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On the Picard group of the stable module category for infinite groups

Ph.D. Thesis

by

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Abstract

We introduce the stable module ∞-category for groups of type Φ as an enhancement of the stable category defined by N. Mazza and P. Symonds. For groups of type Φ which act on a tree, we show that the stable module ∞-category decomposes in terms of the associated graph of groups. For groups which admit a finite-dimensional cocompact model for the classifying space for proper actions, we exhibit a decomposition in terms of the stable module ∞-categories of their finite subgroups. We use these decompositions to implement methods to compute the Picard group of the stable module category. In particular, we provide a description of the Picard group for countable locally finite p-groups. We also deal with groups arising from triangles of groups and, in certain cases, we give a description of the modules that restrict stably to the trivial module on each finite subgroup. In a slightly different direction, we discuss the existence of separable commutative algebra objects of infinite degree in the context of essentially small tensor triangulated categories.

Contents

Introduction

In modular representation theory of finite groups, there is no hope to classify all kG–modules, not even finitely generated ones[1]. As a consequence of Maschke's theorem, there are non-projective modules preventing us to proceed as in ordinary representation theory. For instance, character theory is generally not sufficient to fully understand kG–modules. Instead, a classification up to projectives is desired, but it remains as an ambitious task.

This motivates the philosophy of looking into certain classes of modules that are *small enough to be classified and large enough to be useful*[2]. With this in mind, E. Dade introduced the class of endotrivial modules as a key step to classify general objects, the so-called endopermutation modules (see [Dad78b], [Dad78a], [Thé07]). Endopermutation modules play an important role in the representation theory of finite p-groups; they appear as sources of simple modules for p-solvable groups, and in the description of the source algebra of a nilpotent block, hence the importance of fully understanding these modules (see [Thé95, Chapter 5]).

Endotrivial modules are interesting objects in their own right (see [Car12], [Car17]). The isomorphism classes of endotrivial modules determine an abelian group with the multiplication given by the tensor product over the field; this group in fact, agrees with the Picard group of the stable module category. Their classification has been completed for finite p-groups thanks to the contributions of many authors, concluding with the celebrated work of J. Carlson and J. Thévenaz [CT04], [CT05]. The problem for general finite groups is still open, and it has become an active research area attracting the interest of many mathematicians. Tools to deal with the classification of endotrivial modules have been developed beyond representation theory. Notably, the stable module category has opened a door to use machinery from homotopy theory, which has been applied successfully in the classification of *Sylow-trivial modules*[3] for certain families of finite groups [Gro23], and in the classification of torsion endotrivial modules for finite groups of Lie type (announced in [CGMN22]). Certainly, the sta-

[1]Except in a few exceptional cases.
[2]Quote by E. Dade.
[3]Endotrivial modules that, up to projectives, restrict to the trivial module on a p-Sylow subgroup.

ble module category is omnipresent in modular representation theory of finite groups and not just in the context of endotrivial modules (see [BCR97], [BIK12]), hence its fundamental role in the theory is not surprising.

The modular representation theory of infinite groups is much less known. It is reasonable then, to seek for a protagonist to play the role of the stable module category for finite groups. Of course, there is an additional problem; infinite groups are too *wild*. Fortunately, the class of groups of type Φ introduced by O. Talelli[4] (see [Tal07]) has convenient finiteness properties making it a suitable candidate to explore the theory for infinite groups. In particular, this class contains all groups admitting a finite-dimensional model for the classifying space for proper actions (see [MS19, Corollary 2.6]). For these groups, N. Mazza and P. Symonds constructed a stable category as the largest quotient of the category of modules on which the syzygy functor is invertible [MS19]. This stable category is equipped with a triangulated structure compatible with the tensor product over the ground ring. In other words, it is a tensor triangulated category in the language of Balmer [Bal10]. Naturally, we want to investigate the Picard group of this stable category. However, for infinite groups there are just a few available tools to compute it.

We introduce a homotopy-theoretic interpretation of the stable module category for groups of type Φ as a symmetric monoidal stable ∞-category. For finite groups, this interpretation agrees with the one given in [Mat15]. In particular, for groups of type Φ acting on a tree, we exhibit a decomposition in terms of the fundamental domain of the action and its isotropy groups. No restriction on the size of the isotropy groups is needed for the following result (see Theorem 3.1.3).

Theorem. *Let G be a group of type Φ acting on a tree. Consider the associated graph of groups $\Gamma(G)\colon \Gamma \to \mathrm{Gps}$, that is, G is the fundamental group of $\Gamma(G)$. Then there is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{StMod}(kG) \xrightarrow{\simeq} \varprojlim_{\sigma\in\Gamma^{op}} \mathrm{StMod}(kG_\sigma).$$

This decomposition leads to a spectral sequence which computes the Picard group of the stable module category for groups of type Φ acting on trees. In particular, we provide a more conceptual proof of Theorem 7.1 in [MS19]. That is, we obtain a short exact sequence of abelian groups (see Corollary 4.1.2)

$$0 \to H^1(\Gamma; \pi_1 \circ f) \to T(G) \to H^0(\Gamma; \pi_0 \circ f) \to 0$$

where $T(G)$ denotes the Picard group of the stable module category of G, and f is

[4] This class depends on a ground ring, so a fixed field of prime characteristic is understood.

the composition of *the Picard space functor* $\mathrm{Pic}\colon \mathrm{Cat}_\infty^\otimes \to \mathcal{S}$ (see Definition 4.1.1) and the stable module ∞-category functor $\Gamma^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$ corresponding to the graph of groups $\Gamma(G)\colon \Gamma \to \mathrm{Gps}$ (see Section 3.1 for more details about this functor).

For groups admitting a finite-dimensional cocompact model for the classifying space for proper actions, we follow the ideas in [Mat16] to exhibit a different decomposition of the stable module ∞-category, in this case, in terms of the finite subgroups. The following result summarizes the results in Subsection 3.2.2.

Theorem. *Let G be a group admitting a finite-dimensional cocompact model X for $\underline{E}G$. Let $\mathscr{F}$ be a family of finite subgroups of G which contains the family of finite p-subgroups of G. Then we have an equivalence of symmetric monoidal stable ∞-categories*

$$\mathrm{StMod}(kG) \xrightarrow{\simeq} \varprojlim_{G/H \in \mathcal{O}_{\mathscr{F}}(G)^{op}} \mathrm{StMod}(kH).$$

If additionally the fundamental domain of the action is homeomorphic to the standard n-simplex, we obtain an equivalence of symmetric monoidal stable ∞-categories

$$\mathrm{StMod}(kG) \xrightarrow{\simeq} \varprojlim_{\sigma \in \mathcal{T}^{op}} \mathrm{StMod}(kG_\sigma).$$

where $\mathcal{T}$ denotes the barycentric subdivision of Δ^n.

We provide computations for certain classes of groups. For instance, we use the classification of endotrivial modules for finite p-groups [CT04], [CT05] and Corollary 4.1.2 to determine the Picard group of the stable module category for countable locally finite p-groups. The following theorem summarizes the results of Section 4.2.

Theorem. *Let P be a countable locally finite p-group. Then the following hold.*

(a) If $P = \mathbb{Z}/p^\infty$, then $T(P) \cong \mathbb{Z}/2$.

(b) Let $D_{2^\infty} = \bigcup D_{2^n}$, where D_{2^n} denotes the dihedral group of order 2^n. Then $T(D_{2^\infty}) \cong \mathbb{Z}$.

(c) Let $Q_{2^\infty} = \bigcup Q_{2^n}$, where Q_{2^n} denotes the generalized quaternion group of order 2^n. Then $T(Q_{2^\infty}) \cong \mathbb{Z}/4$.

(d) Suppose that P is artinian and that it admits a tower $Q_1 \leq Q_2 \leq \ldots$ whose union is P and such that Q_n is not cyclic, dihedral, semi-dihedral or generalized quaternion for all $n \geq 1$. Then

$$T(P) = \begin{cases} \mathbb{Z}^r & \text{if } P \text{ has } p\text{-rank at most } 2 \\ \mathbb{Z}^{r+1} & \text{if } P \text{ has } p\text{-rank at least } 3 \end{cases}$$

where r is the number of conjugacy classes of maximal elementary abelian subgroups of P of rank 2.

(e) If P is not artinian, then $T(P) \cong \mathbb{Z}$.

We provide a tool to compute the Picard group for a certain class of groups of type Φ that we call amalgam groups (see Definition 4.4.1). These groups have geometric dimension 2 with respect to the family of finite subgroups. In particular, for an amalgam group G which acts on a tree, we could attempt to use Corollary 4.1.2 to compute the Picard group of G, but it will involve computing invariants of the stable module ∞-category for infinite groups, which could be as hard to compute as the invariants of the stable module ∞-category of G. Hence the importance of the following result (see Theorem 4.4.2).

Theorem. *Let G be a group admitting a 2-dimensional model X for $\underline{E}G$ such that the fundamental domain of the action is homeomorphic to the standard 2-simplex. Consider the associated triangle of groups $\mathcal{T}(G)\colon \mathcal{T} \to Gps$ (see Section 4.4), where $\mathcal{T}$ denotes the barycentric subdivision of Δ^2. Then there is an exact sequence of abelian groups*

$$0 \to H^1(\mathcal{T}; \pi_1 \circ f) \to T(G) \to H^0(\mathcal{T}; \pi_0 \circ f) \to 0$$

where f is the composition of the Picard space functor and the stable module ∞-category functor corresponding to the triangle of groups $\mathcal{T}(G)$. Moreover, if p divides the order of the face group, then the map

$$T(G) \to H^0(\mathcal{T}; \pi_0 \circ f)$$

is an isomorphism.

In a slightly different direction, we approach an open question in [Bal14]. This was motivated by a talk given by Luca Pol at the Hausdorff Research Institute for Mathematics during the author's participation on a special trimester program. Recall that a tensor triangulated category is a triangulated category with a symmetric monoidal structure such that the monoidal product is exact in each variable. In his talk, Luca introduced a method to classify tt-rings of finite degree in certain tensor triangulated categories, and he brought out that no tt-ring of infinite degree is known. By a tt-ring we mean a commutative algebra object A that is separable, that is, the multiplication $A \otimes A \to A$ admits an (A, A)–bilinear section $A \to A \otimes A$. For a tt-ring A, Balmer constructed a tower of A–algebras $A := A^{[1]} \to A^{[2]} \to \dots$ where each $A^{[i+1]}$ is characterized as the $A^{[i]}$–algebra such that $A^{[i]} \otimes_{A^{[i-1]}} A^{[i]}$ splits as the product of $A^{[i]}$ with $A^{[i+1]}$. The degree of a tt-ring A is defined to be the greatest i such that $A^{[i]} \neq 0$ [Bal14, Definition 3.4].

We construct a family of infinite degree tt-rings in Chapter 5, mainly motivated by our study of a decomposition of the stable module category for infinite groups. Our example is quite simple but it has led the author to many interesting research questions. It will appear in this text as Theorem 5.3.1.

Theorem. *For $n \in \mathbb{N}$, let $\mathcal{K}_n$ be a non-trivial essentially small tensor triangulated category, and let $\mathbf{1}_n$ denote the unit of the monoidal product in $\mathcal{K}_n$. Then the tt-ring*

$$(\mathbf{1}_n^{\times n})_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{K}_n$$

has infinite degree.

This thesis is organized as follows. Chapter 1 is focused on background about endotrivial modules for finite groups and the stable module category for groups of type Φ, including the finite case. In Chapter 2 we discuss a symmetric monoidal model structure on the category of modules for groups of type Φ and define the stable module ∞-category. In Chapter 3, we exhibit a decomposition of the stable module ∞-category for groups acting on trees and for groups admitting a finite-dimensional cocompact model for the classifying space for proper actions. Chapter 4 is devoted to computations of the Picard group for countable locally finite groups and amalgam groups. Finally, in Chapter 5 we discuss our example of a tt-ring of infinite degree, and provide some background in order to introduce the problem.

We will work in the setting of ∞-categories and, in general, we will borrow notation and terminology from [Lur17]. We include an appendix that outlines the main concepts and facts about ∞-categories that will be used in the thesis. In particular, when we talk about limits and colimits, we always refer to homotopy limits and homotopy colimits in the ∞-categorical sense, unless we specify otherwise. Sometimes we do not distinguish between an ordinary category and its nerve.

Remark. The results of Chapters 2-4 appear in [Góm23b], and the results of Chapter 5 appear in [Góm23a], both have been submitted for publication.

Chapter 1

Preliminaries

In this chapter we provide some background material about the stable module category for finite groups. We also discuss some results regarding the group of endotrivial modules for finite groups, we mainly focus on results that will be used in Section 4.2 and Section 4.3. In particular, we describe the classification of endotrivial modules for finite p-groups. In addition, we give a short overview of the stable module category for groups of type Φ given by Mazza and Symonds, as well as some of its basic properties.

1.1 The stable module category

Let G be a finite group, and let k be a field of prime characteristic p, where p divides the order of G. Denote by $\mathrm{Mod}(kG)$ the category of all kG–modules, and $\mathrm{mod}(kG)$ the full subcategory of finitely generated kG–modules. Recall that kG is a self-injective ring, hence projective modules agree with injective modules.

Definition 1.1.1. Let $\underline{\mathrm{Mod}}(kG)$ denote the category whose elements are all the kG–modules, and with morphisms given by classes of morphisms in $\mathrm{Mod}(kG)$ under the equivalence relation given by $f \sim g$ if and only if $f - g$ factors through a projective module. Let $\underline{\mathrm{mod}}(kG)$ denote the full subcategory of $\underline{\mathrm{Mod}}(kG)$ on the finitely generated kG–modules. We will refer to $\underline{\mathrm{Mod}}(kG)$ as the *stable module category* of G, and to $\underline{\mathrm{mod}}(kG)$ as the *small stable module category* of G.

Remark 1.1.2. Any projective kG–module P is isomorphic to 0 in the stable module category. The morphism $P \to 0$ is an isomorphism with inverse given by $0 \to P$.

Fix a kG–module M. Let $\Omega(M)$ be the kernel of a surjection $P \to M$, where P is projective. By Schanuel's lemma $\Omega(M)$ is well-defined up to projectives. Let $\Omega^{-1}(M)$ denote the cokernel of a monomorphism $M \to I$, where I is injective. Then $\Omega^{-1}(M)$

is well-defined up to injectives. Consider a morphism $f\colon M \to N$ of kG–modules. Then we have a commutative diagram

$$\begin{array}{ccccc} \Omega(M) & \longrightarrow & P & \longrightarrow & M \\ {\scriptstyle \Omega(f)}\downarrow & & \downarrow & & \downarrow {\scriptstyle f} \\ \Omega(N) & \longrightarrow & P' & \longrightarrow & N \end{array}$$

Hence Ω defines an endofunctor of $\underline{\mathrm{Mod}}(kG)$. Dually, Ω^{-1} defines an endofunctor of $\underline{\mathrm{Mod}}(kG)$. Since projective is equivalent to injective, we obtain self-equivalences

$$\Omega^{-1}, \Omega\colon \underline{\mathrm{Mod}}(kG) \to \underline{\mathrm{Mod}}(kG).$$

They restrict to self-equivalences of the small stable module category.

Remark 1.1.3. A slightly different way to think about the functor $\Omega\colon \underline{\mathrm{Mod}}(kG) \to \underline{\mathrm{Mod}}(kG)$ that, hopefully, clarifies that it is well-defined is as follows. Fix a surjection $P \xrightarrow{\alpha} k$, with P a projective module. Let $\Omega(k)$ denote the kernel of α. Then the functor $\Omega(k) \otimes -\colon \mathrm{Mod}(kG) \to \mathrm{Mod}(kG)$ induces a functor $\Omega(k) \otimes -\colon \underline{\mathrm{Mod}}(kG) \to \underline{\mathrm{Mod}}(kG)$. Note that for a different choice of P and the surjection α, we obtain naturally isomorphic functors. Moreover, for any kG–module M we have that $\Omega k \otimes M$ is isomorphic to $\Omega(M)$ up to projectives. Hence we can define $\Omega(-)$ as the functor $\Omega(k) \otimes -$.

We will use the symbol "$\simeq$" to refer to an isomorphism in the stable module category. We will say that f is a *stable isomorphism* if f is an isomorphism in the stable module category.

Proposition 1.1.4. *Let M, N be two kG–modules. Then $M \simeq N$ in $\underline{\mathrm{Mod}}(kG)$ if and only if there exist projective kG–modules P, Q such that $M \oplus P \cong N \oplus Q$.*

Proof. Suppose that $M \simeq N$ in the stable module category. Then there are homomorphisms $f\colon M \to N$ and $g\colon N \to M$ such that $f \circ g - 1$ and $g \circ f - 1$ factor through a projective, that is, there are commutative triangles

$$\begin{array}{ccc} & P & \\ {\scriptstyle \alpha_1}\nearrow & & \searrow{\scriptstyle \alpha_2} \\ M & \xrightarrow{g \circ f - 1} & M \end{array} \qquad \begin{array}{ccc} & Q & \\ {\scriptstyle \beta_1}\nearrow & & \searrow{\scriptstyle \beta_2} \\ N & \xrightarrow{f \circ g - 1} & N \end{array}$$

with P and Q projectives. Note that the homomorphism $f' = (f, \alpha_1)\colon M \to N \oplus P$ has a retract given by $g' = g - \alpha_2\colon N \oplus P \to M$. Then $M \oplus \mathrm{Ker}(g') \cong N \oplus P$. Note that $f' \circ g' - 1$ factors through the projective $Q \oplus P \oplus P$. A factorization is given by

$$\begin{pmatrix} \beta_2 & 0 & f\alpha_2 \\ 0 & 1 & \alpha_1\alpha_2 \end{pmatrix} \circ \begin{pmatrix} \beta_1 & 0 \\ \alpha_1 g & -1 \\ 0 & -1 \end{pmatrix}.$$

Hence we obtain a commutative diagram

$$\begin{array}{ccccccc} & & & Q \oplus P \oplus P & & & \\ & & \nearrow & & \searrow & & \\ \mathrm{Ker}(g') & \xrightarrow{i} & N \oplus P & \xrightarrow{\quad f' \circ g' - 1 \quad} & N \oplus P & \xrightarrow{p} & \mathrm{Ker}(g') \end{array}$$

where i and p denote the inclusion and the projection, respectively. Since

$$p \circ (f' \circ g' - 1) \circ i = 1$$

we deduce that $\mathrm{Ker}(g')$ is a retract of $Q \oplus P \oplus P$. Thus $\mathrm{Ker}(g')$ is projective. □

We refer to [BCR95, Section 2] for more details about the following proposition.

Proposition 1.1.5. *The stable module category* $\underline{\mathrm{Mod}}(kG)$ *admits a triangulated structure where the shift functor is given by* Ω^{-1}*, and the distinguished triangles are induced by short exact sequences of* kG*–modules. The same holds for* $\underline{\mathrm{mod}}(kG)$*.*

Also, we can identify the stable module category with the homotopy category of $\mathrm{Mod}(kG)$ equipped with a certain model structure. Explicitly, this structure is given as follows (see [Hov99, Section 2.2]):

(1) The cofibrations are the monomorphisms.

(2) The fibrations are the epimorphisms.

(3) The weak equivalences are the stable equivalences, that is, a morphism is a weak equivalence if and only if the induced map in $\underline{\mathrm{Mod}}(kG)$ is an isomorphism.

The stable module category $\underline{\mathrm{Mod}}(kG)$ inherits a symmetric monoidal structure from the one on $\mathrm{Mod}(kG)$ given by the tensor product $\otimes_k$ over the ground field with the diagonal action of G. In fact, $\underline{\mathrm{Mod}}(kG)$ is a tensor-triangulated category in the language of Balmer [Bal10].

1.2 Endotrivial modules for finite groups

Endotrivial modules were introduced by Dade [Dad78b], [Dad78a] in the context of finite p-groups as a fundamental step to classify general objects, the so-called *endopermutation* modules.

Definition 1.2.1. Let G be a finite group. A kG–module M is *endotrivial* if there exists a kG–module N such that $M \otimes N$ is equivalent to the trivial module k up to projectives, that is, $M \otimes N \simeq k$ in the stable module category.

If M is finitely generated, then $M \otimes M^* \cong \mathrm{End}(M)$. In particular, a finitely generated kG–module M is endotrivial if and only if $\mathrm{End}(M) \cong k$ up to projectives. Actually, this is the original motivation for the terminology *endotrivial*. Some of the basic properties of endotrivial modules are given in the following proposition (see [Maz19, Section 2.2]).

Proposition 1.2.2. *Let M be an endotrivial module. Then the following properties hold.*

(i) There exists an indecomposable endotrivial module M_0 so that $M \simeq M_0$ in the stable module category.

(ii) If M, N are endotrivial modules, then $M \otimes N$ is endotrivial as well.

Example 1.2.3. Consider a short exact sequence of kG–modules

$$0 \to \; M \to P \to N \to 0$$

with P projective. Then M is endotrivial if and only if N is endotrivial. Therefore, if M is endotrivial, then $\Omega^n(M)$ is endotrivial for any integer n. In particular, $\Omega^n k$ is endotrivial for any integer n, since k is endotrivial.

Definition 1.2.4. Let G be a finite group. We define the group of endotrivial kG–modules $T(G)$ as the set

$$\{[M] \in \underline{\mathrm{Mod}}(kG) \mid M \text{ is endotrivial}\}$$

of isomorphism classes of endotrivial modules in the stable module category equipped with the tensor product over the ground field k.

Definition 1.2.5. The Picard group $\mathrm{PicGp}(\mathcal{C})$ of a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ is defined as the group of isomorphism classes of objects which are invertible with respect to the tensor product $\otimes$.

Proposition 1.2.6. *The group of endotrivial modules $T(G)$ agrees with the Picard group of the stable module category with the symmetric monoidal structure discussed above.*

One of the earliest related results given by Dade was the classification for abelian p-groups [Dad78a, Theorem 10.1], which motivated the idea that the class of endotrivial modules can be classified.

Theorem 1.2.7. *Let G be an abelian p-group. Then $T(G)$ is cyclic generated by the class of $\Omega(k)$. Explicitly,*

$$T(G) \cong \begin{cases} 0 & \text{if } G \text{ has order at most 2,} \\ \mathbb{Z}/2 & \text{if } G \text{ is cyclic of order at least 3,} \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

A key step toward the classification of endotrivial modules for finite p-groups is the result given by Puig [Pui90] that asserts that the restriction map

$$\mathrm{Res}\colon T(G) \to \prod_E T(E)$$

has finite kernel, where E runs through all the elementary abelian subgroups of G. Thus the group $T(G)$ is finitely generated. Carlson, Mazza and Nakano proved that $T(G)$ is finitely generated for arbitrary finite groups (see [CMN06, Section 3]). As a consequence we can write

$$T(G) = TT(G) \oplus TF(G)$$

where $TT(G)$ and $TF(G)$ denote the torsion part and the free part of $T(G)$, respectively.

The hard work of several authors, including Alperin, Carlson, Dade, Mazza, Puig, Thévenaz, and others (see [Maz19]), led to perhaps the biggest achievement so far concerning this problem: the classification of endotrivial modules for finite p-groups. The description of the group of endotrivial modules for p-groups was completed by Carlson and Thévenaz [CT04], [CT05].

For finite p-groups, the rank of the free part $TF(G)$ was determined by Alperin [Alp01] in terms of the number of conjugacy classes of maximal elementary abelian subgroups of rank at least 2. Thus the main object of study remained the torsion part, $TT(G)$, that turns out to be trivial almost always. The cases where $TT(G)$ is non-trivial occur just for cyclic, quaternion or semi-dihedral groups, and are particularly tricky. Besides those cases, the idea was to find a suitable "detection" family $\mathcal{H}$ of subgroups of G, i.e. a family for which the restriction map

$$\mathrm{Res}\colon T(G) \to \prod_{E \in \mathcal{H}} T(E)$$

is injective. Carlson and Thévenaz proved that the collection of elementary abelian p-subgroups of rank at least 2 and additionally, if $p = 2$, the subgroups of G isomorphic to the quaternion group of order 8, is a desired detection family (see [CT00]).

Theorem 1.2.8. *Let G be a finite p-group. If G is not cyclic of order at least 3, quaternion or semidihedral, then $TT(G)$ is trivial.*

The rank of the free part $TF(G)$ as a free $\mathbb{Z}$–module is given as the number of conjugacy classes of maximal elementary abelian subgroups of rank 2 if the rank of G is 2, or this number plus one if the rank of G is at least 3. Alperin used relative syzygys to provide a construction of endotrivial modules such that the image through the restriction map determines a subgroup of finite index in $\prod_E T(E)$ where E runs through a set of representatives of the conjugacy classes of maximal elementary abelian subgroups (see [Alp01, Theorem 4]).

Theorem 1.2.9. *Let G be a p-group. If G is not cyclic, quaternion, dihedral or semi-dihedral, then*

$$TF(G) = \begin{cases} \mathbb{Z}^n & \text{if } G \text{ has rank at most } 2 \\ \mathbb{Z}^{n+1} & \text{if } G \text{ has rank at least } 3 \end{cases}$$

where n is the number of conjugacy classes of maximal elementary abelian subgroups of G of rank 2.

Actually, Carlson and Thévenaz proved that the modules given by Alperin determine a $\mathbb{Z}$–basis for $TF(G)$. For convenience we shall give the construction and the basic properties satisfied by those modules.

Let $\mathcal{E}_{\geq 2}(G)$ be the poset whose elements are the elementary abelian p-subgroups of G of rank at least 2, with order relation given by inclusion of subgroups. The action of the group G by conjugation on the elements of $\mathcal{E}_{\geq 2}(G)$ determines a poset $\mathcal{E}_{\geq 2}(G)/G$ whose objects are the G–orbits, with order relation induced by subconjugation $\leq_G$. The elements $[E],[F] \in \mathcal{E}_{\geq 2}(G)/G$ are called connected if there exists subgroups $E_1, \ldots, E_n$ in $\mathcal{E}_{\geq 2}(G)$ such that $E \leq_G E_1 \geq_G \ldots \geq_G E_n \leq_G F$. A connected component is an *isolated vertex* if it has no elementary abelian subgroups of p-rank greater that 2. Some useful properties of this poset are summarized in the following theorem (see [Maz19, Section 3.2]).

Theorem 1.2.10. *Let G be a finite group and let S be a p-Sylow subgroup of G.*

(1) *If S has p-rank 2, then each connected component of $\mathcal{E}_{\geq 2}(G)/G$ is an isolated vertex.*

(2) *If S has p-rank at least 3, then $\mathcal{E}_{\geq 2}(G)/G$ contains a unique connected component containing all the conjugacy classes of elementary abelian p-subgroups of rank at least 3 and all the other components are isolated vertices.*

(3) *Suppose that $\mathcal{E}_{\geq 2}(G)/G$ is disconnected. Then a maximal elementary abelian subgroup of S of rank 2 has the form $\langle u\rangle \times Z$, where u is a noncentral element of S of order p, and Z is the unique central subgroup of S of order p. Moreover, $C_S(\langle u\rangle \times Z) = C_S(u) = \langle u\rangle \times Q$, where Q is cyclic, or possibly quaternion if $p = 2$.*

(4) *If $\mathcal{E}_{\geq 2}(G)/G$ is disconnected, then G has rank at most p if p is odd, and at most 4 if $p = 2$.*

(5) *$\mathcal{E}_{\geq 2}(G)/G$ has at most $p+1$ connected components if p is odd, and at most 5 if $p = 2$.*

Definition 1.2.11. Let G be a finite group and let X be a G–set. The *relative syzygy* $\Omega_X(k)$ is the kernel of the augmentation map $kX \to k$. In particular, $\Omega_G(k)$ is just $\Omega(k)$ in the stable module category.

Suppose that G is a nonabelian finite p-group, and if $p = 2$, assume that G is not dihedral, semi-dihedral or generalized quaternion. Let $E_0, \ldots, E_n$ be representatives of the G–orbits of $\mathcal{E}_{\geq 2}(G)/G$ with E_0 a normal elementary abelian subgroup of G of rank 2, where $E_i = Z \times \langle u_i\rangle$ for a non-central subgroup $\langle u_i\rangle$ of order p, for $1 \leq i \leq n$, and Z the unique central subgroup of G of order p. Recall that $C_G(\langle u_i\rangle)$ is of the form $\langle u_i\rangle \times Q_i$, where Q_i is cyclic, or possibly quaternion if $p = 2$. Define N_i as the module:

- $(\Omega_G^{-1}(\Omega_{G/\langle u_i\rangle}(k)))^{\otimes 2}$ if Q_i is cyclic of order at least 3.
- $\Omega_G^{-1}(\Omega_{G/\langle u_i\rangle}(k))$ if $p = 2$ and $|Q_i| = 2$.
- $(\Omega_G^{-1}(\Omega_{G/\langle u_i\rangle}(k)))^{\otimes 4}$ if $p = 2$ and Q_i is quaternion.

These modules are endotrivial and satisfy

$$\mathrm{Res}^G_{E_j}(N_i) \cong \begin{cases} k & \text{if } i \neq j, \\ \Omega_{E_j}^{-2p}(k) & \text{if } i = j \text{ and } Q_i \text{ is cyclic of order at least 3}, \\ \Omega_{E_j}^{-2}(k) & \text{if } i = j \text{ and } Q_i \text{ has order 2}, \\ \Omega_{E_j}^{-8}(k) & \text{if } i = j \text{ and } Q_i \text{ is quaternion}, \end{cases}$$

up to projectives (see [Maz19, Proposition 3.2]). The following theorem corresponds to [CT04, Theorem 7.1].

Theorem 1.2.12. *Let G be a finite p-group that is not cyclic, quaternion, dihedral or semi-dihedral. Then the classes of the kG–modules $\Omega_G(k), N_1, \ldots, N_n$ determine a $\mathbb{Z}$–basis for $TF(G)$.*

We will give now details about the remaining cases and some of the properties of the generators that will be useful in Chapter 4. The only case where $T(G)$ depends on the field k is if G is the quaternion group

$$Q_8 = \langle x, y \mid x^2 = y^2 = (xy)^2, x^4 = 1\rangle$$

of order 8. For an arbitrary field we have that the class of $\Omega_{Q_8}k$ has order 4. If k contains a cubic root ω of the unit, we can construct a 3-dimensional module L where the action is given by

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad y = \begin{pmatrix} 1 & 0 & 0 \\ \omega & 1 & 0 \\ 0 & \omega^2 & 1 \end{pmatrix}$$

which is an endotrivial kQ_8–module. In fact, we can prove that $\Omega^1_{Q_8}(L)$ has order 2. The restriction map $T(Q_8) \to T(\langle x\rangle)$ is surjective and the kernel consists of the classes of the modules

$$k,\ \Omega^2_{Q_8}(k),\ \Omega^{-1}_{Q_8}(L),\ \text{and}\ \Omega^{-1}_{Q_8}(L^*).$$

Since $L \ncong L^*$, the kernel is a four-Klein group. The following corresponds to [CT00, Theorem 6.3].

Theorem 1.2.13. *If k contains cubic roots of unity, then $T(Q_8) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$ generated by the classes of $\Omega_{Q_8}(k)$ and $\Omega_{Q_8}(L)$. If k does not contain cubic roots of unity, then $T(Q_8) \cong \mathbb{Z}/4$ generated by the class of $\Omega_{Q_8}(k)$.*

Convention 1.2.14. For the rest of this section, we refer to the pair in $T(H) \oplus T(H')$ given by the class of M in $T(H)$ and the class of N in $T(H')$ just as *the class of* (M, N) *in* $T(H) \oplus T(H')$.

Let $Q_{2^{n+1}} = \langle x, y \mid x^{2^n} = 1, y^2 = x^{2^{n-1}}, yxy = x^{-1}\rangle$ be the quaternion group of order 2^n, for $n \geq 4$. There is a $(2^n - 1)$-dimensional endotrivial module L such that

$$\operatorname{Res}^G_{\langle x\rangle}(L) = \Omega_{\langle x\rangle}(L).$$

The only subgroups of $Q_{2^{n+1}}$ in the detection family are quaternion groups of order 8. The conjugacy classes of those subgroups are represented by $H = \langle x^{2^{n-2}}, y\rangle$ and $H' = \langle x^{2^{n-2}}, xy\rangle$. The restriction map $\operatorname{Res}\colon T(Q_{2^{n+1}}) \to T(H) \oplus T(H')$ is injective. The class of L restricts to the class of $\pm(\Omega_H(k), \Omega_{H'}(k))$. Moreover, $\Omega^1_{Q_{2^n}}(L)$ restricts to either the class of $(2\Omega_H(k), k)$ or the class of $(k, 2\Omega_{H'}(k))$. The following corresponds to [CT00, Theorem 6.5].

Theorem 1.2.15. *Let $n \geq 4$. Then $T(Q_{2^n}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$ generated by the classes of $\Omega_{Q_{2^n}}(k)$ and $\Omega_{Q_{2^n}}(L)$.*

Let $D_{2^n} = \langle r, s \mid r^{2^{n-1}} = s^2 = (sr)^2 = 1 \rangle$ be the dihedral group of order 2^n, for $n \geq 3$. The group D_{2^n} has two conjugacy classes of elementary abelian groups of rank 2. Consider the representatives $E = \langle rs, r^{2^{n-2}} \rangle$ and $E' = \langle s, r^{2^{n-2}} \rangle$. Then the restriction map $\mathrm{Res}\colon T(D_{2^n}) \to T(E) \oplus T(E')$ is injective. The module $\Omega_{D_{2^n}/\langle y \rangle}(k)$ restricts to the class of $(\Omega_E(k), -\Omega_{E'}(k))$. In this case, the group of endotrivial modules can be described as follows (see [CT00, Theorem 5.4]).

Theorem 1.2.16. *Let $n \geq 3$. Then $T(D_{2^n}) \cong \mathbb{Z}^2$ generated by the classes of $\Omega_{D_{2^n}}(k)$ and $\Omega_{D_{2^n}/\langle y \rangle}(k)$.*

Consider the semidihedral group $SD_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ yxy = x^{2^{n-2}-1} \rangle$, for a fixed $n \geq 4$. The subgroup $E = \langle y, x^{2^{n-2}} \rangle$ is the only elementary abelian subgroup of rank 2 up to conjugacy and $H = \langle yx, x^{2^{n-2}} \rangle$ is the only quaternion subgroup of order 8 up to conjugacy. The restriction map $\mathrm{Res}\colon T(SD_{2^n}) \to T(E) \oplus T(H)$ is injective. The class of the module $L := \Omega_{SD_{2^n}/\langle y \rangle}(k)$ restricts to the class of $(\Omega_E^{-1}(k), \Omega_H^1(k))$. Hence $\Omega^1_{SD_{2^n}}(L)$ restricts to the class of $(k, 2\Omega_H(k))$. In particular, the classes of $\Omega_{SD_{2^n}}(k)$ and $\Omega^1_{SD_{2^n}}(L)$ determine a $\mathbb{Z}$–basis for the group of endotrivial modules $T(SD_{2^n})$ (see [CT00, Theorem 7.1]).

Theorem 1.2.17. *Let $n \geq 4$. Then $T(SD_{2^n}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ generated by the classes of $\Omega_{SD_{2^n}}(k)$ and $\Omega^1_{SD_{2^n}}(L)$.*

The classification of endotrivial modules for finite groups is still an open problem. The general strategy consists in the study of the restriction map $\mathrm{Res}\colon T(G) \to T(S)$, where S is a p-Sylow subgroup of G. The restriction determines a split exact sequence, except in well known cases. The free part $TF(G)$ is determined by the number of connected components of $\mathcal{E}_{\geq 2}(G)/G$. The problem has been reduced to computing the kernel $T(G, S)$ of the restriction map, at least to determine $T(G)$ as an abstract abelian group. In fact, $T(G, S)$ is a finite p'-group[1] containing the group of 1-dimensional kG–modules. If S is a normal subgroup of G, then $T(G)$ can be described as follows (see [Maz19, Theorem 3.10]).

Theorem 1.2.18. *Let G be a finite group with a normal p-Sylow subgroup S. Then there is an exact sequence*

$$0 \to \mathrm{Hom}(G, k^\times) \to T(G) \to T(S)^{G/S} \to 0$$

where $T(S)^{G/S}$ is the subgroup of $T(S)$ generated by the classes of the G–stable endotrivial kS–modules, i.e. kS–modules M such that ${}^gM \cong M$ for all $g \in G$.

[1] A finite group that does not contain p-torsion.

In this case, we can construct a basis for $TF(G)$ from a basis for $TF(S)$ using tensor induction (see [Maz19, Definition 1.3]). Moreover, if $T(S)$ has no torsion, then $TT(G) \cong \mathrm{Hom}(G, k^\times)$. Thus we have to deal with the cases where S is cyclic, semi-dihedral or generalized quaternion separately.

For a general finite group, the restriction map $\mathrm{Res}^G_{N_G(S)}\colon T(G) \to T(N_G(S))$ restricts to an injective group homomorphism $\mathrm{Res}^G_{N_G(S)}\colon T(G,S) \to T(N_G(S),S)$. Since $T(N_G(S),S) = \mathrm{Hom}(N_G(S), k^\times)$ we have

$$\mathrm{Hom}(G, k^\times) \hookrightarrow T(G,S) \hookrightarrow \mathrm{Hom}(N_G(S), k^\times).$$

For instance, if $S \cap {}^gS \neq 1$ for all $g \in G$, then $T(G,S) \cong \mathrm{Hom}(G, k^\times)$.

There are different methods to investigate $T(G,S)$. For instance, character theory (see [Maz19, Section 5.1]), weak homomorphisms [Bal13], and homotopy theory [Gro23]. We will not describe these methods, but we want to highlight Grodal's work since it has been a source of motivation for this work. He identifies the group $T(G,S)$ with the first cohomology group of the p-orbit category with coefficients in the units of the field k.

The p-orbit category of G is the category with objects G/P, for P a non-trivial p-subgroup, and G–maps as morphisms. This category is denoted by $\mathcal{O}_p^*(G)$. The following theorem corresponds to [Gro23, Theorem A].

Theorem 1.2.19. *There exists a group isomorphism*

$$\Psi\colon T(G,S) \to H^1(\mathcal{O}_p^*(G); k^*)$$

with $\Psi(M) = \varphi_M$, where $\varphi_M\colon \mathcal{O}_p^*(G) \to k^*$ maps G/P to the 1-dimensional kG–module $M^P/u_P M$, and $u_P = \sum_{u\in P} u$ is the norm element.

1.3 Stable categories for groups of type Φ

Let $\mathrm{projdim}_{kG} M$ denote the projective dimension of the kG–module M, that is, the shortest possible length of a projective resolution of the module or ∞ if there is no finite resolution.

Definition 1.3.1. Let G be a group. If any kG–module M satisfies that $\mathrm{projdim}_{kG} M$ is finite if and only if $\mathrm{projdim}_{kF} M$ is finite for any finite subgroup F of G, then we say that G is a group of *type* Φ (or Φ_k if we need to emphasize the role of k).

The following result gives us a large collection of groups of type Φ (see [MS19, Proposition 2.5]. In particular, groups of finite virtual cohomological dimension are groups of type Φ, as well as groups that admit a finite-dimensional model for the classifying space for proper actions. Recall that the *finitistic dimension* of the group ring kG is given by

$$\sup\{\mathrm{projdim}_{kG}M \mid \mathrm{projdim}_{kG}M < \infty\}.$$

Proposition 1.3.2. *Let G be a group. If there exists an exact complex of kG–modules*

$$0 \to C_n \to \ldots \to C_0 \to k \to 0$$

such that each C_i is a summand of a sum of modules of the form $k{\uparrow}_H^G$ with H of type Φ and findim$(kH) \leq m$ for a fixed m, then G is of type Φ.

The finitistic dimension of kG is finite if G is a group of type Φ. As a consequence, a free abelian group of infinite rank[2] is not of type Φ.

Let G be a group. The *projective stable module category* $\underline{\mathrm{Mod}}(kG)$ is the category whose objects are kG–modules and morphisms are equivalence classes of homomorphisms under the relation $f \sim g$ if and only if $f - g$ factors through a projective. Note that if G is finite, then this construction is just the stable module category described in Definition 1.1.1.

We have an endofunctor $\Omega\colon \underline{\mathrm{Mod}}(kG) \to \underline{\mathrm{Mod}}(kG)$ given by taking the kernel of a surjection $P \to M$, where P is projective. In particular, for any pair M, N of kG–modules there is a natural map

$$\Omega\colon \underline{\mathrm{Hom}}_{kG}(M, N) \to \underline{\mathrm{Hom}}_{kG}(\Omega M, \Omega N).$$

In general, the functor Ω does not induce a self-equivalence of categories; for infinite groups, projectives rarely agree with injectives.

Definition 1.3.3. Let M, N be kG–modules. The *complete cohomology* is defined via

$$\widehat{\mathrm{Ext}}^r_{kG}(M, N) := \varinjlim_n \mathrm{Hom}_{\underline{\mathrm{Mod}}(kG)}(\Omega^{n+r}M, \Omega^n N).$$

In particular, a kG–module M has finite projective dimension if and only if $\widehat{\mathrm{Ext}}^0_{kG}(M, M) = 0$ (see [Ben97, Lemma 3.1]).

Definition 1.3.4. Let $\underline{\mathrm{StMod}}(kG)$ be the category whose objects are all the kG–modules and the morphisms between two objects M, N are given by complete cohomology

$$\mathrm{Hom}_{\underline{\mathrm{StMod}}(kG)}(M, N) = \widehat{\mathrm{Ext}}^0_{kG}(M, N).$$

[2]The rank of an abelian group is just the rank as a $\mathbb{Z}$–module.

Remark 1.3.5. For groups of type Φ, let $\widehat{H}^i(G;N)$ denote $\widehat{\operatorname{Ext}}^i_{kG}(k,N)$ for all $i \in \mathbb{Z}$. This coincides with generalized Tate-Farrell cohomology defined by Ikenaga [Ike84] when the group has finite generalized cohomological dimension. In particular, for finite groups it coincides with Tate cohomology.

A totally acyclic complex of projectives is an unbounded exact complex of projectives P_* such that $\operatorname{Hom}_{kG}(P_*, Q)$ is acyclic for any projective module Q. Let $\mathrm{K}_{tac}(kG)$ denote the category of totally acyclic complexes of projective kG–modules and chain maps up to homotopy. Let $\mathrm{GP}(kG)$ (resp. $\underline{\mathrm{GP}}(kG)$) denote the full subcategory of $\mathrm{Mod}(kG)$ (resp. $\underline{\mathrm{Mod}}(kG)$) on the *Gorenstein projective modules*, i.e. the modules that are isomorphic to a kernel in a totally acyclic complex of projectives.

For groups of type Φ, any complex with finitely many non-zero homology groups has a complete resolution. Then we have a functor

$$\mathrm{CompRes}\colon D^b(\mathrm{Mod}(kG)) \to K_{tac}(kG)$$

where $D^b(\mathrm{Mod}(kG))$ is the derived category of complexes of kG–modules with only finitely many nonzero homology groups. The kernel of this functor contains the homotopy category of bounded complexes of projective kG–modules which is denoted by $K^b(\mathrm{Proj}(kG))$. Recall that the kernel of a triangulated functor $F\colon \mathcal{C} \to \mathcal{D}$ is the full subcategory of $\mathcal{C}$ whose objects map to zero objects in $\mathcal{D}$. In particular, the kernel of a triangulated functor is a thick subcategory of $\mathcal{C}$, i.e. a triangulated subcategory that is closed under retracts (see [Nee01, Lemma 2.1.5]). Let $D^b(\mathrm{Mod}(kG))/K^b(\mathrm{Proj}(kG))$ denote the Verdier quotient.

The categories $\underline{\mathrm{GP}}(kG)$, $D^b(\mathrm{Mod}(kG))/K^b(\mathrm{Proj}(kG))$ and $K_{\mathrm{tac}}(kG)$ are equivalent as triangulated categories (this holds for more general rings, see [Bel00], [Buc87]). For groups of type Φ, Mazza and Symonds prove that these equivalences factor through $\underline{\mathrm{StMod}}(kG)$ (see [MS19, Theorem 3.9]), and use these equivalences to define a triangulated structure on $\underline{\mathrm{StMod}}(kG)$.

Theorem 1.3.6. *For groups of type Φ, the following categories are equivalent.*

- $\underline{\mathrm{StMod}}(kG)$.
- $D^b(\mathrm{Mod}(kG))/K^b(\mathrm{Proj}(kG))$.
- $K_{tac}(kG)$.
- $\underline{GP}(kG)$.

They are equivalent as triangulated categories, except for $\underline{\mathrm{StMod}}(kG)$.

These equivalences define a triangulated structure on $\underline{\mathrm{StMod}}(kG)$. The distinguished triangles are all the triangles isomorphic to a short exact sequence of modules and the shift Ω^{-1} of M is obtained by taking the kernel in degree -1 of a complete resolution of the module M. In the following theorem, we summarize some of the results of Section 4 and Section 5 in [MS19].

Theorem 1.3.7. *Let G be a group of type Φ. The following properties hold.*

(1) $-\otimes_k M$ *and* $\mathrm{Hom}_k(M,-)$ *induce triangulated functors from* $\underline{\mathrm{StMod}}(kG)$ *to itself, for any module M.*

(2) *The category* $\underline{\mathrm{StMod}}(kG)$ *has products and coproducts.*

(3) *The ring of stable endomorphisms* $\widehat{\mathrm{End}}_{kG}(k)$ *and the group of stable automorphisms* $\widehat{\mathrm{Aut}}_{kG}(k)$ *are commutative.*

(4) *Let $f\colon M \to N$ be a morphism in* $\underline{\mathrm{StMod}}(kG)$. *Then f is a stable isomorphism if and only if $f{\downarrow_F}$ is a stable isomorphism for any finite subgroup F of G.*

1.4 The Picard group of $\underline{\mathrm{StMod}}(kG)$

Recall that the Picard Group $\mathrm{PicGp}(\mathcal{C})$ of a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ is the group of isomorphism classes of objects which are invertible with respect to the tensor product $\otimes$.

We will consider $\underline{\mathrm{StMod}}(kG)$ with the symmetric monoidal structure inherited from the symmetric monoidal structure on $\mathrm{Mod}(kG)$ given by the tensor product over k. In fact, this structure turns $\underline{\mathrm{StMod}}(kG)$ into a tensor triangulated category in the sense of Balmer. In particular, we will write $T(G)$ to denote the Picard group of $\underline{\mathrm{StMod}}(kG)$. Note that if G is finite, this group corresponds to the so-called group of endotrivial modules (see Proposition 1.2.6). In this context, we say that a kG–module is *invertible, or endotrivial,* if $[M]$ belongs to $T(G)$.

Proposition 1.4.1. *Let G be a group of type Φ. Let M be a kG–module. Then M is invertible if and only if its restriction to any finite subgroup is invertible.*

Proof. Let $f\colon M \to N$ be a homomorphism of kG–modules. Consider the distinguished triangle

$$M \xrightarrow{f} N \to C(f) \to \Omega^{-1}M.$$

Then f is an isomorphism in $\underline{\mathrm{StMod}}(kG)$ if and only if its cone $C(f)$ is trivial, that is, if it has finite projective dimension. Since G is of type Φ, finite projective dimension is detected by the family of finite subgroups, and we have that f is an isomorphism in

$\underline{\mathrm{StMod}}(kG)$ if and only if $f{\downarrow_H}$ induces an isomorphism in $\underline{\mathrm{StMod}}(kH)$ for any finite subgroup H of G. In fact, this proves (4) in Theorem 1.3.7.

Consider the evaluation map $\mathrm{ev}\colon M \otimes M^* \to k$. If $M{\downarrow_H}$ is invertible, it is well known that $\mathrm{ev}{\downarrow_H}$ induces an isomorphism in $\underline{\mathrm{StMod}}(kH)$ (for instance, see [MS19, Theorem 6.4]), and then it follows that M is invertible. □

Chapter 2

Stable module ∞-categories

In this chapter we show that the category of kG–modules admits a combinatorial symmetric monoidal model structure, for any group G of type Φ. We will use this model structure on $\mathrm{Mod}(kG)$ to define the stable module ∞-category $\mathrm{StMod}(kG)$.

For a given group G, we let $\mathrm{Mod}(kG)$ denote the category of all kG–modules, and let $\otimes$ denote $\otimes_k$ equipped with the diagonal action of G, unless it is specified otherwise.

2.1 Model categories

Definition 2.1.1. Let B be the set of functions from G to k which take only finitely many different values in k, with the kG–structure given by pointwise scalar multiplication and addition, and G–action given by $g(\varphi)(h) = \varphi(g^{-1}h)$. A kG–module M is said to be cofibrant if $B \otimes_k M$ is a projective module.

Example 2.1.2. Let G be a group of type Φ. Then the following statements hold.

(*i*) Any projective kG–module is cofibrant.

(*ii*) If G is finite, then B is isomorphic to kG. In particular, B is free. Therefore any kG–module is cofibrant.

(*iii*) If M is a cofibrant kG–module, then ΩM is cofibrant.

Definition 2.1.3. Note that B is a commutative k–algebra with pointwise multiplication. Let $\mu\colon B \otimes B \to B$ denote the multiplication map, and let $\iota\colon k \to B$ denote the inclusion of the constant functions. For a kG–module M, let $\Omega^{-1}M$ denote $\mathrm{Coker}(\iota) \otimes M$, that is, the cokernel of the injective map

$$M \xrightarrow{\cong} k \otimes M \xrightarrow{\iota \otimes 1} B \otimes M.$$

Remark 2.1.4. Note that if M is a cofibrant kG–module, then $\Omega^{-1}M$ is cofibrant as well.

Lemma 2.1.5. *Let G be a group of type Φ. Let M, N be cofibrant kG–modules, then the natural map*

$$\underline{\mathrm{Hom}}_{kG}(M,N) \to \widehat{\mathrm{Ext}}^0_{kG}(M,N)$$

is an isomorphism.

Proof. Note that for any cofibrant module M, we have a short exact sequence

$$0 \to \Omega^{-1}\Omega M \to \Omega^{-1}P \to \Omega^{-1}M \to 0$$

obtained from the short exact sequence $0 \to \Omega M \to P \to M \to 0$ by tensoring with $\mathrm{Coker}(\iota)$, where P is projective. Note that $\Omega^{-1}P$ is projective. Moreover, consider a projective module Q mapping onto $\Omega^{-1}M$. Then there is a short exact sequence

$$0 \to \Omega\Omega^{-1}M \to Q \to \Omega^{-1}M \to 0$$

By Schanuel's Lemma we obtain isomorphisms

$$M \simeq \Omega\Omega^{-1}M \simeq \Omega^{-1}\Omega M$$

up to projectives. Hence the result follows. □

Lemma 2.1.6. *Let M be a cofibrant kG–module. If M has finite projective dimension, then M is projective.*

Proof. Since M is cofibrant, by Schanuel's lemma we have

$$M \oplus P \cong \Omega\Omega^{-1}M \oplus Q$$

for some projective modules P, Q. Note that the exact sequence obtained by tensoring with B

$$0 \to B \otimes M \to B \otimes B \otimes M \to B \otimes \Omega^{-1}M \to 0$$

has a splitting given by $\mu \otimes 1_M$, hence $\Omega^{-1}M$ is cofibrant. We define inductively $\Omega^{-n}M$ by $\Omega^{-1}\Omega^{-n+1}M$, for $n \geq 2$. In particular, $\Omega^{-n}M$ is cofibrant for all $n \geq 1$.

Suppose that M has finite projective dimension at most $r > 0$. Then $\Omega^r M$ is projective. On the other hand, note that $\Omega^{-r}M$ has projective dimension at most r. Consider a projective resolution of $\Omega^{-r}M$

$$0 \to \Omega^r(\Omega^{-r}M) \to P_{r-1} \to \ldots \to P_0 \to \Omega^{-r}M$$

By definition of $\Omega^{-r}M$, we can construct an exact sequence

$$0 \to M \to Q_{r-1} \to \ldots \to Q_0 \to \Omega^{-r}M$$

with Q_i projective, for $i \in \{0, \ldots, r-1\}$. By the extended version of Schanuel's Lemma we have that $\Omega^r\Omega^{-r}M \simeq M$ up to projectives, thus M is projective as well. □

The following definition corresponds to [Ben98, Definition 10.1].

Definition 2.1.7. Let G be a group of type Φ, and $f\colon M \to N$ a homomorphism of kG–modules. We say that f is:

(i) a fibration if it is surjective,

(ii) a cofibration if it is injective with cofibrant cokernel,

(iii) a weak equivalence if it is a stable isomorphism, that is, if it is an isomorphism in $\underline{\mathrm{StMod}}(kG)$.

Recall that if G is finite, then any kG–module is cofibrant (see Example 2.1.2). Moreover, by Lemma 2.1.5 we have that $\widehat{\mathrm{Ext}}^0_{kG}(M, N)$ is just $\underline{\mathrm{Hom}}_{kG}(M, N)$. Hence Definition 2.1.7 coincides with the model structure on $\mathrm{Mod}(kG)$ given in [Hov99, Section 2.2].

Lemma 2.1.8. *The following properties hold for a map f of kG–modules.*

(i) f is a trivial cofibration if and only if f is injective with projective cokernel.

(ii) f is a trivial fibration if and only if f is surjective and the kernel has finite projective dimension.

Proof. (i) Assume that f is injective. Consider the exact sequence

$$0 \to M \xrightarrow{f} N \to L \to 0.$$

It defines a distinguished triangle in the stable module category. If f is a trivial cofibration, then the cofibrant module L is trivial in the stable module category, hence it has finite projective dimension. Thus L is projective by Lemma 2.1.6. On the other hand, if f has projective cokernel, then it is clear that f induces a stable isomorphism. Moreover, any projective is cofibrant, hence f is a trivial cofibration. (ii) It follows in a similar fashion. □

Convention 2.1.9. For the rest of this section, we will use the same notation as in [Hov99, Section 2.1]. Moreover, we say that a commutative square

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{g} & N' \end{array}$$

has a *filling* if there is a map $h\colon N \to M'$ such that $h \circ i = f$ and $p \circ h = g$.

Definition 2.1.10. Let G be a group of type Φ. Let $\mathcal{J}$ be the set consisting of the inclusion $0 \to kG$, and let $\mathcal{I}$ be the set containing all the induced maps $I{\uparrow}_F^G \to kG$, where $I \to kF$ is the inclusion of a left ideal I of kF, and F is a finite subgroup of G.

Proposition 2.1.11. *The class $\mathcal{J}$-inj is given by the class of fibrations.*

Proof. Let $f\colon M \to N$ be a map in $\mathrm{Mod}(kG)$. Suppose that f has the right lifting property with respect to the inclusion $0 \to kG$. For each $n \in N$, there exists an homomorphism $\alpha_n\colon kG \to N$ whose image contains n. Thus we have a commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle f} \\ kG & \xrightarrow{\alpha_n} & N \end{array}$$

and let $\gamma\colon kG \to M$ be a filling. It follows that the image of $f \circ \gamma$ contains n, hence f is surjective. Conversely, if f is surjective, then the result follows since kG is projective. □

Proposition 2.1.12. *The class $\mathcal{I}$-inj agrees with the class of trivial fibrations.*

Proof. Let $f\colon M \to N$ be a map in $\mathrm{Mod}(kG)$. Suppose that f is in $\mathcal{I}$-inj. Then f must be a surjection since $0 \to kG$ is in $\mathcal{I}$. Since G is of type Φ, it is enough to prove that $\mathrm{Ker}(f)$ is projective on the restriction to any finite subgroup F of G. Hence it is enough to prove that $f{\downarrow}_F$ is a trivial fibration on $\mathrm{Mod}(kF)$, which is equivalent to proving that $f{\downarrow}_F$ has the left lifting property respect to all the inclusions $I \to kF$ where I is a left ideal of kF. Suppose that we have a commutative diagram

$$\begin{array}{ccc} I & \longrightarrow & M{\downarrow}_F \\ \downarrow & & \downarrow{\scriptstyle f{\downarrow}_F} \\ kF & \longrightarrow & N{\downarrow}_F \end{array}$$

By the restriction-induction adjunction we have a commutative diagram

$$\begin{array}{ccc} I{\uparrow}_F^G & \longrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle f} \\ kG & \longrightarrow & N \end{array}$$

Since $I{\uparrow}_F^G \to kG$ is in $\mathcal{I}$, there is a filling for this diagram. Hence by the adjunction there is a filling for the former diagram. The converse follows in a similar fashion. □

Remark 2.1.13. Given a pullback square

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{g} & N' \end{array}$$

we claim that i is surjective if p is surjective. Recall that the pullback has the form $\{(n, m) \in N \oplus M' \mid g(n) = p(m)\}$. Let $n \in N$. Since p is surjective, there is $m \in M'$ such that $g(n) = p(m)$. It follows that $(n, m) \in L$ and $i(n, m) = n$, hence i is surjective.

Let $f\colon M \to N$ be a homomorphism of kG–modules. Recall that f is *left split* if there is a homomorphism $g\colon N \to M$ such that $g \circ f = 1_M$, and f is *right split* if there is a homomorphism $h\colon N \to M$ such that $f \circ h = 1_N$. If the context is clear, we just say that f is split, and we will refer to g (resp. h) as a *splitting* for f.

Lemma 2.1.14. *Any short exact sequence of kG–modules*

$$0 \to K \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

is split if K has finite projective dimension and N is cofibrant.

Proof. We will prove this lemma by induction on the projective dimension of K. Suppose that K is projective. Consider the following commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\alpha} & M \\ {\scriptstyle i}\downarrow & & \downarrow \\ B \otimes K & \xrightarrow{1\otimes\alpha} & B \otimes M \end{array}$$

Note that $\Omega^{-1}K$ is projective. Hence i splits. On the other hand, the cokernel of the injective map $1 \otimes \alpha$ corresponds to $B \otimes N$ which is projective by Lemma 2.1.6 since N is cofibrant, thus $1 \otimes \alpha$ splits. It follows that α splits as well. For the general case, let P_K and P_N be projectives mapping onto K and N, respectively. The projective module $P_M = P_K \oplus P_N$ maps onto M, thus we have an induced exact sequence

$$0 \to \Omega K \xrightarrow{\Omega\alpha} \Omega M \xrightarrow{\Omega\beta} \Omega N \to 0.$$

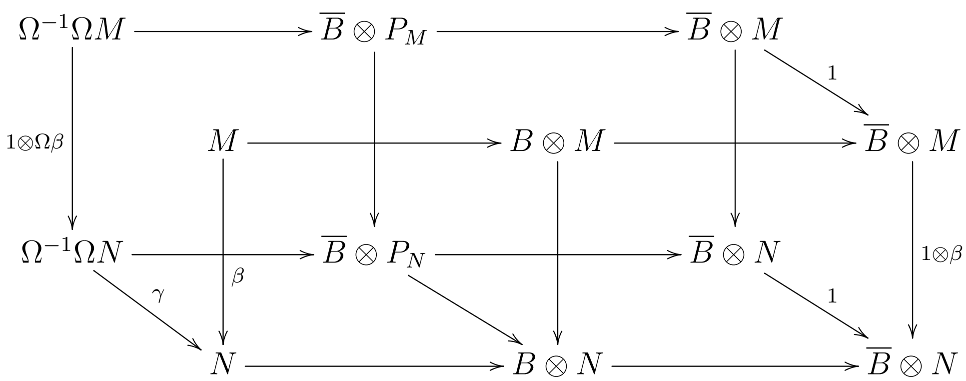
It splits by the inductive hypothesis. Recall that $\Omega^{-1}X$ denotes the module $\mathrm{Coker}(\iota) \otimes X$ for a cofibrant module X, where ι denotes the inclusion of the constant functions from k to B. In order to simplify notation, let $\overline{B}$ denote $\mathrm{Coker}(\iota)$. Since $\overline{B} \otimes P_N$ is projective, we can construct a commutative diagram

$$\begin{array}{ccccc} \Omega^{-1}\Omega N & \longrightarrow & \overline{B} \otimes P_N & \longrightarrow & \overline{B} \otimes N \\ {\scriptstyle \gamma}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\ N & \longrightarrow & B \otimes N & \longrightarrow & \overline{B} \otimes N \end{array}$$

where the rows are short exact sequences. Since $\Omega^{-1}\Omega N$ is isomorphic to N up to projectives, we can assume that P_N is large enough so the map $\gamma\colon \Omega^{-1}\Omega N \to N$ is split surjective. We claim that there is a map $\Omega^{-1}\Omega M \to M$ making the following diagram commutative.

$$\begin{array}{ccc} \Omega^{-1}\Omega M & \longrightarrow & M \\ {\scriptstyle 1\otimes\Omega\beta}\downarrow & & \downarrow{\scriptstyle \beta} \\ \Omega^{-1}\Omega N & \xrightarrow{\gamma} & N \end{array}$$

Note that if this square is commutative, then we obtain a splitting for β, which completes the result. We will construct such a map. Consider the following commutative diagram.



where all the rows are short exact sequences, and P_M is a projective mapping onto M. Construct the pullback square

$$\begin{array}{ccc} L & \longrightarrow & \overline{B}\otimes M \\ \downarrow & & \downarrow{\scriptstyle 1\otimes\beta} \\ B\otimes N & \longrightarrow & \overline{B}\otimes N \end{array}$$

Since the map $B\otimes N \to \overline{B}\otimes N$ is surjective, by Remark 2.1.13 the map $L \to \overline{B}\otimes M$ is surjective as well. Moreover, the kernel of $L \to \overline{B}\otimes M$ agrees with N. Then we can construct a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & B\otimes M & \longrightarrow & \overline{B}\otimes M & \longrightarrow & 0 \\ & & {\scriptstyle\beta}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & \overline{B}\otimes M & \longrightarrow & 0 \end{array}$$

where the rows are short exact sequences. We deduce that the map $B\otimes M \to L$ is surjective. By the universal property of the pullback, we have a map $\overline{B}\otimes P_M \to L$ induced by the maps $\overline{B}\otimes P_M \to \overline{B}\otimes P_N \to B\otimes N$ and $\overline{B}\otimes P_M \to \overline{B}\otimes M$. Hence there is a map $\overline{B}\otimes P_M \to B\otimes M$ making the following triangle commutative.

$$\begin{array}{ccc} & & \overline{B}\otimes P_M \\ & \swarrow\!\!{\scriptstyle --} & \downarrow \\ B\otimes M & \longrightarrow & L \end{array}$$

Moreover, this map makes the right-hand side cube commutative, hence the induced map $\Omega^{-1}\Omega M \to M$ makes the desired square commutative. $\square$

Lemma 2.1.15. *Let N be a cofibrant kG–module. Then any commutative square*

$$\begin{array}{ccc} 0 & \longrightarrow & M' \\ \downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{g} & N' \end{array}$$

has a filling provided that p is a trivial fibration.

Proof. Consider the pullback square of g and p

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{g} & N' \end{array}$$

Since p is surjective, we have that β is surjective by Remark 2.1.13. We obtain a short exact sequence

$$0 \to K \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

and note that K has finite projective dimension since it is isomorphic to the kernel of p. Then this short exact sequence is split by Lemma 2.1.14. This completes the result since a splitting of β followed by f is a filling for the original commutative diagram. $\square$

Remark 2.1.16. Consider the following pushout square

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{g} & N' \end{array}$$

Then p is injective if i is injective. Recall that N' has an explicit description as a quotient of $N \oplus M'$ by the submodule $\{(i(m), -f(m)) \mid m \in M\}$. Let $m' \in M'$. Suppose that $p(m') = 0$. Then there is $m \in M$ such that $(i(m), -f(m)) = (0, m')$. Since i is injective, it follows that $m' = 0$, and hence p is injective as well.

Proposition 2.1.17. *The class of cofibrations agrees with the class $\mathcal{I}$-cof.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{g} & N' \end{array}$$

and suppose that i is a cofibration and p is a trivial fibration. Let X denote the pushout of f and i, and let $\pi\colon N \to L$ denote the cokernel of i. Since i is injective, the morphism $M' \to X$ is injective by Remark 2.1.16, and its cokernel is isomorphic to L. Moreover, note that $X \to N'$ is surjective, since p is surjective. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K' & \longrightarrow & M' & \xrightarrow{p} & N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & K & \longrightarrow & X & \xrightarrow{p'} & N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & \xrightarrow{1} & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where all the columns and rows are exact, and K and K' denote the kernel of p and p' respectively. Note that the cokernel of $K' \to K$ agrees with the cokernel of $M' \to X$ since the upper left square is a pushout. In particular, the short exact sequence $0 \to K' \to K \to L \to 0$ splits by Lemma 2.1.14, since L is cofibrant and K' has finite projective dimension. Then the short exact sequence

$$0 \to M' \to X \to L \to 0$$

splits as well. We obtain a map $\gamma \colon N \to M'$ by composing a splitting of the map $M' \to X$ and the map $N \to X$. Note that this map satisfies $f = \gamma \circ i$. Therefore

$$(g - p \circ \gamma) \circ i = 0$$

hence there is a map $\eta \colon L \to N'$ such that $\eta \circ \pi = g - p \circ \gamma$. By Lemma 2.1.15 we have a solution for the lifting problem

$$\begin{array}{ccc}
0 & \longrightarrow & M' \\
\downarrow & \overset{\xi}{\nearrow} & \downarrow{\scriptstyle p} \\
L & \xrightarrow[\eta]{} & N'
\end{array}$$

Define $h \colon N \to M'$ as $\gamma + \xi \circ \pi$. Note that

$$h \circ i = \gamma \circ i + \xi \circ \pi \circ i = f$$

$$p \circ h = p \circ \gamma + p \circ \xi \circ \pi = (g - \eta \circ \pi) + \eta \circ \pi = g$$

hence h is a solution for the original lifting problem.

Conversely, suppose that $i \colon M \to N$ has the left lifting property with respect to all trivial fibrations. Recall that the natural map $M \to \mathrm{CoInd}(M{\downarrow_H})$ given by the unit of the restriction-coinduction adjunction is injective, where H denotes the trivial group. Then we have a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & \mathrm{CoInd}(M{\downarrow_H}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p'} \\ N & \longrightarrow & 0 \end{array}$$

Note that p' is a trivial fibration since $\mathrm{CoInd}(M{\downarrow_H})$ has finite projective dimension (see [MS19, Lemma 3.13]). Then the diagram has a filling and we obtain that i is injective. It remains to show that the cokernel of i is cofibrant.

We claim that a kG–module Y such that any extension by a module of finite projective dimension is split must be cofibrant. Construct ΩY large enough so that $\Omega^{-1}\Omega Y \to Y$ is a surjective map. Since G is of type Φ, we have that $B \otimes Y$ has finite projective dimension. Hence the kernel of $\Omega^{-1}\Omega Y \to Y$ has finite projective dimension, therefore the map splits. If Y is not cofibrant, then $B \otimes \Omega^{-1}\Omega Y$ has smaller projective dimension than $B \otimes Y$, since the former has the same projective dimension as $B \otimes \Omega Y$. This is a contradiction since $B \otimes Y$ is isomorphic to a direct summand of $B \otimes \Omega^{-1}\Omega Y$. Therefore Y is cofibrant.

Let $\pi\colon N \to L$ denote the cokernel of i. Consider a short exact sequence

$$0 \to K \to X \xrightarrow{p} L \to 0$$

where K has finite projective dimension. In particular, p is a trivial fibration. We will show that this sequence splits. Consider the following commutative diagram

$$\begin{array}{ccc} & & K \\ & & \downarrow \\ M & \xrightarrow{0} & X \\ {\scriptstyle i}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle p} \\ N & \xrightarrow[\pi]{} & L \end{array}$$

Then by the assumption on i we have a filling $\alpha\colon N \to X$ for the square. Since $\alpha \circ i = 0$, there exists a map $\gamma\colon L \to X$ such that $\alpha = \gamma \circ \pi$. Note that γ is a splitting for p. Then the result follows. □

Proposition 2.1.18. *The class $\mathcal{J}$-cof agrees with the class of trivial cofibrations.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{g} & N' \end{array}$$

and suppose that i is a trivial cofibration and p is a fibration. Let $\pi\colon N \to L$ denote the cokernel of i. Since L is projective, we have maps $\alpha\colon N \to M$ and $\beta\colon L \to N$

such that $\alpha \circ i = 1_M$ and $\pi \circ \beta = 1_L$ and $i \circ \alpha + \beta \circ \pi = 1_N$. Moreover, we have a map $\delta\colon L \to M'$ such that $p \circ \delta = g \circ \beta$. The map

$$f \circ \alpha + \delta \circ \pi \colon N \to M'$$

is a filling for the diagram.

On the other hand, suppose that $i\colon M \to N$ has the left lifting property with respect to all fibrations. Consider the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{(1_M,0)} & M \oplus P_N \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{1} & N \end{array}$$

where P_N is a projective mapping onto N. Then we have a map $\beta\colon N \to M \oplus P_N$ such that $\beta \circ i = (1_M, 0)$ and $p \circ \beta = 1_N$. Hence i is a retract of $(1_M, 0)$, and the latter is a trivial cofibration. It is straightforward to verify that i is then a trivial cofibration. □

Theorem 2.1.19. *Let G be a group of type Φ. Then* $\mathrm{Mod}(kG)$ *is a combinatorial model category with respect to the collections of cofibrations, fibrations and weak equivalences described in Definition 2.1.7. Moreover, the functor* $\mathrm{Mod}(kG) \to \underline{\mathrm{StMod}}(kG)$ *induces an equivalence* $\mathrm{Ho}\,\mathrm{Mod}(kG) \to \underline{\mathrm{StMod}}(kG)$.

Proof. Note that every kG–module is small (see [Hov99, Example 2.1.6]). Then the domains of $\mathcal{I}$ (resp. $\mathcal{J}$) are small relative to $\mathcal{I}$-cell (resp. $\mathcal{J}$-cell). Let $\mathcal{W}$ denote the class of weak equivalences. We have proved that $\mathcal{J}$-cell $\subseteq \mathcal{W} \cap \mathcal{J}$-cof, and $\mathcal{I}$-inj $\subseteq \mathcal{W} \cap \mathcal{J}$-inf, and $\mathcal{W} \cap \mathcal{I}$-cof $\subseteq \mathcal{J}$-cof. Hence the result follows from [Hov99, Theorem 2.1.19]. □

For groups of type Φ, the class of cofibrant modules coincides with the class of Gorenstein projective modules (see [DT10, Conjecture A] and [BDT09]), and by the definition of fibration, any module is fibrant. Thus the full subcategory of $\underline{\mathrm{StMod}}(kG)$ on bifibrant[1] modules agrees with $\underline{\mathrm{GP}}(kG)$. Then by general theory of model categories we have the following result (for instance, see [Hov99, Theorem 1.2.10]).

Corollary 2.1.20. *The map* $\underline{\mathrm{Hom}}(M, N) \to \mathrm{Hom}_{\underline{\mathrm{StMod}}(kG)}(M, N)$ *is surjective if M is Gorenstein projective. Moreover, the inclusion of* $\underline{\mathrm{GP}}(kG)$ *into* $\underline{\mathrm{StMod}}(kG)$ *is an equivalence.*

[1] An object that is both fibrant and cofibrant.

2.2 Symmetric monoidal stable ∞-categories

Proposition 2.2.1. *Consider* $\mathrm{Mod}(kG)$ *endowed with the symmetric monoidal structure given by the tensor product over* k*. Then this monoidal structure and the model structure from Definition 2.1.7 make* $\mathrm{Mod}(kG)$ *a symmetric monoidal model category.*

Proof. Recall that for any kG–module M, the functor $-\otimes M$ is exact, and let $\alpha\colon Qk \xrightarrow{\sim} k$ be the cofibrant replacement of the unit k. Then we have that $Qk\otimes M \xrightarrow{\sim} k\otimes M$ is a weak equivalence.

On the other hand, since cofibrations are injective maps with Gorenstein projective cokernel, and tensoring with a Gorenstein projective is Gorenstein projective, we deduce that if f and g are cofibrations then $f\otimes g$ is a cofibration. Moreover, if f is one of the generating cofibrations and g is one of the generating trivial cofibrations, then $f\otimes g$ is a trivial cofibration because it is a trivial cofibration on the restriction to any finite subgroup of G. Then the result follows by [Hov99, Corollary 4.2.5]. □

Recall that any model category has an associated underlying ∞-category (see [Lur17, Def. 1.3.4.15]).

Definition 2.2.2. Let G be a group of type Φ. We define the stable module ∞-category $\mathrm{StMod}(kG)$ as the underlying ∞-category of the model structure on $\mathrm{Mod}(kG)$ from Definition 2.1.7 (c.f. [Mat15, Def. 2.2]).

Proposition 2.2.3. *The stable module* ∞*-category* $\mathrm{StMod}(kG)$ *inherits the structure of a stable homotopy theory in the language of Mathew, that is, it is a presentable, symmetric monoidal stable* ∞*-category, where the tensor product commutes with colimits in each variable.*

Proof. By Theorem 2.1.19 and Proposition 2.2.1, we have that $\mathrm{Mod}(kG)$ is a combinatorial symmetric monoidal model category. Hence $\mathrm{StMod}(kG)$ is presentable and symmetric monoidal by [Lur17, Proposition 1.3.4.22] and [Lur17, Corollary 4.1.7.16]. Moreover, the tensor product on $\mathrm{StMod}(kG)$ commutes with colimits separately in each variable by [Lur17, Lemma 4.1.8.8]. □

Remark 2.2.4. We let $\underline{\mathrm{Hom}}_G(M,N)$ denote the mapping space between objects M,N in $\mathrm{StMod}(kG)$. The homotopy category of $\mathrm{StMod}(kG)$ corresponds to the stable module category $\underline{\mathrm{StMod}}(kG)$ (see Definition 1.3.4), that is, the category whose objects are kG–modules and hom-sets are given by

$$\pi_0\underline{\mathrm{Hom}}_G(M,N)\cong \widehat{\mathrm{Ext}}^0_{kG}(M,N).$$

To finish this chapter, we include comments on the compatibility of these structures with maps induced from inclusions and conjugations, and some of their properties. Let H be a subgroup of G. The inclusion $i\colon H \to G$ induces a symmetric monoidal functor

$$\mathrm{Res}^G_H = i^*\colon\ \mathrm{StMod}(kG) \to \mathrm{StMod}(kH)$$

with left adjoint $i_!$ known as induction, and right adjoint i_* known as coinduction. In particular, we have that Res^G_H preserves all (homotopy) limits and colimits. On the other hand, given an element $g \in G$, we can restrict along the right conjugation map $c_g\colon {}^gH \to H$, so we obtain a functor

$$(c_g)^*\colon\ \mathrm{StMod}(kH) \to \mathrm{StMod}(k{}^gH).$$

Chapter 3

Stable decompositions

In this chapter we exhibit a decomposition of the stable module ∞-category for certain groups of type Φ. For groups of type Φ which act on a tree, a decomposition is given in terms of its associated graph of groups. On the other hand, for groups which admit a finite-dimensional model for the classifying space for proper actions with compact orbit space, we exhibit a decomposition of the stable module ∞-category in terms of its finite subgroups. The latter decomposition is motivated by Mathew's result in [Mat16], where he shows that the stable module ∞-category of a finite group decomposes in terms of the orbit category with isotropy groups in the family of finite p-subgroups.

3.1 Groups acting on trees

In this section, we let G be a group of type Φ acting on a tree T. In particular, G corresponds to the fundamental group of a graph of groups $(G(-), \Gamma)$ (see [DD89, Definition 3.1]). We shall show that the stable module ∞-category of G admits a decomposition in terms of the graph Γ.

Remark 3.1.1. Let Γ be a directed graph. We will consider Γ as a category, still denoted by Γ, in the following fashion.

- The objects are the vertices and edges of the graph Γ.
- The morphisms are given by the incidence maps and the identities. That is, for an edge e we have morphisms $\iota(e) \to e$ and $\tau(e) \to e$.

The category Γ^{op} associated to a directed graph Γ is sometimes referred to as the exit path category of the directed graph.

For instance, consider an amalgamated product $G = A *_C B$ of finite groups. The associated graph of groups $(G(-), \Gamma)$ can be depicted by

$$\begin{array}{ccc} & & B \\ & & \uparrow{\scriptstyle j} \\ A & \xleftarrow{i} & C \end{array}$$

The associated graph Γ (as a category) corresponds to the barycentric subdivision of a segment. We have a corresponding diagram of shape Γ^{op} in $\mathrm{Cat}_\infty^\otimes$ depicted as follows.

$$\begin{array}{ccc} & & \mathrm{StMod}(kA) \\ & & \downarrow{\scriptstyle i^*} \\ \mathrm{StMod}(kB) & \xrightarrow{j^*} & \mathrm{StMod}(kC) \end{array}$$

In this case, we will show that $\mathrm{StMod}(kG)$ is the pullback in $\mathrm{Cat}_\infty^\otimes$ of the diagram above. Let $\mathcal{D}$ be the homotopy pullback of the diagram $\Gamma^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$, and let

$$F\colon \mathrm{StMod}(kG) \to \mathcal{D}$$

denote the comparison functor. In order to show that F is essentially surjective, consider a kA–module M and a kB–module N such that $M \simeq_\varphi N$ in $\underline{\mathrm{StMod}}(kC)$. Following [Sym18, Section 5], we can add projectives to M and N so that φ can be a genuine isomorphism of kC–modules. Let $\overline{C}(\varphi) = M$ as a k-vector space. Fix $m \in \overline{C}(\varphi)$. For $a \in A$ define $a\cdot m = am$ and for $b \in B$ define $b\cdot m = \varphi^{-1}(b\varphi(m))$. This action makes $\overline{C}(\varphi)$ a kG–module. Moreover, note that $\overline{C}(\varphi)|_A = M$ and $\overline{C}(\varphi)|_B \simeq N$. Therefore, F is essentially surjective. On the other hand, consider $M, N \in \mathrm{StMod}(kG)$. It is enough to show that

$$\pi_*\underline{\mathrm{Hom}}_G(M,N) \to \pi_* \mathrm{Hom}_{\mathcal{D}}(F(M), F(N))$$

is an isomorphism. Since $\mathcal{D}$ is the pullback of the diagram $\Gamma^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$, we have that the space $\mathrm{Hom}_{\mathcal{D}}(F(M), F(N))$ corresponds to the homotopy pullback of a diagram of shape Γ^{op}. In particular, we have a long exact sequence

$$\begin{aligned} \ldots \to \pi_n \mathrm{Hom}_{\mathcal{D}}(F(M), F(N)) \to \pi_n\underline{\mathrm{Hom}}_A(M,N) \times \pi_n\underline{\mathrm{Hom}}_B(M,N) \to \\ \to \pi_n\underline{\mathrm{Hom}}_C(M,N) \to \ldots \end{aligned}$$

Moreover, recall that $\pi_*\underline{\mathrm{Hom}}_G(M,N)$ is given by complete cohomology. By [Bro82, Section VII.9] we have a similar long exact sequence to compute the homotopy groups $\pi_*\underline{\mathrm{Hom}}_G(M,N)$. Then we can compare both long exact sequences and by the 5-lemma, we get the desired isomorphism. Hence F is fully faithful and $\mathrm{StMod}(kG) \simeq \mathcal{D}$.

We will extend this idea to exhibit a decomposition of the stable module ∞-

category for any group of type Φ acting on a tree. Let $D\colon I^{\mathrm{op}} \to \mathrm{Top}$ be a diagram. Recall that there is a Bousfield-Kan spectral sequence for computing the homotopy groups of the homotopy limit of D given by

$$E_2^{p,q} = H^p(I; \pi_q D) \Rightarrow \pi_{p-q}(\varprojlim D)$$

where $\pi_q D$ denotes the diagram of shape I given by $i \mapsto \pi_q(D_i)$. The differentials have the form $d^r\colon E_r^{p,q} \to E_r^{p+r,q+r-1}$. Since this sequence has certain convergence issues, we have to place some restrictions in order to avoid these inconveniences. For instance, we can impose that each D_i is path connected with abelian fundamental group for each i to ensure convergence (see [BK72], [Dug08]).

Remark 3.1.2. Let Γ be a directed graph. Let $V\Gamma$ denote the set of vertices and $E\Gamma$ denote the set of edges of Γ. Let $D\colon \Gamma \to \mathrm{Ab}$ be a diagram (Γ viewed as a category, see Remark 3.1.1). Let $\overline{C}^*(\Gamma; D)$ be the following 2-term complex of abelian groups

$$\prod_{v \in V\Gamma} D(v) \xrightarrow{d} \prod_{e \in E\Gamma} D(e)$$

where the differential d is given as follows. For U in the domain,

$$d(U)(e) = D(v \to e)U(v) - D(w \to e)U(w)$$

where e is an edge of the directed graph with initial vertex v and terminal vertex w, and $v \to e$ and $w \to e$ are given by the incidence functions of the graph. Note that $\overline{C}^*(\Gamma; D)$ is quasi-isomorphic to the cochain complex $C^*(\Gamma; D)$ of Γ with coefficients in D. In particular, we can use $\overline{C}^*(\Gamma; D)$ to compute $H^*(\Gamma; D)$.

Theorem 3.1.3. *Let G be a group of type Φ acting on a tree T. Consider the associated graph of groups $\Gamma \to Gps$. Then we have an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{StMod}(kG) \xrightarrow{\simeq} \varprojlim_{\sigma \in \Gamma^{op}} \mathrm{StMod}(kG_\sigma).$$

Proof. Consider the canonical functor

$$F\colon \mathrm{StMod}(kG) \to \varprojlim_{\sigma \in \Gamma} \mathrm{StMod}(kG_\sigma).$$

Let $\mathcal{C}$ denote the right-hand side ∞-category. Fix M, N in $\mathrm{StMod}(kG)$. Let D be the diagram of shape Γ^{op} that maps σ to the mapping space $\underline{\mathrm{Hom}}_{G_\sigma}(M_\sigma, N_\sigma)$. The

Bousfield-Kan spectral sequence for homotopy limits

$$E_2^{p,q} = H^p(\Gamma; \pi_q D)$$

converges to the homotopy groups

$$\pi_{p-q}\underline{\mathrm{Hom}}_{\mathcal{C}}(F(M), F(N))$$

Recall that the homotopy groups $\pi_i \underline{\mathrm{Hom}}_H(X,Y)$ are given by the complete cohomology $\widehat{\mathrm{Ext}}_H^{-i}(X,Y)$. On the other hand, we have an spectral sequence (see [Bro82, Chapter VII]) to compute the homotopy groups of the mapping space $\underline{\mathrm{Hom}}_G(M,N)$ given by

$$E_1^{p,q} = \prod_{\sigma\in T_q} \widehat{\mathrm{Ext}}^p_{kG_\sigma}(M,N) \Rightarrow \widehat{\mathrm{Ext}}^{p+q}_{kG}(M,N)$$

where T_q is a set of representatives of the G–orbits of q-simplices of T. In particular T_0 is in bijection with $V\Gamma$ and T_1 is in bijection with $E\Gamma$. We deduce that F is fully faithful. By [MS19, Lemma 7.1], we have that F is essentially surjective, and hence an equivalence. □

In fact, following [HY17] we can describe a left adjoint of F (hence an inverse) as the composition

$$\mathcal{C} \xrightarrow{H^{\Gamma^{\mathrm{op}}}} \mathrm{StMod}(kG)^{\Gamma^{\mathrm{op}}} \xrightarrow{\lim} \mathrm{StMod}(kG)$$

where lim is the left adjoint of the constant diagram functor, and $H^{\Gamma^{\mathrm{op}}}$ is the left adjoint of the induced functor on the lax limit. In other words, the functor H can be described informally by the formula

$$H((M_\sigma)_{\sigma\in\Gamma}) = \lim_{\sigma\in\Gamma^{\mathrm{op}}} (M \uparrow_{G_\sigma}^G).$$

3.2 Groups admitting a finite-dimensional model for $\underline{E}G$

In this section, we will assume that G admits a finite-dimensional model for the classifying space for proper actions $\underline{E}G$. It has been conjectured that a group G is of type Φ over $\mathbb{Z}$ if and only if it admits a finite-dimensional model for $\underline{E}G$ (see [Tal07, Conjecture A]). We will follow Balmer's ideas in [Bal16] and adapt Mathew's proof of the decomposition of the stable module ∞-category for finite groups over the orbit category (see [Mat16, Section 9]) to exhibit an analogous decomposition for $\mathrm{StMod}(kG)$ in terms of the stable module ∞-category of its finite subgroups.

3.2.1 A monadic adjunction

Proposition 3.2.1. *Let H be a subgroup of G. Then the adjunction given by*

$$\mathrm{Res}\colon \mathrm{Mod}(kG)^{op} \leftrightarrows \mathrm{Mod}(kH)^{op}\colon \mathrm{Ind}$$

is monadic. In particular, we have that $\mathrm{Mod}(kH)^{op}$ is equivalent to the category $\mathrm{Mod}_{\mathrm{Mod}(kG)^{op}}(A'_H)$ of A'_H–modules in $\mathrm{Mod}(kG)^{op}$, where $A'_H = \mathrm{Ind}\circ\mathrm{Res}$.

Proof. By [Bal15, Lemma 2.10] it is enough to exhibit a natural section of the counit $\epsilon\colon 1_{\mathrm{Mod}(kH)} \to \mathrm{Res}\circ\mathrm{Ind}$ (since it is the opposite adjuntion, we are writing the unit of the adjunction $\mathrm{Ind}-\mathrm{Res}$). Recall that the M-component of the counit is given by $m \mapsto 1\otimes m$ for $m \in M$. For a kH–module M, we define

$$\psi_M\colon \mathrm{Res}(\mathrm{Ind}(M)) \to M$$

as the map

$$g\otimes m \mapsto \begin{cases} gm & \text{if } g \in H \\ 0 & \text{if } g \notin H. \end{cases}$$

Note that this defines a natural transformation $\psi\colon \mathrm{Res}\circ\mathrm{Ind} \to 1_{\mathrm{Mod}(kH)}$ such that $\epsilon\psi = 1$, hence the result follows. □

Definition 3.2.2. Let H be a subgroup of G. Let A_H denote the kG–module $k(G/H)$. Define a comultiplication $\mu\colon A_H \to A_H\otimes A_H$ by $\gamma \mapsto \gamma\otimes\gamma$, and a counit $\epsilon\colon A_H \to k$ by the augmentation map.

Proposition 3.2.3. *Let H be a subgroup of G. We have that (A_H, μ, ϵ) defines a separable algebra object on the symmetric monoidal category $\mathrm{Mod}(kG)^{op}$. Moreover, the monad A'_H induced by the adjunction*

$$\mathrm{Res}\colon \mathrm{Mod}(kG)^{op} \leftrightarrows \mathrm{Mod}(kH)^{op}\colon \mathrm{Ind}$$

is isomorphic to the monad induced by $A_H \otimes -$.

Proof. It is straightforward to verify that (A_H, μ, ϵ) defines a coalgebra object in $\mathrm{Mod}(kG)$ and therefore an algebra object in $\mathrm{Mod}(kG)^{\mathrm{op}}$. The separability of A_H will follow from the equivalence, as monads, with A'_H. Recall that we have a natural isomorphism of functors

$$\theta\colon \mathrm{Ind}\circ\mathrm{Res} \to A_H\otimes -$$

where the M-component is given by $\theta_M(g\otimes m) = gH\otimes gm$ and the inverse is given by $\theta_M^{-1}(\gamma\otimes m) = g\otimes g^{-1}m$ for any choice of $g\in\gamma$. Note that we have a compatibility of the units, that is, $\theta_M \circ \epsilon'_M = \epsilon \otimes 1_M$ for any kG–module M. The multiplication μ' of the monad is given by $\mu'(g \otimes m) = g \otimes 1 \otimes m$. Then the following diagram is commutative.

$$\begin{array}{ccc} kG \otimes_{kH} (kG \otimes_{kH} M) & \xrightarrow{\theta_M^2} & k(G/H) \otimes k(G/H) \otimes M \\ \uparrow{\scriptstyle \mu} & & \uparrow{\scriptstyle \mu'} \\ kG \otimes_{kH} M & \xrightarrow[\theta_M]{} & k(G/H) \otimes M \end{array}$$

The map θ_M^2 is given as follows. For any $g, g' \in G$, we have that $\theta_M^2(g \otimes g' \otimes m) = gH \otimes gg'H \otimes gg'm$. Then the result follows. □

In particular, we have an analogous result in the ∞-categorical setting of the previous result (see [MNN17, Proposition 5.29]). Let H be a subgroup of G. We can equip A_H with the structure of an object in $\mathrm{CAlg}(\mathrm{StMod}(kG)^{\mathrm{op}})$. Then there is an equivalence

$$\mathrm{Mod}_{\mathrm{StMod}(kG)^{\mathrm{op}}}(A_H) \simeq \mathrm{StMod}(kH)^{\mathrm{op}}$$

and we can identify the adjunction $\mathrm{StMod}(kG)^{\mathrm{op}} \leftrightarrows \mathrm{Mod}_{\mathrm{StMod}(kG)^{\mathrm{op}}}(A_H)$ with the adjunction $\mathrm{Res}\colon \mathrm{StMod}(kG)^{\mathrm{op}} \leftrightarrows \mathrm{StMod}(kH)^{\mathrm{op}}\colon \mathrm{Ind}$.

3.2.2 Decompositions of the stable module ∞-category

Proposition 3.2.4. *Let G be a group with a finite-dimensional cocompact model X for $\underline{E}G$. Let $\mathscr{F}$ be the family of finite subgroups of G. Then the commutative algebra object*

$$A = \prod_{H \in \mathscr{S}} A_H \in \mathrm{CAlg}(\mathrm{StMod}(kG)^{op})$$

admits descent, where $\mathscr{S}$ is a set of representatives of the G-orbits of $\mathscr{F}$.

Proof. Let $C_*(X)$ denote the chain complex of X with coefficients in k. By the hypothesis on X, the set $\mathscr{S}$ is finite. Since the forgetful functor

$$\mathrm{CAlg}(\mathrm{StMod}(kG)^{\mathrm{op}}) \to \mathrm{StMod}(kG)^{\mathrm{op}}$$

commutes with limits (see [Lur17, Proposition 3.2.2.1]), it follows that A is just a finite product in $\mathrm{StMod}(kG)^{\mathrm{op}}$, and hence a finite coproduct. As a consequence, we have that $C_n(X)$ is a retract of a finite number of copies of A. Therefore $C_n(X)$ is contained in the smallest thick $\otimes$-ideal $\langle A \rangle$ containing A, for any $n \in \mathbb{Z}$. Since X is contractible, we have that the augmented chain complex $\widetilde{C}_*(X)$ is exact, and therefore k is in $\langle A \rangle$. It follows that $\langle A \rangle = \mathrm{StMod}(kG)^{\mathrm{op}}$. □

Remark 3.2.5. Consider the same notation as in the previous proposition. Let $\mathscr{S}'$ denote the set of Sylow p-subgroups of the elements of $\mathscr{S}$. Then the commutative algebra object

$$B = \prod_{H \in \mathscr{S}'} A_H \in \mathrm{CAlg}(\mathrm{StMod}(kG)^{\mathrm{op}})$$

has A as a retract, thus B admits descent as well. This follows since k is a retract of $k{\uparrow}_S^F$ as kF–modules, where F is a finite group and S is a Sylow p-subgroup of F. In the same fashion, we can construct commutative algebra objects in $\mathrm{CAlg}(\mathrm{StMod}(kG)^{\mathrm{op}})$ which admit descent as long as the set of subgroups indexing our commutative algebra object contains a copy of representatives of the Sylow p-subgroups of the elements in $\mathscr{S}$.

Recall that the *orbit category* $\mathcal{O}(G)$ is the category with objects the G-sets of the form G/H where H is a subgroup of G, and the morphisms are given by G-maps. Given a collection $\mathcal{A}$ of subgroups of G, that is, a set of subgroups of G closed under conjugation, we let $\mathcal{O}_{\mathcal{A}}(G) \subseteq \mathcal{O}(G)$ denote the full subcategory spanned by the objects G/H with $H \in \mathcal{A}$.

For a group G admitting a finite-dimensional cocompact model for $\underline{E}G$, we will exhibit a decomposition of the stable module ∞-category in terms of the orbit category $\mathcal{O}_{\mathscr{F}}(G)$, where $\mathscr{F}$ denotes a family of finite subgroups of G which contains the family of finite p-subgroups of G. That is, we define a functor f_0 from $\mathcal{O}_{\mathscr{F}}(G)^{\mathrm{op}}$ to $\widehat{\mathrm{Cat}}_\infty^{\otimes}$ that maps an object G/H to $\mathrm{Mod}_{\mathrm{StMod}(kG)^{\mathrm{op}}}(A_H) \simeq \mathrm{StMod}(kH)^{\mathrm{op}}$ and a morphisms $G/H \xrightarrow{[g]} G/H'$ to the induced restriction functor along the conjugation by g. Hence we will show that $\mathrm{StMod}(kG)^{\mathrm{op}} \simeq \varprojlim f_0$.

The idea is to find a suitable commutative algebra object A in $\mathrm{StMod}(kG)^{\mathrm{op}}$ which satisfies descent, in fact, a commutative algebra object as in Proposition 3.2.4 will work. This give us a decomposition

$$\mathrm{StMod}(kG)^{\mathrm{op}} \simeq \varprojlim_{\Delta} F$$

where F is a cobar resolution of the form $\mathrm{Mod}_{\mathrm{StMod}(kG)^{\mathrm{op}}}(A^{\otimes k})$. Since we want to rewrite this limit, we must be able to reconstruct this cobar resolution from $\mathcal{O}_{\mathscr{F}}(G)^{\mathrm{op}}$ through final functors. The first step is to construct a category $\mathcal{C}$ by formally attaching finite coproducts to the orbit category and then extend f_0 to a functor $f\colon \mathcal{C}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty^{\otimes}$ which preserves the homotopy type of f_0. Moreover, we would like to find and object in $\mathcal{C}$ mapping to $\mathrm{Mod}_{\mathrm{StMod}(kG)^{\mathrm{op}}}(A)$ under f. The final step is to find a *cofinal object* Z in $\mathcal{C}$ that will allow us to construct a cofinal functor $Z^{\bullet+1}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ such that its composition with f agrees with the cobar resolution F from above. In short, we will obtain a chain of equivalences

$$\varprojlim_{\mathcal{O}_{\mathscr{F}}(G)^{\mathrm{op}}} f_0 \simeq \varprojlim_{\mathcal{C}^{\mathrm{op}}} f \simeq \varprojlim_{\Delta^{\mathrm{op}}} f \circ Z^{\bullet+1} = \varprojlim_{\Delta^{\mathrm{op}}} F \simeq \mathrm{StMod}(kG)^{\mathrm{op}}.$$

Theorem 3.2.6. *Let G be a group with a finite-dimensional cocompact model X for $\underline{E}G$. Let $\mathscr{F}$ be a family of finite subgroups of G which contains the family of finite p-subgroups of G. Then there is an equivalence of symmetric monoidal stable ∞-categories*

$$\mathrm{StMod}(kG) \xrightarrow{\simeq} \varprojlim_{G/H \in \mathcal{O}_{\mathscr{F}}(G)^{op}} \mathrm{StMod}(kH).$$

Proof. Let $\mathscr{S}$ denote a set of representatives of the G-orbits of $\mathcal{F}$ and let $A = \prod_{H \in \mathscr{S}} A_H$ be a commutative algebra object in $\mathrm{StMod}(kG)^{\mathrm{op}}$ as in Proposition 3.2.4. Since A admits descent (see Remark 3.2.5), Proposition 3.22 in [Mat16] gives us a decomposition

$$\mathrm{StMod}(kG)^{\mathrm{op}} \simeq \mathrm{Tot}\left(\mathrm{Mod}_{\mathrm{StMod}(kG)^{\mathrm{op}}}(A) \rightrightarrows \mathrm{Mod}_{\mathrm{StMod}(kG)^{\mathrm{op}}}(A^{\otimes 2}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots\right)$$

We will rewrite this limit in terms of the orbit category $\mathcal{O}_{\mathcal{F}}(G)$. Consider the smallest full subcategory $\mathcal{C}$ of

$$\mathcal{P}(\mathcal{O}_{\mathscr{F}}(G)) = \mathrm{Fun}(\mathcal{O}_{\mathscr{F}}(G)^{\mathrm{op}}, \mathcal{S})$$

that contains the essential image of the Yoneda embedding

$$\mathcal{O}_{\mathscr{F}}(G) \xrightarrow{y} \mathcal{P}(\mathcal{O}_{\mathscr{F}}(G))$$

and which is stable under finite coproducts (see [Lur09, Remark 5.3.5.9]). Then we can extend the stable module ∞-category functor

$$f_0 \colon \mathcal{O}_{\mathscr{F}}(G)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}^{\otimes}$$
$$G/H \mapsto \mathrm{Mod}_{\mathrm{StMod}(kG)^{\mathrm{op}}}(A_H) \simeq \mathrm{StMod}(kH)^{\mathrm{op}}$$

to a functor

$$f \colon \mathcal{C}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}^{\otimes}$$

that sends finite coproducts to products. Moreover, the functor f is the right Kan extension of $f_0 = f|_{\mathcal{O}_{\mathscr{F}}(G)^{\mathrm{op}}}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{F}}(G)^{\mathrm{op}} & \xrightarrow{f_0} & \widehat{\mathrm{Cat}}_{\infty}^{\otimes} \\ & {}_{i}\searrow \quad \nearrow_{\mathrm{Ran}_i f_0 = f} & \\ & \mathcal{C}^{\mathrm{op}} & \end{array}$$

and hence an equivalence

$$\varprojlim_{\mathcal{C}^{\mathrm{op}}} f \simeq \varprojlim_{\mathcal{O}_{\mathscr{F}}(G)^{\mathrm{op}}} f_0.$$

In particular, the object $\bigsqcup_{H\in\mathscr{S}} G/H$ in $\mathcal{C}^{\mathrm{op}}$ is mapped under the functor f to

$$\prod_{H\in\mathscr{S}} \mathrm{StMod}(kH) \simeq \mathrm{Mod}_{\mathrm{StMod}(kG)^{\mathrm{op}}}(A).$$

On the other hand, consider the object $Z = \bigsqcup_{H\in\mathscr{S}} G/H \in \mathcal{C}$. Note that any object $Y \in \mathcal{C}$ admits a map $Y \to Z$. By [MNN17, Proposition 6.28] the simplicial object $Z^{\bullet+1}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is cofinal. Moreover, we have that

$$\begin{aligned} \Delta \to \mathcal{C}^{\mathrm{op}} &\to \widehat{\mathrm{Cat}}_\infty^{\otimes} \\ [k] \mapsto Z^{\times k} &\mapsto \mathrm{Mod}_{\mathrm{StMod}(kG)^{\mathrm{op}}}(A^{\otimes k}) \end{aligned}$$

in other words, the cosimplicial diagram $f \circ Z^{\bullet+1}$ is in fact the cobar construction. Hence the result follows. □

Remark 3.2.7. The previous theorem is analogous to Mathew's decomposition for finite groups (see [Mat15, Corollary 9.16]). In particular, we have that the modular representation theory of G is *determined* by its finite p-local information just as in the case of finite groups, at least when G satisfies the hypothesis of Theorem 3.2.6.

It is worth highlighting that working with the orbit category of an infinite group might be not easy, hence it is convenient to have decompositions of the stable module ∞-category in terms of *simpler* categories. For this, the ∞-categorical version of Quillen's Theorem A will play an important role. For simplicity, we will state here the version we need of such a theorem, we refer to [Lur09, Corollary 4.1.3.3] for further details. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor (here as ordinary categories). Then $N(F)\colon N(\mathcal{C}) \to N(\mathcal{D})$ is cofinal if all the slice categories $\mathcal{C}_{d/}$ are weakly contractible for every d in $\mathcal{D}$.

Let X be a finite-dimensional model for $\underline{E}G$ such that the fundamental domain of the action is homeomorphic to a standard simplex Δ^n. Let $\mathcal{T}$ denote the poset of simplices in Δ^n. In fact, $\mathcal{T}$ can be identified with the barycentric subdivision of Δ^n considering the latter as a poset. We have a functor

$$\begin{aligned} \mathcal{T} &\to \mathcal{O}_{\mathcal{F}}(G) \\ \sigma &\mapsto G/G_\sigma \end{aligned}$$

where $\mathcal{F}$ denotes the family of finite subgroups of G. Recall that the elements in $\mathcal{T}$ are simplices in Δ^n, so here G_σ means the isotropy group of the simplex of X

corresponding to the simplex σ of Δ^n.

Proposition 3.2.8. *The functor $\mathcal{T} \to \mathcal{O}_{\mathcal{F}}(G)$ defined above is cofinal.*

Proof. By Quillen's Theorem A [Lur09, Corollary 4.1.3.3] it is enough to verify that $\mathcal{T}_{(G/H)/}$ has weakly contractible nerve, for all $G/H \in \mathcal{O}_{\mathscr{F}}(G)$. Note that $\mathcal{T}_{(G/H)/}$ corresponds to the poset of simplices of Δ^n whose isotropy groups contain H, but this poset has a minimum element, namely the simplex of higher dimension whose isotropy group contains H. Therefore $\mathcal{T}$ has weakly contractible nerve, and the result follows. □

Corollary 3.2.9. *Let X be a finite-dimensional model for $\underline{E}G$ such that the fundamental domain of the action is homeomorphic to the standard simplex Δ^n. Let $\mathcal{T}$ denote the barycentric subdivision of Δ^n. Then there is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{StMod}(kG) \xrightarrow{\simeq} \varprojlim_{\sigma \in \mathcal{T}^{op}} \mathrm{StMod}(kG_\sigma).$$

Proof. This follows by Proposition 3.2.8 and Theorem 3.2.6. □

Chapter 4

Computations of the Picard group

In this chapter we use Corollary 4.1.2 to compute the Picard group of the stable module category for countable locally finite p-groups. We also provide computations for certain countable locally finite groups. We implement a tool to compute the Picard group for amalgam groups (see Definition 4.4.1). Finally, for amalgam groups with trivial face group we provide a construction of the modules that restrict stably to the trivial representation on any finite subgroup.

4.1 The Picard space

Definition 4.1.1. The *Picard space* $\mathrm{Pic}(\mathcal{C})$ of a symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$ is the ∞-groupoid of $\otimes$-invertible objects in $\mathcal{C}$ and equivalences between them.

In other words, the Picard space is an enhancement of the Picard group, since the former encodes the latter as the connected components, but also keeps track of all higher isomorphisms. It is clear that Pic defines a functor from the ∞-category of symmetric monoidal ∞-categories $\mathrm{Cat}^{\otimes}$ to the ∞-category of spaces $\mathcal{S}$. Moreover, we can describe the higher homotopy groups of $\mathrm{Pic}(\mathcal{C})$ as follows.

$$\pi_i \mathrm{Pic}(\mathcal{C}) = \begin{cases} \mathrm{PicGp}(\mathcal{C}) & \text{if } n = 0 \\ (\pi_0 \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}))^{\times} & \text{if } n = 1 \\ \pi_{i-1} \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}) & \text{if } n \geq 2. \end{cases}$$

Moreover, the Picard space functor

$$\mathrm{Pic}\colon \mathrm{Cat}^{\otimes} \to \mathcal{S}$$

commutes with homotopy limits (see [MS16, Proposition 2.2.3]).

If G is a group of type Φ acting on a tree with associated graph Γ, then Theorem 3.1.3 gives us a decomposition of the Picard space

$$\operatorname{Pic}(\operatorname{StMod}(kG)) \xrightarrow{\simeq} \varprojlim_{\sigma \in \Gamma^{\mathrm{op}}} \operatorname{Pic}(\operatorname{StMod}(kG_\sigma)). \tag{4.1}$$

If G is a group of type Φ admitting a finite-dimensional cocompact model X for $\underline{E}G$, then Theorem 3.2.6 gives us a decomposition of the Picard space

$$\operatorname{Pic}(\operatorname{StMod}(kG)) \xrightarrow{\simeq} \varprojlim_{G/H \in \mathcal{O}_{\mathscr{F}}(G)^{\mathrm{op}}} \operatorname{Pic}(\operatorname{StMod}(kH)) \tag{4.2}$$

where $\mathscr{F}$ denotes a family of finite subgroups containing the family of finite p-subgroups of G. If additionally the fundamental domain of the action of G is homeomorphic to the standard simplex Δ^n, we obtain an easier decomposition by Corollary 3.2.9

$$\operatorname{Pic}(\operatorname{StMod}(kG)) \xrightarrow{\simeq} \varprojlim_{G/H \in \mathcal{T}^{\mathrm{op}}} \operatorname{Pic}(\operatorname{StMod}(kH)) \tag{4.3}$$

where $\mathcal{T}$ denotes the barycentric subdivision of Δ^n. We will use the spectral sequence of Bousfield-Kan for homotopy limits to compute the Picard group of the stable module category. In particular, for the stable module ∞-category of a group G of type Φ, the higher homotopy groups of the Picard space can be described as follows.

$$\pi_i \operatorname{Pic}(\operatorname{StMod}(kG)) = \begin{cases} T(G) & \text{if } n = 0 \\ \widehat{\operatorname{Aut}}_G(k) & \text{if } n = 1 \\ \widehat{H}^{1-i}(G,k) & \text{if } n \geq 2 \end{cases}$$

where $\widehat{\operatorname{Aut}}_G(k)$ denotes the group of automorphisms of k in the stable module category $\underline{\operatorname{StMod}}(kG)$. In particular, for groups acting on trees we can describe the Picard group as an extension of abelian groups (c.f. [MS19, Theorem 7.4]).

Corollary 4.1.2. *Let G be a group of type Φ which acts on a tree. Consider the associated graph of groups $\Gamma \to Gps$. Then we have a short exact sequence of abelian groups*

$$0 \to H^1(\Gamma; \pi_1 \circ \operatorname{Pic} \circ \operatorname{StMod}) \to T(G) \to H^0(\Gamma; \pi_0 \circ \operatorname{Pic} \circ \operatorname{StMod}) \to 0.$$

where StMod *denotes the stable module ∞-category functor associated to the graph of groups (see Section 3.1).*

Proof. By Equation (4.1), we have an spectral sequence

$$E_2^{p,q} = H^p(\Gamma; \pi_q \circ \mathrm{Pic} \circ \mathrm{StMod}) \Rightarrow \pi_{p-q}\mathrm{Pic}(\mathrm{StMod}(kG)).$$

Since Γ is a graph, it is concentrated in two consecutive columns. Then the spectral sequence collapses at page two and the result follows. □

Remark 4.1.3. Note that $H^0(\Gamma; \pi_0 \circ \mathrm{Pic} \circ \mathrm{StMod})$ corresponds to the kernel of the map

$$\prod_{v \in V\Gamma} T(G_v) \xrightarrow{\mathrm{Res}-\mathrm{Res}_f} \prod_{e \in E\Gamma} T(G_e)$$

where $\mathrm{Res} - \mathrm{Res}_f$ is defined as follows. Fix $(M_v)_{v \in V\Gamma} \in \prod_{v \in V\Gamma} T(G_v)$. Then

$$\mathrm{Res} - \mathrm{Res}_f (M_v)_{v \in V\Gamma} = (N_e)_{e \in E\Gamma}$$

where $N_e = M_{\iota(e)}\downarrow_{G_e} - f_e^*(M_{\tau(e)})\downarrow_{G_e}$, and f_e^* is the functor induced by the morphism $G_e \to G_{\tau(e)}$ in the graph of groups. On the other hand, $H^1(\Gamma; \pi_0 \circ \mathrm{Pic} \circ \mathrm{StMod})$ corresponds to the cokernel of the map

$$\prod_{v \in VY} \widehat{\mathrm{Aut}}_{G_v}(k) \xrightarrow{\mathrm{Res}-\mathrm{Res}_f} \prod_{e \in EY} \widehat{\mathrm{Aut}}_{G_e}(k)$$

defined in a similar fashion. This agrees with [MS19, Theorem 7.4].

4.2 Countable Locally Finite p-Groups

Recall that a group G is called *a locally finite p–group* if every finitely generated subgroup is a finite p–group. The following result is well known (see for example [KW73, Lemma 1.A.9]).

Proposition 4.2.1. *Let G be a locally finite group. Then G is countable if and only if there is an ascending chain of finite subgroups*

$$G_1 \leq G_2 \leq G_3 \leq \dots$$

such that

$$G = \bigcup_{n \geq 1} G_n.$$

In this case, we will say that $G_1 \leq G_2 \leq G_3 \leq \dots$ is a tower for G.

Proposition 4.2.2. *Let G be a countable locally finite group. Consider a tower $G_1 \leq G_2 \leq \dots$ of finite subgroups of G. Then the following hold.*

(i) G acts on a tree with isotropy groups in the family $\{G_n\}_{n\geq 1}$.

(ii) If p divides the order of G_r for some $r \geq 1$, then

$$T(G) \cong \varprojlim T(G_n)$$

where the maps in the inverse system are given by the restrictions maps.

In particular, note that part (i) of this proposition gives us that any countable locally finite group is a group of type Φ.

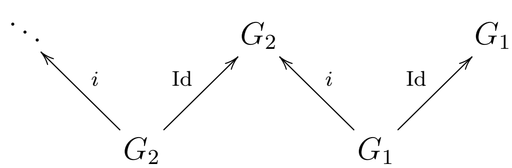
Proof. The first part follows by [Ike84, Example 3]. For convenience we will give the construction of the tree T: define the vertex set VT as the disjoint union of the sets G/G_n for $n \geq 1$. The edges are given by the canonical maps $G/G_n \to G/G_{n+1}$, that is, if mG_n is a vertex, then there is an edge from the corresponding vertex to mG_{n+1}.

The graph T is path connected since any vertex will be mapped to the trivial coset eventually. Moreover, it is clear that there are no loops, hence T is a tree. The action of G on the tree T is induced by the action of G on G/G_n by multiplication. For an edge (mG_n, mG_{n+1}), the action is given by

$$g \cdot (mG_n, mG_{n+1}) = ((gm)G_n, (gm)G_{n+1})$$

for $g \in G$. Note that the stabilizer of the vertex mG_n is isomorphic to G_n.

For the second part, note that the fundamental domain for the action of G on T corresponds to a ray. Then the associated graph of groups $\Gamma \to \mathrm{Gps}$ can be depicted as follows.



where i denotes the inclusion $G_n \to G_{n+1}$. Then the diagram

$$\mathrm{StMod}(-)\colon \Gamma^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$$

simplifies to

$$\ldots \xrightarrow{\mathrm{Res}} \mathrm{StMod}(kG_3) \xrightarrow{\mathrm{Res}} \mathrm{StMod}(kG_2) \xrightarrow{\mathrm{Res}} \mathrm{StMod}(kG_1)$$

By Corollary 4.1.2, we have the following exact sequence of abelian groups.

$$0 \to H^1(\Gamma; \pi_1 \mathrm{StMod}) \to T(G) \to H^0(\Gamma; \pi_0 \mathrm{StMod}) \to 0$$

Note that $\pi_1 \operatorname{StMod}(kG_n) \cong \widehat{\operatorname{Aut}}_{G_n}(k) \cong k^\times$ for all $n \geq r$, thus $\pi_1 \operatorname{StMod}$ is eventually constant. Hence we have that $H^1(\Gamma, \pi_1 \operatorname{StMod})) \cong H^1(|\Gamma|, k^\times) = 0$. On the other hand, recall that

$$H^0(\Gamma; \pi_0 \operatorname{StMod}) = \varprojlim \pi_0(\operatorname{StMod}(kG_n)) = \varprojlim T(G_n)$$

and the result follows. □

We will first consider the case when the group is artinian, that is, if its subgroups satisfy the descending chain condition.

Definition 4.2.3. A group P is called *a discrete p-toral group* if it fits in an extension

$$1 \to K \to P \to S \to 1$$

where K is isomorphic to a finite product of copies of $\mathbb{Z}/p^\infty$ and S is a finite p-group.

Note that $\mathbb{Z}/p^\infty$ is an artinian locally finite p-group. Since these properties are preserved by finite products and finite extensions, we deduce that any discrete p-toral group is an artinian locally finite p-group. The converse also holds (see [BLO07, Proposition 1.2]). Hence a group is a locally finite p-group if and only if it is a discrete p-toral group. As a consequence of this characterization, it follows that the class of discrete p-toral groups is closed under subgroups, quotients and extensions by discrete p-toral groups. Moreover, if P is a discrete p-toral group, then it contains finitely many conjugacy classes of elementary abelian p-subgroups and finitely many conjugacy classes of subgroups of order p^n for $n \geq 0$ (see [BLO07, Lemma 1.4]).

We will use these properties of discrete p-toral groups and the description of the restriction maps in the case of finite p-groups to determine $T(P)$ in terms of the P–conjugacy classes of maximal elementary abelian subgroups of P of rank 2. The result will be analogous to the case of finite p-groups; for almost all the cases the group of invertible modules is an abelian free group. We shall deal separately with discrete p-toral groups that admit a tower of cyclic, dihedral, semidihedral or quaternion groups. We will describe these cases first.

Proposition 4.2.4. *The following hold.*

(a) Let $P = \mathbb{Z}/p^\infty$. Then $T(P) \cong \mathbb{Z}/2$.

(b) Let $D_{2^\infty} = \bigcup D_{2^n}$, where D_{2^n} denotes the dihedral group of order 2^n. Then $T(D_{2^\infty}) \cong \mathbb{Z}$.

(c) Let $Q_{2^\infty} = \bigcup Q_{2^n}$, where Q_{2^n} denotes the generalized quaternion group of order 2^n. Then $T(Q_{2^\infty}) \cong \mathbb{Z}/4$.

Proof. (a) In this case P admits a tower of cyclic groups

$$\mathbb{Z}/p \leq \mathbb{Z}/p^2 \leq \dots$$

Since $T(\mathbb{Z}/2)$ is trivial and $T(\mathbb{Z}/p^m) \cong \mathbb{Z}/2$ generated by $[\Omega(k)]$, for $p^m > 2$ (see [Dad78a, Corollary 8.8]), we deduce that the restriction map $\mathrm{Res}\colon T(\mathbb{Z}/p^{m+1}) \to T(\mathbb{Z}/p^m)$ is the identity, for all $m > 1$. Hence the inverse system $(T(\mathbb{Z}/p^m), \mathrm{Res})$ is eventually constant. Then $T(P)$ is isomorphic to $\mathbb{Z}/2$.

(b) Fix the following presentation for the dihedral group D_{2^n}.

$$\langle r, s \mid r^{2^{n-1}} = s^2 = (sr)^2 = 1 \rangle$$

Then we will consider $D_{2^{n-1}}$ as the subgroup of D_{2^n} generated by r^2 and s.

Recall that $T(D_{2^n}) = \langle \Omega_{D_{2^n}}, [L] \rangle \cong \mathbb{Z}^2$ (see [CT00, Section 5]). Let $\Omega_{D_{2^{n-1}}}, [L']$ be the standard generators of $T(D_{2^{n-1}})$. It is clear that

$$\mathrm{Res}([\Omega_{D_{2^n}}]) = [\Omega_{D_{2^{n-1}}}].$$

Set $F = \langle r^2 s, r^{2^{n-2}} \rangle$ and $F' = \langle s, r^{2^{n-2}} \rangle$, which are representatives of the two conjugacy classes of maximal elementary abelian subgroups of $D_{2^{n-1}}$. Since F and F' are conjugate in D_{2^n}, we have

$$\begin{aligned} \mathrm{Res}\colon T(D_{2^n}) &\to T(F) \oplus T(F') \\ [L] &\mapsto (-\Omega_F, -\Omega_{F'}) \end{aligned}$$

(see [CT00, Theorem 5.4]). The map $\mathrm{Res}\colon T(D_{2^n}) \to T(F) \oplus T(F')$ factors through $T(D_{2^{n-1}})$ so we have a commutative triangle

$$\begin{array}{ccc} T(D_{2^n}) & \longrightarrow & T(D_{2^{n-1}}) \\ & \searrow & \downarrow \\ & & T(F) \oplus T(F') \end{array}$$

and by the detection theorem (see [CT00, Conjecture 2.6]), the vertical map is injective. We deduce that

$$\begin{aligned} \mathrm{Res}\colon T(D_{2^n}) &\to T(D_{2^{n-1}}) \\ m\Omega_{D_{2^n}} + n[L] &\mapsto (m-n)\Omega_{D_{2^{n-1}}} \end{aligned}$$

Hence the result follows.

(c) Fix the following presentation for the generalized quaternion group $Q_{2^{n+1}}$

$$\langle x, y \mid x^{2^n} = 1, y^2 = x^{2^{n-1}}, yxy = x^{-1} \rangle$$

We will identify Q_{2^n} with the subgroup $\langle x^2, y\rangle$ of $Q_{2^{n+1}}$.

Recall that $T(Q_{2^{n+1}}) = \langle \Omega_{Q_{2^{n+1}}}, [\Omega^1_{Q^{n+1}}(L)]\rangle \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$ (see [CT00, Section 6]). Consider the restriction map

$$\operatorname{Res}\colon T(Q_{2^{n+1}}) \to T(H) \oplus T(H')$$

where $H = \langle x^{2^{n-2}}, y\rangle$ and $H' = \langle x^{2^{n-2}}, xy\rangle$ are representatives of the two conjugacy classes of quaternion subgroups of order 8. Then

$$\Omega^1_{Q_{2^{n+1}}}(L) \mapsto (2\Omega_H, 0) \text{ or } (0, 2\Omega_{H'})$$

under the previous restriction map [CT00, Theorem 6.5]. Let $F = \langle (x^2)^{2^{n-3}}, y\rangle$ and $F' = \langle (x^2)^{2^{n-3}}, x^2y\rangle$ be representatives of the two conjugacy classes of quaternion subgroups of Q_{2^n} of order 8. Note that F and F' are $Q_{2^{n+1}}$-conjugate, hence

$$\begin{aligned} \operatorname{Res}\colon T(Q_{2^{n+1}}) &\to T(H) \oplus T(H') \\ \Omega_{Q_{2^{n+1}}} &\mapsto (\Omega_F, \Omega_{F'}) \\ [\Omega^1_{Q_{2^{n+1}}}(L)] &\mapsto (0,0) \text{ or } (2\Omega_F, 2\Omega_{F'}) \end{aligned}$$

Since the map $\operatorname{Res}\colon T(Q_{2^{n+1}}) \to T(F) \oplus T(F')$ factors through $T(Q_{2^n})$ we have a commutative diagram

$$\begin{array}{ccc} T(Q_{2^{n+1}}) & \longrightarrow & T(Q_{2^n}) \\ & \searrow & \downarrow \\ & & T(H) \oplus T(H') \end{array}$$

and by the detection theorem, the vertical arrow is injective. We deduce that

$$\begin{aligned} \operatorname{Res}\colon T(Q_{2^{n+1}}) &\to T(Q_{2^n}) \\ (m\Omega_{Q_{2^{n+1}}}, n[\Omega^1_{Q^{n+1}}(L)]) &\mapsto ((n+2m)\Omega_{Q^{2^n}}, 0) \text{ or} \\ &\quad\ (m\Omega_{Q^{2^n}}, 0) \end{aligned}$$

Therefore $\varprojlim T(Q_{2^n}) \cong \mathbb{Z}/4$. □

Remark 4.2.5. The maximal subgroups of a semi-dihedral group are generalized quaternion, dihedral and cyclic groups and none of them contain a semi-dihedral group as a subgroup (see [Gor80, Theorem 4.3]). Then a locally finite group that admits a tower of semi-dihedral groups is a semi-dihedral group, hence a finite group.

Proposition 4.2.6. *Let P be a discrete p-toral group that admits a tower $Q_1 \leq Q_2 \leq \ldots$ such that Q_n is not cyclic, dihedral, semi-dihedral or quaternion for all $n \geq 1$. Then*

$$T(P) = \begin{cases} \mathbb{Z}^r & \text{if } P \text{ has } p\text{-rank at most 2} \\ \mathbb{Z}^{r+1} & \text{if } P \text{ has } p\text{-rank at least 3} \end{cases}$$

where r is the number of conjugacy classes of maximal elementary abelian subgroups of P of rank 2.

Proof. Since Q_n is not cyclic, semi-dihedral or generalized quaternion for all $n \geq 1$, we have that $T(Q_n)$ is a free abelian group and its rank is determined by the connected components of $\mathcal{E}_{\geq 2}(Q_n)/Q_n$, the poset of Q_n–orbits of elementary abelian subgroups of p-rank at least 2. Recall that we have a finite number of P–conjugacy classes of maximal elementary abelian subgroups of rank 2. Fix representatives $E_1, \ldots, E_r$ of these classes. We can assume that E_i is a subgroup of Q_1 and has the form $E_i = \langle u_i \rangle \times Z$, where Z is the unique central subgroup of Q_1 of order p and $\langle u_i \rangle$ is a non-central subgroup of Q_1 of order p, for $1 \leq i \leq r$ (see [Maz19, Section 3.3]).

Suppose that P has p-rank at least 3. Hence we can assume that Q_1 has p-rank at least 3. For $n \geq 1$, choose elementary abelian subgroups $E_0^n, \ldots, E_{r+s_n}^n$ of rank 2 which are representatives of the connected components of $\mathcal{E}_{\geq 2}(Q_n)/Q_n$. We can assume that $E_0^n = E_0$ for a fixed elementary abelian subgroup in the big component of $\mathcal{E}_{\geq 2}(Q_1)/Q_1$, that is, the connected component that contains all the elementary abelian subgroups of Q_1 of rank at least 3.

We can assume that $E_i^n = E_i$ for $1 \leq i \leq r$, and $E_i^n = Z \times \langle u_i^n \rangle$ for a non-central subgroup $\langle u_i^n \rangle$ of Q_n of order p, for $r+1 \leq i \leq r+s_n$. Then there exist endotrivial modules $N_1^n, .., N_{r+s_n}^n$ such that

$$\mathrm{Res}^{Q_n}_{E_j^n}(N_i^n) \cong \begin{cases} k \oplus (\text{proj}) & \text{if } i \neq j, \\ \Omega^{-2p}_{E_j^n}(k) \oplus (\text{proj}) & \text{if } i = j \text{ and } C_{Q_n}(u_i^n)/\langle u_i^n \rangle \text{ is cyclic of order } \geq 3, \\ \Omega^{-2}_{E_j^n}(k) \oplus (\text{proj}) & \text{if } i = j \text{ and } C_{Q_n}(u_i^n)/\langle u_i^n \rangle \text{ has order 2}, \\ \Omega^{-8}_{E_j^n}(k) \oplus (\text{proj}) & \text{if } i = j \text{ and } C_{Q_n}(u_i^n)/\langle u_i^n \rangle \text{ is quaternion.} \end{cases}$$

for $0 \leq j \leq r+s_n$ and $1 \leq i \leq r+s_n$ (see [Maz19, Section 3.3]).

Since we have only r classes of P–conjugation of maximal elementary abelian subgroups or rank 2, the subgroup E_i^n must be in the same Q_m–orbit of some E_j in $\mathcal{E}_{\geq 2}(Q_m)/Q_m$ for some $m \geq n$, and some $j = 0, \ldots, r$. We can suppose that this holds for $m = n+1$. In particular, we have that

$$\mathrm{Res}^{Q_{n+1}}_{E_j^n}([N_i^{n+1}]) = [k]$$

for $r < i \leq r + s_{m+1}$ and $0 \leq j \leq r + s_m$. Then it follows that $\mathrm{Res}^{Q_{n+1}}_{Q_n}$ is trivial on the N^n_j-components for $r < j \leq r + s_n$. On the other hand, note that we can find a large enough k such that

$$C_{Q_n}(u_i^n)/\langle u_i^n \rangle \cong C_{Q_{n+1}}(u_i^{n+1})/\langle u_i^{n+1} \rangle$$

for all $n \geq k$ since $C_{Q_n}(u_i) \leq C_{Q_{n+1}}(u_i)$. We can suppose that $k = 1$. Then we have that $\mathrm{Res}^{Q_{n+1}}_{Q_n}[N^{n+1}_j] = [N^n_j]$ for $0 \leq j \leq r$.

For $j \geq 1$, define $\pi_j \colon \mathbb{Z}^{r+1} \to T(Q_j)$ as the inclusion on the generators N^n_j for $0 \leq j \leq r$. It is straightforward to show that $(\mathbb{Z}^{r+1}, \pi_i)$ is the limit of the inverse system $\{T(Q_n)\}$. Then the result holds. The case where P has p-rank at most 2 is analogous. □

The following result is analogous to the description of finite p-groups whose group of endotrivial modules is infinite cyclic. For abelian p-groups this is precisely the main theorem given by Dade in [Dad78a]. See also [Maz19, Theorem 3.5].

Corollary 4.2.7. *Let P be a discrete p-toral group. If one of the following conditions holds, then $T(P) \cong \mathbb{Z}$.*

(1) *P is an abelian group of p-rank at least 2.*

(2) *P has p-rank at least $p + 1$ if p is odd or at least 5 if $p = 2$.*

Proof. If (1) holds, then P admits a tower of p-abelian groups of rank at least 2. The group of endotrivial modules of such groups is infinite cyclic by [Dad78a, Theorem 10.1], we deduce that the restriction maps are all the identity. If (2) holds, the result follows in a similar fashion by [Maz19, Theorem 3.5]. □

Let P be a discrete p-toral group. Let $P_1 \leq P_2 \leq \ldots$ be a tower for P. If P_n is cyclic, dihedral, semi-dihedral or quaternion for just a finite number of n, then we can always consider ignore the first few terms and re-index the tower so that P satisfies the hypothesis of Proposition 4.2.6. If P_n is cyclic, dihedral, semi-dihedral or quaternion for an infinite number of n, then we can extract a tower so that all the terms are of the same type, hence P would be isomorphic to one of the groups in Proposition 4.2.4. Thus we have covered completely the class of artinian countable locally finite p-groups.

The following result completes the classification of the Picard group for the class of countable locally finite p-groups.

Proposition 4.2.8. *Let P be a countable locally finite p-group. If P is not artinian, then $T(P) \cong \mathbb{Z}$.*

Proof. By Lemma 3.1 in [KW73], we have that P contains an infinite elementary abelian subgroup. Hence there is a tower $P_1 \leq P_2 \leq P_3 \leq \ldots$ so that P_n has p-rank at least $p+4$, for all $n \geq 1$. By [Maz19, Theorem 3.5] We have $T(P_n) \cong \mathbb{Z}$, for all $n \geq 1$. We deduce that $\{T(P_n)\}$ is constant, and the result follows. □

4.3 Countable Locally Finite Groups

Remark 4.3.1. For general finite groups, the description of the group of endotrivial modules by generators and relations is not complete. Hence it is more elaborated to identify the restriction maps $\mathrm{Res}\colon T(G_{n+1}) \to T(G_n)$ in an inverse system for a countable locally finite group G. A different approach is to study the restriction map $\mathrm{Res}\colon T(G) \to T(S)$ where S is a maximal p-subgroup of G. However, we need to be careful since there are locally finite groups whose maximal p-subgroups are not all isomorphic (see [KW73, Section 1.D] for a discussion).

Definition 4.3.2. Let G be a group. We say that G is *p-artinian* if any p-subgroup of G is artinian.

Let G be a p-artinian countable locally finite group. In this case, a maximal p-subgroup of G plays the role of a Sylow p-subgroup. In particular, any two maximal p-subgroups of G are isomorphic (see [KW73, Theorem 3.7]). Consider a tower of finite groups $G_1 \leq G_2 \leq \ldots$ of G. Set S_1 a p-Sylow subgroup of G_1. For each $n \geq 2$ we can find a p-Sylow subgroup S_n of G_n such that $S_{n-1} \leq S_n$. Then we obtain a ascending chain S_n of finite p-subgroups of G. Then $S = \cup S_n$ is a maximal p-subgroup. Moreover, note that S is a discrete p-toral group.

By Proposition 4.2.2 we know that $T(G)$ agrees with the projective limit of the inverse system $\{T(G_n)\}$ with maps induced by the restriction maps. We would like to give a better description of this projective limit.

For $n \geq 1$ consider the restriction map $\mathrm{Res}\colon T(G_n) \to T(S_n)$ and denote its image by $\overline{T}(S_n)$ and its kernel by $T(G_n, S_n)$. Then we have a short exact sequence of abelian groups

$$0 \to T(G_n, S_n) \to T(G_n) \to \overline{T}(S_n) \to 0.$$

Hence we obtain a short exact sequence of inverse systems. The following sequence is exact since $T(G_n, S_n)$ is finite for all $n \geq 1$, hence we can use the Mittag-Leffler condition for the vanishing of the $\varprojlim^1$.

$$0 \to \varprojlim_n T(G_n, S_n) \to T(G) \to \varprojlim_n \overline{T}(S_n) \to 0.$$

We can identify $\varprojlim_n \overline{T}(S_n)$ as a subgroup of $T(S)$. Moreover, this group agrees with the image of the restriction map Res: $T(G) \to T(S)$ and as a consequence we obtain that $\mathrm{Ker}(\mathrm{Res}) \cong \varprojlim_n T(G_n, S_n)$. We will denote this group by $T(G, S)$.

As we mentioned before, we have the inconvenience that we do not have an explicit description of the restriction maps for arbitrary finite groups as in the case of finite p-groups. Thus the best we can do is considering special cases of countable locally finite groups. We will conclude this section with a couple of examples where we are able to determine $T(G)$ as an abstract group.

Example 4.3.3. Let G be an abelian countable locally finite group that has p-rank at least 2. Suppose additionally that G is p-artinian. Let S be a maximal p-subgroup of G constructed from a tower $\{S_n\}$ of Sylow p-subgroups as above. Since G is of p-rank at least 2, we can prove that $T(S)$ is infinite cyclic. Hence, we deduce that $\varprojlim \overline{T}(S_n) \cong T(S)$. On the other hand, we have that $T(G_n, S_n) \cong \mathrm{Hom}(G_n, k^\times)$. It follows that

$$T(G) \cong \mathbb{Z} \oplus \mathrm{Hom}(G, k^\times).$$

Example 4.3.4. We say that a locally finite group G is *p-nilpotent* if any p-subgroup of G is nilpotent. Let G be a locally finite group that is p-artinian and p-nilpotent. Then we have a tower $G_1 \leq G_2 \leq \ldots$ where G_n is a finite p-nilpotent group for any $n \geq 1$. By [CMT11, Theorem 3.3] we have a short exact sequence of abelian groups

$$0 \to \mathrm{Hom}(G_n, k^\times) \to T(G_n) \to T(S_n) \to 0$$

where S_n is a Sylow p-subgroup of G_n, for $n \geq 1$. Then

$$T(G) \cong \mathrm{Hom}(G, k^\times) \oplus T(S)$$

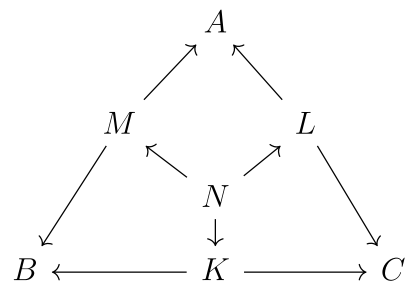
where S is a maximal p-subgroup of G.

4.4 Amalgam Groups

In this section, we let G be a group with geometric dimension two for the family of finite groups. In particular, we are interested in the case where the fundamental domain of the action is homeomorphic to the standard 2-simplex.

Definition 4.4.1. We will say that G is an *amalgam group*, if it admits a 2-dimensional model X for $\underline{E}G$ such that the fundamental domain of the action is homeomorphic to the standard 2-simplex. In particular, amalgam groups are groups of type Φ.

Let $\mathcal{T}$ denote the barycentric subdivision of Δ^2. A triangle of groups is a functor $\mathcal{T} \to \mathrm{Gps}$ that can be depicted as commutative diagram of groups



where all the maps are injective maps, these diagrams are also known as *punctured cubes*. The groups A, B and C are called vertex groups, K, L and M are called edge groups and N is called the face group. In other words, it corresponds to a functor from the barycentric subdivision of the standard 2-simplex that associates a group to each vertex, an injective morphism to each edge satisfying a compatibility condition for the composition on each face (see [FP97, Section 1]).

Any amalgam group defines a triangle of groups where the groups correspond to the isotropy groups of the vertices, edges and face, and the maps are given by inclusions. However, this assignation is not bijective, there are triangles of groups that do not correspond to an amalgam group. If the triangle of groups is non-positively curved, then the fundamental group of the triangle of groups is an amalgam group (see [Hae92], [Sta91]).

Theorem 4.4.2. *Let G be an amalgam group and $\mathcal{T} \to Gps$ be its associated triangle of groups. Then we have an exact sequence of abelian groups*

$$0 \to H^1(\mathcal{T}; \pi_1 \circ f) \to T(G) \to H^0(\mathcal{T}; \pi_0 \circ f) \to 0$$

where f is the composition of the Picard space functor and the stable module ∞-category functor corresponding to the triangle of groups $\mathcal{T} \to Gps$. Moreover, if p divides the order of the face group, then the map

$$T(G) \to H^0(\mathcal{T}; \pi_0 \circ f)$$

is an isomorphism.

Proof. By Equation 4.3, we have a spectral sequence

$$E_2^{p,q} = H^p(\mathcal{T}; \pi_q \circ \mathrm{Pic} \circ \mathrm{StMod}) \Rightarrow \pi_{p-q}(\mathrm{Pic}(\mathrm{StMod}(kG)))$$

and recall that $\pi_0 \circ \mathrm{Pic} \circ \mathrm{StMod}(kG)$ corresponds to the Picard group $T(G)$. Let σ

be a simplex in the 2-dimensional model X for $\underline{E}G$. We have

$$\pi_1(\mathrm{Pic}(\mathrm{StMod}(kG_\sigma))) = \widehat{H}^0(G_\sigma;k) = \begin{cases} k^\times & \text{if } p \text{ divides } |G_\sigma| \\ 0 & \text{otherwise.} \end{cases}$$

$$\pi_2(\mathrm{Pic}(\mathrm{StMod}(kG_\sigma))) = \widehat{H}^{-1}(G_\sigma;k) = \begin{cases} k & \text{if } p \text{ divides } |G_\sigma| \\ 0 & \text{otherwise.} \end{cases}$$

Suppose first that p does not divide the order of the face group. Then it is clear that $H^2(\mathcal{T};\pi_q \circ \mathrm{Pic} \circ \mathrm{StMod})$ is trivial for $q = 1, 2$, and the result follows. On the other hand, if p divides the order of the face group, then $\pi_q \circ \mathrm{Pic} \circ \mathrm{StMod}$ is a constant diagram for $q = 1, 2$, therefore $H^2(\mathcal{T};\pi_1 \circ \mathrm{Pic} \circ \mathrm{StMod}) \cong H^2(|\mathcal{T}|;k^\times) = 0$ and $H^2(\mathcal{T};\pi_2 \circ \mathrm{Pic} \circ \mathrm{StMod}) = H^2(|\mathcal{T}|;k) = 0$. Hence the result follows. □

Example 4.4.3. Consider the Coxeter group $G = \Delta^*(2,4,4)$ of isometries of the Euclidean plane generated by the reflections across the sides of a triangle with angles $\pi/2, \pi/4$ and $\pi/4$. In this case, G is an amalgam group arising from a non-positively curved triangle of groups with trivial face group, with each edge group isomorphic to $\mathbb{Z}/2$ and with vertex groups isomorphic to the dihedral groups D_8, D_{16} and D_{16}, where D_n denotes the dihedral group of order n.

In particular, the diagram $\pi_1 \circ \mathrm{Pic} \circ \mathrm{StMod}$ is constant. It follows that $H^1(\mathcal{T};\pi_1 \circ \mathrm{Pic} \circ \mathrm{StMod}) \cong H^1(|\mathcal{T}|;k^\times) \cong k^\times$. Moreover, $T(\mathbb{Z}/2) = 0$, hence

$$T(G) \cong T(D_8) \oplus T(D_{16}) \oplus T(D_{16}) \oplus k^\times \cong \mathbb{Z}^6 \oplus k^\times.$$

4.5 Locally trivial modules for amalgam groups

In this section, we will discuss a different approach to compute the Picard group for amalgam groups with trivial face group. In particular, we will focus on modules that are stably isomorphic to the trivial module after restriction to any finite subgroup. We will provide a construction of these modules.

Definition 4.5.1. Let G be a group of type Φ. We say that a kG–module M is *locally trivial* if the restriction to any finite subgroup of G is stably isomorphic to the trivial module k, that is, if $M{\downarrow_H} \simeq k$ for any finite subgroup H of G. Let $T_{\mathrm{loc}}(G)$ denote the group of isomorphism classes of locally trivial kG–modules equipped with the tensor product.

Remark 4.5.2. Note that any locally trivial kG–module is invertible. In fact, the group of locally trivial modules corresponds to the kernel of the restriction map

$$\operatorname{Res}\colon T(G) \to \prod_H T(H)$$

where H runs through the family of finite subgroups of G.

Let G be an amalgam group. Recall that, by definition, there is a 2-dimensional model X for $\underline{E}G$ such that the fundamental domain of the action Y is homeomorphic to the standard 2-simplex. Let VY, EY, FY be the sets of vertices, edges and faces of Y, respectively. Consider the following complex of abelian groups

$$\prod_{v\in VY} \widehat{\operatorname{Aut}}_{G_v}(k) \xrightarrow{d^1} \prod_{e\in EY} \widehat{\operatorname{Aut}}_{G_e}(k) \xrightarrow{d^2} \prod_{f\in FY} \widehat{\operatorname{Aut}}_{G_f}(k)$$

with d^1 and d^2 given by

$$(\varphi_v)_{v\in VY} \mapsto (\varphi_{\tau(e)}^{-1}{\downarrow_{G_e}}\varphi_{\iota(e)}{\downarrow_{G_e}})_{e\in EY}$$

and

$$(\varphi_e)_{e\in EY} \mapsto (\varphi_{f_2}{\downarrow_{G_f}}\varphi_{f_0}{\downarrow_{G_f}}\varphi_{f_1}^{-1}{\downarrow_{G_f}})_{f\in FY}$$

respectively, where f_j denotes the jth face of f. We will omit the restriction of an isomorphism when it is clear from the context. We call $\operatorname{Ker}(d^2)$ the *group of 1-cocycles* and $\operatorname{Im}(d^1)$ the *group of 1-coboundaries*. Define $\check{H}^1(G)$ as the quotient group of the 1-cocycles over the 1-coboundaries. We will construct an assignment

$$\sigma\colon T_{\text{loc}}(G) \to \check{H}^1(G)$$

and we will show that this is an isomorphism when G has trivial face group.

Let M be a locally trivial kG–module. Choose a stable isomorphism of kG_v–modules $\xi_v\colon M \to k$, for each $v \in VY$. Let $\varphi_e := \xi_{\tau(e)}\xi_{\iota(e)}^{-1}$, which is an element in $\widehat{\operatorname{Aut}}_{G_e}(k)$. Note that

$$\sigma(M,\xi_v) := (\varphi_e)_{e\in EY} \in \prod_{e\in EY} \widehat{\operatorname{Aut}}_{G_e}(k)$$

is a 1-cocycle. We define $\sigma([M])$ as the class of $\sigma(M,\xi_v)$. We will show that this definition is independent of the representative of $[M]$ and the choice of ξ_v. Let $\xi'_v\colon M \to k$ be a stable isomorphism of kG_v–modules. Consider $\tau_v := \xi'_v\xi_v^{-1} \in \widehat{\operatorname{Aut}}_{G_v}(k)$ for each $v \in VY$. We have that

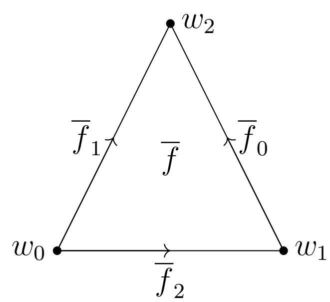
$$\sigma(M,\xi)\sigma(M,\xi')^{-1} = d^1((\tau_v)_{v\in VY})$$

hence σ does not depend on ξ. On the other hand, let N be stably isomorphic to M, and let $\psi\colon M \to N$ be a stable isomorphism. Set $\xi'\colon N \to k$ as $\xi'_v := \xi_v\psi^{-1}$. Then $\sigma(M,\xi) = \sigma(N,\xi')$.

In order to show that σ is a surjective map, we will exhibit a right inverse of σ. We describe first the main idea in the following argument. Given an element φ in $\check{H}^1(G)$, we will define a G–equivariant local coefficient system that depends on φ and show that its chain complex determines a locally trivial module via the equivalence of categories $\underline{\mathrm{StMod}}(kG) \to D^b(kG)/K^b(\mathrm{Proj}(kG))$.

Let X be a 2-dimensional model for $\underline{E}G$ such that the fundamental domain of the action Y is homeomorphic to the standard 2-simplex. Let VX, EX, FX be the sets of vertices, edges and faces of X, respectively. For each vertex $v \in VX$, let $\overline{v}$ denote the only vertex of Y in the same G–orbit of v. Fix $t_v \in G$ such that $t_v\overline{v} = v$, and take $t_v = 1$ if $\overline{v} = v$. We will use the same notation to refer the only edge (resp. face) of Y in the same G–orbit of a given edge (resp. face) of X.

Consider the 2-simplex Y with an orientation and labels for the vertices, edges and face as follows:



Let k_{w_i} denote the trivial kG_{w_i}–module for $i = 0, 1, 2$. We obtain a module for any vertex of the complex X by setting $k_v := t_v \otimes k_{\overline{v}}$ as the kG_v–module with the same action as in the restriction of the induced module $kG \otimes_{kG_{\overline{v}}} k$. Explicitly, $g(t_v \otimes m) = t_v \otimes (t_v^{-1}gt_v)m$ for $g \in G_v$. Note that the action of G on the direct sum of the modules k_v permutes the summands. Let $k_e := k_{\iota(e)}{\downarrow_{G_e}}$ for $e \in EX$ and $k_f := k_{f_0}{\downarrow_{G_f}}$ for $f \in FX$.

Let $\varphi \in \check{H}^1(G)$. Consider a representative $(\varphi_{\overline{f}_0}, \varphi_{\overline{f}_1}, \varphi_{\overline{f}_2})$ of φ. Recall that φ_e is an element in $\widehat{\mathrm{Aut}}_{G_e}(e)$ for any edge e, and hence it corresponds to a unit of k. Moreover, $(\varphi_{\overline{f}_0}, \varphi_{\overline{f}_1}, \varphi_{\overline{f}_2})$ satisfies the condition $\varphi_{\overline{f}_0}\varphi_{\overline{f}_2} = \varphi_{\overline{f}_1}$. We can extend this collection of automorphisms indexed by the edges of Y to a collection of automorphisms indexed by the edges of X. For any $e \in EX$, we can define a kG_e–isomorphism as follows

$$\begin{aligned}\varphi_e\colon k_{\iota(e)} &\to k_{\tau(e)}\\ t_{\iota(e)} \otimes m &\mapsto t_{\tau(e)} \otimes \varphi_{\bar{e}}(m)\end{aligned}$$

so that these maps satisfy the cocycle condition $\varphi_{f_1} = \varphi_{f_0}\varphi_{f_2}$ for any face $f \in FX$, where f_j denotes the jth face of f. Let Vect_k denote the category of k-vector spaces. We will define a G–equivariant local coefficient system of k-vector spaces $\mathcal{F}_\varphi \colon X \to \mathrm{Vect}_k$ as follows (we refer to [Ben98, Section 7.1] for more details about local coefficient systems). For a simplex $x \in X$, set $\mathcal{F}_\varphi(x) := k_x$. Let e be an edge of X, and let v denote a vertex of e.

$$\mathcal{F}_\varphi(v \to e) = \begin{cases} 1_{k_e} & \text{if } v = \iota(e), \\ \varphi_e & \text{if } v = \tau(e). \end{cases}$$

For a face f of X, and an edge e of f, we let

$$\mathcal{F}_\varphi(e \to f) = \begin{cases} \varphi_e & \text{if } e = f_0, \\ \varphi_{f_1} & \text{if } e = f_1, \\ 1_{k_f} & \text{if } e = f_2. \end{cases}$$

Note that $\mathcal{F}_\varphi$ sends morphisms to isomorphisms of vector spaces. Hence $\mathcal{F}_\varphi$ is a G–twisted local coefficient system. The associated chain complex $C_*(X; \mathcal{F}_\varphi)$ is given by (see [Ben98, Definition 7.3.1])

$$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

where the terms of this chain complex correspond to the kG–modules

$$C_0 = \bigoplus_{v \in VX} k_v, \quad C_1 = \bigoplus_{e \in EX} k_e, \quad C_2 = \bigoplus_{f \in FX} k_f$$

that is

$$C_0 \cong \bigoplus_{j=0}^{2} k_{w_j}\!\uparrow_{G_{w_j}}^{G}, \quad C_1 \cong \bigoplus_{j=0}^{2} k_{\overline{f}_j}\!\uparrow_{G_{\overline{f}_j}}^{G}, \quad C_2 \cong k_{\overline{f}}\!\uparrow_{G_{\overline{f}}}^{G}$$

with differentials given as follows. If $x \in \mathcal{F}_\varphi(f)$ with f a face of X and $y \in \mathcal{F}_\varphi(e)$ with e and edge of X, then

$$d_2(x) = \sum_{i=0}^{2} (-1)^i \mathcal{F}_\varphi(f_i \to f)(x)$$

and

$$d_1(y) = \mathcal{F}_\varphi(\tau(e) \to e)(y) - \mathcal{F}_\varphi(\iota(e) \to e)(y).$$

Proposition 4.5.3. *Let G be an amalgam group. Then $\sigma \colon T_{loc}(G) \to \check{H}^1(G)$ is a surjective map.*

Proof. First, we claim that for an element $\varphi \in \check{H}^1(G)$, the complex $C_*(X; \mathcal{F}_\varphi)$ described above determines a locally trivial module through the equivalence of categories

$$\underline{\mathrm{StMod}}(kG) \to D^b(kG)/K^b(\mathrm{Proj}(kG)).$$

Let v be a vertex of X. Consider the G–twisted coefficient system $\mathcal{F}'_\varphi$ on v induced by the inclusion $v \to X$. Since X is a model for $\underline{E}G$, it is G_v–contractible. It follows that

$$\mathcal{F}_\varphi(v) \simeq C_*(v; \mathcal{F}'_\varphi) \xrightarrow{\simeq} C_*(X; \mathcal{F})$$

is a quasi-isomorphism since G–twisted coefficient systems are hG–homotopy invariants (see [Gro23, Subsection 3.2]). Hence the claim follows. Moreover, since the equivalence of categories mentioned above is compatible with restriction and conjugation, we deduce that $\sigma(C_*(X; \mathcal{F}_\varphi))$ corresponds to φ in $\check{H}^1(G)$. Therefore σ is surjective. $\square$

Proposition 4.5.4. *Let G be an amalgam group with trivial face group. Then $\sigma\colon T_{loc}(G) \to \check{H}^1(G)$ is a group isomorphism.*

We need to show some auxiliary results first, so we will leave the proof of this proposition until the end of this section. Let G be an amalgam group with a 2-dimensional model X for $\underline{E}G$. Consider the action of G on the 1-skeleton $X^{(1)}$ of X. We can follow [DD89, Section I.9] to construct the *universal covering group* Γ of G. Recall that Γ satisfies the following properties:

(a) Γ acts properly on a tree T.

(b) T is the universal covering space of $X^{(1)}$.

(c) The graphs of groups T/Γ and X/G are isomorphic.

(d) There is a short exact sequence of groups

$$1 \to \pi_1(X^{(1)}) \to \Gamma \to G \to 1$$

where $\pi_1(X^{(1)})$ denotes the fundamental group of $X^{(1)}$. Note that $\pi_1(X^{(1)})$ is a free group.

Moreover, Γ is a group of type Φ since T is a 1-dimensional model for $\underline{E}\Gamma$.

Proposition 4.5.5. *Let G be an amalgam group with trivial face group and X the associated 2-dimensional simplicial complex. Let Γ be the universal covering group of G. Then the inflation functor given by the quotient homomorphism $p\colon \Gamma \to G$*

$$\mathrm{Inf}\colon \mathrm{Mod}(kG) \to \mathrm{Mod}(k\Gamma)$$

is exact and maps modules of finite projective dimension to modules of finite projective dimension. Moreover, Inf *is a fully faithful functor.*

Proof. The exactness is straightforward. For the second part, let M be a kG–module of finite projective dimension. Let H be a finite subgroup of Γ. Since the kernel of p is a free group, we deduce that $p|_H\colon H \to p(H)$ is a group isomorphism. Note that $\mathrm{Inf}(M){\downarrow_H} \cong M{\downarrow_{p(H)}}$ as kH–modules, hence $\mathrm{Inf}(M){\downarrow_H}$ is projective. Since Γ is a group of type Φ, we deduce that $\mathrm{Inf}(M)$ has finite projective dimension. On the other hand, recall that extension of scalars

$$kG \otimes_{k\Gamma} -\colon \mathrm{Mod}(k\Gamma) \to \mathrm{Mod}(kG)$$

is a left adjoint of Inf. Moreover, it is isomorphic to the coinvariants functor

$$(-)_{\pi_1(X^{(1)})}\colon \mathrm{Mod}(k\Gamma) \to \mathrm{Mod}(kG).$$

In particular, it is clear that the counit of the adjunction $(-)_{\pi_1(X^{(1)})} \dashv \mathrm{Inf}$ is an isomorphism. Hence Inf is fully faithful □

Proposition 4.5.6. *Let G be an amalgam group with trivial face group. Then the inflation functor induces an injective group homomorphism* $\mathrm{Inf}\colon T(G) \to T(\Gamma)$.

Proof. By Proposition 4.5.5, we have that Inf defines an exact functor of stable module categories. Note that inflation is a strongly monoidal functor and maps invertible modules to invertible modules, so we have a homomorphism $T(G) \to T(\Gamma)$. We claim that inflation reflects stable isomorphisms. Let $M \to N$ be a homomorphism of kG–modules satisfying that $\mathrm{Inf}(M) \to \mathrm{Inf}(N)$ is a stable isomorphism. Let H be a finite subgroup of Γ. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Inf}(M){\downarrow_H} & \longrightarrow & \mathrm{Inf}(N){\downarrow_H} \\ \downarrow & & \downarrow \\ M{\downarrow_{p(H)}} & \longrightarrow & N{\downarrow_{p(H)}} \end{array}$$

where the vertical maps are isomorphisms of kH–modules, and the top map is a stable isomorphism. Then the bottom map is a stable isomorphism as well. The claim follows because stable isomorphisms are detected by the family of finite subgroups of G. Since Inf is fully faithful and reflects stable isomorphisms, we deduce that $\mathrm{Inf}\colon T(G) \to T(\Gamma)$ is injective. □

Proof of Proposition 4.5.4. Consider the universal covering group Γ of G. Recall that we have an analogous definition of $\check{H}^1(\Gamma)$ and the map $\sigma\colon T_{\mathrm{loc}}(\Gamma) \to \check{H}^1(\Gamma)$ is an isomorphism (see [MS19, Theorem 7.4]). Note that the following diagram is commutative.

$$\begin{array}{ccc} T_{\mathrm{loc}}(G) & \xrightarrow{\sigma} & \check{H}^1(G) \\ {\scriptstyle \mathrm{Inf}}\downarrow & & \downarrow \\ T_{\mathrm{loc}}(\Gamma) & \xrightarrow[\sigma]{} & \check{H}^1(\Gamma) \end{array}$$

where the map $\check{H}^1(G) \to \check{H}^1(\Gamma)$ is induced by the quotient map $\Gamma \to G$. The vertical left map is injective by Proposition 4.5.6. Hence the top map is an injective homomorphism. By Proposition 4.5.3, the map σ is surjective, and hence it is an isomorphism. $\square$

Chapter 5

Separable algebra objects of infinite degree

In this chapter we give background material on tensor triangular geometry in order to introduce the degree of a tt-ring as well as some of its properties. All known tt-rings in the literature have finite degree and it is an open question in [Bal14] whether any tt-ring in an essentially small tensor-triangulated category must have finite degree. We construct e a family of infinite degree tt-rings, giving a negative answer to this question. In fact, this family extends to a family of infinite degree rigid-compact tt-rings in the framework of rigidly-compactly generated tensor triangulated categories. We refer to [Bal05] and [Bal10] for more details about the introductory material in this chapter.

5.1 Set-up

Recall that an *essentially small category* is a category such that the collection of isomorphism classes of its objects determines a set. This is the only restriction that we will place on the categories in this section, so we will work in full generality as in [Bal05].

Definition 5.1.1. A *tensor triangulated category* $(\mathcal{K}, \otimes, \mathbf{1})$ is a triangulated category $\mathcal{K}$ with a symmetric monoidal structure $\mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$ and unit $\mathbf{1} \in \mathcal{K}$, such that the tensor product $\otimes$ is exact in each variable. A *tensor triangulated* functor $F \colon \mathcal{K} \to \mathcal{L}$ is an exact functor that is strongly monoidal and preserves the unit.

Let $\mathcal{J}$ be a non-empty full subcategory of a tensor triangulated category $\mathcal{K}$. Suppose that $\mathcal{J}$ is a *triangulated subcategory* of $\mathcal{K}$, that is, $\mathcal{J}$ is closed under cones and the suspension functor. Recall that:

- $\mathcal{J}$ is a *thick subcategory* if it is closed under retracts. That is, if $a \oplus b$ belong to $\mathcal{J}$, then a and b belong to $\mathcal{J}$.
- $\mathcal{J}$ is a *tensor-ideal* if $a \otimes b \in \mathcal{J}$ provided that $a \in \mathcal{J}$ and $b \in \mathcal{K}$.
- $\mathcal{J}$ is *prime* if it is a proper thick tensor-ideal such that $a \otimes b \in \mathcal{J}$ implies that $a \in \mathcal{J}$ or $b \in \mathcal{J}$.

Let $\mathcal{K}$ be an essentially small tensor triangulated category. Let $\mathrm{Spc}(\mathcal{K})$ denote the set of all primes $\mathcal{P} \subset \mathcal{K}$. The *support* $\mathrm{supp}(a)$ of an object $a \in \mathcal{K}$ is the set of all primes $\mathcal{P} \subset \mathcal{K}$ such that $a \notin \mathcal{P}$. The complements $U(a) := \mathrm{Spc}(\mathcal{K}) \backslash \mathrm{supp}(a)$, for all $a \in \mathcal{K}$, define an open basis for a topology on $\mathrm{Spc}(\mathcal{K})$. The *Balmer spectrum* of $\mathcal{K}$ is $\mathrm{Spc}(\mathcal{K})$ equipped with this topology.

The support $\mathrm{supp}(a)$ defines a *support datum on* $\mathcal{K}$. That is, it is an assignment from $\mathcal{K}$ to the set of closed subsets $\mathrm{Cl}(\mathrm{Spc}(\mathcal{K}))$ of $\mathrm{Spc}(\mathcal{K})$, which satisfies the following properties.

- $\mathrm{supp}(0) = \emptyset$ and $\mathrm{supp}(\mathbf{1}) = \mathcal{K}$.
- $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$.
- $\mathrm{supp}(\Sigma a) = \mathrm{supp}(a)$.
- If $a \to b \to c \to \Sigma a$ is a distinguished triangle in $\mathcal{K}$, then $\mathrm{supp}(a) \subseteq \mathrm{supp}(b) \cup \mathrm{supp}(c)$.
- $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$.

Then the Balmer spectrum $\mathrm{Spc}(\mathcal{K})$ together with $\mathrm{supp}(-)\colon \mathcal{K} \to \mathrm{Cl}(\mathrm{Spc}(\mathcal{K}))$ is the final support datum on $\mathcal{K}$. That is, if X is a topological space together an assignment $\sigma\colon \mathcal{K} \to \mathrm{Cl}(X)$ satisfying the above properties, then there exists a unique continuous map $\varphi\colon X \to \mathrm{Spc}(\mathcal{K})$ such that $\sigma(a) = \varphi^{-1}(\mathrm{supp}(a))$ for $a \in \mathcal{K}$ (see [Bal05, Theorem 3.2]).

Remark 5.1.2. The Balmer spectrum $\mathrm{Spc}(\mathcal{K})$ of an essentially small tensor triangulated category $\mathcal{K}$ is a spectral space. That is, the following properties hold.

- $\mathrm{Spc}(\mathcal{K})$ is quasi-compact and T_0.
- The topology on $\mathrm{Spc}(\mathcal{K})$ has a basis of quasi-compact open subsets.
- The collection of quasi-compact open subsets is closed under finite intersections.
- $\mathrm{Spc}(\mathcal{K})$ is *sober*. That is, any non-empty closed and irreducible subset of $\mathrm{Spc}(\mathcal{K})$ has a unique generic point.

We refer to [DST19, Chapter 1] for more details about spectral topological spaces.

Let $F\colon \mathcal{K} \to \mathcal{L}$ be a tensor triangulated functor. The assignment $\mathcal{P} \to F^{-1}(\mathcal{P})$ defines a spectral map[1] $\varphi\colon \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$. In other words, $\mathrm{Spc}(-)$ defines a contravariant functor from the category of essentially small triangulated categories to the category of spectral spaces.

Example 5.1.3. Let G be a finite group. Let k be a field of prime characteristic p dividing the order of G. Then the stable module category $\underline{\mathrm{stmod}}(kG)$, which is obtained from the category of finitely generated kG–modules by factoring out projective modules, is an essentially small tensor triangulated category; the symmetric monoidal structure is inherited by the one in the category of kG–modules given by the tensor product $\otimes_k$ over the ground field endowed with the diagonal action of G. In this case, the Balmer spectrum corresponds to the projective support variety $\mathrm{Proj}\, H^{\bullet}(G; k)$ (see [BCR97, Theorem 3.4]).

5.2 Degree of a tt-ring

A *commutative algebra object* A in a tensor triangulated category $\mathcal{K}$ is an associative commutative monoid (A, μ, η) internal to the symmetric monoidal category $(\mathcal{K}, \otimes, \mathbf{1})$, that is, an object A in $\mathcal{K}$ together with a multiplication $\mu\colon A \otimes A \to A$ and a unit $\eta\colon \mathbf{1} \to A$ such that the appropriate diagrams for associativity, commutativity and unit are commutative. Let (A, μ, η) be an algebra object in $\mathcal{K}$. An A–module (M, h) is an object M together with an A–*action* $h\colon A \otimes M \to M$ such that the diagrams for associativity and unit are commutative. An *A-linear morphism* of A–modules $f\colon (M, h) \to (M', h')$ is a morphism $f\colon M \to M'$ in $\mathcal{K}$ such that $f \circ h = h' \circ 1_A \otimes f$. Let $\mathrm{Mod}_{\mathcal{K}}(A)$ denote the category of A–modules and A–linear morphisms. There is an adjunction

$$F_A\colon \mathcal{K} \leftrightarrows \mathrm{Mod}_{\mathcal{K}}(A)\colon U_A$$

given by *extension of scalars* $F_A\colon \mathcal{K} \to \mathrm{Mod}_{\mathcal{K}}(A)$ that sends x to $(A \otimes x, \mu \otimes 1_x)$, and the *forgetful* functor $U_A\colon \mathrm{Mod}_{\mathcal{K}}(A) \to \mathcal{K}$.

Definition 5.2.1. A *tt-ring* (A, μ, η) in a tensor triangulated category $\mathcal{K}$ is an algebra object that is *separable*, i.e., the multiplication μ admits an (A, A)–bilinear section $\sigma\colon A \to A \otimes A$.

In particular, the category of A–modules $\mathrm{Mod}_{\mathcal{K}}(A)$ remains tensor triangulated. The triangulated structure in $\mathrm{Mod}_{\mathcal{K}}(A)$ is such that both extension of scalars $F_A\colon \mathcal{K} \to$

[1] A spectral map is a continuous map such that the inverse image of any quasi-compact open set is quasi-compact open.

$\mathrm{Mod}_{\mathcal{K}}(A)$ and the forgetful functor $U_A\colon \mathrm{Mod}_{\mathcal{K}}(A) \to \mathcal{K}$ are exact (see [Bal14, Section 1]). The monoidal structure in the category of A–modules is given as follows. Let $\mathrm{Free}_{\mathcal{K}}(A)$ denote the *Kleisli category*, i.e., the full subcategory of $\mathrm{Mod}_{\mathcal{K}}(A)$ on the *free* A–modules $F_A(x)$. Define $\otimes_A\colon \mathrm{Free}_{\mathcal{K}}(A) \times \mathrm{Free}_{\mathcal{K}}(A) \to \mathrm{Free}_{\mathcal{K}}(A)$ by $F_A(x) \otimes_A F_A(y) := F_A(x \otimes y)$. By the separability of A, we can prove that the co-unit of the Eilenberg-Moore adjunction given by tensoring with A is split surjective. In particular, any A-module is a retract of a free module: for instance M is a retract of $F_A(U_A(M))$. In fact, $\mathrm{Mod}_{\mathcal{K}}(A)$ is the idempotent completion of $\mathrm{Free}_{\mathcal{K}}(A)$. Hence we obtain a symmetric monoidal product $\otimes_A$ for $\mathrm{Mod}_{\mathcal{K}}(A)$.

Moreover, extension of scalars is a tt-functor, that is, it is exact and strongly monoidal (see [Bal14, Section 1]). For tt-rings the projection formula holds (see [Bal14, Proposition 1.1]):

$$U_A(x \otimes_A F_A(y)) \cong U_A(x) \otimes y$$

in $\mathcal{K}$, for all $x \in \mathrm{Mod}_{\mathcal{K}}(A)$ and all $y \in \mathcal{K}$.

A *morphism of tt-rings* $\alpha\colon A \to B$ is a morphism in $\mathcal{K}$ compatible with the multiplication and units of A and B. In this case, we say that B is an *A–algebra*. The following is Theorem 2.4 in [Bal14].

Theorem 5.2.2. *Let A be a tt-ring in a tensor triangulated category $\mathcal{K}$. Then there exists an isomorphism of tt-rings*

$$\alpha\colon A \otimes A \to A \times B$$

for some tt-ring B in $\mathcal{K}$ such that $\mathrm{pr}_1 \circ \alpha$ is precisely the multiplication μ of A. In fact, B is characterized as the unique A–algebra, up to isomorphism, with this property.

This theorem is the main ingredient to define the notion of degree for tt-rings. Let $A^{[0]} := \mathbf{1}$ and $A^{[1]} := A$. For $n \geq 1$, define $A^{[n+1]}$ as the $A^{[n]}$–algebra for which there exists an isomorphism of $A^{[n]}$–algebras

$$\alpha_n\colon A^{[n]} \otimes_{A^{[n-1]}} A^{[n]} \to A^{[n]} \times A^{[n+1]}$$

with $\mathrm{pr}_1 \circ \alpha = \mu_{A^{[n]}}$, where $\otimes_{A^{[n-1]}}$ denotes the tensor product on $\mathrm{Mod}_{\mathcal{K}}(A^{[n-1]})$.

Definition 5.2.3. The tt-ring A has *finite degree* $n \geq 0$ if $A^{[m]} = 0$ for all $m \geq n+1$ (equivalent to $A^{[n+1]} = 0$). The tt-ring has *infinite degree* if $A^{[n]} \neq 0$ for all $n \geq 0$. Let $\deg(A) \in \mathbb{N} \cup \{\infty\}$ denote the degree of A.

Theorem 5.2.4. *Let A be a tt-ring. The following hold.*

(a) Let $F\colon \mathcal{K} \to \mathcal{L}$ be a tt-functor. Then $deg(F(A)) \leq deg(A)$.

(b) A has finite degree if and only if $q_{\mathcal{P}}(A)$ has finite degree in $\mathcal{K}_{\mathcal{P}}$ for every prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$, where $\mathcal{K}_{\mathcal{P}}$ denotes the idempotent completion of the Verdier quotient $\mathcal{K}/\mathcal{P}$ and $q_{\mathcal{P}}$ denotes the composition $\mathcal{K} \to \mathcal{K}/\mathcal{P} \to \mathcal{K}_{\mathcal{P}}$.

Proof. (a) We claim that $F(A)^{[n]} \cong F(A^{[n]})$ for $n \geq 0$. Applying F to the isomorphism of $A^{[n]}$–algebras as in Definition 5.2.3 we have

$$F(\alpha_n)\colon F(A^{[n]}) \otimes_{F(A^{[n-1]})} F(A^{[n]}) \to F(A^{[n]}) \times F(A^{[n+1]})$$

and the claim follows by induction on n. If A has infinite degree there is nothing to prove. Suppose that A has finite degree n. Then $F(A)^{[n+1]} \cong F(A^{[n+1]}) = 0$, and $\deg(F(A)) \leq \deg(A)$.

(b) Note that $A^{[n+1]} \simeq \Sigma^{-1}\mathrm{cone}(\mu_{A^{[n]}})$ for $n \geq 0$. This follows from the isomorphism of $A^{[n]}$–algebras of Definition 5.2.3 and the octahedral axiom for instance. Then $\mathrm{supp}(A^{[n+1]}) = \mathrm{supp}(\Sigma^{-1}\mathrm{cone}(\mu_{A^{[n]}})) \subseteq \mathrm{supp}(A^{[n]})$. Let $\mathcal{P}$ be a prime in $\mathcal{K}$. If $q_{\mathcal{P}}(A^{[n]}) = 0$, then $P \in U(A^{[n]})$ for $n \geq 0$. Hence the collection $U(A^{[n]})$, for $n \geq 0$ defines an open cover of $\mathrm{Spc}(\mathcal{K})$. Since $\mathrm{Spc}(\mathcal{K})$ is quasi-compact, we have a finite number of open sets $U(A^{[n]})$ covering $\mathrm{Spc}(\mathcal{K})$. By the previous comment we have $U(A^{[n]}) \subseteq U(A^{[n+1]})$, thus there exists m such that $U(A^{[m]}) = \mathrm{Spc}(\mathcal{K})$. It follows that $A^{[m]} = 0$. □

5.3 Examples of tt-rings of infinite degree

For $i \in \mathbb{N}$, let K_i be a non-trivial essentially small tensor triangulated category. Define

$$\mathcal{K} := \prod_{i \in \mathbb{N}} \mathcal{K}_i.$$

It is clear that $\mathcal{K}$ is essentially small; the product of small skeletons in each component defines a small skeleton of $\mathcal{K}$. We give $\mathcal{K}$ a triangulated structure component-wise. We endow $\mathcal{K}$ with a symmetric monoidal structure component-wise. In particular, $\mathcal{K}$ is a non-trivial essentially small tensor triangulated category.

Theorem 5.3.1. *Let $\mathcal{K}_n$ and $\mathcal{K}$ as above and let $\mathbb{1}_n$ denote the monoidal unit of $\mathcal{K}_n$. Then the tt-ring*

$$A := (\mathbb{1}_n^{\times n})_{n \in \mathbb{N}} \in \mathcal{K}$$

has infinite degree with the component-wise tt-ring structure.

Proof. It is clear that A is a tt-ring with component-wise multiplication, and a component-wise bilinear section. On the other hand, by the definition of $\mathcal{K}$, the projection functor

$$\mathrm{pr}_n \colon \mathcal{K} \to \mathcal{K}_n$$

is a tensor triangulated functor for each $n \geq 0$. In particular, $\mathrm{pr}_n(A) = \mathbf{1}^{\times n}$ which has finite degree n (see [Bal14, Theorem 3.9]). Then A has infinite degree, otherwise it contradicts Theorem 5.2.4 (see [Bal14, Theorem 3.7]). □

Remark 5.3.2. By Theorem 5.2.4, it follows that there exists a prime $\mathcal{P}$ in $\mathcal{K}$ such that the tt-ring $q_{\mathcal{P}}(A)$ has infinite degree in $\mathcal{K}_{\mathcal{P}}$. Then placing the adjective *local* on an essentially small tensor triangulated category is not enough to guarantee that tt-rings have finite degree.

At first glance, our example of a tt-ring of infinite degree seems to live in an artificial tensor triangulated category. However, it is possible to find this type of example in practice, for instance in the study of the stable module categories for infinite groups. Recall that an object x in a tensor triangulated category $\mathcal{K}$ is *dualizable* if there exists an object y in $\mathcal{K}$, called *a dual of* x, and morphisms

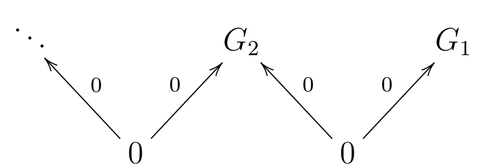
- $\mathrm{ev} \colon x \otimes y \to \mathbf{1}$, called *evaluation*, and
- $\mathrm{coev} \colon \mathbf{1} \to y \otimes x$, called *coevaluation*

such that the compositions

$$x \simeq x \otimes \mathbf{1} \xrightarrow{1_x \otimes \mathrm{coev}} x \otimes y \otimes x \xrightarrow{\mathrm{ev} \otimes 1_x} x$$
$$y \simeq \mathbf{1} \otimes y \xrightarrow{\mathrm{coev} \otimes 1_y} y \otimes x \otimes y \xrightarrow{1_y \otimes \mathrm{ev}} y$$

correspond to the respective identities. A *dualizable* object in a symmetric monoidal ∞-category $\mathcal{C}$ is an object that is dualizable in the homotopy category of $\mathcal{C}$.

Example 5.3.3. Let G be the fundamental group of the following graph of finite groups.



where G_n is a fixed finite group, for all $n \geq 1$. In other words, the group G corresponds to the free product of the groups G_n. In particular, G is a group of type Φ. By Theorem 3.2.6, we have an equivalence of symmetric monoidal stable ∞-categories

$$\mathrm{StMod}(kG) \simeq \prod_{n \in \mathbb{N}} \mathrm{StMod}(kG_n)$$

Note that dualizable objects in StMod(kG) are detected component-wise via this equivalence. In other words, we have a similar decomposition for the dualizable part of StMod(kG) (i.e., the symmetric monoidal, stable ∞-category on the dualizable objects of StMod(kG)). Moreover, this factorization induces a product decomposition at the level of homotopy categories. Hence the homotopy category of StMod(kG) satisfies the hypothesis of Theorem 5.3.1.

In practice, essentially small tensor triangulated categories arise as the dualizable part of a *bigger* tensor triangulated category which, for instance, admits small coproducts, just as in Example 5.3.3. Then we can consider tt-rings in a tensor triangulated category which sits inside a bigger one. In particular, the framework of rigidly-compactly generated tensor triangulated categories has been extensively studied (see for instance [BHS21]). In fact, all tt-rings that have been proved to have finite degree in [Bal14, Section 4] sit in the dualizable part a rigidly-compactly generated tensor triangulated category. In the rest of this section we will investigate rigid-compact tt-rings in the setting of rigidly-compactly generated tensor triangulated categories.

Recall that an object x in a triangulated category $\mathcal{K}$ with small coproducts is *compact* if the functor $\mathrm{Hom}(x,-)$ commutes with small coproducts. In particular, the subcategory $\mathcal{K}^c$ of compact objects remains triangulated.

Definition 5.3.4. A tensor triangulated category $\mathcal{K}$ is *rigidly-compactly generated* if $\mathcal{K}^c$ is essentially small, the smallest triangulated subcategory containing $\mathcal{K}^c$ which is closed under small coproducts is $\mathcal{K}$, and the class of compact objects coincides with the class of dualizable objects. In this case, $\mathcal{K}^c$ remains tensor triangulated.

Remark 5.3.5. For a general tensor triangulated category $\mathcal{K}$ with small coproducts, compact objects are not necessarily dualizable, and vice versa, dualizable objects are not necessarily compact. However, if the monoidal unit of $\mathcal{K}$ is compact, then any dualizable object in $\mathcal{K}$ is compact. This follows from the fact that a dualizable object x and its dual y determine adjoint functors $x\otimes - \dashv y\otimes -$. Then for any set of objects $\{t_i\}_{i\in I}$ in $\mathcal{K}$ we have that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}}\left(x, \amalg_{i\in I} t_i\right) &\simeq \mathrm{Hom}_{\mathcal{K}}\left(\mathbf{1}, \amalg_{i\in I}(t_i\otimes y)\right)\\ &\simeq \amalg_{i\in I}\mathrm{Hom}_{\mathcal{K}}(\mathbf{1}, t_i\otimes y)\\ &\simeq \amalg_{i\in I}\mathrm{Hom}_{\mathcal{K}}(x, t_i).\end{aligned}$$

Example 5.3.6. For more details about the following categories we refer to [Bal10].

- The stable module category $\underline{\mathrm{StMod}}(kG)$ of a finite group G is a rigidly-compactly generated tensor triangulated category. In this case, the compact part corresponds to the small stable module category $\underline{\text{stmod}}(kG)$, that is, the full subcategory of $\underline{\text{StMod}}(kG)$ on the finitely generated kG–modules.

- The stable homotopy category of topological spectra SH is a rigidly-compactly generated tensor triangulated category. The tensor product corresponds to the smash product, and the unit is the sphere spectrum. The compact objects in SH are finite spectra, hence the compact part of SH is the Spanier-Whitehead stable homotopy category on pointed CW-complexes which is denoted by SH^{ω}.

- Let X be a quasi-compact and quasi-separated scheme. Then the derived category D^{perf} on the perfect complexes over X is the compact part of the category $D_{\text{QCoh}(X)}(X)$ of complexes of $\mathcal{O}_X$–modules with quasi-coherent cohomology which is a rigidly-compactly generated tensor triangulated category.

Example 5.3.7. The following results are proved in [Bal14, Section 4].

- Let G be a finite group. Then any tt-ring in $\underline{\text{stmod}}(kG)$ has finite degree.

- Any tt-ring in the compact part of SH has finite degree.

- Let X be a quasi-compact and quasi-separated scheme. Then any tt-ring in $D^{\text{perf}}(X)$ has finite degree.

We might think these are the conditions we should impose on a tensor triangulated category to guarantee that any tt-ring has finite degree. We will see in the following example that this is not the case. Let 2-Ring denote the ∞-category of of essentially small, symmetric monoidal, stable ∞-categories with exact tensor product in each variable. We refer to [Mat16, Definition 2.14]) for further details about this ∞-category.

Example 5.3.8. For $i \in \mathbb{N}$, let $\mathcal{K}_i$ be a non-trivial rigid 2-ring. Define $\mathcal{K} := \prod_{i\in\mathbb{N}} \mathcal{K}_i$ in 2-Ring. Note that $\mathcal{K}$ is a rigid 2-ring. Let $\mathcal{L}$ denote the Ind-completion of $\mathcal{K}$ which is a stable homotopy theory. In particular, the compact objects of $\mathcal{L}$ are precisely the elements of $\mathcal{K}$. Since the inclusion functor

$$\mathcal{K} \hookrightarrow \mathcal{L}$$

is strongly monoidal, we deduce that any compact element in $\mathcal{L}$ is dualizable. Therefore the homotopy category of $\mathcal{L}$ is a rigidly-compactly generated tensor triangulated category. In particular, we can construct a tt-ring in the dualizable part of $\mathcal{L}$, just as in Theorem 5.3.1, which has infinite degree.

Appendix A

Basic concepts from higher algebra

In this appendix we give an overview of the theory of ∞-categories. We aim to provide the basic ∞-categorical notions leading to stable homotopy theories and descent theory. We omit technical details and all the proofs. For more details, we refer to [Lur17] and [Mat16].

A.1 ∞-categories, limits and colimits

Recall that the *simplex category* Δ is the category whose objects are the linearly ordered sets $[n] := \{0, \dots, n\}$, for $n \geq 0$, and the set of morphisms $\mathrm{Hom}_\Delta([m], [n])$ consists of all monotone maps $f\colon [m] \to [n]$. By monotone, we mean that if $0 \leq i \leq j \leq m$, then $f(i) \leq f(j)$.

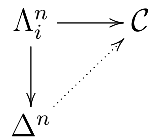
A *simplicial set* is a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$. The category of simplicial sets sSet is $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$. The *standard n-simplex* Δ^n is the simplicial set

$$\Delta^n\colon \Delta \to \mathrm{Set}$$

$$[m] \mapsto \mathrm{Hom}_\Delta([m], [n]).$$

For $n \leq 1$, and $0 \leq i \leq n$, the *ith horn* Λ_i^n of the standard n-simplex is the subsimplicial set of Δ^n such that $\Lambda_i^n[m] \subseteq \Delta^n[m]$ corresponds to the set of morphisms $f\colon [m] \to [n]$ whose image does not contain at least one element of $\{0, \dots, n\}\backslash\{i\}$ In other words, the ith-horn of the standard n-simplex is the union of all *faces* except the ith face.

Definition A.1.1. An ∞*-category* is a simplicial set $\mathcal{C}\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$, which has the right lifting property with respect to inner horn inclusions, that is, inclusions $\Lambda_i^n \to \Delta^n$ with $0 < i < n$. In other words, for any morphism $\Lambda_i^n \to \mathcal{C}$ there is a morphism $\Delta^n \to \mathcal{C}$ such that the diagram



is commutative.

Example A.1.2. Let K be a Kan complex. Then K is an ∞-category. In particular, given a topological space X, the singular complex $\mathrm{Sing}(X)$ is a Kan complex, hence an ∞-category. Recall that the n-simplices of $\mathrm{Sing}(X)$ are the continuous maps $|\Delta^n| \to X$, where $|\Delta^n|$ is the *geometric n-simplex.* In fact, any Kan complex is homotopy equivalent to the singular complex of a topological space.

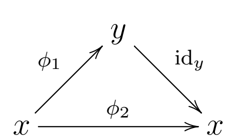
Example A.1.3. Let $\mathcal{C}$ be a small category. The *nerve* $N(\mathcal{C})$ of $\mathcal{C}$ is a simplicial set whose n-simplices are given by n composable morphisms in $\mathcal{C}$. More precisely, an n-simplex of $N(\mathcal{C})$ is a chain

$$x_0 \to x_1 \to \ldots \to x_n.$$

In particular, $N(\mathcal{C})$ is an ∞-category. There is a characterization of simplicial sets obtained as the nerve of a small category. Let K be a simplicial set. Then there exists a small category $\mathcal{C}$ such that K is isomorphic to $N(C)$ (as simplicial sets) if and only if there is a unique way to lift any map $\Lambda_i^n \to K$ with respect to horn inclusions $\Lambda_i^n \to \Delta^n$ for $0 < i < n$.

Let Cat_∞^1 denote the category of (small) ∞-categories. That is, the full subcategory of sSet on the ∞-categories. In particular, a functor between ∞-categories is just a simplicial map between the underlying simplicial sets. The *∞-category of ∞-categories* Cat_∞ is the coherent nerve of the the simplicial category on ∞-categories and hom-simplicial sets given by the maximal ∞-groupoid in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, see [Lan21, Section 2.1] for further details. Let $\mathcal{S}$ denote the coherent nerve of the simplicial category on Kan complexes and hom-simplicial sets given by the internal hom-set, see [Lan21, Definition 1.3.41]. We will say that $\mathcal{S}$ is the *∞-category of spaces.*

Let $\mathcal{C}$ be an ∞-category. Recall that the *homotopy category* $\mathrm{Ho}\,\mathcal{C}$ of $\mathcal{C}$ is the category whose objects are the vertices (0-simplices) of $\mathcal{C}$, and the morphisms between two objects x, y are given by homotopy classes of edges (1-simplices) $\phi\colon x \to y$. Two edges $\phi_1\colon x \to y$, $\phi_2\colon x \to y$ are *homotopic* if there exists a *homotopy* from x to y, that is a 2-simplex $\sigma\colon \Delta^2 \to \mathcal{C}$, which can be depicted as follows:



and the composition in $\operatorname{Ho}\mathcal{C}$ is induced by the filling of the horns $\Lambda^2_1 \to \mathcal{C}$.

Remark A.1.4. Let x and y denote a pair of vertices of a simplicial set $\mathcal{C}$. We define the simplicial set $\operatorname{Hom}_{\mathcal{C}}(x, y)$ as the pullback

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}}(x,y) & \longrightarrow & \operatorname{Fun}(\Delta^1, \mathcal{C}) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{(x,y)} & \mathcal{C} \times \mathcal{C} \end{array}$$

In particular, if $\mathcal{C}$ is an ∞-category, then $\operatorname{Hom}_{\mathcal{C}}(x, y)$ is a Kan complex. In this case, $\operatorname{Hom}_{\mathcal{C}}(x, y)$ is known as the *space of morphisms from x to y*.

Now that we have the notion of mapping spaces we can give a characterization of functors between ∞-categories which are equivalences. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. We say that

- the functor F is *essentially surjective* if the induced functor

  $$\operatorname{Ho} F\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$$

  is essentially surjective,
- the functor F is *fully faithful* if for each pair of objects x, y in $\mathcal{C}$, the map

  $$\operatorname{Hom}_{\mathcal{C}}(x, y) \to \operatorname{Hom}_{\mathcal{D}}(F(x), F(y))$$

  is a weak equivalence of spaces.

Then the functor F is an *equivalence* of ∞-categories if and only if it is essentially surjective and fully faithful.

Definition A.1.5. Let $\mathcal{C}$ be an ∞-category. Let x be an object in $\mathcal{C}$. We say that x is an *initial object* if the mapping space $\operatorname{Hom}_{\mathcal{C}}(x, y)$ is contractible for any object $y \in \mathcal{C}$. We say that x is a *final object* if $\operatorname{Hom}(y, x)$ is contractible for any $y \in \mathcal{C}$.

Recall that the *join* $S \star T$ of two simplicial sets S and T is the simplicial set whose value on a non-empty, linearly ordered finite set J is

$$\coprod_{I \cup I' = J} S(I) \times T(I')$$

where the coproduct runs over decompositions $I \cup I'$ of J into disjoint sets I, I' such that every element of I is smaller than every element of I'.

Example A.1.6. Let K be a simplicial set. Then $\Delta^0 \star K$ corresponds to attaching an *initial* vertex to K, and $K \star \Delta^0$ corresponds to attaching a *final* vertex to K. In particular $\Delta^1 \times \Delta^1 \cong \Lambda^2_0 \star \Delta^0 \cong \Delta^0 \star \Lambda^2_2$.

Let K be a simplicial set. Let $p\colon K \to \mathcal{C}$ be a diagram of shape K in an ∞-category $\mathcal{C}$. The *overcategory* $\mathcal{C}_{/p}$ is the ∞-category with n-simplices

$$(\mathcal{C}_{/p})_n := \hom_p(\Delta^n \star K, \mathcal{C})$$

where the right-hand side denotes the subset of morphisms $f\colon \Delta^n \star K \to \mathcal{C}$ such that $f|_K = p$. Similarly, the *undercategory* $\mathcal{C}_{p/}$ is the ∞-category with n-simplices

$$(\mathcal{C}_{p/})_n := \hom_p(K \star \Delta^n, \mathcal{C})$$

where the right-hand side denotes the subset of morphisms $f\colon K \star \Delta^n \to \mathcal{C}$ such that $f|_K = p$ (see [Lur17, Section 2.1.2] for a discussion).

Definition A.1.7. Let $p\colon K \to \mathcal{C}$ be a diagram in an ∞-category $\mathcal{C}$. A *limit* of p is a final object in $\mathcal{C}_{/p}$, and a *colimit* of p is an initial object in $\mathcal{C}_{p/}$.

A.2 Stable ∞-categories and exact functors

Definition A.2.1. We say that an object x in an ∞-category is a *zero object* if it is both initial and final.

If a zero object exists, then it is unique up to contractible choice. In particular, if x is a zero object, we write $x = 0$.

Definition A.2.2. Let $\mathcal{C}$ be an ∞-category with a zero object. We say that $\mathcal{C}$ is a *stable ∞-category* if it admits finite limits and colimits, and a commutative square is a pullback if and only if it is a pushout[1].

Let $\mathcal{C}$ be an ∞-category with a zero object. A commutative diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$

$$\begin{array}{ccc} x & \xrightarrow{\alpha} & y \\ \downarrow & & \downarrow{\scriptstyle\beta} \\ 0 & \longrightarrow & z \end{array}$$

is called a *fiber sequence* if it is a pullback. In this case, we write $\mathrm{fib}(\beta) = x$ and call it the fiber of β. Dually, a commutative diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ depicted as above is called a *cofiber sequence* if it is a pushout. In this case, we write $\mathrm{cofib}(\alpha) = z$ and call it the cofiber of α.

Definition A.2.3. Let $\mathcal{C}$ be an ∞-category with a zero object. Assume that $\mathcal{C}$ admits all finite limits and colimits and let $x \in \mathcal{C}$. The *suspension* Σx of x is the cofiber of the morphism $x \to 0$. Dually, the *loop* Ωx of x is the fiber of the morphism $0 \to x$. These constructions define functors $\Sigma, \Omega\colon \mathcal{C} \to \mathcal{C}$.

[1]Note that this is a property of an ∞-category rather than of the extra structure.

Let $\mathcal{C}$ be a stable ∞-category. Then the functors $\Sigma, \Omega\colon \mathcal{C} \to \mathcal{C}$ are mutually inverse equivalences. Note that in a stable ∞-category, fiber sequences coincide with cofiber sequences. In particular, for $x \in X$ we have a cofiber sequence

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & \Sigma x \end{array}$$

that is also a fiber sequence, thus it follows that $\Omega\Sigma x \simeq x$. Dually, we have that $\Sigma\Omega x \simeq x$.

Definition A.2.4. Let $\mathcal{C}$ be an ∞-category with a zero object. A *distinguished triangle* in $\mathrm{Ho}\,\mathcal{C}$ is a sequence

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x$$

such that, for certain zero objects 0 and $0'$, there exists a diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$ of the form

$$\begin{array}{ccccc} x & \xrightarrow{\tilde{f}} & y & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \tilde{g}} & & \downarrow \\ 0' & \longrightarrow & z & \xrightarrow[\tilde{h}]{} & w \end{array}$$

such that both squares are pushouts, $\tilde{f}$ and $\tilde{g}$ represent f and g, respectively, and the following diagram is commutative.

$$\begin{array}{ccc} z & \xrightarrow{\tilde{h}} & w \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle \simeq} \\ & & \Sigma x \end{array}$$

The following corresponds to [Lur17, Theorem 1.1.2.14].

Theorem A.2.5. *Let $\mathcal{C}$ be a stable ∞-category. Then the suspension functor of Definition A.2.3 and the class of distinguished triangles of Definition A.2.4 define a triangulated structure on the homotopy category* $\mathrm{Ho}\,\mathcal{C}$.

In practice, all triangulated categories of interest arise as the homotopy category of a stable ∞-category, and there is a dictionary for translating between them. For instance, the shift functor corresponds to the suspension functor, the distinguished triangles correspond to fiber and cofiber sequences, and the mapping cone to the cofiber. We can consider stable ∞-categories as enhancements of triangulated categories; functoriality of cones is a notable improvement.

Example A.2.6. Let $\mathcal{S}_*^{\mathrm{fin}}$ denote the ∞-category of finite pointed spaces. The ∞-

category of finite spectra $\mathrm{Sp}^{\mathrm{fin}}$ is defined as the colimit of the sequence

$$\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \dots$$

in Cat_∞. The ∞-*category of spectra* is the Ind-completion of $\mathrm{Sp}^{\mathrm{fin}}$. The stable ∞-category of spectra is a stable ∞-category and its homotopy category corresponds to the classical stable homotopy category. (See [Lur17, Section 1.4].)

Let $\mathcal{W}$ be a collection of morphisms in an ∞-category $\mathcal{C}$. The ∞-category *obtained from $\mathcal{C}$ by inverting the morphisms in $\mathcal{W}$* is an ∞-category $\mathcal{C}[\mathcal{W}^{-1}]$ with a map $f\colon \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ which satisfies the following property. For any ∞-category $\mathcal{D}$, composition with the map f induces an equivalence between $\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D})$ and the the subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ on the functors which invert the morphisms in $\mathcal{W}$.

Definition A.2.7. Let $\mathcal{C}$ be a model category. The *underlying ∞-category of $\mathcal{C}$* is given by $\mathrm{N}(\mathcal{C})[\mathcal{W}^{-1}]$, where $\mathcal{W}$ denotes the collection of weak equivalences in $\mathcal{C}$ (see [Lur17, Definition 1.3.4.15] for further details).

Example A.2.8. Let G be a finite group and let k be a field of prime characteristic p dividing the order of G. The stable module ∞-category $\mathrm{StMod}(kG)$ is the ∞-categorical localization of the category of kG–modules $\mathrm{Mod}(kG)$ at the class of stable isomorphisms (See [Mat15, Section 2]). Then $\mathrm{StMod}(kG)$ is a stable ∞-category and its homotopy category corresponds to the stable homotopy category obtained from $\mathrm{Mod}(kG)$ by quotienting out the projective modules.

Example A.2.9. Let $\mathcal{A}$ be a Grothendieck category. The *unbounded derived ∞-category $\mathcal{D}(\mathcal{A})$* of $\mathcal{A}$ is the ∞-categorical localization of the nerve of the category of chain complexes $\mathrm{Ch}(\mathcal{A})$ at the quasi-isomorphisms.

Definition A.2.10. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories.

- F is *left exact* if it commutes with finite limits.
- F is *right exact* if it commutes with finite colimits.
- F is *exact* if it is right and left exact.

If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between stable ∞-categories, then F is exact if and only if it preserves all fiber and cofiber sequences. In particular, the identity functor of any stable ∞-category is exact, and the composition of exact functors is exact. Moreover, exact functors between stable ∞-categories induce triangulated functors between the respective triangulated homotopy categories.

Let $\mathrm{Cat}_\infty^{\mathrm{st}}$ denote the subcategory of Cat_∞ spanned by stable ∞-categories and exact functors.

Definition A.2.11. An ∞-category $\mathcal{C}$ is called *accessible* if there exists a regular cardinal κ and a small category $\mathcal{C}'$, such that $\mathcal{C}$ is obtained from $\mathcal{C}'$ by freely adjoining κ-filtered colimits, also known as the Ind_κ-completion (see [Lur17, Section 5.3]). We say that $\mathcal{C}$ is *presentable* if it is accessible and admits all colimits.

In particular, any presentable ∞-category is equivalent to the underlying ∞-category of a combinatorial simplicial model category [Lur17, Proposition A.3.7.6]. Moreover, the underlying ∞-category of a combinatorial model category is a presentable ∞-category [Lur17, Proposition 1.3.4.22].

The following is an ∞-categorical version of the adjoint functor theorem [Lur17, Corollary 5.5.2.9]

Theorem A.2.12. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories.*

- *F admits a right adjoint if and only if it preserves small colimits.*
- *F admits a left adjoint if and only if it preserves all limits and it is accessible (that is, it preserves κ-filtered colimits for some regular cardinal κ).*

Example A.2.13. Let G be a finite group and let k be a field of prime characteristic p dividing the order of G. Define the stable module ∞-category $\mathrm{stmod}(kG)$ as the ∞-categorical localization of the category of finitely generated kG–modules $\mathrm{mod}(kG)$ at the stable isomorphisms. In this case, we have that $\mathrm{stmod}(kG)$ is a small ∞-category and $\mathrm{StMod}(kG) = \mathrm{Ind}(\mathrm{stmod}(kG))$. Moreover, the ∞-category $\mathrm{StMod}(kG)$ admits all colimits. It follows that $\mathrm{StMod}(kG)$ is a presentable ∞-category.

On the other hand, consider a subgroup H of G. Then the restriction functor $\mathrm{Res}\colon \mathrm{Mod}(kG) \to \mathrm{Mod}(kH)$ induces an exact functor

$$\mathrm{Res}\colon \mathrm{StMod}(kG) \to \mathrm{StMod}(kH).$$

In fact, it has a left adjoint given by extension of scalars

$$kG \otimes_{kH} -\colon \mathrm{StMod}(kH) \to \mathrm{StMod}(kH).$$

Note that the restriction functor also has a right adjoint given by coinduction.

Remark A.2.14. The inclusion $\mathrm{Cat}_\infty^{\mathrm{st}} \subseteq \mathrm{Cat}_\infty$ preserves all small limits [Lur17, Theorem 1.1.4.4] and all small filtered colimits [Lur17, Proposition 1.1.4.6], hence limits and filtered colimits in $\mathrm{Cat}_\infty^{\mathrm{st}}$ can be computed in Cat_∞. For instance, consider the diagram in $\mathrm{Cat}_\infty^{\mathrm{st}}$ depicted by

$$\begin{array}{ccc} & & \mathcal{C} \\ & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{E} \end{array}$$

Then the homotopy pullback $\mathcal{C}'$ in Cat_∞ is automatically a stable ∞-category.

Let Pr^L denote the ∞-category of presentable ∞-categories and left-adjoint (colimit-preserving) functors. Dually, let Pr^R denote the ∞-category of presentable ∞-categories and right-adjoint (limit-preserving) functors (see [Lur17, Definition 5.5.3]). Let $\mathrm{Pr}^L_{\mathrm{st}} \subset \mathrm{Pr}^L$ and $\mathrm{Pr}^R_{\mathrm{st}} \subset \mathrm{Pr}^R$ denote the full subcategories spanned by stable ∞-categories. Computations of limits in these ∞-categories reduce to computations of limits in Cat_∞ since the inclusions $\mathrm{Pr}^L_{\mathrm{st}} \subset \mathrm{Pr}^L \subsetneq \mathrm{Cat}_\infty$ and $\mathrm{Pr}^R_{\mathrm{st}} \subset \mathrm{Pr}^R \subsetneq \mathrm{Cat}_\infty$ preserve all limits.

Let $F\colon I \to \mathrm{Pr}^R$ be a diagram. For each map $f\colon i \to j$, the right-adjoint $R_f\colon = F(f)\colon F(j) \leftrightarrows F(i)$ has a left adjoint $L_f\colon F(i) \leftrightarrows F(j)$, hence we have an adjunction of ∞-categories

$$L_f\colon F(j) \leftrightarrows F(i)\colon R_f.$$

An object x in $\varinjlim_I F$ is the following data (see [Mat16, Section 2]).

- An object x_i in $F(i)$, for each $i \in I$.
- An isomorphism $x_j \to R_f(x_i)$, for each $f\colon i \to j$ in I.
- Higher homotopies and coherences.

A.3 Stable homotopy theories

The ∞-category $\mathrm{Pr}^L_{\mathrm{st}}$ has a *symmetric monoidal structure* given as follows (see [Lur17, Section 4.8]).

Definition A.3.1. Let $\mathcal{C}$, $\mathcal{D}$ be presentable stable ∞-categories. The *tensor product* $\mathcal{C} \otimes \mathcal{D}$ is the presentable ∞-category defined by the universal property

$$\mathrm{Hom}_{\mathrm{Pr}^L_{\mathrm{st}}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

where $\mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ consists of functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ preserving colimits in both entries.

Recall that a *commutative algebra object* (X, μ, η) in a symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$ is an object X in $\mathcal{C}$ together a multiplication map $\mu\colon X \otimes X \to X$ and a unit map $\eta\colon \mathbb{1} \to X$ satisfying the analogous axioms of a commutative algebra object (in the classical setting of symmetric monoidal categories) up to coherent homotopy. We refer to [Lur17, Section 2.4.2] for further discussion. In particular, symmetric monoidal ∞-categories correspond to commutative algebra objects in the ∞-category of ∞-categories. Moreover, there is an ∞-category $\mathrm{CAlg}(\mathcal{C})$ of commutative algebra objects in a given symmetric monoidal ∞-category $\mathcal{C}$.

Definition A.3.2. A *stable homotopy theory* $\mathcal{C}$ is a commutative algebra object in $\mathrm{Pr}^L_{\mathrm{st}}$, that is, a presentable symmetric monoidal stable ∞-category with bicontinuous tensor product (see [Mat16, Definition 2.14]).

Example A.3.3. The derived ∞-category $D(R)$ of a commutative ring is a stable homotopy theory. The tensor product corresponds to the derived tensor product.

Example A.3.4. The stable module ∞-category $\mathrm{StMod}(kG)$ for a finite group G is a stable homotopy theory. The tensor product is induced by the tensor product on $\mathrm{Mod}(kG)$, that is, the tensor product $\otimes_k$ over the ground field k with the diagonal action of G.

Let $\mathcal{C}$ be a stable homotopy theory. Let X be a commutative algebra object in $\mathcal{C}$. There is an ∞-category $\mathrm{Mod}_{\mathcal{C}}(X)$ of X–module objects in $\mathcal{C}$. The ∞-category $\mathrm{Mod}_{\mathcal{C}}(X)$ is a stable homotopy theory with the relative X–linear tensor product (see [Lur17, Section 4.5]).

A.4 Descent

Let $\mathcal{C}$ be a stable homotopy theory. A $\otimes$*-ideal* is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ that is stable, idempotent-complete which satisfies that $X \otimes Y$ is in $\mathcal{I}$ provided that $X \in \mathcal{C}$ and $Y \in \mathcal{I}$. A subcategory $\mathcal{D} \subset \mathcal{C}$ is *thick* if $\mathcal{D}$ is closed under finite limits, finite colimits and retracts. A *thick* $\otimes$*-ideal* $\mathcal{D}$ of $\mathcal{C}$ is a thick subcategory that in addition is a $\otimes$-ideal.

Definition A.4.1. Let $\mathcal{C}$ be a stable homotopy theory. A commutative algebra object X in $\mathcal{C}$ is *descendable* (or *admits descent*) if $\mathcal{C}$ is the smallest thick $\otimes$-ideal containing A.

A commutative algebra object X in a stable homotopy theory $\mathcal{C}$ defines an adjunction

$$F_X \colon \mathcal{C} \leftrightarrows \mathrm{Mod}_{\mathcal{C}}(X) \colon U_X$$

where F_X is given by tensoring by X, and U_X is the forgetful functor. If additionally X admits descent, then the adjuntion is monadic and the canonical functor

$$\mathcal{C} \to \mathrm{Tot}\left(\mathrm{Mod}_{\mathcal{C}}(X) \rightrightarrows \mathrm{Mod}_{\mathcal{C}}(X \otimes X) \substack{\to\\ \to\\ \to} \dots\right)$$

is an equivalence [Mat16, Proposition 3.22].

Definition A.4.2. Let A be an object in a stable homotopy theory $\mathcal{C}$. A map $f \colon X \to Y$ in $\mathcal{C}$ is *A–zero* if $A \otimes X \xrightarrow{1_A \otimes f} A \otimes Y$ is nullhomotopic in $\mathcal{C}$.

A *tensor ideal* $\mathcal{I}$ in a tensor triangulated category $\mathcal{C}$ is a collection of maps which satisfies the following properties. The class of maps $X \to Y$ in $\mathcal{I}$ is a subgroup of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that for any triple (f, g, h) of composable maps in $\mathcal{C}$ and any object X in $\mathcal{C}$, the composition $h \circ g \circ f$ and the map $1_X \otimes g$ are in $\mathcal{I}$ provided that g is in $\mathcal{I}$. The collection $\mathcal{I}_A$ of A–zero maps in the homotopy category $\mathrm{Ho}\,\mathcal{C}$ is a tensor ideal. The following result corresponds to Proposition 3.27 in [Mat16].

Proposition A.4.3. *Let A be a commutative algebra object in a stable homotopy theory $\mathcal{C}$. Then A admits descent if and only $\mathcal{I}_A^n = 0$ for some $n \geq 0$, where $\mathcal{I}_A^n$ denotes the smallest tensor ideal containing the compositions of n consecutive A–zero maps.*

Example A.4.4. Let G be a finite group. For a subgroup H of G, let A_H denote the commutative algebra object $\prod_{G/H} k$ in $\mathrm{StMod}(kG)$ (the G–action permutes the factors). Then the commutative algebra object

$$\prod_H A_H$$

where the product runs over all the p-subgroups of G, admits descent (see [Bal15, Theorem 4.3] and [Mat16, Section 4]).

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