X_{p-2},$$

where $X_i = X_{\mathbb{Z}/p,(\det)^i}$, $i = 0, \ldots, p-2$. In particular, $\mathbb{B}\mathbb{Z}/2$ is indecomposable.

Example 3.2.9. Let $G = \Sigma_p$ and $Q = \mathbb{Z}/p$. The Sylow p-subgroups of G are isomorphic to Q, and we can compute

$$\dim_{\mathbb{F}_p} \mathbb{S}^{\triangleright}_{\mathbb{Z}/p,(\det)^i}(\Sigma_p) = \begin{cases} 1, & \text{if } i = 0, p-2, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, $\mathbb{B}G^{\wedge}_p \simeq X_0 \vee X_{p-2}$.

3.3 THE STABLE MARTINO-PRIDDY CONJECTURE

By the discussion above, the homotopy type of $\mathbb{B}G^{\wedge}_p$ is determined by the multiplicities $n_{Q,V}(G)$, and they in turn are determined by the evaluation $\mathbb{S}_{Q,V}(G)$. However, there is no general formula to compute $\dim_{\mathbb{F}_p}\mathbb{S}_{Q,V}(G)$ in practice for an arbitrary finite group G. In the rest of this chapter, we study the problem of detecting in algebraic terms when two p-completed classifying spectra $\mathbb{B}G^{\wedge}_p$ and $\mathbb{B}H^{\wedge}_p$ are homotopy equivalent. We recall from Chapter 1 that $\mathbb{B}G \simeq \bigvee_{q||G|} \mathbb{B}G^{\wedge}_q$. Therefore, given a p-group Q, it follows that

$$[\mathbb{B}Q, \mathbb{B}G] \cong \bigoplus_{q||G|} [\mathbb{B}Q, \mathbb{B}G^{\wedge}_q] = [\mathbb{B}Q, \mathbb{B}G^{\wedge}_p], \tag{3.3.1}$$

since the other summands are trivial. We can therefore use isomorphism (3.2.5) to obtain

$$\mathbb{Z}^{\wedge}_p \widetilde{B}^{\triangleleft}(Q,G) \cong [\mathbb{B}Q, \mathbb{B}G^{\wedge}_p]. \tag{3.3.2}$$

Let $J(Q,G)$ denote the $\mathbb{Z}^{\wedge}_p$-submodule of $\mathbb{Z}^{\wedge}_p \widetilde{B}^{\triangleleft}(Q,G)$ generated by the classes $\widetilde{[P,\alpha]}$, with $|\alpha(P)| < |Q|$. The quotient $\mathbb{Z}^{\wedge}_p \widetilde{B}^{\triangleleft}(Q,G)/J(Q,G)$ is isomorphic to $\mathbb{Z}^{\wedge}_p\mathrm{Inj}(Q,G)$ as a $\mathbb{Z}^{\wedge}_p\mathrm{Out}(Q)$-module, where

$$\mathrm{Inj}(Q,G) = \{[f] \in \mathrm{Rep}(Q,G) \mid f \text{ is a monomorphism}\}.$$

When $\mathbb{B}G^{\wedge}_p$ is homotopy equivalent to $\mathbb{B}H^{\wedge}_p$, we have by isomorphisms (3.3.2) and (3.2.5) (after tensoring with $\mathbb{F}_p$) that

$$\mathbb{F}_p B^{\triangleleft}(Q,G) \cong \mathbb{F}_p B^{\triangleleft}(Q,H) \text{ as } \mathbb{F}_p\mathrm{Out}(Q)\text{-modules}, \tag{3.3.3}$$

for every finite p-group Q. Passing to the quotients induces an isomorphism

$$\mathbb{F}_p\mathrm{Inj}(Q,G) \cong \mathbb{F}_p\mathrm{Inj}(Q,H) \text{ as } \mathbb{F}_p\mathrm{Out}(Q)\text{-modules,} \tag{3.3.4}$$

for every finite p-group Q. Then the isomorphism in Equation (3.3.4) is a necessary condition for $\mathbb{B}G_p^\wedge$ and $\mathbb{B}H_p^\wedge$ to be homotopy equivalent. Now we ask if it is sufficient too, and this was answered affirmatively by Martino-Priddy [34, Theorem 1].

Theorem 3.3.5 (Martino-Priddy)**.** *Given two finite groups G, H, the following are equivalent:*

(1) *$\mathbb{B}G_p^\wedge$ and $\mathbb{B}H_p^\wedge$ are homotopy equivalent.*

(2) $\mathbb{F}_p\mathrm{Rep}(Q,G) \cong \mathbb{F}_p\mathrm{Rep}(Q,H)$ *as* $\mathbb{F}_p\mathrm{Out}(Q)$*-modules for every finite p-group Q.*

(3) $\mathbb{F}_p\mathrm{Inj}(Q,G) \cong \mathbb{F}_p\mathrm{Inj}(Q,H)$ *as* $\mathbb{F}_p\mathrm{Out}(Q)$*-modules for every finite p-group Q.*

Condition (3) implies that G and H have isomorphic Sylow p-subgroups. Although it was only mentioned in the proof of [34, Theorem. 1.1], there is another equivalent condition:

(4) $\mathbb{F}_p\mathrm{Cen}(Q,G) \cong \mathbb{F}_p\mathrm{Cen}(Q,H)$ as $\mathbb{F}_p\mathrm{Out}(Q)$-modules for every finite p-group Q, where $\mathrm{Cen}(Q,G) = \{[f] \in \mathrm{Inj}(Q,G) \mid f(Q) \text{ is } p\text{-centric in } G\}$.

In the original proof, the interesting implication is $(4) \Rightarrow (1)$. Its proof required defining an abstract matrix [34, Theorem 3.3] that allowed the authors to describe the multiplicities of the indecomposable stable summands of $\mathbb{B}G_p^\wedge$ using the notion of *linked summands*. Implication $(1) \Rightarrow (2)$ is quite similar to $(1) \Rightarrow (3)$ outlined above, while $(2) \Rightarrow (3)$ is technical. On the other hand, some years later, Ragnarsson pointed out that the proof $(3) \Rightarrow (4)$ needed a correction, and the same authors proposed a new proof in [36]. The author finds some arguments of the proof in [36] unclear (see Section 3.5).

In the same spirit of Webb [70] (see comments below Remark 3.2.7), we use biset functors to refine [34, Theorem 1], in some sense. To be precise, we will show that condition (4) implies

($1^\bullet$) $G \cong H$ in $\mathbb{Z}_p^\wedge\widetilde{B}^{\Delta_p}$,

where $\mathbb{Z}_p^\wedge\widetilde{B}^{\Delta_p}$ is the p-local bifree $\mathbb{Z}_p^\wedge$-Burnside category (see Section 3.4). As we will see in the next section, condition ($1^\bullet$) is apparently stronger than (1) (see comments below Proposition 3.4.9), but Theorem 3.3.5 asserts that they would all be equivalent.

3.4 STABLE MAPS BETWEEN p-COMPLETED CLASSIFYING SPACES

Since a homotopy equivalence from $\mathbb{B}G_p^\wedge$ to $\mathbb{B}H_p^\wedge$ corresponds to an isomorphism in $[\mathbb{B}G_p^\wedge, \mathbb{B}H_p^\wedge]$, it would be desirable to have, via a convenient version of the Segal conjecture, an isomorphism of $\mathbb{Z}_p^\wedge$-linear categories

$$\widetilde{\alpha} : \mathcal{D} \to \mathbb{B}_p^\wedge\mathrm{Grp}, \tag{3.4.1}$$

for some $\mathcal{D}$, where $\mathbb{B}_p^\wedge\mathrm{Grp}$ is the category with objects the finite groups and morphisms $[\mathbb{B}G_p^\wedge, \mathbb{B}H_p^\wedge]$, and $\mathcal{D}(G, H)$ should be of algebraic nature. Of course, the idea is to reformulate the isomorphism (3.2.5) so that it works for an arbitrary finite group G (recall isomorphism (3.3.1) as well).

The first candidate for $\mathcal{D}$ could be a (reduced) admissible Burnside category $\mathbb{Z}_p^\wedge\widetilde{B}^{\mathcal{S}}$, but isomorphisms in admissible R-Burnside categories, for any ring R, detect group isomorphisms, i.e. $G \cong H$ in $RB^{\mathcal{S}}$ if and only if $G \cong H$ as groups [63, Proposition 4.3]. The same is true after reducing. However, we recall from Example 1.1.6 that in general the homotopy type of $\mathbb{B}G_p^\wedge$ does not determine G up to isomorphism.

In other words, our candidate for a $\mathbb{Z}_p^\wedge$-linear category $\mathcal{D}$ cannot be a reduced admissible Burnside category, although the nature of $[\mathbb{B}G_p^\wedge, \mathbb{B}H_p^\wedge]$ and the particular case when G is a p-group in isomorphism (3.2.5) suggest that in general $\mathcal{D}(G, H)$ should be a submodule of $\mathbb{Z}_p^\wedge\widetilde{B}^{\triangleleft}(G, H)$ encoding the p-local structures of G and H. This problem was successfully addressed by Ragnarsson, whose paper [57] provides the main source for the following definition.

Definition 3.4.2. Let G, H be finite groups and R a commutative ring with identity. The *p-local left-free Burnside module of G and H*, denoted $RB^{\triangleleft_p}(G, H)$ (resp. the *p-local bifree Burnside module of G and H*, denoted $RB^{\Delta_p}(G, H)$) is the R-submodule of $RB^{\triangleleft}(G, H)$ (resp. $RB^{\Delta}(G, H)$) freely generated by classes $[P, \varphi]$, where P is a p-subgroup of G and $\varphi : P \to H$ is a homomorphism (resp. a monomorphism).

The following table summarizes the above definitions on Burnside submodules:

Burnside submodule	**Notation**	R-**basis**
Left-free Burnside module	$RB^{\triangleleft}(G,H)$	$[K,\varphi]$
Bifree Burnside module	$RB^{\Delta}(G,H)$	$[K,\varphi]$, φ is a monomorphism
p-local left-free Burnside module	$RB^{\triangleleft_p}(G,H)$	$[K,\varphi]$, K is a p-group
p-local bifree Burnside module	$RB^{\Delta_p}(G,H)$	$[K,\varphi]$, K is a p-group, φ is a monomorphism

Note that we cannot speak of the subcategory $RB^{\triangleleft_p}$ of RB since the identity morphism $[G,\mathrm{id}_G]$ of $RB(G,G)$ is not in $RB^{\triangleleft_p}(G,G)$, unless G is a p-group. The same can be said about RB^{Δ_p}.

The following version of the Segal conjecture [57, Theorem 4.1], not established by the time of [5, 33, 34], will be one of our main tools in reaching the goal of this chapter.

Theorem 3.4.3 (Ragnarsson)**.** *Let G,H be finite groups. The map*

$$\mathbb{Z}_p^\wedge B^{\triangleleft_p}(G,H) \xrightarrow{\alpha} [(\mathbb{B}_+G)_p^\wedge, (\mathbb{B}_+H)_p^\wedge],$$

which sends the class $[P,\varphi]$ to the composite $(\mathbb{B}_+G)_p^\wedge \xrightarrow{\mathrm{tr}_{P_p^\wedge}} (\mathbb{B}_+P)_p^\wedge \xrightarrow{\mathbb{B}\varphi_p^\wedge} (\mathbb{B}_+H)_p^\wedge$ is an isomorphism of $\mathbb{Z}_p^\wedge$-modules.

Passing to the respective quotients, in the following corollary we obtain the $\mathbb{Z}_p^\wedge$-module $\mathcal{D}(G,H)$ mentioned in the introduction of this section, thus enhancing isomorphism (3.3.2).

Corollary 3.4.4 ([57, Corollary 3.2])**.** *The map α above induces an isomorphism of $\mathbb{Z}_p^\wedge$-modules*

$$\mathbb{Z}_p^\wedge \widetilde{B}^{\triangleleft_p}(G,H) \to [\mathbb{B}G_p^\wedge, \mathbb{B}H_p^\wedge].$$

However, there remains the question of whether $\mathbb{Z}_p^\wedge \widetilde{B}^{\triangleleft_p}$ is a category or not. In fact, we ask this question for more general coefficient rings. Let R be a *p-local ring*, meaning that any integer in R coprime to p is invertible (e.g. $\mathbb{Z}_p^\wedge, \mathbb{Z}_{(p)}, \mathbb{F}_p$). There is a ring homomorphism

$$\pi : \mathbb{Z}_{(p)} \to R$$

that sends $\frac{m}{n}$ to $m1_R.(n1_R)^{-1}$. In that case $RB^{\lhd p}$ and RB^{Δ_p} meet the requirements to be categories. Indeed, composition is not an issue and the only remaining requirement is the existence of an identity in $RB^{\lhd p}(G,G)$ and $RB^{\Delta_p}(G,G)$, for every finite group G. For convenience, we first treat the case $R = \mathbb{Z}_{(p)}$. The following proposition appears in [57, Proposition 5.3].

Proposition 3.4.5 (Ragnarsson)**.** *Let G be a finite group and $\{P_1,\dots,P_n\}$ be a poset of p-subgroups of G, partially ordered by G-subconjugacy, where we pick one representative P_i for each G-conjugacy class, so that $i \leq j$ if $P_i \geq_G P_j$. The virtual biset*

$$1_G^p = \sum_j a_j[P_j,\iota_j]_G^G$$

is the multiplicative identity of $\mathbb{Z}_{(p)}B^{\lhd p}(G,G)$, where $\iota_j : P_j \to G$ is the inclusion, and the coefficients a_j satisfy the equations

$$\sum_j a_j \frac{|N_G(P_i,P_j)|}{|P_j|} = 1 \tag{3.4.6}$$

for $i = 1,\dots,n$.

By definition of 1_G^p, it is consequently, also the identity of $\mathbb{Z}_{(p)}B^{\Delta_p}(G,G)$. The same virtual biset works for $R = \mathbb{Z}_p^\wedge$ (seeing $\mathbb{Z}_{(p)} \subset \mathbb{Z}_p^\wedge$), $R = \mathbb{F}_p$ (after reducing mod p the coefficients a_j) and more generally any p-local ring, but for simplicity we will restrict to these three cases. Similarly, it follows that $R\widetilde{B}^p$ and $R\widetilde{B}^{\Delta_p}$, with $R = \mathbb{Z}_p^\wedge, \mathbb{Z}_{(p)}, \mathbb{F}_p$, are categories.

Observation 3.4.7. In general, we do not have $1_S^p = [S,\mathrm{id}]_S^S$, but we will see in Remark 3.4.11 that $M(1_S^p) = M([S,\mathrm{id}])$, when $R = \mathbb{F}_p$ and M is cohomological.

Remark 3.4.8. By the above discussion, $\mathbb{B}G_p^\wedge$ is homotopy equivalent to $\mathbb{B}H_p^\wedge$ if and only if $G \cong H$ in $\mathbb{Z}_p^\wedge\widetilde{B}^{\lhd p}$. In fact, we prefer to work with the unreduced category $\mathbb{Z}_p^\wedge B^{\lhd p}$, and it is not hard to see that both conditions are equivalent to $G \cong H$ in $\mathbb{Z}_p^\wedge B^{\lhd p}$. At last, we can reduce mod p the coefficients without loss of information, i.e. $\mathbb{B}G_p^\wedge \simeq \mathbb{B}H_p^\wedge$ if and only if $G \cong H$ in $\mathbb{F}_pB^{\lhd p}$ (see Proposition A.2.1). On the other hand, it is clear that we can instead use $\mathbb{F}_pB^{\rhd p}$ and make the analogous assertion.

Thus, $(4) \Rightarrow (1)$ in Theorem 3.3.5 follows from the next proposition.

Proposition 3.4.9. *If $\mathbb{F}_p\mathrm{Cen}(Q,G) \cong \mathbb{F}_p\mathrm{Cen}(Q,H)$ as $\mathbb{F}_p\mathrm{Out}(Q)$-modules, for every finite p-group Q, then $G \cong H$ in $\mathbb{F}_pB^{\Delta_p}$.*

Proposition 3.4.9 is a refinement of (4) ⇒ (1) since, in general, $RB^{\triangle p}$ is a proper subcategory of $RB^{\triangleleft p}$.

For a proof of Proposition 3.4.9, it will be necessary to study the functors $\mathbb{F}_pB^{\Delta_p}(G,-)$. In fact, it is not hard to check that $\mathbb{F}_pB^{\Delta_p}(G,-)$ is a global Mackey functor. In general, $\mathbb{F}_pB^{\Delta_p}(G,-)$ does not agree with the representable global Mackey functor $\mathbb{F}_pB^{\Delta}(G,-)$, unless G is a p-group. However, a Yoneda lemma argument still works: For the following proposition and henceforth in this chapter, the coefficients are taken modulo p.

Lemma 3.4.10. *Let $\mathrm{Mack}_{\mathbb{F}_p}$ denote the category of global Mackey functors over $\mathbb{F}_p$. Then $\mathrm{Mack}_{\mathbb{F}_p}(\mathbb{F}_pB^{\Delta_p}(G,-),M) \cong M(G)$ for each cohomological global Mackey functor M.*

Proof. Given a cohomological Mackey functor M, we send $\Phi \in \mathrm{Hom}_{\mathrm{Mack}_{\mathbb{F}_p}}(\mathbb{F}_pB^{\Delta_p}(G,-),M)$ to $\left[\frac{|G|}{|N_G(S)|}\right]^{-1}\Phi_G(1_G^p) \in M(G)$, where S is a p-Sylow subgroup of G. It is not hard to check that this map is injective, like in Yoneda's lemma. The subtle difference appears when proving surjectivity: given $v \in M(G)$, we define $\Psi \in \mathrm{Hom}_{\mathrm{Mack}_{\mathbb{F}_p}}(\mathbb{F}_pB^{\Delta_p}(G,-),M)$ by $\Psi_L(X) = M(X)(v)$, for each group L and each $X \in \mathbb{F}_pB^{\Delta_p}(G,L)$. Now,

$$\begin{aligned}
\left[\frac{|G|}{|N_G(S)|}\right]^{-1}\Psi_G(1_G^p) &= \left[\frac{|G|}{|N_G(S)|}\right]^{-1} M\left(\sum_j a_j[P_j,\iota_j]\right)(v) \\
&= \left[\frac{|G|}{|N_G(S)|}\right]^{-1} \sum_j a_j M([P_j,\iota_j])(v) \\
&= \left[\frac{|G|}{|N_G(S)|}\right]^{-1} \sum_j a_j [G:P_j] v \\
&= \left[\frac{|G|}{|N_G(S)|}\right]^{-1} a_1 \frac{|G|}{|S|} v \\
&= v.
\end{aligned}$$

Indeed, we used that M is cohomological in the third equality, the coefficients are reduced modulo p in the forth equality, and $a_1 = \left[\frac{|N_G(S)|}{|S|}\right]^{-1}$ in the fifth equality, by Equation (3.4.6). □

Remark 3.4.11. By the arguments given in the Lemma 3.4.10, $M(1_S^p) = \left[\frac{|S|}{|N_S(S)|}\right]1_{M(S)} = 1_{M(S)}$, for a p-group S and a cohomological Mackey functor M.

Proposition 3.4.12. *Let S be a Sylow p-subgroup of G. Then $\mathbb{F}_pB^{\Delta_p}(G,-)$ is a retract of $\mathbb{F}_pB^{\Delta}(S,-)$.*

Proof. The biset $[S,\iota]_S^G \circ [S,\mathrm{id}]_G^S$ corresponds to

$$(\mathbb{B}_+G)_p^\wedge \xrightarrow{\mathrm{tr}_{S_p^\wedge}} (\mathbb{B}_+S)_p^\wedge \xrightarrow{\mathbb{B}\iota_p^\wedge} (\mathbb{B}_+G)_p^\wedge,$$

which induces multiplication by $[G:S]$ in $H^*(BG;\mathbb{F}_p)$. This map in cohomology is an isomorphism since $[G:S]$ is invertible in $\mathbb{F}_p$. By definition of p-completion , $\mathbb{B}\iota_p^\wedge \circ \mathrm{tr}_{S_p^\wedge}$ is a stable homotopy equivalence, hence $[S,\iota]_S^G \circ [S,\mathrm{id}]_G^S$ is an isomorphism in $\mathbb{F}_pB^{\Delta_p}(G,G)$ (see Remark 3.4.8). We denote its inverse as Ω. The composite

$$\mathbb{F}_pB^{\Delta_p}(G,-) \xrightarrow{([S,\mathrm{id}]_G^S)_*} \mathbb{F}_pB^{\Delta_p}(S,-) \xrightarrow{(\Omega\circ[S,\iota]_S^G)_*} \mathbb{F}_pB^{\Delta_p}(G,-)$$

is the identity morphism, exhibiting $\mathbb{F}_pB^{\Delta_p}(G,-)$ as a retract of $\mathbb{F}_pB^{\Delta}(S,-)$. □

We recall that $\mathbb{F}_pB^{\Delta}(S,-) \cong \bigoplus_{(Q,V)} (\mathbb{P}_{Q,V}^{\Delta})^{n_{Q,V}(S)}$ by [70, Lemma 5.3], where each $\mathbb{P}_{Q,V}^{\Delta}$ is the projective cover of $\mathbb{S}_{Q,V}^{\Delta}$, and each Q is isomorphic to a subgroup of S. By Proposition 3.4.12 and [71, Proposition 7.4.1], we have

$$\mathbb{F}_pB^{\Delta_p}(G,-) \cong \bigoplus_{(Q,V)} (\mathbb{P}_{Q,V}^{\Delta})^{n_{Q,V}(G)},$$

where

$$n_{Q,V}(G) = \frac{\dim_{\mathbb{F}_p}\mathrm{Mack}_{\mathbb{F}_p}(\mathbb{F}_pB^{\Delta_p}(G,-),\mathbb{S}_{Q,V}^{\Delta})}{\dim_{\mathbb{F}_p}\mathrm{End}_{\mathbb{F}_p\mathrm{Out}(Q)}(V)}.$$

But each $\mathbb{S}_{Q,V}^{\Delta}$ is cohomological, since Q is a p-group. Then

$$n_{Q,V}(G) = \frac{\dim_{\mathbb{F}_p}\mathbb{S}_{Q,V}^{\Delta}(G)}{\dim_{\mathbb{F}_p}\mathrm{End}_{\mathbb{F}_p\mathrm{Out}(Q)}(V)},$$

by Lemma 3.4.10. In summary, we just proved the following result.

Proposition 3.4.13. *Let G be a finite group with S a Sylow p-subgroup. Then*

$$\mathbb{F}_pB^{\Delta_p}(G,-) \cong \bigoplus_{(Q,V)} (\mathbb{P}_{Q,V}^{\Delta})^{n_{Q,V}(G)},$$

where (Q,V) runs over the isomorphism classes of seeds with Q isomorphic to a subgroup of S and

$$n_{Q,V}(G) = \frac{\dim_{\mathbb{F}_p} \mathbb{S}^{\Delta}_{Q,V}(G)}{\dim_{\mathbb{F}_p} \mathrm{End}_{\mathbb{F}_p\mathrm{Out}(Q)}(V)}.$$

We can rephrase Proposition 3.4.10 and Proposition 3.4.12 using $\triangleright_p$ instead of Δ_p and obtain a result probably interesting on its own.

Proposition 3.4.14. *Let G be a finite group with S a Sylow p-subgroup. Then*

$$\mathbb{F}_p B^{\triangleright_p}(G,-) \cong \bigoplus_{(Q,V)} (\mathbb{P}^{\triangleright}_{Q,V})^{n_{Q,V}(G)},$$

where (Q,V) runs over the isomorphism classes of seeds with Q isomorphic to a subgroup of S and

$$n_{Q,V}(G) = \frac{\dim_{\mathbb{F}_p} \mathbb{S}^{\triangleright}_{Q,V}(G)}{\dim_{\mathbb{F}_p} \mathrm{End}_{\mathbb{F}_p\mathrm{Out}(Q)}(V)}.$$

We can consider Proposition 3.4.14 as a refinement of [70, Theorem 6.2], discussed in Remark 3.2.7. Indeed, [70, Theorem 6.2] follows easily from Proposition 3.4.14.

The group ring $\mathbb{F}_p\mathrm{Out}(Q)$ satisfies the following lemma.

Lemma 3.4.15 ([34, Lemma 4.2, Corollary 4.3])**.** *Let x, y be elements of $\mathbb{F}_p\mathrm{Out}(Q)$ such that $\mathbb{F}_p\mathrm{Out}(Q)x \cong \mathbb{F}_p\mathrm{Out}(Q)y$ as $\mathbb{F}_p\mathrm{Out}(Q)$-modules and M be any finitely generated left $\mathbb{F}_p\mathrm{Out}(Q)$-module. Then $xM \cong yM$ as $\mathbb{F}_p$-vector spaces. More generally, if $\bigoplus_x \mathbb{F}_p\mathrm{Out}(Q)x \cong \bigoplus_y \mathbb{F}_p\mathrm{Out}(Q)y$ as $\mathbb{F}_p\mathrm{Out}(Q)$-modules, then $\bigoplus_x xM \cong \bigoplus_y yM$ as $\mathbb{F}_p$-vector spaces.*

Proof of Proposition 3.4.9. To prove that $G \cong H$ in $\mathbb{F}_p B^{\Delta_p}$, it will suffice to show that $\mathbb{F}_p B^{\Delta_p}(G,-)$ and $\mathbb{F}_p B^{\Delta_p}(H,-)$ are isomorphic in $\mathrm{Mack}_{\mathbb{F}_p}$, since

$$\mathrm{Mack}_{\mathbb{F}_p}(\mathbb{F}_p B^{\Delta_p}(H,-), \mathbb{F}_p B^{\Delta_p}(G,-)) \cong \mathbb{F}_p B^{\Delta_p}(G,H).$$

The above isomorphism follows because $\mathbb{F}_p B^{\Delta_p}(G,-)$, as a direct sum of functors $\mathbb{P}^{\Delta}_{Q,V}$, which are cohomological, is cohomological as well. Indeed, each $\mathbb{P}^{\Delta}_{Q,V}$ fits into an exact sequence $\mathbb{P}^{\Delta}_{Q,V} \to \mathbb{S}^{\Delta}_{Q,V} \to 0$, so the fact that $\mathbb{P}^{\Delta}_{Q,V}$ is cohomological follows from [68, Lemma 2.1] and Proposition 2.4.25, which shows that $\mathbb{S}^{\Delta}_{Q,V}$ is cohomological. The hypothesis implies that G and H have isomorphic Sylow p-subgroups, let us say S. Thus both $\mathbb{F}_pB^{\Delta_p}(G,-)$ and $\mathbb{F}_pB^{\Delta_p}(H,-)$ are retracts of $\mathbb{F}_pB^{\Delta_p}(S,-)$ by Proposition 3.4.12. In this way,

$$\bigoplus_{(Q,V)} (\mathbb{P}^{\Delta}_{Q,V})^{n_{Q,V}(G)} \cong \mathbb{F}_pB^{\Delta_p}(G,-) \cong \mathbb{F}_pB^{\Delta_p}(H,-) \cong \bigoplus_{(Q,V)} (\mathbb{P}^{\Delta}_{Q,V})^{n_{Q,V}(H)}$$

if and only if $n_{Q,V}(G) = n_{Q,V}(H)$ for each seed (Q,V). In turn, this holds if and only if

$$\dim_{\mathbb{F}_p}\mathbb{S}^{\Delta}_{Q,V}(G) = \dim_{\mathbb{F}_p}\mathbb{S}^{\Delta}_{Q,V}(H),$$

as we show now. Indeed, by Proposition 2.4.19:

$$\mathbb{S}^{\Delta}_{Q,V}(G) \cong \bigoplus_{\substack{\alpha:Q\to L\\ L\leq_G G}} \mathrm{tr}_L^{N_G(L)}({}^{L}V) = \bigoplus_{\substack{\alpha:Q\to L\\ L\leq_G G}} W_L({}^{L}V).$$

Now,

$$W_L = \sum_{x\in N_G(L)/L} c_x = \left|\frac{C_G(L)}{Z(L)}\right| \sum_{\sigma\in\mathrm{Out}_G(L)} \sigma.$$

However, $\left|\frac{C_G(L)}{Z(L)}\right| = 0$ in $\mathbb{F}_p$ unless L is p-centric in G. Therefore, writing $\overline{W}_L$ for $\sum_{\sigma\in\mathrm{Out}_G(L)} \sigma$, we have

$$\mathbb{S}^{\Delta}_{Q,V}(G) \cong \bigoplus_{\substack{\alpha:Q\to L\\ L\leq_G G \text{ is } p\text{−centric}}} \overline{W}_L({}^{L}V). \tag{3.4.16}$$

On the other hand,

$$\mathbb{F}_p\mathrm{Cen}(Q,G) \cong \bigoplus_{L\leq_G G \text{ is } p\text{-centric}} \mathbb{F}_p\left[\frac{\mathrm{Out}(L)}{\mathrm{Out}_G(L)}\right] \cong \bigoplus_{L\leq_G G \text{ is } p\text{-centric}} \mathbb{F}_p\mathrm{Out}(Q)\overline{W}_L.$$

Condition (2) means

$$\bigoplus_{L\leq_G G \text{ is } p\text{-centric}} \mathbb{F}_p\mathrm{Out}(Q)\overline{W}_L \cong \bigoplus_{L'\leq_H H \text{ is } p\text{-centric}} \mathbb{F}_p\mathrm{Out}(Q)\overline{W}_{L'},$$

hence we can apply Lemma 3.4.15 to isomorphism (3.4.16) to obtain $\mathbb{S}^{\Delta}_{Q,V}(G) \cong \mathbb{S}^{\Delta}_{Q,V}(H)$ as $\mathbb{F}_p$-vector spaces, as desired. □

3.5 A CLOSE-UP VIEW OF MARTINO-PRIDDY'S PROOF

In this section, we revisit the proof of $(3) \Rightarrow (4)$ in Theorem 3.3.5 given in [36]. Recall that Martino and Priddy introduced this (unpublished) preprint to correct the original proof [34, Proposition 4.5].

As a part of the motivation for this section, we specify an argument by Martino-Priddy that remains unclear to us (see comments below Proposition 3.5.3). Moreover, we improve Proposition 3.4.9 (see Theorem 3.5.5) and finally explore a couple of consequences of our results if [34, Proposition 4.5] is assumed.

Given a p-group Q, let $n\,\mathrm{Cen}(Q,G)$ be the complement of $\mathrm{Cen}(Q,G)$ in $\mathrm{Inj}(Q,G)$. Let $L \leq G$ such that $Q \cong L$ and p divides $|C_G(L)/Z(L)|$, i.e. L is not p-centric in G. Let $\widetilde{L}$ be a Sylow p-subgroup of $L \cdot C_G(L)$. Choosing a convenient representative of G-conjugacy L, we have that $\widetilde{L}$ is simply $L \cdot C_S(L)$ (see Section). In any case $|L| < |\widetilde{L}|$ by definition. An equivalence $s : \widetilde{L}_1 \xrightarrow{\sim} \widetilde{L}_2$ between such groups will be an isomorphism such that $s(L_1) = L_2$. Let $\{\widetilde{Q}_k\}$ be a set of representatives of equivalence classes of such p-subgroups.

We define

$$\mathrm{Cen}_{Q_j}\left(\widetilde{Q}_j, \widetilde{Q}_k, G\right) = \left\{\beta : \widetilde{Q}_j \to G \;\middle|\; [\beta] \in \mathrm{Cen}\left(\widetilde{Q}_j, G\right), \widetilde{\beta\left(Q_j\right)} \sim \widetilde{Q}_k\right\} \Big/ G,$$

We also define a set of monomorphisms

$$R\left(Q', \widetilde{Q}\right) = \left\{\alpha : Q' \to \widetilde{Q} \mid \alpha\left(Q'\right) = Q\right\}$$

and a group of automorphisms

$$\mathrm{Aut}(\widetilde{Q} \mid Q) = \{\alpha \in \mathrm{Aut}(\widetilde{Q}) \mid \alpha(Q) = Q\}.$$

Observe that $R\left(Q', \widetilde{Q}\right)$ and $\mathrm{Cen}_Q\left(\widetilde{Q}, \widetilde{Q}_k, G\right)$ are $\mathrm{Aut}(\widetilde{Q} \mid Q)$-sets since the image of Q is invariant. The following two results appear in [36].

Lemma 3.5.1. *There is an isomorphism of* $\mathrm{Out}(Q)$*-sets*

$$\psi : \coprod_k R\left(Q, \widetilde{Q}_k\right) \times_{\mathrm{Aut}(\widetilde{Q}_k|Q_k)} \mathrm{Cen}_{Q_k}\left(\widetilde{Q}_k, \widetilde{Q}_k, G\right) \to n\,\mathrm{Cen}(Q,G) \qquad (3.5.2)$$

given by composition, i.e. $\psi(\gamma \times [\delta]) = [\delta\gamma]$, *and which is natural in* G.

Isomorphism (3.5.2) and its proof are clear, but it is not clear what Martino and Priddy meant by naturality in G of this isomorphism.

Proposition 3.5.3. *If* $\mathbb{F}_p \operatorname{Inj}(Q, G) \cong \mathbb{F}_p \operatorname{Inj}(Q, G')$ *as* $\mathbb{F}_p\operatorname{Out}(Q)$*-modules for all* p*-groups* Q *then* $\mathbb{F}_p \operatorname{Cen}\left(\widetilde{Q}_j, G\right) \cong \mathbb{F}_p \operatorname{Cen}\left(\widetilde{Q}_j, G'\right)$ *as* $\operatorname{Out}\left(\widetilde{Q}_j\right)$ *modules for all* j *and*

$$\mathbb{F}_p \operatorname{Cen}_{Q_j}\left(\widetilde{Q}_j, \widetilde{Q}_j, G\right) \cong \mathbb{F}_p \operatorname{Cen}_{Q_i}\left(\widetilde{Q}_j, \widetilde{Q}_j, G'\right)$$

as $\operatorname{Aut}\left(\widetilde{Q_j} \mid Q_j\right)$*-modules for all* j.

Observation 3.5.4. Here we see that Martino and Priddy implicitly assume that if $\mathbb{F}_p \operatorname{Inj}(Q, G) \cong \mathbb{F}_p \operatorname{Inj}(Q, G')$ as $\mathbb{F}_p\operatorname{Out}(Q)$ modules, then the set of representatives $\{\widetilde{Q}_k\}$, in principle defined a priori for either G or G', works equally well for the other, i.e. the same family $\{\widetilde{Q}_k\}$ can be used in isomorphism (3.5.2) for both G and G'.

Under the hypothesis of Proposition 3.5.3 and using Lemma 3.5.1, we have the following commutative diagram

$$\begin{array}{ccc}
\bigoplus_k \mathbb{F}_p R\left(Q, \widetilde{Q}_k\right) \otimes_{\mathbb{F}_p \operatorname{Aut}\left(\widetilde{Q}_k \mid Q_k\right)} \mathbb{F}_p \operatorname{Cen}_{Q_k}\left(\widetilde{Q}_k, \widetilde{Q}_k, G\right) & \xrightarrow{\cong} & \mathbb{F}_p n\operatorname{Cen}(Q, G) \\
\Big\downarrow \cong & & \Big\downarrow \\
\bigoplus_k \mathbb{F}_p R\left(Q, \widetilde{Q}_k\right) \otimes_{\mathbb{F}_p \operatorname{Aut}\left(\widetilde{Q}_k \mid Q_k\right)} \mathbb{F}_p \operatorname{Cen}_{Q_k}\left(\widetilde{Q}_k, \widetilde{Q}_k, G'\right) & \xrightarrow{\cong} & \mathbb{F}_p n\operatorname{Cen}(Q, G')
\end{array}$$

This would imply that $\mathbb{F}_p n\operatorname{Cen}(Q, G) \cong \mathbb{F}_p n\operatorname{Cen}(Q, G')$ as $\mathbb{F}_p\operatorname{Out}(Q)$-modules, consequently $\mathbb{F}_p\operatorname{Cen}(Q, G) \cong \mathbb{F}_p\operatorname{Cen}(Q, G')$ as $\mathbb{F}_p\operatorname{Out}(Q)$-modules by complement and cancellation.

In view of this situation, we find it convenient to give, independently of Martino-Priddy's argument, a proof of the following theorem.

Theorem 3.5.5. *Given two finite groups* G, H, *the following are equivalent:*

(1) $G \cong H$ *in* $\mathbb{F}_p B^{\Delta_p}$

(2) $\mathbb{F}_p\operatorname{Cen}(Q, G) \cong \mathbb{F}_p\operatorname{Cen}(Q, H)$ *as* $\mathbb{F}_p\operatorname{Out}(Q)$*-modules, for every finite* p*-group* Q.

Of course, (2) ⇒ (1) is Proposition 3.4.9. Before giving the proof of (1) ⇒ (2), let us recall Remark 2.1.6: We remarked that $\mathbb{S}^{\mathcal{S}}_{Q,V}$ still makes sense when V is not simple. What is more, for $\mathcal{S} = \triangleright, \Delta$, its alternative definition by Webb [70] also works when V is not simple. To prove that both definitions are equivalent (see Lemma 2.4.11 and comments below it) we used the fact that V is simple. However, in the particular case $\mathcal{S} = \Delta$, the isomorphism

$$\mathbb{S}^{\Delta}_{Q,V}(G) \cong \bigoplus_{\substack{\alpha:Q\to L\\ L\leq_G G}} W_L({}^L V) \tag{3.5.6}$$

is valid more generally when V is a cyclic $\mathbb{F}_p\mathrm{Out}(Q)$-module. Indeed, in [7, Proposition 15] Bouc gives a proof of this isomorphism, where the only fact the proof needs about V is that it is generated by a single element v_0 as an $\mathbb{F}_p\mathrm{Out}(Q)$-module. In particular, we can take $V = \mathbb{F}_p\mathrm{Out}(Q)$ in isomorphism (3.5.6).

Proof of (1) ⇒ (2). An isomorphism from G to H in $\mathbb{F}_p B^{\Delta}_p$ induces an isomorphism

$$\mathbb{S}^{\Delta}_{Q,V}(G) \to \mathbb{S}^{\Delta}_{Q,V}(H) \tag{3.5.7}$$

as $\mathbb{F}_p$-modules, for any finite p-group Q, since $\mathbb{S}^{\Delta}_{Q,V}$ is cohomological by Proposition 2.4.25.

Consider $V = \mathbb{F}_p\mathrm{Out}(Q)$ as an $(\mathbb{F}_p\mathrm{Out}(Q), \mathbb{F}_p\mathrm{Out}(Q))$-bimodule, so that

$$L_{Q,\mathbb{F}_p\mathrm{Out}(Q)}(G) = \mathbb{F}_p B^{\Delta}(Q,G) \otimes_{\mathbb{F}_p\mathrm{Out}(Q)} \mathbb{F}_p\mathrm{Out}(Q)$$

has a right $\mathbb{F}_p\mathrm{Out}(Q)$-module structure and consequently so does $\mathbb{S}^{\Delta}_{Q,\mathbb{F}_p\mathrm{Out}(Q)}(G)$. Equation (4.7.16) displays an isomorphism of $\mathbb{F}_p\mathrm{Out}(Q)$-modules (recall the description of $\mathbb{S}_{Q,V}$ in Section 2.1). Now, we have

$$\begin{aligned}
\mathbb{S}_{Q,\mathbb{F}_p\mathrm{Out}(Q)}(G) &\cong \bigoplus_{L\leq_G G \text{ is } p\text{-centric}} \overline{W}_L \mathbb{F}_p\mathrm{Out}(L)\\
&\cong \bigoplus_{L\leq_G G \text{ is } p\text{-centric}} \mathbb{F}_p\mathrm{Out}(L)/\mathbb{F}_p\mathrm{Out}_G(L)\\
&\cong \mathbb{F}_p\mathrm{Cen}(Q,G),
\end{aligned}$$

where all isomorphisms are isomorphisms of right $\mathbb{F}_p\mathrm{Out}(Q)$-modules. We then obtain an isomorphism of $\mathbb{F}_p\mathrm{Out}(Q)$-modules

$$\mathbb{F}_p\mathrm{Cen}(Q,G) \to \mathbb{F}_p\mathrm{Cen}(Q,H),$$

as desired. □

Moreover, analogously to Remark 3.4.8, we have

$$G \cong H \text{ in } \mathbb{F}_pB^{\triangle_p} \text{ if and only if } G \cong H \text{ in } \mathbb{Z}_p^\wedge\widetilde{B}^{\triangle_p}. \tag{3.5.8}$$

Now, we devote the last paragraphs of this section to make a brief digression of what would follow as a consequence of [34, Proposition 4.5] and our results. Homotopically, the first consequence is that there is an arbitrary homotopy equivalence between $\mathbb{B}G_p^\wedge$ and $\mathbb{B}H_p^\wedge$ if and only if there is another, more particular, homotopy equivalence between $\mathbb{B}G_p^\wedge$ and $\mathbb{B}H_p^\wedge$, induced by an element of $\mathbb{Z}_p^\wedge\widetilde{B}^{\Delta_p}(G,H)$ via the Segal conjecture. At the moment, we are not aware of any homotopical description of that kind of equivalences.

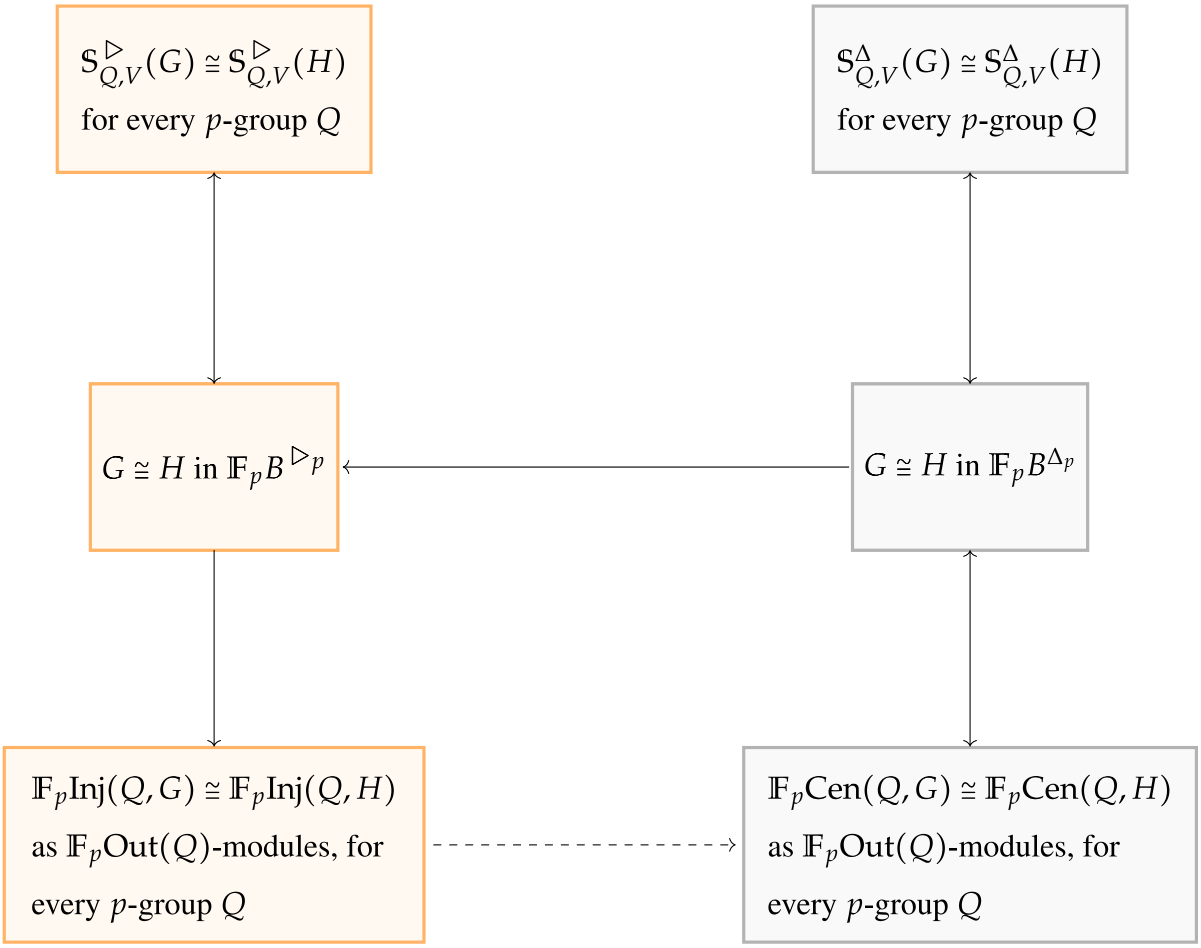
On the algebraic side, in [46, Chapter 2], the author asked if $\mathbb{Z}_{(p)}B^{\lhd\, p}$ and $\mathbb{Z}_{(p)}B^{\Delta_p}$ have the same isomorphism classes. Changing the coefficient ring $\mathbb{Z}_{(p)}$ to either $\mathbb{Z}_p^\wedge$ or $\mathbb{F}_p$, we would be able to affirmatively answer this question. Indeed, it follows from Theorem 3.3.5, Proposition 3.4.9 and the discussion above in this section.

Corollary 3.5.9. *Let $R = \mathbb{Z}_p^\wedge$ or $\mathbb{F}_p$. Then $RB^{\lhd\, p}$ has the same isomorphism classes as RB^{Δ_p}.*

3.6 AN OVERVIEW

To summarize Chapter 3, we recall some different situations for a couple of finite groups G, H and how they are related according to the arguments displayed throughout this chapter.

In the following figure, direction of the arrows mean implication, so bidirectional arrows mean equivalence. We achieved all these implications independently of [34], except the dashed arrow.



Now, assuming Martino-Priddy's argument discussed in Section 3.5, they all are equivalent to $\mathbb{B}G^{\wedge}_p \simeq \mathbb{B}H^{\wedge}_p$, by Ragnarsson's version of the Segal conjecture. We end this chapter with another consequence of $(3) \Rightarrow (4)$ in Theorem 3.3.5, which easily follows from the above diagram.

Corollary 3.6.1. *Let G, H be finite groups. The following are equivalent:*

(1) $\mathbb{S}^{\triangleright}_{Q,V}(G) \cong \mathbb{S}^{\triangleright}_{Q,V}(H)$ *for every finite p-group Q.*

(2) $\mathbb{S}^{\Delta}_{Q,V}(G) \cong \mathbb{S}^{\Delta}_{Q,V}(H)$ *for every finite p-group Q.*

4

ON THE STABLE HOMOTOPY TYPE OF p-LOCAL FINITE GROUPS

Our goal in this chapter is to, in some sense, extend the theory of biset functors for finite groups to saturated fusion systems. As a result, we will be able to generalize or reformulate some classical results by Webb and our previous results in Chapter 3.

4.1 FUSION SYSTEMS

In this section we study the notion of saturated fusion systems. They were originally introduced by Puig [53] under the name of Frobenius systems in the context of modular representation theory and by Broto, Levi and Oliver [18] in the context of homotopy theory. More precisely, Broto, Levi and Oliver defined p-local finite groups as generalizations of finite groups and their p-completed classifying spaces. A p-local finite group $(\mathcal{F}, \mathcal{L})$ [18, Definition 1.8] is informally a saturated fusion system $\mathcal{F}$ together with a space, a priori not determined by $\mathcal{F}$, but by a centric linking system $\mathcal{L}$ associated to $\mathcal{F}$. Such a space is $|\mathcal{L}|^\wedge_p$, the p-completed geometric realization of $\mathcal{L}$ as a small category, which is called the classifying space of $(\mathcal{F}, \mathcal{L})$.

Definition 4.1.1. Let S be a finite p-group. A *fusion system over S* is a category $\mathcal{F}$ whose objects are the subgroups $P \leq S$ and the morphism sets $\mathcal{F}(P, Q)$ satisfy:

1. For all $P, Q \leq S$, we have $\mathrm{Hom}_S(P, Q) \subseteq \mathcal{F}(P, Q) \subseteq \mathrm{Inj}(P, Q)$.
2. Every morphism $\varphi \in \mathcal{F}(P, Q)$ factors as $P \xrightarrow{\varphi} \varphi(P) \hookrightarrow Q$ in $\mathcal{F}$, with $P \xrightarrow{\varphi} \varphi(P)$ being an isomorphism in $\mathcal{F}$.

Example 4.1.2. Given a finite group G and a Sylow p-subgroup S of G, the *fusion system of G over S*, denoted $\mathcal{F}_S(G)$, is defined by $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) := \mathrm{Hom}_G(P,Q)$.

We say that two subgroups P,Q of S are $\mathcal{F}$-conjugate if there is an isomorphism $\varphi : P \to Q$ in $\mathcal{F}$.

Definition 4.1.3. Let $\mathcal{F}$ be a fusion system over S. A subgroup $P \leq S$ is *fully $\mathcal{F}$-centralized* if $|C_S(P)| \geq |C_S(P')|$ for every P' which is $\mathcal{F}$-conjugate to P. Similarly, P is *fully $\mathcal{F}$-normalized* if $|N_S(P)| \geq |N_S(P')|$ for every P' which is $\mathcal{F}$-conjugate to P. A fusion system $\mathcal{F}$ is *saturated* if it satisfies the following conditions:

1. If P is fully $\mathcal{F}$-normalized, then P is also fully $\mathcal{F}$-centralized and $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$.

2. If $\varphi \in \mathcal{F}(P,Q)$ is such that $\varphi(P)$ is fully $\mathcal{F}$-centralized, then φ extends to a morphism $\overline{\varphi} \in \mathcal{F}(N_\varphi, Q)$, where

$$N_\varphi = \{x \in N_S(P) \mid \varphi \circ c_x = c_y \circ \varphi \in \mathcal{F}(P,S), \text{ for some } y \in N_S(\varphi(P))\}.$$

The additional axioms for saturated fusion systems are aimed at imitating the Sylow theorems and, at least from the homotopical viewpoint, only the saturated fusion systems are relevant. Indeed, these axioms are necessary to prove the properties expected for $B\mathcal{F}$ (see [18] for more details).

Example 4.1.4. The fusion system $\mathcal{F}_S(G)$ of a finite group over its p-Sylow subgroup S is the archetypal example of a saturated fusion system.

Definition 4.1.5. Let $\mathcal{F}_1, \mathcal{F}_2$ be fusion systems over S_1, S_2, respectively. We say that $\mathcal{F}_1$ is *isomorphic to* $\mathcal{F}_2$, and denote it by $\mathcal{F}_1 \cong \mathcal{F}_2$, if there exists a *fusion-preserving isomorphism* $\lambda : S_1 \to S_2$, that is, an isomorphism $\lambda : S_1 \to S_2$ such that

$$\varphi \in \mathcal{F}_1(P,Q) \text{ if and only if } \lambda\varphi\lambda^{-1} \in \mathcal{F}_2(\lambda(P), \lambda(Q)).$$

Definition 4.1.6. We say that a fusion system is *realizable* if there exists a finite group G, with Sylow p-subgroup S, such that $\mathcal{F} \cong \mathcal{F}_S(G)$. When $\mathcal{F}$ is not realizable, we say that $\mathcal{F}$ is *exotic*.

For the prime $p = 2$, the first known examples of exotic (saturated) fusion system are essentially due to Solomon [67]. They are fusion systems over the 2-Sylow subgroup of $\mathrm{Spin}_7(q)$ for any odd prime power q, denoted by $\mathcal{F}_{\mathrm{Sol}(q)}$ [31]. On the other hand, Dwyer and Wilkerson [24] constructed the 2-compact group $\mathrm{DI}(4)$ (roughly speaking, a homotopical analogue of a compact Lie group at the prime 2) which is exotic (it cannot be realized by a compact Lie group, in the sense of [42, Example 2.2]). One of the motivations of Dwyer and Wilkerson was to find a space that realizes the mod 2 Dickson invariants of rank 4 through its cohomology ring, this is achieved by the classifying space $B\mathrm{DI}(4)$, i.e.

$$H^*(B\mathrm{DI}(4); \mathbb{F}_2) \cong H^*((\mathbb{Z}/2)^4; \mathbb{F}_2)^{\mathrm{GL}(4,2)}.$$

Inspired by these works, Benson [6] observed that $\mathrm{DI}(4)$ subsumes the information of $\mathcal{F}_{\mathrm{Sol}(q)}$ for each q. Then he was led to consider a likely "classifying space for the non-existing Solomon group $\mathrm{Sol}(q)$". More precisely, Benson defined $B\mathrm{Sol}(q)$ as the homotopy pullback of the diagram

$$\begin{array}{ccc} B\mathrm{Sol}(q) & \longrightarrow & B\mathrm{DI}(4) \\ \downarrow & & \downarrow \Delta \\ B\mathrm{DI}(4) & \xrightarrow{(1,\Psi^q)} & B\mathrm{DI}(4) \times B\mathrm{DI}(4) \end{array}$$

where Ψ^q is the Adams operation corresponding to q, in the sense of [45]. As a result, he predicted that the saturation axioms would lead to the existence of a classifying space for a fusion system.

Remark 4.1.7. There are other notions of realisability for fusion systems, but the notion we just discussed is the strongest one. In fact, any fusion system over a p-group S is realizable by a finite pregroup P containing S as a Sylow p-subgroup [30, Corollary 4.15]. The pregroup P induces a universal group $U(P)$ [30, Definition 2.6], in such a way that

$$\mathcal{F} \cong \mathcal{F}_S(P) \cong \mathcal{F}_S(U(P)).$$

However, $U(P)$ is often infinite.

The following example displays the classification of saturated fusion systems over abelian p-groups, up to isomorphism.

Example 4.1.8. Let S be an abelian p-group and $\mathcal{F}$ a saturated fusion system over S. It is known that $\mathcal{F} \cong \mathcal{F}_S(S \rtimes W)$ for some finite p'-group $W \leq \operatorname{Aut}(S)$. In particular, there are no exotic fusion systems over abelian p-groups. Moreover, given two such groups W_1, W_2, we have

$$\mathcal{F}_S(S \rtimes W_1) \cong \mathcal{F}_S(S \rtimes W_2)$$

if and only if W_1 and W_2 are conjugate in $\operatorname{Aut}(S)$, see for instance [55, Proposition 2.1.3].

Definition 4.1.9. Given a saturated fusion system $\mathcal{F}$ over S and $P \leq S$, we say that P is *$\mathcal{F}$-centric* if $C_S(P') = Z(P')$, for every P' which is $\mathcal{F}$-conjugate to P. On the other hand, we say that P is *$\mathcal{F}$-radical* if $\operatorname{Out}_{\mathcal{F}}(P)$ has no nontrivial normal p-subgroups.

The notion of $\mathcal{F}$-centric subgroups generalizes that of p-centric subgroups studied in Chapter 3.

Proposition 4.1.10 ([17, Lemma A.5])**.** *Let G be a finite group and S a p-Sylow subgroup of G. Then $P \leq S$ is p-centric in G if and only if P is $\mathcal{F}_S(G)$-centric.*

The following theorem, due to Alperin, tells us that, to recover the whole fusion system $\mathcal{F}$, we only need to know a certain family of subgroups of S, which are well-behaved with respect to $\mathcal{F}$, and their $\mathcal{F}$-automorphism groups.

Theorem 4.1.11 ([18, Theorem A.10])**.** *Let $\mathcal{F}$ be a saturated fusion system over S. Then for each isomorphism $\psi \in \mathcal{F}(P, P')$, there exists a sequence of subgroups of S*

$$P = P_0, P_1, \ldots, P_k = P' \quad \textit{and} \quad Q_1, Q_2, \ldots, Q_k$$

and morphisms $\psi_i \in \operatorname{Aut}_{\mathcal{F}}(Q_i)$, such that

- *Q_i is $\mathcal{F}$-fully normalized, $\mathcal{F}$-radical and $\mathcal{F}$-centric for each i,*
- *$P_{i-1}, P_i \leq Q_i$ and $\psi_i(P_{i-1}) = P_i$ for each i, and*
- *$\psi = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1$.*

We refer to the $\mathcal{F}$-fully normalized, $\mathcal{F}$-radical and $\mathcal{F}$-centric subgroups of S simply as *$\mathcal{F}$-Alperin subgroups*.

Example 4.1.12 (The Ruiz-Viruel exotic fusion systems)**.** Let E be the extraspecial group of order 7^3 and exponent 7. The group E has a presentation

$$E =< x, y, z \mid x^7 = y^7 = z^7 = 1, xz = zx, yz = zy, [x, y] = z > .$$

Given a fusion system $\mathcal{F}$ over E, the only possible $\mathcal{F}$-Alperin proper subgroups are $V_0, \ldots, V_7$, where $V_i =< z, xy^i >$, with $0 \le i \le 6$ and $V_7 =< z, y >$. Each of these subgroups is isomorphic to $\mathbb{Z}/7 \times \mathbb{Z}/7$. To determine $\mathcal{F}$, it suffices to know $\mathrm{Out}_{\mathcal{F}}(E)$ and the $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-Alperin proper subgroups [64, Corollary 4.4]. Particularly, if $\mathcal{F}$ is one of the three exotic Ruiz-Viruel fusion systems RV1, RV2 or RV3, then all V_i are $\mathcal{F}$-Alperin. The following table displays the structure of these saturated fusion systems.

Fusion system	$\mathrm{Out}_{\mathcal{F}}(E)$	$\lvert\mathcal{F}\text{-Alperin}\rvert$	$\mathrm{Aut}_{\mathcal{F}}(V)$
RV1	$(\mathbb{Z}6 \times \mathbb{Z}/6) \rtimes \mathbb{Z}/2$	$6+2$	$\mathrm{SL}_2(7) \rtimes \mathbb{Z}/2, \mathrm{GL}_2(7)$
RV2	$D_{16} \times \mathbb{Z}/3$	4+4	$\mathrm{SL}_2(7) \rtimes \mathbb{Z}/2, \mathrm{SL}_2(7) \rtimes \mathbb{Z}/2$
RV3	$SD_{32} \times \mathbb{Z}/3$	8	$\mathrm{SL}_2(7) \rtimes \mathbb{Z}/2$

The expression $m + n$ in the third column indicates that there is an $\mathcal{F}$-conjugacy class with m $\mathcal{F}$-Alperin proper subgroups and another one with n.

As we said earlier in this section, Benson predicted the existence of a classifying space for a saturated fusion system. The pursuit of such a space was fruitful: Broto, Levi and Oliver generalized the Martino-Priddy conjecture [18, Theorem 7.4] by showing that the homotopy type of $|\mathcal{L}|^\wedge_p$ determines $\mathcal{F}$, up to isomorphism. However, it remained the question if, given $\mathcal{F}$, a centric linking system for $\mathcal{F}$ always exists. This issue was finally resolved by Chermak [21]: he proved both the existence and uniqueness of a centric linking system $\mathcal{L}$ associated to $\mathcal{F}$, up to isomorphism. Therefore, because of the existence and uniqueness up to homotopy equivalence of $|\mathcal{L}|^\wedge_p$, we can now call it the *classifying space of* $\mathcal{F}$ and denote it by $B\mathcal{F}$. For illustrative purposes, we briefly recall the idea behind the construction of $B\mathcal{F}$: for any saturated fusion system over S, we have a functor

$$\begin{aligned} B : \mathcal{O}(\mathcal{F}^c) &\to \mathrm{HoTop} \\ P &\longmapsto BP, \\ [\varphi] &\longmapsto [B\varphi], \end{aligned}$$

where $\mathcal{O}(\mathcal{F}^c)$ is the *centric orbit category of* $\mathcal{F}$, with objects the $\mathcal{F}$-centric subgroups of S and morphism sets $\mathcal{F}(P,Q)/\mathrm{Inn}(Q)$. The saturation axioms on $\mathcal{F}$ make it possible to lift B up to natural isomorphism to a functor

$$\widetilde{B} : \mathcal{O}(\mathcal{F}^c) \to \mathrm{Top},$$

in an essentially unique way [49, Theorem B]. In this manner, we have [18, Proposition 2.2]

$$\mathrm{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \widetilde{B} \simeq |\mathcal{L}|.$$

One can prove that $|\mathcal{L}|$ is p-good [18, Proposition 1.12] and, after p-completing, we have

$$\left(\mathrm{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \widetilde{B}\right)^{\wedge}_p \simeq B\mathcal{F}.$$

In particular, $B\mathcal{F}$ is p-complete. If $\mathcal{F} = \mathcal{F}_S(G)$, there is a homotopy equivalence $B\mathcal{F} \simeq BG^{\wedge}_p$.

4.2 THE BURNSIDE RING OF A FUSION SYSTEM

Let S be a finite p-group and $\mathcal{F}$ a fusion system over S. Given a finite S-set X and φ in $\mathcal{F}(P,S)$, we denote by ${}_{P,\varphi}X$ the set X, with P acting by $p \cdot x := \varphi(p) \cdot x$. We say that X is *$\mathcal{F}$-stable* if

$${}_{P,\varphi}X \cong {}_{P,\iota_P} X \tag{4.2.1}$$

as P-sets, for every $\varphi \in \mathcal{F}(P,S)$, where $\iota_P : P \to S$ is the inclusion. More generally, if X is a virtual S-set, we can analogously define ${}_{P,\varphi}X \in B(P)$ and say that X is $\mathcal{F}$-stable if it satisfies

$${}_{P,\varphi}X = {}_{P,\iota_P} X \tag{4.2.2}$$

in $B(P)$, for every $\varphi \in \mathcal{F}(P,S)$. It is not hard to check that $\mathcal{F}$-stability is preserved by sums, additive inverses and products in $B(S)$.

Definition 4.2.3 (Reeh [60])**.** The subring of $B(S)$ consisting of $\mathcal{F}$-stable elements is called the *Burnside ring of* $\mathcal{F}$ and denoted $B(\mathcal{F})$.

Remark 4.2.4. If $\mathcal{F}$ is saturated, we could equivalently define $B(\mathcal{F})$ as the isomorphic image in $B(S)$ of the Grothendieck group of the subsemiring $B_+(\mathcal{F}) \subset B_+(S)$ consisting of

$\mathcal{F}$-stable sets, under the monomorphism ι induced by the inclusion $B_+(\mathcal{F}) \hookrightarrow B_+(S)$, see [60, Proposition 4.4].

Given an $\mathcal{F}$-stable element X, a property of special interest is that (recall Definition 3.1.1) $\Phi_P(X) = \Phi_Q(X)$ for all $\mathcal{F}$-conjugate subgroups P, Q (see [60, Lemma 4.1]). This allows us to define, in analogy to Definition 3.1.1, the *mark homomorphism*

$$\Phi^{\mathcal{F}} : B(\mathcal{F}) \xrightarrow{\prod_{P \leq_{\mathcal{F}} S} \Phi_P} \prod_{P \leq_{\mathcal{F}} S} \mathbb{Z}$$

and $\Omega(\mathcal{F}) := \prod_{P \leq_{\mathcal{F}} S} \mathbb{Z}$, the *ghost ring of* $B(\mathcal{F})$. As expected, $\Phi^{\mathcal{F}}$ is injective [60, Theorem B].

Let $\mathcal{F}$ be a saturated fusion system. Using the mark homomorphism and its properties, Reeh [60] picks a representative P of each $\mathcal{F}$-conjugacy class and constructs an $\mathcal{F}$-stable element α_P, such that $B(\mathcal{F})$ is freely generated as a $\mathbb{Z}$-module by $\{\alpha_P\}_P$. Moreover, each α_P lives in $B_+(S)$ (it is not only a virtual S-set but an honest S-set) and it is irreducible (it cannot be expressed as a disjoint union of $\mathcal{F}$-stable sets).

Although the proof of the existence of the sets α_P is constructive, its expression as a linear combination of transitive bisets $[S/P]$ is not immediate. More details can be found in [25].

4.3 THE p-LOCAL BURNSIDE RING OF A FUSION SYSTEM

Throughout this section, $\mathcal{F}$ is a saturated fusion system. In accordance with the discussion above, $RB(\mathcal{F}) := R \otimes B(\mathcal{F})$ is a free R-module with rank the number of $\mathcal{F}$-conjugacy classes of subgroups of S. For our purposes, we will concentrate on the case $R = \mathbb{Z}_{(p)}$ and we call $B(\mathcal{F})_{(p)} := \mathbb{Z}_{(p)} B(\mathcal{F})$ the *p-local Burnside ring of $\mathcal{F}$*. In this context, we can find another more convenient basis $\{\beta_P\}_P$, whose properties will be outlined below.

In [61], Reeh defines a stabilization map $\pi_{\mathcal{F}} : B(S)_{(p)} \to B(S)_{(p)}$, in the sense that any virtual S-set X becomes $\mathcal{F}$-stable via π and π leaves X unaffected if X is already $\mathcal{F}$-stable. The following results in the rest of this section appear in [61].

Theorem 4.3.1 ([61, Theorem A]). *There exists an epimorphism $\pi_{\mathcal{F}} : B(S)_{(p)} \to B(\mathcal{F})_{(p)}$ of $B(\mathcal{F})_{(p)}$-modules, where $B(S)_{(p)} := \mathbb{Z}_{(p)}B(S)$, such that*

$$\Phi_Q(\pi_{\mathcal{F}}(X)) = \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X),$$

for all $Q \leq S$ and all $X \in B(S)_{(p)}$.

When $\mathcal{F}$ is clear from the context, we simply denote $\pi_{\mathcal{F}}$ by π.

Our goal in the rest of this section is to find an explicit formula for $\beta_P := \pi_{\mathcal{F}}([S/P])$. For that, we will need the following lemma.

Lemma 4.3.2 ([61, Lemma 4.1]). *Let $\mathcal{F}$ be a saturated fusion system over S, and let $P \leq S$ be fully $\mathcal{F}$-normalized. Then the number $|[P]_{\mathcal{F}}|$ of $\mathcal{F}$-conjugates of P equals $\frac{|S|}{|N_S P|} \cdot k$ for some positive integer k coprime to p. Under the same assumptions we also have $|\mathcal{F}(P,S)| = |\mathrm{Aut}_{\mathcal{F}}(P)| \cdot |[P]_{\mathcal{F}}| = \frac{|S|}{|C_S P|} \cdot k'$, for some positive integer k' coprime to p.*

Given a commutative ring R, we say that *a divides b in R* if $ar = b$, for some $r \in R$.

Lemma 4.3.3. *Let $P, Q \leq S$, then $|[Q]_{\mathcal{F}}|$ divides $\left|[Q']_S\right|$ in $\mathbb{Z}_{(p)}$ for all $Q' \sim_{\mathcal{F}} Q$; and furthermore*

$$\Phi_Q(\beta_P) = \frac{|\mathcal{F}(Q,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q,S)|} \in \mathbb{Z}_{(p)}.$$

Proof. We can always find a fully normalized subgroup Q_0 such that $Q_0 \sim_{\mathcal{F}} Q$. By Lemma 4.3.2, we have $|[Q]_{\mathcal{F}}| = |[Q_0]_{\mathcal{F}}| = \frac{|S|}{|N_S(Q_0)|} \cdot k$, with $p \nmid k$. On the other hand, $\left|[Q']_S\right| = \frac{|S|}{|N_S(Q')|}$. Since $p^n = |N_S(Q')| \leq |N_S(Q_0)| = p^m$, we have

$$\frac{\left|[Q']_S\right|}{|[Q]_{\mathcal{F}}|} = \frac{p^m}{p^n k} = \frac{p^{m-n}}{k} \in \mathbb{Z}_{(p)}.$$

Therefore $|[Q]_{\mathcal{F}}|$ divides $\left|[Q']_S\right|$ in $\mathbb{Z}_{(p)}$.

This leads to the following sequence of equalities:

$$\begin{aligned}
\frac{1}{|[Q]_{\mathcal{F}}|}\sum_{Q'\in[Q]_{\mathcal{F}}}\Phi_{Q'}([S/P]) &= \sum_{[Q']_S\subseteq[Q]_{\mathcal{F}}}\frac{\left|[Q']_S\right|}{|[Q]_{\mathcal{F}}|}\Phi_{Q'}([S/P]) \\
&= \frac{1}{|[Q]_{\mathcal{F}}|}\sum_{[Q']_S\subseteq[Q]_{\mathcal{F}}}\frac{|S|}{|N_S(Q')|}\cdot\frac{|N_S(Q',P)|}{|P|} \\
&= \frac{|S|}{|P|\cdot|[Q]_{\mathcal{F}}|}\sum_{[Q']_S\subseteq[Q]_{\mathcal{F}}}\frac{|N_S(Q',P)|}{|N_S(Q')|} \\
&= \frac{|S|}{|P|\cdot|[Q]_{\mathcal{F}}|}\sum_{[Q']_S\subseteq[Q]_{\mathcal{F}}}\left|\left\{R\in[Q']_S \mid R\le P\right\}\right| \\
&= \frac{|S|}{|P|\cdot|[Q]_{\mathcal{F}}|}\left|\{R\in[Q]_{\mathcal{F}} \mid R\le P\}\right| \\
&= \frac{|S|}{|P|\cdot|[Q]_{\mathcal{F}}|\cdot|\mathrm{Aut}_{\mathcal{F}}(Q)|}\left|\{R\in[Q]_{\mathcal{F}} \mid R\le P\}\right|\cdot|\mathrm{Aut}_{\mathcal{F}}(Q)| \\
&= \frac{|\mathcal{F}(Q,P)|\cdot|S|}{|P|\cdot|\mathcal{F}(Q,S)|}
\end{aligned}$$

In other words, $\Phi_Q(\beta_P) = \dfrac{|\mathcal{F}(Q,P)|\cdot|S|}{|P|\cdot|\mathcal{F}(Q,S)|}$, as desired. □

Now, we are ready to exhibit a formula for β_P.

Proposition 4.3.4. *For each $P \le S$, the element $\beta_P \in B(\mathcal{F})_{(p)}$ is given by the following $\mathbb{Z}_{(p)}$-linear combination of transitive S-sets:*

$$\beta_P = \sum_{[R]_S}\frac{|R|\cdot|S|}{|N_S R|\cdot|P|}\left(\sum_{R\le Q\le S}\frac{|\mathcal{F}(Q,P)|}{|\mathcal{F}(Q,S)|}\cdot\mu(R,Q)\right)[S/R].$$

In particular β_P contains no copies of $[S/R]$ unless R is $\mathcal{F}$-subconjugate to P.

Proof. By definition of the mark homomorphism Φ and the canonical basis $\{e_Q\}$, we have

$$\Phi(\beta_P) = \sum_{[Q]_S}\Phi_Q(\beta_P)\cdot e_Q = \sum_{Q\le S}\frac{|N_S Q|}{|S|}\cdot\Phi_Q(\beta_P)\cdot e_Q$$

We then apply the formula from Equation (3.1.4) for the inverse of Φ and get

$$\begin{aligned}\beta_P &= \Phi^{-1}\left(\sum_{Q\leq S}\frac{|N_SQ|}{|S|}\cdot\Phi_Q(\beta_P)\cdot e_Q\right)\\ &= \sum_{Q\leq S}\frac{|N_SQ|}{|S|}\cdot\Phi_Q(\beta_P)\cdot\frac{1}{|N_SQ|}\left(\sum_{R\leq Q}\mu(R,Q)\cdot|R|\cdot[S/R]\right)\\ &= \sum_{R\leq S}\frac{|R|}{|S|}\left(\sum_{R\leq Q\leq S}\Phi_Q(\beta_P)\cdot\mu(R,Q)\right)[S/R]\\ &= \sum_{[R]_S}\frac{|R|}{|N_SR|}\left(\sum_{R\leq Q\leq S}\Phi_Q(\beta_P)\cdot\mu(R,Q)\right)[S/R]\\ &= \sum_{[R]_S}\frac{|R|\cdot|S|}{|N_SR|\cdot|P|}\left(\sum_{R\leq Q\leq S}\frac{|\mathcal{F}(Q,P)|}{|\mathcal{F}(Q,S)|}\cdot\mu(R,Q)\right)[S/R].\end{aligned}$$

If R is not $\mathcal{F}$-subconjugate to P, then $|\mathcal{F}(Q,P)| = 0$ for all $R \leq Q \leq S$, and hence the coefficient of $[S/R]$ above becomes zero. □

Remark 4.3.5. The formula $c_{[R]}(\beta_P) = \dfrac{1}{\Phi_R([S/R])}\left(\sum_{R\leq Q\leq S}\Phi_Q(\beta_P)\cdot\mu(R,Q)\right)$ displayed in the proof above will be helpful in the next section.

4.4 THE CHARACTERISTIC IDEMPOTENT OF A SATURATED FUSION SYSTEM

Definition 4.4.1. Let $\mathcal{F}_1, \mathcal{F}_2$ be fusion systems over S_1, S_2 respectively. An element $X \in B(S_1, S_2)$ is called right $\mathcal{F}_1$-stable if for every $P_1 \leq S_1$ and every $\varphi \in \mathcal{F}(P_1, S_1)$, we have:

$$X \circ [P_1, \varphi]_{P_1}^{S_1} = X \circ [P_1, \mathrm{incl}]_{P_1}^{S_1}.$$

Similarly, X is called left $\mathcal{F}_2$-stable if for every $P_2 \leq S_2$ and every $\phi \in \mathcal{F}_2(P_2, S_2)$, we have

$$[\phi(P_2), \phi^{-1}]_S^P \circ X = [P_2, \mathrm{id}_{P_2}]_{S_2}^{P_2} \circ X.$$

When $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$, we simply say X is $\mathcal{F}$-stable if it is both right and left $\mathcal{F}$-stable.

Definition 4.4.2. Let $\mathcal{F}_i$ be a fusion system over S_i, $i = 1, 2$. We denote by $B(\mathcal{F}_1, \mathcal{F}_2) \subset B(S_1, S_2)$ the submodule of right $\mathcal{F}_1$-stable, left $\mathcal{F}_2$-stable elements. It is called *the Burnside module of* $\mathcal{F}_1$ *and* $\mathcal{F}_2$. When $X \in B(\mathcal{F}_1, \mathcal{F}_2)$, we simply say that X is $(\mathcal{F}_1, \mathcal{F}_2)$-*stable*.

It is easy to see that two fusion systems $\mathcal{F}_1, \mathcal{F}_2$ over S_1, S_2, respectively, induce a fusion system $\mathcal{F}_1 \times \mathcal{F}_2$ over $S_1 \times S_2$ and $\mathcal{F}_1 \times \mathcal{F}_2$ is saturated if $\mathcal{F}_1, \mathcal{F}_2$ so are [18, Lemma 1.5].

Proposition 4.4.3 ([61, Lemma 5.6])**.** *Let $\mathcal{F}_1, \mathcal{F}_2$ be fusion systems over S_1, S_2 respectively and $X \in B(S_1, S_2)$. Then X is $(\mathcal{F}_1, \mathcal{F}_2)$-stable if and only if it is $(\mathcal{F}_2 \times \mathcal{F}_1)$-stable when viewed as an element of $B(S_2 \times S_1)$. Therefore, $B(\mathcal{F}_1, \mathcal{F}_2)$ and $B(\mathcal{F}_2 \times \mathcal{F}_1)$ are isomorphic as abelian groups.*

In particular $X \in B(S, S)$ is right (resp. left) $\mathcal{F}$-stable if and only if it is $(S \times \mathcal{F})$-stable (resp. $(\mathcal{F} \times S)$-stable).

As desired, compatibility with the composition works: If $X \in B(\mathcal{F}_1, \mathcal{F}_2)$ and $Y \in B(\mathcal{F}_2, \mathcal{F}_3)$, then $Y \circ X \in B(\mathcal{F}_1, \mathcal{F}_3)$. We might wish that Burnside modules for fusion systems define a category like for finite groups. Unfortunately, it is not the case as we cannot expect the existence of identities in $B(\mathcal{F}_i, \mathcal{F}_i)$ in general. However, it is viable for saturated fusion systems if we extend the coefficient ring. The following paragraphs will outline this idea.

Definition 4.4.4 (Linckelmann-Webb)**.** We say that $\Omega \in B^{\Delta}(S, S)$ is a *characteristic biset of* $\mathcal{F}$ if it satisfies the following conditions:

(a) Ω is a linear combination of some $[P, \varphi]$, with $\varphi \in \mathcal{F}(P, S)$.

(b) Ω is $\mathcal{F}$-stable.

(c) $\varepsilon(X)$ is prime to p.

Example 4.4.5. If $\mathcal{F} = \mathcal{F}_S(G)$, the class $[G]$ is a characteristic biset.

The following result appears in [18, Proposition 5.5].

Proposition 4.4.6. *For any saturated fusion system, there is a characteristic biset.*

A saturated fusion system $\mathcal{F}$ may have several characteristic bisets. We note that Definition 4.4.1 still works if we consider bisets with coefficients in $\mathbb{Z}_{(p)}$ instead of $\mathbb{Z}$. Similarly with Definition 4.4.4 if we replace condition (c) by

(c') $\varepsilon(X)$ is invertible in $\mathbb{Z}_{(p)}$.

The advantage of taking characteristic bisets with coefficients in $\mathbb{Z}_{(p)}$ is that now, among all the characteristic bisets of a saturated fusion system $\mathcal{F}$, there is only one that is idempotent [56, Proposition 4.9, Proposition 5.6].

Theorem 4.4.7 (Ragnarsson)**.** *Every saturated fusion system $\mathcal{F}$ has a unique idempotent characteristic element $\omega_{\mathcal{F}}$ i.e., $\omega_{\mathcal{F}} \circ \omega_{\mathcal{F}} = \omega_{\mathcal{F}}$, which is called the characteristic idempotent of $\mathcal{F}$.*

Example 4.4.8. If $\mathcal{F} = \mathcal{F}_S(S)$, the trivial fusion system over S, then $\omega_{\mathcal{F}} = [S, \mathrm{id}]$.

Example 4.4.9. Let $\mathcal{F}$ be the fusion system of $G = \Sigma_4$ over $S = D_8$, we have $[\Sigma_4] = [D_8, \mathrm{id}] + [V, c_{(1\ 2)}]$, and $\omega_{\mathcal{F}} = [D_8, \mathrm{id}] + \frac{1}{3}[V, c_{(1\ 2)}] - \frac{1}{3}[V, \mathrm{incl}]$, where $V \leq D_8$ is isomorphic to the Klein group.

Reeh [61] proves that $\omega_{\mathcal{F}}$ coincides with $\beta_{\Delta(S)} = \pi_{\mathcal{F}\times S}([S \times S/\Delta(S)])$. In contrast, Ragnarsson [56] proves the existence of $\omega_{\mathcal{F}}$ in $\mathbb{Z}_p^{\wedge}B(S,S)$ by a limit argument and its uniqueness by showing that a certain system of linear equations has a unique solution. Then he realizes that the coefficients of $\omega_{\mathcal{F}}$ actually live in $\mathbb{Z}_{(p)}$. Reeh's approach is more concrete, indeed, he proves independently of [56], that $\beta_{\Delta(S)}$ is $\mathcal{F}$-characteristic, idempotent and unique with these features.

In turn, $\beta_{\Delta(s)}$ coincides with $\pi_{\mathcal{F}\times\mathcal{F}}([S \times S/\Delta(S)])$ because they have the same image in the mark homomorphism from $B(S,S)_{(p)}$. Hence we can denote them both $\beta_{\Delta(S)}$ without danger of confusion.

Remark 4.4.10. From now on, any fusion system $\mathcal{F}$ we deal with is saturated.

Proposition 4.4.11. *Let $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$. Then*

$$\Phi_{\Delta(P,\varphi)}(\beta_{\Delta(S)}) = \frac{|S|}{|\mathcal{F}(P,S)|}, \tag{4.4.12}$$

whereas $\Phi_D(\beta_{\Delta(S)}) = 0$ for all other subgroups $D \leq S \times S$.

Proof. By Lemma 4.3.3, we have

$$\Phi_D(\beta_{\Delta(S)}) = \frac{|\mathcal{F}\times\mathcal{F}(D,\Delta(S))| \cdot |S \times S|}{|\mathcal{F}\times\mathcal{F}(D, S\times S)| \cdot |S|} = \frac{|\mathcal{F}\times\mathcal{F}(D,\Delta(S))| \cdot |S|}{|\mathcal{F}\times\mathcal{F}(D, S\times S)|}.$$

But $|\mathcal{F}\times\mathcal{F}(D,\Delta(S))| = 0$ unless $D \leq_{\mathcal{F}\times\mathcal{F}} \Delta(S)$. Any subgroup of $\Delta(S)$ has the form $\Delta(Q,\iota)$, and any subgroup of $S\times S$ which is $(\mathcal{F}\times\mathcal{F})$-conjugate to $\Delta(P,\iota)$ is of the form $\Delta(P,\varphi)$, with $\varphi\in\mathcal{F}(P,S)$. Thus, we can assume $D=\Delta(P,\varphi)$ and

$$\begin{aligned}\Psi_{\Delta(P,\varphi)}(\beta_{\Delta(S)}) &= \frac{|\mathcal{F}\times\mathcal{F}(\Delta(P,\varphi),\Delta(S))|\cdot|S\times S|}{|\mathcal{F}\times\mathcal{F}(\Delta(P,\varphi),S\times S)|\cdot|S|}\\ &= \frac{|\mathcal{F}(P,S)|\cdot|S|}{|\mathcal{F}(P,S)|\cdot|\mathcal{F}(P,S)|} = \frac{|S|}{|\mathcal{F}(P,S)|},\end{aligned}$$

as claimed. □

Corollary 4.4.13. *Let R,T be subgroups of S. Then*

$$\Phi_{\Delta(P,\varphi)}([R,\mathrm{id}]^R_S\circ\omega_{\mathcal{F}}\circ[T,\iota]^S_T) = \begin{cases}\dfrac{|S|}{|\mathcal{F}(P,S)|}, & \text{if } \varphi\in\mathcal{F}(P,R), \text{ with } P\leq T\\[2ex] 0, & \text{otherwise.}\end{cases}$$

Theorem 4.4.14 (Reeh)**.** *Let $\mathcal{F}$ be a saturated fusion system over S. The equality*

$$\omega_{\mathcal{F}} = \sum_{\substack{[P,\varphi]\\ \varphi\in\mathcal{F}(P,S)}} \frac{|S|}{\Phi_{\Delta(P,\varphi)}([P,\varphi]^S_S)}\left(\sum_{P\leq Q\leq S}\frac{|\{\psi\in\mathcal{F}(Q,S)\mid \psi|_P=\varphi\}|}{|\mathcal{F}(Q,S)|}.\mu(P,Q)\right)[P,\varphi]^S_S$$

holds in $B(S,S)_{(p)}$.

Proof. We recall from Remark 4.3.5 that

$$\begin{aligned}c_{\Delta(P,\varphi)}(\omega_{\mathcal{F}}) &= \frac{1}{\Phi_{\Delta(P,\varphi)}([P,\varphi])}\left(\sum_{\Delta(P,\varphi)\leq D}\Phi_D(\omega_{\mathcal{F}})\cdot\mu(\Delta(P,\varphi),D)\right)\\ &= \frac{1}{\Phi_{\Delta(P,\varphi)}([P,\varphi])}\left(\sum_{\Delta(P,\varphi)\leq\Delta(Q,\psi)}\frac{|S|}{|\mathcal{F}(Q,S)|}\cdot\mu(\Delta(P,\varphi),\Delta(Q,\psi))\right)\\ &= \frac{|S|}{\Phi_{\Delta(P,\varphi)}([P,\varphi])}\left(\sum_{P\leq Q}\frac{|\{\psi\in\mathcal{F}(Q,S)|\psi|_P=\varphi\}|}{|\mathcal{F}(Q,S)|}\cdot\mu(P,Q)\right),\end{aligned}$$

since $\mu(\Delta(P,\varphi),\Delta(Q,\psi)) = |\{\psi\in\mathcal{F}(Q,S)|\psi|_P=\varphi\}|\cdot\mu(P,Q)$. We used Proposition 4.4.11 for the second equality. □

Corollary 4.4.15. *Let $\mathcal{F}$ be a saturated fusion system over S and $R, T \leq S$. Then*

$$c_{[T,\varphi]_T^R}([R,\mathrm{id}]_S^R \circ \omega_{\mathcal{F}} \circ [T,\iota]_T^S) = \begin{cases} \dfrac{|S|}{\Phi_{\Delta(T,\varphi)}([T,\varphi])} \cdot \dfrac{1}{|\mathcal{F}(T,S)|}, & \text{if } \varphi \in \mathcal{F}(T,R), \\ 0, & \text{otherwise.} \end{cases}$$

The following proposition implies that characteristic idempotents work as identities in their respective Burnside modules.

Proposition 4.4.16 ([59, Proposition 2.4.6])**.** *Let $\mathcal{F}_1, \mathcal{F}_2$ be saturated fusion systems over S_1, S_2 respectively. Then $X \in B(S_1, S_2)$ is right $\mathcal{F}_1$-stable if and only if $X \circ \omega_{\mathcal{F}_1} = X$ and left $\mathcal{F}_2$-stable if and only if $\omega_{\mathcal{F}_2} \circ X = X$.*

Consequently, any $(\mathcal{F}_1, \mathcal{F}_2)$-stable element is of the form $\omega_{\mathcal{F}_2} \circ X \circ \omega_{\mathcal{F}_1}$, for some $X \in B(S_1, S_2)$. We record this fact by writing $B(\mathcal{F}_1, \mathcal{F}_2) = \omega_{\mathcal{F}_2} \circ B(S_1, S_2) \circ \omega_{\mathcal{F}_1}$.

4.5 CLASSIFYING SPECTRA OF SATURATED FUSION SYSTEMS

Given a characteristic element Ω of $\mathcal{F}$, we take its image $\widetilde{\Omega}$ in $[\mathbb{B}S, \mathbb{B}S]$ via the composite

$$\mathbb{Z}_p^\wedge B^{\triangleleft}(S,S) \xrightarrow{\text{red}} \mathbb{Z}_p^\wedge \widetilde{B}^{\triangleleft}(S,S) \xrightarrow{\cong} [\mathbb{B}S, \mathbb{B}S]. \tag{4.5.1}$$

The *classifying spectrum of $\mathcal{F}$*, denoted $\mathbb{B}\mathcal{F}$, is defined via the mapping telescope:

$$\mathbb{B}\mathcal{F} := \mathrm{Tel}(\mathbb{B}S \xrightarrow{\widetilde{\Omega}} \mathbb{B}S \xrightarrow{\widetilde{\Omega}} \mathbb{B}S \cdots).$$

This spectrum does not depend on Ω, up to homotopy. Indeed, $\mathbb{B}\mathcal{F}$ is, independently of the chosen characteristic biset Ω, homotopy equivalent to the suspension spectrum of $B\mathcal{F}$, the classifying space of $\mathcal{F}$ [18, Section 5]. In particular, we can choose $\Omega = \omega_{\mathcal{F}}$, and this choice has the advantage that $\widetilde{\omega_{\mathcal{F}}}$ is a homotopy idempotent. It follows that $\mathbb{B}\mathcal{F}$ is a homotopy summand of $\mathbb{B}S$ [44, Section 4].

Example 4.5.2. Let $\mathcal{F} = \mathcal{F}_S(G)$. We know that $\Omega = [G]$ is a characteristic biset, thus $\mathcal{B}\mathcal{F}_S(G) \simeq \Sigma^\infty B\mathcal{F}_S(G) \simeq \mathbb{B}G_p^\wedge$.

In the following example, we will see that the homotopy type of $\mathbb{B}\mathcal{F}$ does not suffice to determine $\mathcal{F}$, up to isomorphism.

Example 4.5.3. Let S be a p-group and W_1, W_2 be finite p'-groups that act faithfully on S. We define $G_i := S \ltimes W_i$ and $\mathcal{F}_i = \mathcal{F}_S(G_i)$, $i = 1,2$. By assumptions on W_1, W_2, they can be viewed as subgroups of $\mathrm{Out}(S)$. Indeed, we have

$$W_i \cong \mathrm{Out}_{\mathcal{F}_i}(S),\ i = 1,2.$$

For simplicity, we assume $W_i = \mathrm{Out}_{\mathcal{F}_i}(S)$. In [34, Theorem 1.5], the authors show, independently of Theorem 3.3.5, that $(\mathbb{B}G_1)^\wedge_p \simeq (\mathbb{B}G_2)^\wedge_p$ if and only if W_1 is *pointwise conjugate to* W_2 in $\mathrm{Out}(S)$, that is, if there is a bijection $\alpha : W_1 \to W_2$ such that

$$\alpha(w) = g_w w g_w^{-1},$$

for $g_w \in \mathrm{Out}(S)$ depending on $w \in W_1$. Of course, this condition is weaker than W_1 being conjugate to W_2 in $\mathrm{Out}(S)$.

On the other hand, $(BG_1)^\wedge_p \simeq (BG_2)^\wedge_p$ if and only if W_1 is conjugate to W_2 in $\mathrm{Out}(S)$, by the Martino-Priddy conjecture and some extra work as in Example 4.1.8. Concretely, in [34, Example 5.2] it was found that $G = (\mathbb{Z}/2)^{27} \rtimes (\mathbb{Z}/3)^3$ and $H = (\mathbb{Z}/2)^{27} \rtimes \mathrm{U}_3(\mathbb{F}_3)$ fit into the situation above, since the regular representations

$$(\mathbb{Z}/3)^3 \to GL(27,2)$$
$$\mathrm{U}_3(\mathbb{F}_3) \to GL(27,2)$$

are faithful. Recall that $GL(27,2) \cong \mathrm{Aut}((\mathbb{Z}/2)^{27})$. Thus, G, H satisfy that $\mathbb{B}G^\wedge_2 \simeq \mathbb{B}H^\wedge_2$, although $BG^\wedge_2 \not\simeq BH^\wedge_2$, since $(\mathbb{Z}/3)^3$ and $\mathrm{U}_3(\mathbb{F}_3)$ are pointwise conjugate in $GL(27,2)$, but these groups are not even isomorphic, hence they cannot be conjugate.

More generally, the criteria above apply to any pair of finite groups with normal Sylow p-subgroups. Indeed, let G be a group with a normal (hence unique) Sylow p-subgroup S. By the Schur-Zassenhaus Theorem, G is a semidirect product of S and G/S. Moreover, the quotient group

$$G' := G/O_{p'}(G)$$

has the same p-fusion as G, with the advantage that G' is isomorphic to $S \rtimes W$, for some p'-group W which acts faithfully on S.

4.6 BURNSIDE CATEGORIES FOR SATURATED FUSION SYSTEMS

We can now define a Burnside category for (saturated) fusion systems: Let R be a p-local ring, the ring homomorphism $\pi : \mathbb{Z}_{(p)} \to R$ induces a change-of-coefficients ring homomorphism from $\mathbb{Z}_{(p)}B(S,S)$ to $RB(S,S)$, also denoted by π.

Definition 4.6.1. Let R be a p-local ring. We define the *Burnside category for fusion systems* RB whose objects are all fusion systems over finite p-groups and morphisms

$$RB(\mathcal{F}_1, \mathcal{F}_2) = \pi(\omega_{\mathcal{F}_2}) \circ RB(S_1, S_2) \circ \pi(\omega_{\mathcal{F}_1}).$$

More generally, for any admissible pair $(\mathcal{D}, \mathcal{S})$ with $\mathcal{D}$ containing the class of finite p-groups and $\mathcal{S}$ containing Δ, we analogously define the $(\mathcal{D},\mathcal{S})$-*Burnside category for fusion systems* $RB^{(\mathcal{D},\mathcal{S})}$.

For our purposes, we will concentrate on the cases $\mathcal{S} = \mathrm{fin}_p$, the class of finite p-groups, and $\mathcal{D} = \lhd$ or Δ. In this context, the category $RB^{(\mathrm{fin}_p, \lhd)}$ is simply denoted by $RB^{\lhd}$. Similarly, the category $RB^{(\mathrm{fin}_p, \Delta)}$ is denoted by RB^{Δ}. We would like to have an R-basis for $RB^{\lhd}(\mathcal{F}_1, \mathcal{F}_2)$. A natural way to achieve is via the following definition.

Definition 4.6.2 (Ragnarsson)**.** Let S_1, S_2 be finite p-groups. Given $[P,\gamma], [Q,\rho] \in RB^{\lhd}(S_1, \mathcal{S}_2)$ and $\mathcal{F}_1, \mathcal{F}_2$ fusion systems over S_1, S_2, respectively, we say that $[P,\gamma]$ is $(\mathcal{F}_1,\mathcal{F}_2)$-*subconjugate to* $[Q,\rho]$ and write

$$[P,\gamma] \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} [Q,\rho]$$

if there exist $\varphi_1 \in \mathcal{F}_1(P,Q)$ and $\varphi_2 \in \mathcal{F}_2(\gamma(P), \rho(Q))$ such that the following diagram

$$\begin{array}{ccc} P & \xrightarrow{\gamma} & \gamma(P) \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\varphi_2} \\ Q & \xrightarrow[\rho]{} & \rho(Q) \end{array}$$

commutes. We say that $[P,\gamma]$ is $(\mathcal{F}_1,\mathcal{F}_2)$-*conjugate to* $[Q,\rho]$ and write

$$[P,\gamma] \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\sim} [Q,\rho]$$

or simply $[P,\gamma] \sim [Q,\rho]$, if $[P,\gamma] \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} [Q,\rho]$ and $[Q,\rho] \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} [P,\gamma]$.

Proposition 4.6.3 ([56, Proposition 5.1])**.** *Let $\mathcal{F}_1,\mathcal{F}_2$ be fusion systems over S_1,S_2, respectively. If $[P,\gamma]$ is $(\mathcal{F}_1,\mathcal{F}_2)$-conjugate to $[Q,\rho]$, then*

$$\omega_{\mathcal{F}_2} \circ [P,\gamma] \circ \omega_{\mathcal{F}_1} = \omega_{\mathcal{F}_2} \circ [Q,\rho] \circ \omega_{\mathcal{F}_1}.$$

The above result gives the recipe to prove the following result.

Proposition 4.6.4 ([56, Proposition 5.2])**.** *Let $\mathcal{F}_1,\mathcal{F}_2$ be fusion systems over S_1,S_2, respectively, and let $\{[P_i,\gamma_i] \mid i \in I\}$ be a set of representatives of $(\mathcal{F}_1,\mathcal{F}_2)$-conjugacy classes. Then the set*

$$\{\omega_{\mathcal{F}_2} \circ [P_i,\gamma_i] \circ \omega_{\mathcal{F}_1} \mid i \in I\}$$

is an R-basis of $RB^{\triangleleft}(\mathcal{F}_1,\mathcal{F}_2)$.

Naturally, Proposition 4.6.4 can be adapted to right-free or bifree Burnside modules. Now, Definition 4.4.1 leads naturally to the following definition.

Definition 4.6.5. Let $\mathcal{F}$ be a fusion system over S, and let M be either a deflation functor or a global Mackey functor over a p-local ring R. An element $x \in M(S)$ is $\mathcal{F}$-*stable* if for every $P \leq S$ and every $\varphi \in \mathcal{F}(P,S)$ we have

$$M([P,\varphi]_P^S)(x) = M([P,\mathrm{incl}]_P^S)(x)$$

in the contravariant case, or

$$M([\varphi(P),\varphi^{-1}]_S^P)(x) = M([P,\mathrm{id}_P]_S^P)(x)$$

in the covariant case. The submodule of $\mathcal{F}$-stable elements is denoted $M(\mathcal{F})$.

The following theorem, due to Ragnarsson, tells us that the $\mathcal{F}$-stable elements are those that are invariant under the action of ω_F.

Theorem 4.6.6 ([58, Theorem 4.10])**.** *An element $x \in M(S)$ is $\mathcal{F}$-stable if and only if $M(\omega_{\mathcal{F}})(x) = x$.*

Alternatively, thanks to Theorem 4.6.6 and the fact that $\omega_{\mathcal{F}} \circ \omega_{\mathcal{F}} = \omega_{\mathcal{F}}$, we can equivalently define $M(\mathcal{F})$ as

$$\mathrm{Im}(M(S) \xrightarrow{M(\omega_{\mathcal{F}})} M(S)).$$

For any finite group G, let us consider the corepresentable deflation functor $RB^{\lhd}(G,-)$. By Theorem 4.6.6, it follows that $RB^{\lhd}(G,\mathcal{F}) = \omega_{\mathcal{F}} \circ RB^{\lhd}(G,S)$. Thus, $RB^{\lhd}(-,\mathcal{F})$ is a contravariant deflation functor, and $RB^{\lhd}(-,\mathcal{F}) = \omega_{\mathcal{F}} \circ RB^{\lhd}(-,S)$.

Proposition 4.6.7. *The inflation functor $RB^{\lhd}(-,\mathcal{F})$ satisfies the following properties:*

(1) $RB^{\lhd}(-,\mathcal{F})$ is projective.

(2) $\mathrm{Hom}_{\mathrm{Fun}(RB^{\lhd},\mathrm{Mod}_R)}(RB^{\lhd}(-,\mathcal{F}),M) \cong M(\mathcal{F})$.

(3) $RB^{\lhd}(-,\mathcal{F})$ is generated by its value at S.

Proof. For (1), we use the fact that $RB^{\lhd}(-,\mathcal{F}) \hookrightarrow RB^{\lhd}(-,S) \xrightarrow{\omega_{\mathcal{F}}\circ -} RB^{\lhd}(-,\mathcal{F})$ is the identity. Then $RB^{\lhd}(-,\mathcal{F})$ is a retract of $RB^{\lhd}(-,S)$. The result follows because $RB^{\lhd}(-,S)$ is projective.

In turn, property (2) follows from an argument analogous to Yoneda's lemma. Indeed, the map

$$\mathrm{Hom}_{\mathrm{Fun}(RB^{\lhd},\mathrm{Mod}_R)}(RB^{\lhd}(-,\mathcal{F}),M) \to M(\mathcal{F})$$

sending a morphism of deflation functors (i.e. a natural transformation) $\Psi : RB^{\lhd}(-,\mathcal{F}) \to M$ to $\Psi_{\mathcal{F}}(\omega_{\mathcal{F}})$ is bijective. For (3), if $\theta \in RB^{\lhd}(H,\mathcal{F})$, then $\theta = \omega_{\mathcal{F}} \circ \beta$ for some $\beta \in RB^{\lhd}(H,S)$. The morphism $-\circ\beta : RB^{\lhd}(S,\mathcal{F}) \to RB^{\lhd}(H,\mathcal{F})$ sends $\omega_{\mathcal{F}}$ to θ. □

As $RB^{\lhd}(-,\mathcal{F})$ is a retract of $RB^{\lhd}(-,S)$, and $\mathrm{End}(\mathbb{P}_{H,V})$ is a local ring [71, Corollary 11.1.5], all indecomposable projective summands $\mathbb{P}_{Q,V}$ of $RB^{\lhd}(-,\mathcal{F})$ form part of are those summands of $RB^{\lhd}(-,S)$.

Proposition 4.6.8. *Let R be $\mathbb{Z}_p^{\wedge}$ or $\mathbb{F}_p$ and $\mathcal{F}$ as above. Then*

$$RB^{\lhd}(-,\mathcal{F}) \cong \bigoplus_{(Q,V)} \mathbb{P}_{Q,V}^{n_{Q,V}(\mathcal{F})},$$

with (Q, V) running over representatives of isomorphism classes of seeds with Q isomorphic to a subgroup of S, where

$$n_{Q,V}(\mathcal{F}) = \frac{\dim \mathbb{S}_{Q,V}(\mathcal{F})}{\dim \operatorname{End}_{R\mathrm{Out}(Q)}(V)},$$

and dimensions are taken over $\mathbb{F}_p$.

Proof. From the properties of a projective cover we have

$$n_{Q,V}(\mathcal{F}) = \frac{\dim \operatorname{Hom}(RB^{\triangleleft}(-,\mathcal{F}), \mathbb{S}_{Q,V})}{\dim \operatorname{End}_{R\mathrm{Out}(Q)}(V)} = \frac{\dim \mathbb{S}_{Q,V}(\mathcal{F})}{\dim \operatorname{End}_{R\mathrm{Out}(Q)}(V)},$$

as desired. □

This generalizes the finite group case. Indeed, $\mathbb{S}_{Q,V}(\mathcal{F}_S(G)) \cong \mathbb{S}_{Q,V}(G)$ since $\mathbb{S}_{Q,V}$ is cohomological by Proposition 2.4.25.

Proposition 4.6.9. *Let $\mathcal{F}_1, \mathcal{F}_2$ be saturated fusion systems over S_1, S_2, respectively. Then there is an isomorphism of $\mathbb{Z}_p^\wedge$-modules*

$$\mathbb{Z}_p^\wedge \widetilde{B}^{\triangleleft}(\mathcal{F}_1, \mathcal{F}_2) \cong [\mathbb{B}\mathcal{F}_1, \mathbb{B}\mathcal{F}_2].$$

Proof. By [44, Lemma 4.1], we have $[Y, Y'] \cong e_{Y'} \circ [\mathbb{B}S_1, BS_2] \circ e_Y$. In particular,

$$[\mathbb{B}\mathcal{F}_1, \mathbb{B}\mathcal{F}_2] \cong \widetilde{\omega_{\mathcal{F}_2}} \circ [\mathbb{B}S_1, \mathcal{B}S_2] \circ \widetilde{\omega_{\mathcal{F}_1}}.$$

On the other hand, we recall that

$$\mathbb{Z}_p^\wedge \widetilde{B}^{\triangleleft}(\mathcal{F}_1, \mathcal{F}_2) \cong \omega_{\mathcal{F}_2} \circ \mathbb{Z}_p^\wedge \widetilde{B}^{\triangleleft}(S_1, S_2) \circ \omega_{\mathcal{F}_1}.$$

The Segal conjecture (recall Corollary 3.4.4) is compatible with composition, so the claim follows. □

Corollary 4.6.10. *Let $\mathcal{F}_1, \mathcal{F}_2$ be as above. Then there is an isomorphism of $\mathbb{Z}_p^\wedge$-modules*

$$\mathbb{Z}_p^\wedge B^{\triangleleft}(\mathcal{F}_1, \mathcal{F}_2) \cong [(\mathbb{B}_+\mathcal{F}_1)_p^\wedge, (\mathbb{B}_+\mathcal{F}_2)_p^\wedge].$$

4.7 BISET FUNCTORS FOR SATURATED FUSION SYSTEMS

Throughout this section, R will denote a p-local ring.

Definition 4.7.1. An *$\mathcal{S}$-biset functor for fusion systems over finite p-groups* with values in Mod_R is a (covariant or contravariant) functor F from $RB^{\mathcal{S}}$ to Mod_R. In particular, when $\mathcal{S} = \triangleleft$ (resp. $\mathcal{S} = \Delta$), we call F a *deflation functor* (resp. a *global Mackey functor*).

Clearly, a biset functor for finite groups (actually, for finite p-groups is enough) induces a biset functor for fusion systems. Indeed, given $H : RB^{\mathcal{S}} \to \mathrm{Mod}_R$, a biset functor for fusion systems $\widetilde{H}$ is induced, defined by taking $\widetilde{H}(\mathcal{F}) := H(\mathcal{F})$, the $\mathcal{F}$-stable submodule of $H(S)$ and for $Y \in RB^{\mathcal{S}}(\mathcal{F}_1, \mathcal{F}_2)$, we define

$$\widetilde{H}(Y) := H(\omega_{\mathcal{F}_2}) \circ H(X) \circ H(\omega_{\mathcal{F}_1}),$$

where $Y = \omega_{\mathcal{F}_2} \circ X \circ \omega_{\mathcal{F}_1}$, with $X \in RB(S_1, S_2)$.

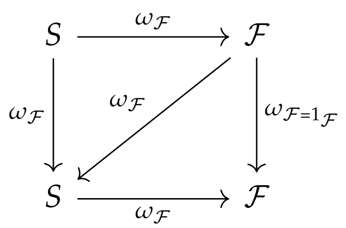
When using the trivial fusion system $\mathcal{F}_S(S)$, we denote $H(\mathcal{F}_S(S))$ simply by $H(S)$.

Proposition 4.7.2. *Let H be a biset functor for fusion systems. For any fusion system $\mathcal{F}$ over S, we have*

$$H(\mathcal{F}) \cong \omega_{\mathcal{F}}.H(S) := H(\omega_{\mathcal{F}})(H(S))$$

as R-modules, viewing $\omega_{\mathcal{F}}$ as a morphism in $RB^{\mathcal{C}}(\mathcal{F}_S(S), \mathcal{F}_S(S)) = RB^{\mathcal{C}}(S, S)$.

Proof. As a morphism, $\omega_{\mathcal{F}}$ can be considered to have source and target S or $\mathcal{F}$, as depicted in the commutative diagram below:



Since the composite

$$H(\mathcal{F}) \xrightarrow{H(\omega_{\mathcal{F}})} H(S) \xrightarrow{H(\omega_{\mathcal{F}})} H(\mathcal{F}) \tag{4.7.3}$$

equals $H(1_{\mathcal{F}}) = 1_{H(\mathcal{F})}$, we have $H(\mathcal{F}) \cong \mathrm{Im}(H(\mathcal{F}) \xrightarrow{H(\omega_{\mathcal{F}})} H(S))$. But

$$H(S) \xrightarrow{H(\omega_{\mathcal{F}})} H(\mathcal{F}) \xrightarrow{H(\omega_{\mathcal{F}})} H(S) \text{ equals } H(S) \xrightarrow{H(\omega_{\mathcal{F}})} H(S)$$

and $H(S) \xrightarrow{H(\omega_{\mathcal{F}})} H(\mathcal{F})$ is surjective by Equation (4.7.3). Thus $\mathrm{Im}(H(\mathcal{F}) \xrightarrow{H(\omega_{\mathcal{F}})} H(S)) = \mathrm{Im}(H(S) \xrightarrow{H(\omega_{\mathcal{F}})} H(S))$. □

Proposition 4.7.2 confirms the intuition that $H(\mathcal{F})$ coincides with the submodule of $\mathcal{F}$-stable elements of $H(S)$ defined in Section 4.6, also denoted $H(\mathcal{F})$. Therefore,

$$RB(-, \mathcal{F}_S(G)) \cong RB(-, G).$$

Let $\mathcal{F}$ be a fusion system over S and V an $RB(\mathcal{F},\mathcal{F})$-module. We can naturally define $\mathbb{S}_{\mathcal{F},V} = L_{\mathcal{F},V}/J_{\mathcal{F},V}$, where

$$L_{\mathcal{F},V}(\mathcal{H}) = RB(\mathcal{F},\mathcal{H}) \otimes_{RB(\mathcal{F},\mathcal{F})} V,$$

and

$$J_{\mathcal{F},V}(\mathcal{H}) = \left\{ \sum_i \varphi_i \otimes v_i \in L_{\mathcal{F},V}(\mathcal{H}) \;\middle|\; \forall \psi \in RB(\mathcal{H},\mathcal{F}), \sum_i \varphi_i\psi \otimes v_i = 0 \right\}.$$

As expected, if V is simple, then so is $\mathbb{S}_{\mathcal{F},V}$.

In particular, for the following proposition, we assume that $R = \mathbb{F}_p$, Q is a finite p-group and V is a simple $\mathbb{F}_p\mathrm{Out}(Q)$-module.

Theorem 4.7.4. *There is an isomorphism $\mathbb{S}^{\Delta}_{Q,V}(\mathcal{F}) \cong \bigoplus_L W_L({}^L V)$, where the direct sum runs over $\mathcal{F}$-fully normalized subgroups $L \le S$ which are isomorphic to Q, taken up to $\mathcal{F}$-conjugation, and $W_L = k\left(\sum_{\sigma\in\mathrm{Out}_{\mathcal{F}}(L)} \sigma\right)$ for some $k \in \mathbb{F}_p$.*

Proof. Since V is simple, we can choose a single generator v for V as an $\mathbb{F}_p\mathrm{Out}(Q)$-module. Thus, we have an isomorphism $\mathbb{S}^{\Delta}_{Q,V}(\mathcal{F}) \cong \mathbb{F}_pB^{\Delta}(Q,\mathcal{F})/N$, where

$$N = \{\varphi \in \mathbb{F}_pB^{\Delta}(Q,\mathcal{F}) \mid (\psi\varphi)\cdot v = 0, \text{ for all } \psi \in \mathbb{F}_pB^{\Delta}(\mathcal{F},Q)\}.$$

We would like to find an alternative description for the elements of N. Indeed, let $\varphi = \omega_{\mathcal{F}}\widetilde{\varphi} \in N$. Then

$$0 = \psi\varphi = \widetilde{\psi}\omega_{\mathcal{F}}\omega_{\mathcal{F}}\widetilde{\varphi} = \widetilde{\psi}\omega_{\mathcal{F}}\widetilde{\varphi}, \tag{4.7.5}$$

for all $\psi = \widetilde{\psi}\omega_{\mathcal{F}} \in \mathbb{F}_p B^{\Delta}(\mathcal{F}, Q)$. By linearity, it suffices to assume that $\widetilde{\psi} = [P,\alpha]_S^Q$. Thus, if $\widetilde{\varphi} = \sum \lambda_{[R,\beta]}[R,\beta]_Q^S$, we have $\widetilde{\psi} \circ \omega_{\mathcal{F}} \circ \widetilde{\varphi} = \sum \lambda_{[R,\beta]}[P,\alpha] \circ \omega_{\mathcal{F}} \circ [R,\beta]$. But the products

$$([P,\alpha] \circ \omega_{\mathcal{F}} \circ [R,\beta]) \cdot v \tag{4.7.6}$$

are trivial unless $R = Q$ and $\alpha(P) = Q$, so we can assume that $R = Q$ and $P = \gamma(Q)$, where $\gamma : Q \to S$ is a monomorphism and $\alpha = \gamma^{-1} : \gamma(Q) \to Q$. In other words, we can assume

$$[P,\alpha] = [\gamma(Q), \gamma^{-1}] \text{ and } [R,\beta] = [Q,\beta].$$

Thus, we can simply write λ_β for $\lambda_{[Q,\beta]}$. Hence, Equation (4.7.5) takes now the form

$$0 = \left(\sum \lambda_\beta [\gamma(Q), \gamma^{-1}] \circ \omega_{\mathcal{F}} \circ [Q,\beta]\right) \cdot v. \tag{4.7.7}$$

On the other hand, we have that

$$[Q,\beta] = [\beta(Q), \iota]_{\beta(Q)}^S \circ [Q,\beta]_Q^{\beta(Q)}$$

and

$$[\gamma(Q), \gamma^{-1}] = [\gamma(Q), \gamma^{-1}]_{\gamma(Q)}^Q \circ [\gamma(Q), \mathrm{id}]_S^{\gamma(Q)}.$$

We recall by Corollary 4.4.15 that $[\gamma(Q), \mathrm{id}] \circ \omega_{\mathcal{F}} \circ [\beta(Q), \iota]$ equals

$$\left(\sum_{[\beta(Q),\sigma]} \frac{|S|}{\Phi_{\Delta(\beta(Q),\sigma)}([\beta(Q),\sigma])} \cdot \frac{1}{\mathcal{F}(\beta(Q), S)} [\beta(Q),\sigma]_{\beta(Q)}^{\gamma(Q)}\right) + \cdots, \tag{4.7.8}$$

where the exhibited sum runs over the classes $[\beta(Q),\sigma]_{\beta(Q)}^{\gamma(Q)}$ with $\sigma \in \mathcal{F}(\beta(Q), \gamma(Q))$ and the rest of summands are annihilated by v. Therefore, we have that $([\gamma(Q), \gamma^{-1}] \circ \omega_{\mathcal{F}} \circ [Q,\beta]) \cdot v$ equals

$$\left(\sum_{\substack{[\beta(Q),\sigma] \\ \sigma \in \mathcal{F}(\beta(Q),\gamma(Q))}} \frac{|S|}{\Phi_{\Delta(\beta(Q),\sigma)}([\beta(Q),\sigma])} \cdot \frac{1}{|\mathcal{F}(\beta(Q), S)|} [Q, \gamma^{-1}\sigma\beta]_Q^Q\right) \cdot v.$$

In particular $([\gamma(Q),\gamma^{-1}]\circ\omega_{\mathcal{F}}\circ[Q,\beta])\cdot v$ vanishes if $\beta(Q)$ is not $\mathcal{F}$-conjugate to $\gamma(Q)$. If we choose $\sigma_0\in\mathcal{F}(\beta(Q),\gamma(Q))$ and set $\rho=\sigma_0\beta$, then

$$\omega_{\mathcal{F}}\circ[Q,\beta]=\omega_{\mathcal{F}}\circ[Q,\rho]$$

by Proposition 4.6.3. This has the advantage that $\rho=\gamma\theta$, for some $\theta\in\mathrm{Aut}(Q)$. Therefore without loss of generality we can assume that $\beta=\rho=\gamma\theta$ so that $\beta(Q)=\gamma(Q)$ and $\sigma\in\mathrm{Aut}_{\mathcal{F}}(Q)$. In consequence, $[\gamma(Q),\gamma^{-1}]\circ\omega_{\mathcal{F}}\circ[Q,\beta]$ equals

$$\sum_{[\sigma]\in\mathrm{Out}_{\mathcal{F}}(\gamma(Q))}\frac{|S|}{\Phi_{\Delta(\gamma(Q),\sigma)}([\gamma(Q),\sigma])}\cdot\frac{1}{|\mathcal{F}(\gamma(Q),S)|}[Q,\gamma^{-1}\sigma\gamma]\circ[Q,\theta] \tag{4.7.9}$$

Note that $\Phi_{\Delta(\gamma(Q),\sigma)}([\gamma(Q),\sigma])=|Z(\gamma(Q))|$ does not depend on σ, hence sum (4.7.9) equals

$$\frac{|S|}{|Z(\gamma(Q))|}\cdot\frac{1}{|\mathcal{F}(\gamma(Q),S)|}\sum_{[\sigma]\in\mathrm{Out}_{\mathcal{F}}(\gamma(Q))}[Q,\gamma^{-1}\sigma\gamma]\circ[Q,\theta].$$

We can assume that $\gamma(Q)$ is fully $\mathcal{F}$-normalized. Then $|\mathcal{F}(\gamma(Q),S)|=\frac{|S|}{|C_S(\gamma(Q))|}k'$, with k' coprime to p, by Lemma 4.3.2. Therefore

$$\begin{aligned}([\gamma(Q),\gamma^{-1}]\circ\omega_{\mathcal{F}}\circ[Q,\beta])\cdot v&=\left(\frac{|S|}{|Z(\gamma(Q))|}\cdot\frac{|C_S(\gamma(Q))|}{|S|k'}\sum_{[\sigma]\in\mathrm{Out}_{\mathcal{F}}(\gamma(Q))}[Q,\gamma^{-1}\sigma\gamma]\circ[Q,\theta]\right)\cdot v\\&=\left(\frac{|C_S(\gamma(Q))|}{|Z(\gamma(Q))|}\cdot\frac{1}{k'}\sum_{[\sigma]\in\mathrm{Out}_{\mathcal{F}}(\gamma(Q))}[Q,\gamma^{-1}\sigma\gamma]\circ[Q,\theta]\right)\cdot v.\end{aligned}$$

If $\gamma(Q)$ is $\mathcal{F}$-centric, then $C_S(\gamma(Q))=Z(Q)$. Thus

$$\begin{aligned}([\gamma(Q),\gamma^{-1}]\circ\omega_{\mathcal{F}}\circ[Q,\beta])\cdot v&=\left(k\sum_{[\sigma]\in\mathrm{Out}_{\mathcal{F}}(\gamma(Q))}[Q,\gamma^{-1}\sigma\gamma]\circ[Q,\theta]\right)\cdot v\\&=\left(k\sum_{[\sigma]\in\mathrm{Out}_{\mathcal{F}}(\gamma(Q))}[\gamma^{-1}\sigma\gamma]\right)\cdot([\theta].v),\end{aligned}$$

where

$$k=\frac{|C_S(\gamma(Q))|}{|Z(\gamma(Q))|}\cdot\frac{1}{k'}=\frac{|S|}{|\mathcal{F}(\gamma(Q),S)||Z(\gamma(Q))|}. \tag{4.7.10}$$

At last, Equation (4.7.7) becomes

$$0 = \big([\gamma(Q),\gamma^{-1}] \circ \omega_{\mathcal{F}} \circ \sum \lambda_\beta [Q,\beta]\big) \cdot v = \left(k \sum_{[\sigma]\in \mathrm{Out}_{\mathcal{F}}(\gamma(Q))} [\gamma^{-1}\sigma\gamma]\right) \cdot \big(\sum \lambda_{[\theta]}[\theta].v\big), \tag{4.7.11}$$

where $\lambda_{[\theta]} := \lambda_\beta = \lambda_{[Q,\beta]}$. We now define a homomorphism

$$\mathbb{F}_p B^\Delta(Q,\mathcal{F}) \xrightarrow{R} \bigoplus_L W_L({}^L V)$$

that sends $\omega_{\mathcal{F}} \circ [R,\gamma]$ to zero if $R \neq Q$ and, for each $\mathcal{F}$-fully normalized representative of $L \leq S$ of $\mathcal{F}$-conjugation, we choose $\omega_{\mathcal{F}} \circ [Q,\gamma]$ with $\gamma(Q) = L$ and send it to

$$\left(k \sum_{[\sigma]\in \mathrm{Out}_{\mathcal{F}}(\gamma(Q))} [\gamma^{-1}\sigma\gamma]\right) \cdot v \in W_{\gamma(Q)}({}^{\gamma(Q)}V)$$

with k as in Equation (4.7.10) and, for any other $\omega_{\mathcal{F}} \circ [Q,\gamma']$ with $\gamma'(Q) = L$, we have $\omega_{\mathcal{F}} \circ [Q,\gamma'] = \omega_{\mathcal{F}} \circ [Q,\gamma] \cdot [\theta]$, for some $[\theta] \in \mathrm{Out}(Q)$ and define

$$R(\omega_{\mathcal{F}} \circ [Q,\gamma']) = \left(k \sum_{[\sigma]\in \mathrm{Out}_{\mathcal{F}}(\gamma(Q))} [\gamma^{-1}\sigma\gamma]\right) \cdot ([\theta] \cdot v) \in W_{\gamma(Q)}({}^{\gamma(Q)}V),$$

then extend linearly. This homomorphism is surjective and has kernel N. Hence

$$\mathbb{S}^\Delta_{Q,V}(\mathcal{F}) \cong \bigoplus_L W_L({}^L V),$$

as desired. □

Particularly, we note from Equation (4.7.10) that $k = 0$ unless $L := \gamma(Q)$ is $\mathcal{F}$-centric. Hence, $W_L({}^L V) = 0$ unless L is $\mathcal{F}$-centric and $W_L({}^L V) \cong \overline{W}_L({}^L V)$ if L is $\mathcal{F}$-centric, where

$$\overline{W}_L = \sum_{[\tau]\in \mathrm{Out}_{\mathcal{F}}(L)} [\tau].$$

Therefore, alternatively we have

$$\mathbb{S}^\Delta_{Q,V}(\mathcal{F}) \cong \bigoplus_{L \text{ is } \mathcal{F}\text{-centric}} \overline{W}_L({}^L V). \tag{4.7.12}$$

Remark 4.7.13. Our proof of Theorem 4.7.4 does not use that V is simple but generated by a single element as an $\mathbb{F}_p\mathrm{Out}(Q)$-module, therefore the isomorphism there works more generally when V is cyclic, analogously to the finite groups case.

Example 4.7.14. In the case $\mathcal{F} = \mathrm{RV1}$ (see Example 4.1.12), it is known that $\mathrm{Out}(E) \cong \mathrm{GL}_2(7)$. We recall from Example 2.4.22 the simple $\mathrm{GL}_2(7)$-modules, in particular, $M_{6,0} \cong \mathbb{F}_7\{x^6, x^5y, \ldots, xy^5, y^6\}$. Moreover, $\mathrm{Out}_{\mathcal{F}}(E) = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \mid a,b,c,d \in \mathbb{F}_7\right\}$ by Example 4.1.12. Then $\overline{W}_E.x^6 = \overline{W}_E.y^6 = 6x^6 + 6y^6$, whereas $\overline{W}_E.x^iy^j = 0$ for $0 < i,j < 6$. Therefore $\dim_{\mathbb{F}_7}\overline{W}_E.M_{6,0} = 1$.

For the following result, we define $\mathrm{Rep}(Q,\mathcal{F}) := \mathrm{Hom}(Q,S)/\sim$, with $\rho \sim \rho'$ if and only if there exists $\chi \in \mathcal{F}(\rho(Q),\rho'(Q))$ such that $\rho' = \chi \circ \rho$. Similarly, $\mathrm{Cen}(P,\mathcal{F}) \subset \mathrm{Rep}(P,\mathcal{F})$ is defined by taking only the classes of monomorphisms with $\mathcal{F}$-centric image.

Theorem 4.7.15. *The following conditions are equivalent*

(1) $\mathcal{F}_1 \cong \mathcal{F}_2$ *in* $\mathbb{F}_pB^{\Delta}$.

(2) $\mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_1) \cong \mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_2)$ *as* $\mathbb{F}_p\mathrm{Out}(Q)$*-modules, for every finite* p*-group* Q.

Proof. $(2) \Rightarrow (1)$ The proof follows the idea of Theorem 3.5.5: we will show that $\mathbb{F}_pB^{\Delta}(-,\mathcal{F}_1)$ and $\mathbb{F}_pB^{\Delta}(-,\mathcal{F}_2)$ are isomorphic in $\mathrm{Mack}_{\mathbb{F}_p}$, since

$$\mathrm{Mack}_{\mathbb{F}_p}(\mathbb{F}_pB^{\Delta}(-,\mathcal{F}_1),\mathbb{F}_pB^{\Delta}(-,\mathcal{F}_2)) \cong \mathbb{F}_pB^{\Delta}(\mathcal{F}_1,\mathcal{F}_2),$$

The hypothesis implies that $\mathcal{F}_1$ and $\mathcal{F}_2$ have isomorphic Sylow p-subgroups, thus both $\mathbb{F}_pB^{\Delta}(-,\mathcal{F}_1)$ and $\mathbb{F}_pB^{\Delta}(-,\mathcal{F}_2)$ are retracts of $\mathbb{F}_pB^{\Delta}(-,S)$. In this way,

$$\bigoplus_{(Q,V)} (\mathbb{P}^{\Delta}_{Q,V})^{n_{Q,V}(\mathcal{F}_1)} \cong \mathbb{F}_pB^{\Delta}(-,\mathcal{F}_1) \cong \mathbb{F}_pB^{\Delta}(-,\mathcal{F}_2) \cong \bigoplus_{(Q,V)} (\mathbb{P}^{\Delta}_{Q,V})^{n_{Q,V}(\mathcal{F}_2)}$$

if and only if $n_{Q,V}(\mathcal{F}_1) = n_{Q,V}(\mathcal{F}_2)$ for each pair (Q,V). In turn, the latter holds if and only if

$$\dim_{\mathbb{F}_p}\mathbb{S}^{\Delta}_{Q,V}(\mathcal{F}_1) = \dim_{\mathbb{F}_p}\mathbb{S}^{\Delta}_{Q,V}(\mathcal{F}_2).$$

We will show this. Indeed, by Theorem 4.7.4 and comments below, we have

$$\mathbb{S}^{\Delta}_{Q,V}(\mathcal{F}_1) \cong \bigoplus_{L} \overline{W}_L({}^LV),$$

where $\overline{W}_L = \sum_{[\sigma]\in \mathrm{Out}_{\mathcal{F}(L)}} \sigma$, with L running over $\mathcal{F}_1$-centric, $\mathcal{F}_1$-fully normalized representatives of $\mathcal{F}_1$-conjugacy classes of subgroups of S which are isomorphic to Q.

On the other hand, $\mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_1) \cong \bigoplus_{L\leq_{\mathcal{F}_1} S \text{ is } \mathcal{F}-\text{centric}} \mathbb{F}_p\mathrm{Out}(Q)\overline{W}_L$. Condition (2) means

$$\bigoplus_{L\leq_{\mathcal{F}_1} S \text{ is } \mathcal{F}_1-\text{centric}} \mathbb{F}_p\mathrm{Out}(Q)\overline{W}_L \cong \bigoplus_{L'\leq_{\mathcal{F}_2} S \text{ is } \mathcal{F}_2-\text{centric}} \mathbb{F}_p\mathrm{Out}(Q)\overline{W}_{L'},$$

hence we can apply Lemma 3.4.15 to equation (4.7.12) to obtain

$$\mathbb{S}^{\Delta}_{Q,V}(G) \cong \mathbb{S}^{\Delta}_{Q,V}(\mathcal{F}_2)$$

as $\mathbb{F}_p$-vector spaces, as expected.

(1) ⇒ (2) An isomorphism from $\mathcal{F}_1$ to $\mathcal{F}_2$ in $\mathbb{F}_pB^{\Delta}_p$ induces an isomorphism

$$\mathbb{S}^{\Delta}_{Q,V}(\mathcal{F}_1) \to \mathbb{S}^{\Delta}_{Q,V}(H) \tag{4.7.16}$$

as $\mathbb{F}_p$-modules, for any finite p-group Q, since $\mathbb{S}^{\Delta}_{Q,V}$ is cohomological by Proposition 2.4.25.

Essentially as in the proof of Theorem 3.5.5, $\mathbb{S}^{\Delta}_{Q,\mathbb{F}_p\mathrm{Out}(Q)}(\mathcal{F}_1)$ is a right $\mathbb{F}_p\mathrm{Out}(Q)$-module. It follows that isomorphism (4.7.16) is an isomorphism of $\mathbb{F}_p\mathrm{Out}(Q)$-modules. Now, we have

$$\begin{aligned}
\mathbb{S}_{Q,\mathbb{F}_p\mathrm{Out}(Q)}(\mathcal{F}_1) &\cong \bigoplus_{L\leq_{\mathcal{F}_1} S \text{ is } \mathcal{F}-\text{centric}} \overline{W}_L\mathbb{F}_p\mathrm{Out}(L) \\
&\cong \bigoplus_{L\leq_{\mathcal{F}_1} S \text{ is } \mathcal{F}-\text{centric}} \mathbb{F}_p\mathrm{Out}(L)/\mathbb{F}_p\mathrm{Out}_{\mathcal{F}_1}(L) \\
&\cong \mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_1),
\end{aligned}$$

where all isomorphisms are isomorphisms of right $\mathbb{F}_p\mathrm{Out}(Q)$-modules. We then obtain an isomorphism of $\mathbb{F}_p\mathrm{Out}(Q)$-modules

$$\mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_1) \to \mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_2),$$

as desired. □

By reasoning analogously to Remark 3.4.8, we have that Theorem 4.7.15 leads to the following consequence.

Corollary 4.7.17. *If* $\mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_1) \cong \mathbb{F}_p\mathrm{Cen}(Q,\mathcal{F}_2)$ *as* $\mathbb{F}_p\mathrm{Out}(Q)$*-modules, for every finite* p*-group* Q*, then* $\mathbb{B}\mathcal{F}_1 \simeq \mathbb{B}\mathcal{F}_2$*.*

We end this thesis by giving some evidence of [34, Proposition 4.5], that is, the implication $(3) \Rightarrow (4)$ in Theorem 3.3.5. Let G, H be finite groups with normal Sylow p-subgroups. Following Example 4.5.3, we can assume that $G = S \rtimes W$, for some p'-group W acting faithfully on S, without losing information of its p-fusion, and analogously for $H = S' \rtimes W'$.

Now, let us assume that G and H satisfy condition (3) in Theorem 3.3.5. Hence, we can assume that G and H have S in common as their Sylow p-subgroup. In particular, we have $H = S \rtimes W'$. Regarding G, we have

$$\mathbb{F}_p\mathrm{Inj}(Q,G) \cong \mathbb{F}_p\mathrm{Inj}(Q,S) \otimes_{\mathbb{F}_p\mathrm{Out}(S)} \mathbb{F}_p\left[\frac{\mathrm{Out}(S)}{W}\right] \tag{4.7.18}$$

and

$$\mathbb{F}_p\mathrm{Cen}(Q,G) \cong \mathbb{F}_p\mathrm{Cen}(Q,S) \otimes_{\mathbb{F}_p\mathrm{Out}(S)} \mathbb{F}_p\left[\frac{\mathrm{Out}(S)}{W}\right], \tag{4.7.19}$$

and analogously for H. Now, by hypothesis, we have isomorphisms

$$\mathbb{F}_p\left[\frac{\mathrm{Out}(S)}{W}\right] \cong \mathbb{F}_p\mathrm{Inj}(S,G) \cong \mathbb{F}_p\mathrm{Inj}(S,H) \cong \mathbb{F}_p\left[\frac{\mathrm{Out}(S)}{W'}\right] \tag{4.7.20}$$

as $\mathbb{F}_p\mathrm{Out}(S)$-modules. Therefore, the dependence of $\mathbb{F}_p\mathrm{Cen}(Q,G)$ on $\mathbb{F}_p\left[\frac{\mathrm{Out}(S)}{W}\right]$ in isomorphism (4.7.19), and analogously for $\mathbb{F}_p\mathrm{Cen}(Q,H)$, guarantees that

$$\mathbb{F}_p\mathrm{Cen}(Q,G) \cong \mathbb{F}_p\mathrm{Cen}(Q,H),$$

as desired. In particular, we are able to affirmatively answer O'Hare's question (see the conclusion of Section 3.5) for finite groups with normal Sylow p-subgroups, extending the family given in [46, Proposition 2.4].

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A

APPENDIX

A.1 THE MOBIUS FUNCTION

Let $(P, \leq)$ be a poset. We say that $(P, \leq)$ is *locally finite* if all intervals

$$[x, z] = \{y \in P \mid x \leq y \leq z\}$$

are finite. When the order $\leq$ is known, we simply denote $(P, \leq)$ by P.

Definition A.1.1. Let P be a poset. The *incidence algebra of P* is the set

$$I(P) = \{f : P \times P \to \mathbb{R} \mid f(x, y) = 0 \text{ unless } x \leq y\},$$

with the following operations:

-(Addition): pointwise

$$(f + g)(x, z) = f(x, z) + g(x, z)$$

-(Multiplication): convolution

$$(f * g)(x, z) = \sum_{x \leq y \leq z} f(x, y) g(y, z)$$

-Multiplicative identity: $e(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$

Proposition A.1.2. *A function $f \in I(P)$ is invertible if and only if $f(x, x) \neq 0$ for all $x \in P$.*

The *Mobius function of* P, denoted by μ, is defined as the inverse of the *zeta function* ζ, with

$$\zeta(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, ζ is invertible by Proposition A.1.2. The function μ can inductively constructed as follows:

$$\begin{aligned} \mu(x,x) &= 1, \\ \mu(x,y) &= -\sum_{x \leq z < y} \mu(x,z). \end{aligned}$$

A.2 LIFTING OF ISOMORPHISMS

The following result is proved in [69], but it is not contained in any published source to the best of the author's knowledge. Therefore we reproduce the proof from [69] here for completeness.

Proposition A.2.1. *Let* C *be an additve category, such that* $C(x,y)$ *is a finitely generated abelian group, for every* $x,y \in \mathrm{Ob}(C)$*. Then any isomorphism in* $\mathbb{F}_p C(x,y)$ *lifts to an isomorphism in* $\mathbb{Z}_p^\wedge C(x,y)$*, for every* $x,y \in \mathrm{Ob}(C)$*.*

Proof. Given an isomorphism $\overline{f} \in \mathbb{F}_p C(x,y)$, we claim that any lift $f \in \mathbb{Z}_p^\wedge C(x,y)$ of $\overline{f}$ is an isomorphism. Indeed, if $g \in \mathbb{Z}_p^\wedge C(y,x)$ is a lift of $\overline{g} = \overline{f}^{-1}$, we know that $gf = 1 - p\varphi$ for some $\varphi \in \mathbb{Z}_p^\wedge C(x,x)$. By hypothesis, it follows that $\mathbb{Z}_p^\wedge C(x,x) \cong C(x,x)_p^\wedge$, thus it is p-complete (recall Remark 1.4.4). The series $\psi = \sum_{i=0}^{\infty} p^i \varphi^i$ converges in $\mathbb{Z}_p^\wedge C(x,x)$ and gives a two-sided inverse of gf, so f has a left inverse ψg, and g has a right inverse $f\psi$. Swapping the roles of f and g shows that f has a right inverse as well, so f is invertible. □