THE MONOTONE SECANT CONJECTURE IN THE REAL SCHUBERT CALCULUS

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Abstract. The Monotone Secant Conjecture posits a rich class of polynomial systems, all of whose solutions are real. These systems come from the Schubert calculus on flag manifolds, and the Monotone Secant Conjecture is a compelling generalization of the Shapiro Conjecture for Grassmannians (Theorem of Mukhin, Tarasov, and Varchenko). We present the Monotone Secant Conjecture, explain the massive computation evidence in its favor, and discuss its relation to the Shapiro Conjecture.

1. Introduction

A system of real polynomial equations with finitely many solutions has some, but likely not all, of its solutions real. In fact, sometimes the structure of the equations leads to upper bounds [2, 11] ensuring that not all solutions can be real. The Shapiro Conjecture and the Monotone Secant Conjecture posit a family of systems of polynomial equations with the extreme property of having all their solutions be real.

The Shapiro Conjecture asserts that a zero-dimensional intersection of Schubert subvarieties of a flag manifold consists only of real points provided that the Schubert varieties are given by flags tangent to (osculating) a real rational normal curve. Eremenko and Gabrielov gave a proof in the special case the Grassmannian of codimension-two planes [3, 5]. The general case for Grassmannians was established by Mukhin, Tarasov, and Varchenko [13, 14]. A complete story of this conjecture and its proof can be found in [18].

The Shapiro conjecture is false for non-Grassmannian flag manifolds, but in a very interesting manner. This failure was first noticed in [16] and systematic computer experimentation suggested a correction, the Monotone Conjecture [15, 17], that appears to be valid for flag manifolds. Eremenko, Gabrielov, Shapiro, and Vainshtein [6] proved a result that implied the Monotone Conjecture for some manifolds of two-step flags. This result concerned codimension-two subspaces that meet flags which are secant to the rational normal curve along disjoint intervals. This suggested the Secant Conjecture, which asserts that an

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intersection of Schubert varieties in a Grassmannian is transverse with all points real, provided that the Schubert varieties are defined by flags secant to a rational normal curve along disjoint intervals. This was posed and evidence was presented for its validity in [9].

The Monotone Secant Conjecture is a common extension of both the Monotone Conjecture and the Secant Conjecture. We give here the simplest open instance, expressed as a system of polynomial equations in local coordinates for the variety of flags $E_2 \subset E_3$ in $\mathbb{C}^5$, where $\dim E_i = i$. Let $x_1,\ldots,x_8$ be indeterminates and consider the polynomials

$$(1.1)\quad f(s,t,u;x) \;:=\; \det\begin{pmatrix} 1 & 0 & x_1 & x_2 & x_3\\ 0 & 1 & x_4 & x_5 & x_6\\ \hline 1 & s & s^2 & s^3 & s^4\\ 1 & t & t^2 & t^3 & t^4\\ 1 & u & u^2 & u^3 & u^4\end{pmatrix}, \quad g(v,w;x) \;:=\; \det\begin{pmatrix} 1 & 0 & x_1 & x_2 & x_3\\ 0 & 1 & x_4 & x_5 & x_6\\ 0 & 0 & 1 & x_7 & x_8\\ \hline 1 & v & v^2 & v^3 & v^4\\ 1 & w & w^2 & w^3 & w^4\end{pmatrix},$$

which depend upon parameters s,t,u and v,w respectively.

Conjecture 1.1. *Let $s_1 < t_1 < u_1 < \cdots < s_4 < t_4 < u_4 < v_1 < w_1 < \cdots < v_4 < w_4$ be real numbers. Then the system of polynomial equations*

$$(1.2)\qquad \begin{aligned} f(s_1,t_1,u_1;x) \;=\; f(s_2,t_2,u_2;x) \;=\; f(s_3,t_3,u_3;x) \;=\; f(s_4,t_4,u_4;x) \;&=\; 0\\ g(v_1,w_1;x) \;=\; g(v_2,w_2;x) \;=\; g(v_3,w_3;x) \;=\; g(v_4,w_4;x) \;&=\; 0\end{aligned}$$

has twelve solutions, and all of them are real.

Geometrically, the equation $f(s,t,u;x) = 0$ is the condition that a general 2-plane E_2 (spanned by the first two rows of the matrix) meets the 3-plane which is secant to the rational normal curve $\gamma\colon y \mapsto (1,y,y^2,y^3,y^4)$ at the points $\gamma(s),\gamma(t),\gamma(u)$. Similarly, the equation $g(v,w;x) = 0$ is the condition that a general 3-plane E_3 meets the 2-plane secant to γ at the points $\gamma(v),\gamma(w)$. The monotonicity hypothesis is that the four 3-planes given by s_i,t_i,u_i are secant along intervals $[s_i,u_i]$ which are pairwise disjoint and occur before the intervals $[v_i,w_i]$ where the 2-planes are secant. If the order of the intervals $[s_4,u_4]$ and $[v_1,w_1]$ is switched, the evaluation is no longer monotone. We tested 1,000,000 instances of Conjecture 1.1, finding only real solutions. In contrast, we tested 7,000,000 with the monotonicity condition relaxed, finding instances in which not all solutions were real.

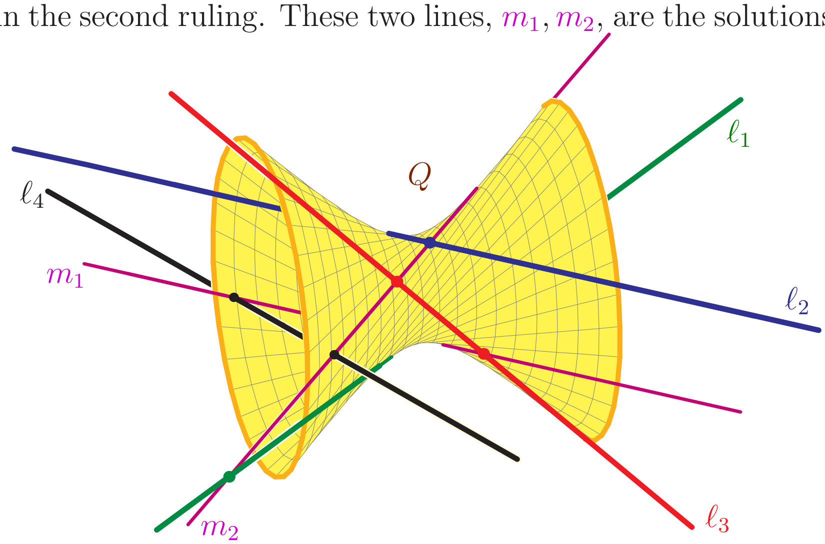
We formulate the Monotone Secant Conjecture, explain its relation to the other conjectures, and present overwhelming computational evidence that supports its validity. These data are obtained in an experiment we are conducting on a supercomputer at Texas A&M University whose day job is Calculus instruction. Our data can be viewed online; these data and the code can be accessed from our website [8]. The design and execution of this kind of large-scale experiment was described in [10].

This paper is organized as follows. In Section 2 we illustrate the rich geometry behind Schubert problems and we make use of the classical problem of four lines to depict the Monotone Secant Conjecture. Section 3 provides a primer on flag manifolds, states the Shapiro, Secant, and Monotone conjectures and there we state in detail the Monotone Secant

Conjecture. In Section 4 we discuss the results collected from the observations of our data. and we give a brief guide to our data. Lastly, in Section 5 we describe the methods we used to test the conjecture

2. The problem of four lines

The classical problem of four lines asks for the finitely many lines m that meet four given general lines $\ell_1, \ell_2, \ell_3, \ell_4$ in (projective) three-space. Three general lines ℓ_1, ℓ_2, ℓ_3 lie in one ruling of a doubly-ruled quadric surface Q, with the other ruling consisting of all lines that meet the first three. The line ℓ_4 meets Q in two points, and through each of these points there is a line in the second ruling. These two lines, m_1, m_2, are the solutions to this problem.



If the lines $\ell_1, \ell_2, \ell_3, \ell_4$ are real, then so is Q, but the intersection of Q with ℓ_4 may consist either of two real points or of a complex conjugate pair of points. In the first case, the problem of four lines has two real solutions, while in the second, it has no real solutions.

The Shapiro Conjecture asserts that if the four given lines are tangent to a rational normal curve, then both solutions are real. We illustrate this. Set $\gamma(t) := (6t^2-1, \frac{7}{2}t^3+\frac{3}{2}t, -\frac{1}{2}t^3+\frac{3}{2}t)$, a rational normal curve $\gamma\colon \mathbb{R} \to \mathbb{R}^3$. Write $\ell(t)$ for the line tangent at the point $\gamma(t)$. Our given lines will be $\ell(-1), \ell(0), \ell(1)$, and $\ell(s)$ for $s \in (0,1)$. The first three lines lie on the quadric Q defined by $x^2 - y^2 + z^2 = 1$. The line $\ell(s)$ meets the quadric in two real points, as illustrated in Figure 1, giving two real solutions to this instance of the problem of four lines.

For the problem of four lines, the Secant Conjecture replaces the four tangent lines by four lines that are secant to γ. Suppose that the four are close to four tangents in that the intervals along γ given by their points of secancy are disjoint. Figure 2 shows three lines secant to γ along disjoint intervals and the quadric Q that they lie upon. Their intervals of secancy are disjoint from the indicated interval I. Any line secant along I will meet Q in two points, giving two real solutions to this instance of the Secant Conjecture.

Figure 2 also illustrates the Monotone Secant Conjecture. Consider flags of a line lying on a plane, $m \subset M$. We require that the line m meets three fixed lines secant to γ and that the

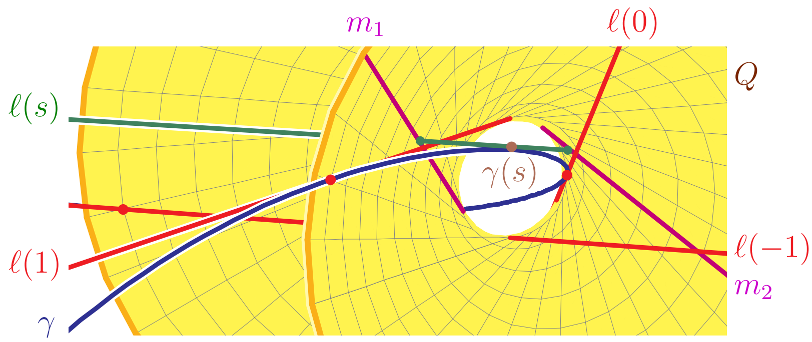


FIGURE 1. Four tangent lines give two real solutions.

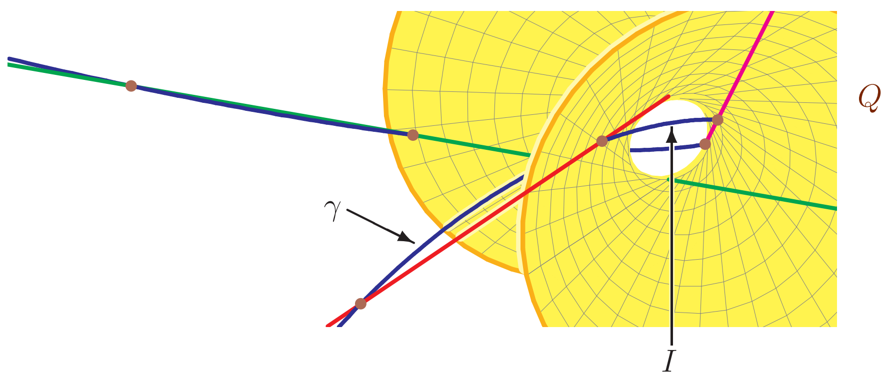


FIGURE 2. The problem of four secant lines.

plane M meets two points, $\gamma(s)$ and $\gamma(t)$, of γ. Since the plane M contains the two points $\gamma(s)$ and $\gamma(t)$, it contains the secant line they span, $\ell(s,t)$. But the line m also lies in M, and therefore it must also meet the secant line $\ell(s,t)$, in addition to the three original secant lines. If the three original secant lines are the three lines in Figure 2 and the points $\gamma(s)$,$\gamma(t)$ lie in the interval I, then there will be two real lines m meeting all four secant lines, and for each line m, the plane M is the span of m and $\ell(s,t)$.

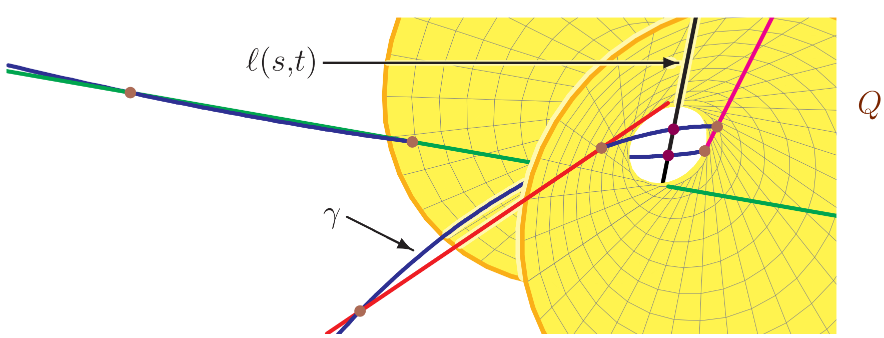


FIGURE 3. A non-monotone evaluation.

If the points $\gamma(s)$ and $\gamma(t)$ are chosen as in Figure 3, so that the secant line $\ell(s,t)$ does not meet the quadric Q, then the solutions will not be real. Thus the positions of the points $\gamma(s)$,$\gamma(t)$ relative to the other intervals of secancy affect whether or not the solutions are real. The schematic in Figure 4 illustrates the relative positions of the secancies along γ (which is homeomorphic to the circle). The idea behind the Monotone Secant Conjecture is

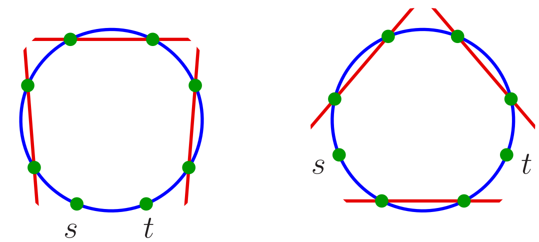


FIGURE 4. Schematic for the secancies.

to attach to each interval the dimension of that part of the flag, 1 for m and 2 for M, which it affects. Then the schematic on the left has labels 1,1,1,2,2, reading clockwise, starting just past the point s, while the schematic on the right reads 1,1,2,1,2. In the first, the labels increase monotonically, while in the second, they do not.

3. Background

We develop the background for the statement of the Monotone Secant Conjecture, defining flag varieties and their Schubert problems. Fix positive intgers $\alpha := (a_1 < \cdots < a_k)$ and n with $a_k < n$. A *flag $E_\bullet$ of type α* is a sequence of subspaces

$$E_\bullet \ :\ \{0\} \subset E_{a_1} \ \subset\ E_{a_2} \ \subset\ \cdots\ \subset\ E_{a_k} \ \subset\ \mathbb{C}^n\,, \qquad \text{where } \dim(E_{a_i}) = a_i\,.$$

The set of all such flags forms the *flag manifold* $\mathbb{F}\ell(\alpha;n)$, which has dimension $\dim(\alpha) := \sum_{i=1}^{k}(n-a_i)(a_i-a_{i-1})$, where $a_0 := 0$. When $\alpha = (a)$ is a singleton, $\mathbb{F}\ell(\alpha;n)$ is the *Grassmannian* of a-planes in $\mathbb{C}^n$, written $\mathrm{Gr}(a,n)$. Flags of type $1 < 2 < \cdots < n-1$ in $\mathbb{C}^n$ are *complete*. The positions of flags of type α relative to a fixed complete flag $F_\bullet$ stratify $\mathbb{F}\ell(\alpha;n)$ into cells whose closures are *Schubert varieties*. These positions are indexed by certain permutations. The *descent set $\delta(\sigma)$* of a permutation $\sigma \in S_n$ is the set of numbers i such that $\sigma(i) > \sigma(i{+}1)$. Given a permutation $\sigma \in S_n$ with descent set a subset of α, set $r_\sigma(i,j) :=\ |\{l \leq i \mid j + \sigma(l) > n\}|$. Then the Schubert variety $X_\sigma F_\bullet$ is

$$X_\sigma F_\bullet \ =\ \{E_\bullet \in \mathbb{F}\ell(\alpha;n) \mid \dim E_{a_i} \cap F_j \geq r_\sigma(a_i,j),\ i = 1,\ldots,k,\ j = 1,\ldots,n\}.$$

Flags $E_\bullet$ in $X_\sigma F_\bullet$ have position σ relative to $F_\bullet$. A permutation σ with descent set contained in α is a *Schubert condition* on flags of type α. The Schubert variety $X_\sigma F_\bullet$ is irreducible with codimension $\ell(\sigma) := |\{i < j \mid \sigma(i) > \sigma(j)\}|$. A *Schubert problem* for $\mathbb{F}\ell(\alpha;n)$ is a list of Schubert conditions $(\sigma_1,\ldots,\sigma_m)$ for $\mathbb{F}\ell(\alpha;n)$ satisfying $\ell(\sigma_1) + \cdots + \ell(\sigma_m) = \dim(\alpha)$.

Given a Schubert problem $(\sigma_1,\dots,\sigma_m)$ for $\mathbb{F}\ell(\alpha;n)$ and complete flags $F^1_\bullet,\dots,F^m_\bullet$, consider the intersection

$$X_{\sigma_1}F^1_\bullet \cap \cdots \cap X_{\sigma_m}F^m_\bullet\,. \tag{3.1}$$

When the flags are in general position, this intersection is transverse and zero-dimensional [12], and it consists of all flags $E_\bullet \in \mathbb{F}\ell(\alpha;n)$ having position σ_i relative to $F^i_\bullet$, for each $i = 1,\dots,m$. Such a flag $E_\bullet$ is a *solution* to the Schubert problem.

The degree of a zero-dimensional intersection (3.1) is independent of the choice of the flags and we call this number $d(\sigma_1,\dots,\sigma_m)$ the *degree* of the Schubert problem. When the intersection is transverse, the number of solutions to a Schubert problem equals its degree.

When the flags $F^1_\bullet,\dots,F^m_\bullet$ are real, the solutions to the Schubert problem need not be real. The Monotone Secant Conjecture posits a method to select the flags $F_\bullet$ so that all solutions are real, for a certain class of Schubert problems.

Let $\gamma\colon \mathbb{R}\to\mathbb{R}^n$ be a rational normal curve, which is affinely equivalent to the moment curve $\gamma(t) := (1,t,t^2,\dots,t^{n-1})$. A flag $F_\bullet$ is *secant along an interval I* of γ if every subspace in the flag is spanned by its intersection with I. A list of flags $F^1_\bullet,\dots,F^m_\bullet$ secant to γ is *disjoint* if the intervals of secancy are pairwise disjoint. Disjoint flags are naturally ordered by order in which their intervals of secancy lie within $\mathbb{R}$.

A permutation σ is *Grassmannian* of *type $\delta(\sigma) := a_i$* if its only descent is at position a_i. A *Grassmannian Schubert problem* is one that involves only Grassmannian Schubert conditions. A list of disjoint secant flags $F^1_\bullet,\dots,F^m_\bullet$ is *monotone* with respect to a Grassmannian Schubert problem $(\sigma_1,\dots,\sigma_m)$ if the function $F^i_\bullet \mapsto \delta(\sigma_i)$ is monotone; in other words, if

$$\delta(\sigma_i) \;<\; \delta(\sigma_j) \implies F^i < F^j\,, \qquad \text{for all } i,j\,.$$

Monotone Secant Conjecture 3.1. *For any Grassmannian Schubert problem $(\sigma_1,\dots,\sigma_m)$ on the flag manifold $\mathbb{F}\ell(\alpha;n)$ and any disjoint secant flags $F^1_\bullet,\dots,F^m_\bullet$ that are monotone with repsect to the Schubert problem, the intersection*

$$X_{\sigma_1}F^1_\bullet \cap X_{\sigma_2}F^2_\bullet \cap \cdots \cap X_{\sigma_m}F^m_\bullet$$

is transverse with all points real.

Conjecture 1.1 is the Monotone Secant Conjecture for a Schubert problem on $\mathbb{F}\ell(2,3;5)$ involving the Schubert conditions $\sigma := (1\,3\,2\,4\,5)$ and $\tau := (1\,2\,4\,3\,5)$. Then $\delta(\sigma) = 2$, $\delta(\tau) = 3$, and $\ell(\sigma) = \ell(\tau) = 1$, so that $(\sigma,\sigma,\sigma,\sigma,\tau,\tau,\tau,\tau) = (\sigma^4,\tau^4)$ is a Schubert problem for $\mathbb{F}\ell(2,3;5)$, as $\dim(\mathbb{F}\ell(2,3;5)) = 8$. The corresponding Schubert varieties are

$$\begin{aligned} X_\sigma F_\bullet &= \{E_\bullet \in \mathbb{F}\ell(2,3;5) \mid \dim E_2\cap F_3 \geq 1\}\,,\\ X_\tau F_\bullet &= \{E_\bullet \in \mathbb{F}\ell(2,3;5) \mid \dim E_3\cap F_2 \geq 1\}\,,\end{aligned}$$

that is, the set of flags $E_\bullet$ whose 2-plane E_2 meets a fixed 3-plane F_3 non-trivially, and the set of $E_\bullet$ where E_3 meets a fixed 2-plane F_2 non-trivially, respectively. For $s,t,u,v,w\in\mathbb{R}$, let $F_3(s,t,u)$ be the linear span of $\gamma(s)$, $\gamma(t)$, and $\gamma(u)$ and $F_2(v,w)$ be the linear span of $\gamma(v)$ and

$\gamma(w)$; these are a secant 3-plane and a secant 2-plane to the rational normal curve, respectively. The condition $f(s,t,u;x)=0$ of Conjecture 1.1 implies that $E_\bullet \in X_\sigma F_\bullet(s,t,u)$, where we ignore the larger subspaces in the flag $F_\bullet(s,t,u)$. Similarly, the condition $g(v,w;x)=0$ implies that $E_\bullet \in X_\tau F_\bullet(v,w)$. Lastly, the condition on the ordering of the points s_i,t_i,u_i,v_i,w_i in Conjecture 1.1 implies that the flags $F_\bullet(s_i,t_i,u_i)$ and $F_\bullet(v_i,w_i)$ are disjoint secant flags that are monotone with respect to this Schubert problem.

Three conjectures that have driven progress in enumerative real algebraic geometry are specializations of the Monotone Secant Conjecture. Observe that in the Grassmannian $\mathrm{Gr}(a;n)$, any list of disjoint secant flags $F_\bullet^1,\ldots,F_\bullet^m$ is monotone with respect to any Schubert problem $(\sigma_1,\ldots,\sigma_m)$, as all the conditions have the same descent. In this way, the Monotone Secant Conjecture reduces to the Secant Conjecture, when the flag manifold is a Grassmannian.

Secant Conjecture 3.2. *For any Schubert problem $(\sigma_1,\ldots,\sigma_m)$ on a Grassmannian $\mathrm{Gr}(a;n)$ and any disjoint secant flags $F_\bullet^1,\ldots,F_\bullet^m$, the intersection*

$$X_{\sigma_1}F_\bullet^1 \cap X_{\sigma_2}F_\bullet^2 \cap \cdots \cap X_{\sigma_m}F_\bullet^m$$

is transverse with all points real.

We studied this conjecture in a large-scale experiment whose results are described in [9], solving 1,855,810,000 instances of 703 Schubert problems on 13 different Grassmannians, verifying the Secant Conjecture in each of the 448,381,157 instances checked. This took 1.065 terahertz years of computing.

The limit of any family of flags whose intervals of secancy shrink to a point $\gamma(t)$ is the *osculating flag $F_\bullet(t)$*. This is the flag whose j-dimensional subspace is the span of the first j derivatives $\gamma(t)$, $\gamma'(t),\ldots,\gamma^{(j-1)}(t)$ of γ at t. In this way, the limit of the Monotone Secant Conjecture, as the secant flags become osculating flags, is a similar conjecture where we replace monotone secant flags by monotone osculating flags.

Monotone Conjecture 3.3. *For any Schubert problem $(\sigma_1,\ldots,\sigma_m)$ on the flag manifold $\mathbb{F}\ell(\alpha;n)$ and any flags $F_\bullet^1,\ldots,F_\bullet^m$ osculating a rational normal curve γ at points that are monotone with respect to the Schubert problem, the intersection*

$$X_{\sigma_1}F_\bullet^1 \cap X_{\sigma_2}F_\bullet^2 \cap \cdots \cap X_{\sigma_m}F_\bullet^m$$

is transverse with all points real.

Ruffo, et al. [15] formulated and studied this conjecture, establishing special cases and giving substantial experimental evidence in support of it.

The specialization of the Monotone Secant Conjecture that both restricts to the Grassmannian and to osculating flags is the Shapiro Conjecture which was posed around 1995 by Boris Shapiro and Michael Shapiro, studied in [16], and for which proofs were given by Eremenko and Gabrielov for $\mathrm{Gr}(n-2;n)$ [5] and in complete generality by Mukhin, Tarasov, and Varchenko [13, 14].

Shapiro Conjecture 3.4. *For any Schubert problem* $(\sigma_1, \ldots, \sigma_m)$ *in* $\mathrm{Gr}(a; n)$ *and any distinct real numbers* $t_1, \ldots, t_m$*, the intersection*

$$X_{\sigma_1} F_\bullet(t_1) \cap X_{\sigma_2} F_\bullet(t_2) \cap \cdots \cap X_{\sigma_m} F_\bullet(t_m)$$

is transverse with all points real.

4. Results

The Secant Conjecture (like the Shapiro Conjecture before it) cannot hold for flag manifolds. The monotonicity condition seems to correct this failure in both conjectures. Here, we give more details on the relation of the Monotone Conjecture to the Monotone Secant Conjecture, and then discuss some of our data in an ongoing experiment testing both conjectures.

4.1. The Monotone Conjecture is the limit of the Monotone Secant Conjecture.

The osculating plane $F_i(s)$ is the unique i-dimensional subspace having maximal order of contact with the rational normal curve γ at the point $\gamma(s)$, and therefore it is a limit of secant flags.

Proposition 4.1. *Let* $\{s_1^{(j)} \ldots, \ldots, s_i^{(j)}\}$ *for* $j = 1,2,\ldots$ *be a sequence of lists of* i *distinct complex numbers with the property that for each* $p = 1, \ldots, i$*, we have*

$$\lim_{j\to\infty} s_p^{(j)} = s,$$

for some number s*. Then,*

$$\lim_{j\to\infty} \mathrm{span}\{\gamma(s_1^{(j)}), \ldots, \gamma(s_i^{(j)})\} = F_i(s).$$

As explained in the provious section, the Monotone Conjecture is implied by the Monotone Secant Conjecture by this proposition. There is a partial converse which follows from a standard limiting argument.

Theorem 4.2. *Let* $(\sigma_1, \ldots, \sigma_m)$ *be a Schubert problem on* $\mathbb{F}\ell(a; n)$ *for which the Monotone Conjecture holds. Then, for any distinct real numbers that are monotone with respect to* $(\sigma_1, \ldots, \sigma_m)$*, there exists a number* $\epsilon > 0$ *such that, if for each* $i = 1, \ldots, m$*,* $F_\bullet^i$ *is a flag secant to* γ *along an interval of length* ϵ *containing* t_i*, then the intersection*

$$X_{\sigma_1} F_\bullet^1 \cap X_{\sigma_2} F_\bullet^2 \cap \cdots \cap X_{\sigma_m} F_\bullet^m$$

is transverse with all points real.

4.2. **Experimental evidence for the Monotone Secant Conjecture.** While its relation to existing conjectures led to positing the Monotone Secant Conjecture, we believe the immense weight of empirical evidence is the strongest support for it. Our ongoing experiment is testing this conjecture and related notions for many computable Schubert problems. As of 4 February 2011, we have solved 4,090,490,116 instances of 775 Schubert problems. About 4.5% of these (176,809,563) were instances of the Monotone Secant Conjecture, and in every case, it was verified by symbolic computation. Other computations tested the Monotone conjecture for comparison. The remaining instances involved disjoint secant flags, but with the monotonicity condition violated.

Table 1 shows the data we obtained for the Schubert problem (σ^4,τ^4) with 12 solutions on the Flag manifold $\mathbb{F}\ell(2,3;5)$ introduced in Conjecture 1.1. We computed 8,000,000 instances

Real Solutions

Necklace \	0	2	4	6	8	10	12	Total
22223333							1000000	1000000
22233233			6	68210	181738	415395	334651	1000000
22322333			70	134436	357068	322668	185758	1000000
22332233			147	267567	399979	216682	115625	1000000
22323323		354	23116	100299	313296	374515	188420	1000000
22323233		11148	316401	419371	186548	54634	11898	1000000
22232333		31172	95108	153468	336276	249805	134171	1000000
23232323	295403	284925	276937	99691	34520	7807	717	1000000
Total	295403	327599	711785	1243042	1809425	1641506	1971240	8000000

TABLE 1. Necklaces vs. real solutions for (σ^4,τ^4) in $\mathbb{F}\ell(2,3;5)$.

of this problem, all involving flags that were secant to the rational normal curve along disjoint intervals. This took 15.058 gigahertz-years. The columns are indexed by even integers numbers from 0 to 12, indicating the number of real solutions. The rows are indexed by *necklaces*, which are sequences $\{\delta(\sigma_1),\dots,\delta(\sigma_m)\}$, where $\delta(\sigma_i)$ denotes the unique descent of the Grassmannian condition σ_i, as described in Section 3. Therefore, in our example, a 2 represents the condition on the two-plane E_2 given by the permutation $\sigma = (1\,3\,2\,4\,5)$, and similarly a 3 represents the condition on E_3 given by the permutation $\tau = (1\,2\,4\,3\,5)$.

In Table 1, the first row labeled with 22223333 represents tests of the Monotone Secant conjecture, since the only entries are in the column for 12 real solutions, the Monotone Secant conjecture was verified in 1,000,000 instances. This is the only row with only real solutions.

Compare this to the 8,000,000 instances of the same Schubert problem, but with osculating flags. These data are presented in Table 2. This computation took 67.460 gigahertz-days. Both tables are similar with nearly identical "inner borders", except for the shaded box in Table 2.

4.3. **Lower bounds and inner borders.** Probably the most enigmatic phenomenon that we observe in our data is the presence of an "inner border" for may geometric problems, as we have pointed out in example of Table 1. That is, for some necklaces (besides the

Real Solutions

Necklace \	0	2	4	6	8	10	12	Total
22223333							1000000	1000000
22233233			514	123534	290754	291572	293626	1000000
22322333			765	132416	310881	291640	264298	1000000
22332233			4467	108818	430805	251237	204673	1000000
22323323			59935	201234	333260	274979	130592	1000000
22323233		3697	127290	215573	332693	210303	110444	1000000
22232333		12857	68514	113824	207927	212245	384633	1000000
23232323	24493	62798	279201	198460	258211	121806	55031	1000000
Total	24493	79352	540686	1093859	2164531	1653782	2443297	8000000

TABLE 2. Necklaces vs. real solutions for (σ^4, τ^4) in $\mathbb{F}\ell(2,3;5)$.

monotone ones), there appears to be a lower bound on the number of real solutions. We do not understand this phenomenon, even conjecturally.

Another common phenomenon is that for many problems, there are always at least some solutions real, for any necklace. (Note that the last rows of Tables 1 and 2 had instanecs with no real solutions). Table 3 displays an example of this for a Schubert problem on $\mathbb{F}\ell(2,3;6)$ involving three conditions $W := (1\,3\,2\,4\,5\,6)$ and five involving $X := (1\,2\,4\,3\,5\,6)$ that has 21

Real Solutions

Necklace \	1	3	5	7	9	11	13	15	17	19	21	Total
WWWXXXXX											80000	80000
WWXWXXXX							921	16549	26267	14475	21788	80000
WWXXWXXX						39	1208	24559	39013	13947	1234	80000
WXWXXWXX						612	9544	43256	23583	2927	78	80000
WXWXWXXX						3244	19887	31931	13688	3632	7618	80000
Total						3895	31560	116295	102551	34981	110718	400000

TABLE 3. Enumerative Problem $W^3X^5 = 21$ on $\mathbb{F}\ell(2,3;6)$

solutions. Very prominently, it appears that at least 11 of the solutions will always be real.

These lower bounds and inner borders were observed in the computations studying the Monotone Conjecture [15]. Eremenko and Gabrielov established lower bounds for the Wronski map [4] in Schubert calculus for the Grassmannian, and more recently, Azar and Gabrielov [1] established lower bounds for some instances of the Monotone Conjecture which were observed in [15].

5. Method

Our experimentation was possible as instances of Schubert problems are simple to model on a computer. The procedure we use may be semi-automated and run on supercomputers. We will not describe how this automation is done, for that is the subject of the paper [10]; instead, we explain here the computations we are performing.

For a Schubert condition σ_i, a fixed flag instantiating σ_i is secant to the rational normal curve γ along some t_i points. Thus, for a Schubert problem $(\sigma_1, \ldots, \sigma_m)$, we need $t := t_1 + \cdots + t_m$ points of γ for the given Schubert problem. We construct secant flags by choosing t points on γ. With the flags selected, we formulate the Schubert problem as a system of equations by a choice of local coordinates, whose common zeroes represent the solutions to the Schubert problem in the local coordinates. This was illustrated in the Introduction when Conjecture 1.1 was presented. We direct the reader to [7, 15, 16] for details. We then eliminate all but one variable from the equations, obtaining an *eliminant*. If the eliminant is square-free and has degree equals to the expected number of complex solutions (this is easily verified;) then, the Shape Lemma concludes that the number of real roots of the eliminant equals the number of real solutions to the Schubert problem.

Instead of solving the Schubert problem, we may determine its number of real solutions in specific instances with software tools using implementations of Sturm sequences. If the software is reliably implemented, which we believe, then this computation provides a proof that the given instance has the computed number of real solutions to the original Schubert problem.

For each Schubert problem, we perform these steps thousands to millions of times, starting by selecting t points in γ and constructing flags secant to the rational normal curve and distributed with respect to a necklace. For every Schubert problem we have a list of necklaces with the first always being the necklace that creates an instance of the Monotone Secant conjecture, and the rest being necklaces where the monotonicity is broken.

Our experiment is not only testing the Monotone Secant Conjecture, but is also testing the Monotone Conjecture by just taking a point of tangency to γ instead of an interval of secancy. With this, we are not only extending those computations made by Ruffo, et al. [15], but also comparing the results with those from the Montone Secant Conjecture in order to understand both conjectures deeply.

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